ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 4, Pages 91–100 (November 23, 1998)
S 1079-6762(98)00053-5

CROFTON FORMULAS IN PROJECTIVE FINSLER SPACES

J. C. ÁLVAREZ PAIVA AND E. FERNANDES

(Communicated by Dmitri Burago)

Abstract. We extend the classical Crofton formulas in Euclidean integral geometry to Finsler metrics on $\mathbb{R}^n$ whose geodesics are straight lines.

1. Introduction

Hilbert's fourth problem asks to construct and study all metric structures on convex subsets of $\mathbb{R}P^n$ such that the straight line segment is the shortest curve joining two points. Hilbert's motivation is given in his presentation of the problem (see [10]): his close friend Minkowski had defined what we now call finite-dimensional Banach spaces and Hilbert himself had modified the Cayley-Klein construction of hyperbolic geometry to yield a family of metric spaces where the straight lines are geodesics. Moreover, Hilbert believed that the study of metric spaces where the geodesics are straight lines "will throw a new light upon the idea of distance, as well as upon other elementary ideas."

Busemann, who called the metrics appearing in Hilbert's fourth problem *projective metrics*, proposed an integral-geometric construction which is inspiringly simple. Here is his construction of projective metrics on $\mathbb{R}^n$ (see [6]):

Let Φ be a smooth (possibly signed) measure on the space of hyperplanes such that if x, y, and z are three noncollinear points, then the measure of the set of hyperplanes intersecting the wedge formed by the segments xy and yz is strictly positive. The Φ-distance between two points is defined as one-half the measure of all hyperplanes intersecting the line segment joining them.

Explicit formulas can be given if the space of cooriented hyperplanes in $\mathbb{R}^n$ is identified with the manifold $S^{n-1} \times \mathbb{R}$ by assigning to each pair $(\xi, r) \in S^{n-1} \times \mathbb{R}$ the hyperplane $\{x \in \mathbb{R}^n : r = \xi \cdot x\}$ cooriented by ξ. The measure on $S^{n-1} \times \mathbb{R}$ which descends to the measure Φ on the space of hyperplanes without coorientation can be written as $\nu|\Omega \wedge dr|$, where ν is a smooth function and Ω is the standard volume form on S^{n-1}. If we define the function $L : T\mathbb{R}^n \to \mathbb{R}$ by the formula

$$L(x, v) = \frac{1}{4} \int_{\xi \in S^{n-1}} |\xi \cdot v| \nu(\xi, \xi \cdot x)\Omega,$$

then the Φ-distance between any two points x and y is given as the infimum of the numbers $\int L(\dot{\gamma}(t))dt$, where γ ranges over all smooth curves joining x and y.

Received by the editors August 08, 1998.
1991 *Mathematics Subject Classification.* Primary 53C65; Secondary 53C60.
Key words and phrases. Crofton formulas, Hilbert's fourth problem, Finsler geometry.
Partially supported by a *credit aux chercheurs* from the FNRS.

Example. Taking $\nu : S^1 \times \mathbb{R} \to \mathbb{R}$ to be the function $\nu(\theta, r) := 1 + r^2$, we obtain

$$L(x_1, x_2, v_1, v_2) = \frac{1}{3\sqrt{v_1^2 + v_2^2}} \cdot (2x_1x_2v_1v_2 + (3 + 2x_1^2 + x_2^2)v_1^2 + (3 + 2x_2^2 + x_1^2)v_2^2).$$

Remarkably, for any smooth projective metric there exists a smooth (possibly signed) measure on the space of hyperplanes such that the length of any line segment equals the measure of the set of hyperplanes intersecting it (see [12, 14] and [3]).

Do there exist measures Φ_{n-k}, $1 \leq k \leq n-1$, on the space of $(n-k)$-flats such that the volume of any domain contained in a k-flat equals the measure of all $(n-k)$-flats intersecting it?

In the case of finite-dimensional Banach spaces this question is due to Busemann (see [7]). To make sense of it we need to define the volume of a k-dimensional submanifold in a projective metric space. The definition will depend solely on the fact that the function L is a *Finsler metric* on $\mathbb{R}^n$ (see [3]):

Definition 1.1. Let M be a manifold and let $TM\backslash 0$ denote its tangent bundle with the zero section deleted. A Finsler metric on M is a smooth positive function

$$L : TM\backslash 0 \to \mathbb{R}$$

with the following properties :

- $L(tv_m) = |t|L(v_m)$, for any real number t and any nonzero vector $v_m \in T_mM$.
- For each $m \in M$, the set $\{v_m \in T_mM \,|\, L(v_m) = 1\} \subset T_mM$ is a smooth quadratically convex hypersurface.

Note that the restriction of a Finsler metric to the tangent space of any submanifold defines a Finsler metric on the submanifold.

In order to define the volume of a Finsler manifold (M, L), let $D^*_mM \subset T^*_mM$ be the dual to the convex set $\{v_m \in T_mM : L(v_m) < 1\} \subset T_mM$ and define the *unit co-disc bundle*, $D^*M \subset T^*M$, as the the union of all the D^*_mM, $m \in M$.

Definition 1.2. The *volume* of an n-dimensional Finsler manifold (M, L) is the symplectic volume of its unit co-disc bundle divided by the volume of the Euclidean n-dimensional unit ball. The k-volume of a k-dimensional submanifold is the volume of the submanifold with its induced Finsler metric.

We are now in a position to announce our main result:

Theorem 1.1 (Crofton formulas for projective Finsler spaces)**.** *Let L be a Finsler metric on $\mathbb{R}^n$ whose geodesics are straight lines and let k, $1 \leq k \leq n-1$, be a natural number. There exists a smooth (possibly signed) measure Φ_{n-k} on the manifold $H_{n,n-k}$ of $(n-k)$-flats such that if $N \subset \mathbb{R}^n$ is an immersed k-dimensional submanifold, then*

$$vol_k(N) = \frac{1}{\epsilon_k} \int_{\lambda \in H_{n,n-k}} \#(N \cap \lambda)\Phi_{n-k}, \tag{1}$$

where ϵ_k is the volume of the Euclidean unit ball of dimension k.

To construct the measures Φ_{n-k}, $1 \leq k < n-1$, from the measure Φ_{n-1}, consider the fibration

$$(\underbrace{H_{n,n-1} \times \cdots \times H_{n,n-1}}_{k \text{ times}})\backslash\Delta \xrightarrow{\pi} H_{n,n-k},$$

where

$$\Delta = \{(\lambda_1, \ldots, \lambda_{n-k}) \in (H_{n,n-1})^k : \dim(\lambda_1 \cap \cdots \cap \lambda_{n-k}) > k\},$$

and

$$\pi(\lambda_1, \ldots, \lambda_{n-k}) := \lambda_1 \cap \cdots \cap \lambda_{n-k}.$$

We define Φ_{n-k} as the pushforward of the product measure on $(H_{n,n-1})^k \setminus \Delta$.

This definition of the measures Φ_{n-k}, together with the Crofton formulas and an easy application of Fubini's theorem, yields the following result:

Theorem 1.2. *Let L be a Finsler metric on $\mathbb{R}^n$ whose geodesics are straight lines and let K be a smooth compact convex hypersurface in $\mathbb{R}^n$. For any $\lambda \in H_{n,n-k}$, $1 \le k \le n-1$, let $vol_{(n-k-1)}(K \cap \lambda)$ denote the $(n-k-1)$-volume of $K \cap \lambda$. If Φ_{n-k} is the volume density of $H_{n,n-k}$ appearing in the Crofton formula, then there exists a constant c, independent of K, such that*

$$vol_{(n-1)}(K) = c \cdot \int_{\lambda \in H_{n,n-k}} vol_{(n-k-1)}(K \cap \lambda)\Phi_{n-k}. \tag{2}$$

The Crofton formula (1) was proved in the case of hypermetric, finite-dimensional Banach spaces by Schneider and Wieacker (see [13]). This is precisely the case where the associated measure on the space of hyperplanes is nonnegative. It is likely that the proof in [13] can be extended to cover all finite-dimensional Banach spaces. We remark that our proof, which involves the symplectic structure on the space of geodesics of a projective Finsler space, only covers those Banach spaces whose unit spheres are smooth and quadratically convex.

Remarks. 1. The volume of a Finsler manifold given here is based on the Holmes-Thompson volume of submanifolds of finite-dimensional Banach spaces (see [11] and [15]).

2. Trivial modifications of Theorems 1.1 and 1.2 hold for projective metrics on any open convex subset of $\mathbb{R}P^n$, including $\mathbb{R}P^n$ itself. In this announcement we have restricted our considerations to $\mathbb{R}^n$ in order to give shorter and clearer statements.

The first author is happy to acknowledge many fruitful discussions with I. M. Gelfand.

2. Crofton formula for double fibrations

In [9] Gelfand and Smirnov gave a first step in establishing the relation between the two main currents in integral geometry: the integral geometry of the Blaschke school and that of the Gelfand school. The main result of the present section can be taken as a point of departure in the simplification and extension of their work.

Let us begin by recalling some preliminary notions: double fibrations and densities.

Definition 2.1. A double fibration is a diagram of manifolds

$$\begin{array}{ccccc} & & A & & \\ & \overset{\pi_1}{\swarrow} & & \overset{\pi_2}{\searrow} & \\ B & & & & \Gamma \end{array}$$

with the following properties:

- $\pi_1 : A \to B$, $\pi_2 : A \to \Gamma$ are fiber bundles.
- The map $\pi_1 \times \pi_2 : A \to B \times \Gamma$ is a smooth embedding.
- For each $b \in B$ and for each $\gamma \in \Gamma$, the sets $B_\gamma := \pi_1(\pi_2^{-1}(\gamma)) \subset B$ and $\Gamma_b := \pi_2(\pi_1^{-1}(b)) \subset \Gamma$ are smooth submanifolds.

Definition 2.2. A k-*density* φ on a smooth manifold M is a real-valued function of a point $x \in N$ and of k linearly independent vectors $v = (v_1, \dots, v_k)$ in T_xM such that, if $v' = (v'_1, \dots, v'_k)$ is another set of r linearly independent vectors which generates the same k-dimensional subspace of T_xM, i.e., $v' = Av$ with $A \in GL(k, \mathbb{R})$, then

$$\varphi(x; v') = |\det A| \varphi(x; v).$$

The absolute value of a differential form of degree k is a k-density. A Finsler metric is a 1-density.

Because densities change as the absolute value of the determinant, they can be integrated over unoriented and nonorientable manifolds. Moreover, they can be pulled back under smooth maps in exactly the same way as differential forms.

To any double fibration $B \xleftarrow{\pi_1} A \xrightarrow{\pi_2} \Gamma$ one associates a number of integral transforms which send functions, differential forms, or densities on Γ to similar objects on B. The definition of the transforms is surprisingly simple:

Take a density (or form) Φ on Γ, pull it back to A and push it forward to B by fiber integration.

Remark that the fiber integration of a form or density is not always defined. In the rest of the paper we shall make the implicit assumption that the fibers of $\pi_1 : A \to B$ are compact and, when working with forms, that the fibers are oriented.

Proposition-Definition 2.1. *Let $B \xleftarrow{\pi_1} A \xrightarrow{\pi_2} \Gamma$ be a double fibration. Let k be the dimension of the fibers of $\pi_1 : A \to B$ and let Φ be an m-density (resp. m-form) on Γ with $m \geq k$. The Gelfand transform of Φ is the $(m-k)$-density (resp. $(m-k)$-form) $\pi_{1*}\pi_2^*\Phi$.*

Classical integral-geometric transforms such as those of Radon, John, and Funk are particular cases of this construction.

We now describe a second construction involving double fibrations and densities.

Let $B \xleftarrow{\pi_1} A \xrightarrow{\pi_2} \Gamma$ be a double fibration with $\dim(B) = n$ and $\dim(B_\gamma) = n-k$. If Φ is a top-order density on Γ, we define the following functional on the space of k-dimensional submanifolds of B:

$$S_\Phi(N) := \int_{\gamma \in \Gamma} \#(B_\gamma \cap N)\Phi.$$

These functionals were first considered by Busemann (see [7] and [8]) in the particular case of the double fibration associated to $(n-k)$-flats in $\mathbb{R}^n$. Also in this particular case, Gelfand and Smirnov proved that for any top-order density Φ on the space of $(n-k)$-flats there exists a k-density φ on $\mathbb{R}^n$ such that

$$\int_{\lambda \in H_{n,n-k}} \#(\lambda \cap N)\Phi = \int_N \varphi.$$

Moreover, they showed through explicit formulas that the assignment $\Phi \mapsto \varphi$ is an integral-geometric transform in the sense of Radon, John, and Gelfand. Underlying these formulas is the following general, and apparently new, result:

Theorem 2.1 (Crofton formula for double fibrations)**.** *Let* $B \xleftarrow{\pi_1} A \xrightarrow{\pi_2} \Gamma$ *be a double fibration with* $\dim(B) = n$ *and* $\dim(B_\gamma) = n - k$*. If* Φ *is a top-order density on* Γ *and* $N \subset B$ *is a* k*-dimensional submanifold, then*

$$S_\Phi(N) = \int_\Gamma \#(N \cap B_\gamma)\Phi = \int_N \pi_{1*}\pi_2^*\Phi. \tag{3}$$

Functorial properties of the Gelfand transform play an important role in the proof of the main theorem.

Definition 2.3. A morphism between two double fibrations $B \xleftarrow{\pi_1} A \xrightarrow{\pi_2} \Gamma$ and $B' \xleftarrow{\pi_1'} A' \xrightarrow{\pi_2'} \Gamma'$ is a commutative diagram of fibrations

$$\begin{array}{ccccc} B & \xleftarrow{\pi_1} & A & \xrightarrow{\pi_2} & \Gamma \\ \downarrow{\scriptstyle \rho_B} & & \downarrow{\scriptstyle \rho_A} & & \downarrow{\scriptstyle \rho_\Gamma} \\ B' & \xleftarrow{\pi_1'} & A' & \xrightarrow{\pi_2'} & \Gamma' \end{array}$$

Theorem 2.2. *Let*

$$\begin{array}{ccccc} B & \xleftarrow{\pi_1} & A & \xrightarrow{\pi_2} & \Gamma \\ \downarrow{\scriptstyle \rho_B} & & \downarrow{\scriptstyle \rho_A} & & \downarrow{\scriptstyle \rho_\Gamma} \\ B' & \xleftarrow{\pi_1'} & A' & \xrightarrow{\pi_2'} & \Gamma' \end{array}$$

be a morphism of double fibrations such that for every point $a' \in A'$ *the map* π_2 *restricted to* $\rho_A^{-1}(a')$ *is a diffeomorphism onto* $\rho_\Gamma^{-1}(\pi_2'(a'))$*. If* Φ *is a density or differential form on* Γ*, then*

$$\rho_{B*}\pi_{1*}\pi_2^*\Phi = \pi_{1*}'(\pi_2')^*\rho_{\Gamma*}\Phi.$$

Theorem 2.3. *Let* $B \xleftarrow{\pi_1^1} A_1 \xrightarrow{\pi_2^1} \Gamma_1, B \xleftarrow{\pi_1^2} A_2 \xrightarrow{\pi_2^2} \Gamma_2$*, and* $B \xleftarrow{\pi_1^3} A_3 \xrightarrow{\pi_2^3} \Gamma_3$ *be double fibrations and let* $\rho : \Gamma_1 \times \Gamma_2 \to \Gamma_3$ *be a fibration satisfying the following condition:*

The points $\gamma_1 \in \Gamma_1$ *and* $\gamma_2 \in \Gamma_2$ *are incident to* $b \in B$ *if and only if* $\rho(\gamma_1, \gamma_2) \in \Gamma_3$ *is incident to* b*.*

If Ω_1 *and* Ω_2 *are top-order forms on* Γ_1 *and* Γ_2 *and if* $p_1 : \Gamma_1 \times \Gamma_2 \to \Gamma_1$*,* $p_2 : \Gamma_1 \times \Gamma_2 \to \Gamma_2$ *are the canonical projections, then*

$$(\pi_1^3)_*(\pi_2^3)^*\rho_*(p_1^*\Omega_1 \wedge p_2^*\Omega_2) = (\pi_1^1)_*(\pi_2^1)^*\Omega_1 \wedge (\pi_1^2)_*(\pi_2^2)^*\Omega_2.$$

3. Integral geometry and symplectic geometry

Let (M, L) be a Finsler manifold such that its space of oriented geodesics is a manifold $G(M)$. Let S^*M denote the *unit co-sphere bundle* of M and let $\pi : S^*M \to G(M)$ be the canonical projection which sends a given unit covector to the geodesic which has this covector as initial condition.

Proposition-Definition 3.1 ([4] and [5])**.** *Let* (M, L) *be a Finsler manifold whose space of oriented geodesics,* $G(M)$*, is a smooth manifold and let*

$$\begin{array}{ccc} S^*M & \xrightarrow{i} & T^*M \\ \downarrow{\scriptstyle \pi} & & \\ G(M) & & \end{array}$$

be the canonical projection onto $G(M)$ and the canonical inclusion into T^*M. If ω_0 is the standard symplectic form on T^*M, then there is a unique symplectic form ω on $G(M)$ which satisfies the equation $\pi^*\omega = i^*\omega_0$.

The case $k = n-1$, that of the volume of hypersurfaces, in Theorem 1.1 follows from a more general result:

Theorem 3.1 ([1]). *Let M be an n-dimensional Finsler manifold whose space of oriented geodesics, $G(M)$, is a smooth manifold. If $N \subset M$ is an immersed hypersurface and if ω^{n-1} denotes the Liouville volume form on $G(M)$, then*

$$vol(N) = \frac{1}{2\epsilon_{n-1}} \cdot \int_{\gamma \in G(M)} \#(\gamma \cap N)|\omega^{n-1}|,$$

where ϵ_{n-1} is the volume of the Euclidean unit ball of dimension $n-1$.

The symplectic structure on the space of oriented straight lines, $H^+_{n,1}$, induced by a projective Finsler metric on $\mathbb{R}^n$ has a simple characterization:

Theorem 3.2 ([2]). *Let L be a projective Finsler metric on $\mathbb{R}^n$ and let ω be the symplectic form on the space of oriented lines induced by L. The 2-form ω is characterized, up to sign, by the following properties:*

- *The form ω is closed.*
- *The form ω is odd: if $a : H^+_{n,1} \to H^+_{n,1}$ denotes the involution that changes the orientation of the lines, then $a^*\omega = -\omega$.*
- *For any point $x \in \mathbb{R}^n$, the pull-back of ω to the submanifold of all oriented lines passing through x is identically zero.*
- *If x and y are two distinct points in $\mathbb{R}^n$ and Π is any 2-flat containing them, then the integral of $|\omega|$ over the set of all oriented lines lying on Π and intersecting the segment xy equals four times the distance between x and y.*

Recall from the introduction that if L is a projective Finsler metric on $\mathbb{R}^n$, then it is given by the formula

$$L(x, v) = \frac{1}{4} \int_{\xi \in S^{n-1}} |\xi \cdot v| \nu(\xi, \xi \cdot x)\Omega,$$

for some smooth function ν on $S^{n-1} \times \mathbb{R}$, which we identify with the space of oriented hyperplanes $H^+_{n,n-1}$.

In order to write the symplectic form ω in terms on ν, we consider the form $\Omega_{n-1} := \nu\Omega \wedge dr$ and the double fibration

$$\begin{array}{ccc} & A & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ H^+_{n,1} & & H^+_{n,n-1} \end{array}$$

where A is the set $\{(l, \lambda) \in H^+_{n,1} \times H^+_{n,n-1} : l \subset \lambda\}$. We remark that the fibers of this double fibration inherit a natural orientation from the orientation of $\mathbb{R}^n$ and those of the subspaces involved and that it makes sense to compute the form $\pi_{1*}\pi_2^*\Omega_{n-1}$. It follows from Theorem 3.2 that $\pi_{1*}\pi_2^*\Omega_{n-1} = \omega$.

Theorem 3.3. *The Gelfand transform of $\Omega_{n-1} := \nu\Omega \wedge dr$ associated to the double fibration $H^+_{n,1} \xleftarrow{\pi_1} A \xrightarrow{\pi_2} H^+_{n,n-1}$ equals the symplectic form induced by the Finsler metric L.*

We shall now use the form Ω_{n-1} to construct top-order forms Ω_{n-k} on the spaces of oriented $(n-k)$-flats, $H^+_{n,n-k}$. To do this consider the fibration

$$(\underbrace{H^+_{n,n-1} \times \cdots \times H^+_{n,n-1}}_{k \text{ times}}) \backslash \Delta \xrightarrow{\pi} H^+_{n,n-k},$$

where

$$\Delta = \{(\lambda_1, \ldots, \lambda_{n-k}) \in H_{n,n-1} \times \cdots \times H_{n,n-1} \mid \dim(\lambda_1 \cap \cdots \cap \lambda_{n-k}) > k\},$$

and

$$\pi(\lambda_1, \ldots, \lambda_{n-k}) := \lambda_1 \cap \cdots \cap \lambda_{n-k}.$$

The orientation of the intersection is given by that of $\mathbb{R}^n$ and those of the hyperplanes $\lambda_i, 1 \leq i \leq k$. Since this fibration is oriented, we may define the push-forward of forms and set $\Omega_{n-k} := \pi_* \Omega^k_{n-1}$.

Let us also consider the double fibration

$$\begin{array}{ccc} & A & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ H^+_{n,1} & & H^+_{n,n-k} \end{array}$$

where $H^+_{n,n-k}$ denotes the space of oriented $(n-k)$-flats and A is the set $\{(l, \zeta) \in H^+_{n,1} \times H^+_{n,n-k} : l \subset \zeta\}$. Like the case $k = 1$, the fibers are oriented and it makes sense to compute the form $\pi_{1*}\pi_2^*\Omega_{n-k}$.

The previous theorem together with the functorial properties of the Gelfand transform imply the following

Theorem 3.4. *The Gelfand transform of Ω_{n-k} associated to the double fibration $H^+_{n,1} \xleftarrow{\pi_1} A \xrightarrow{\pi_2} H^+_{n,n-k}$ equals the $2k$-form ω^k. In particular, $\Omega_1 = \omega^{n-1}$.*

Let us clarify the relation between the forms Ω_{n-k} and the measures Φ_{n-k} defined in the introduction.

Note that if M is an oriented manifold of dimension n, there is a simple correspondence between differential n-forms and n-densities: if Ψ is an n-form on M, we define an n-density Φ also on M by letting $\Phi(x, v_1, \ldots, v_n) = \Psi_x(v_1, \ldots, v_n)$ if $v_1, \ldots, v_n$ is positively oriented and letting $\Phi(x, v_1, \ldots, v_n) = -\Psi_x(v_1, \ldots, v_n)$ if not. This construction coincides with sending a top-order form to its absolute value only if the form never changes sign.

It is easy to see that if we apply this procedure to the form Ω_{n-k} on the oriented manifold $H^+_{n,n-k}$, we will obtain a top-order density which descends to Φ_{n-k} on the space of unoriented $(n-k)$-flats.

This section ends with one last construction relating the forms Ω_{n-k} to the powers of the symplectic form ω.

Let Λ be an oriented $(k+1)$-flat, let $H^+_1(\Lambda)$ be the set of oriented lines lying on Λ, and let Δ_Λ be the set of oriented $(n-k)$-flats which do not intersect Λ transversely. We define a fibration $p : H^+_{n,n-k} \setminus \Delta_\Lambda \to H^+_1(\Lambda)$ by assigning to each $\zeta \in H^+_{n,n-k} \setminus \Delta_\Lambda$ the line $\zeta \cap \Lambda$. The orientation of this line is determined as follows: if $\Pi \subset \Lambda$ is any oriented k-flat, then the line $\zeta \cap \Lambda$ is oriented so that its intersection number with Π in Λ coincides with the intersection number of ζ with Π in $\mathbb{R}^n$.

Since the fibration p is oriented, it makes sense to compute $p_*\Omega_{n-k}$. The following result follows easily from the last theorem.

Proposition 3.1. *The form $p_*\Omega_{n-k}$ equals the pull back of the form ω^k to the set of all oriented lines contained in Λ.*

If we regard Λ as a Finsler space with the Finsler metric it inherits from the projective Finsler metric on $\mathbb{R}^n$, then Λ is a projective Finsler space on its own right. The symplectic form on its space of geodesics, $H_1^+(\Lambda)$, coincides with the pull back of ω to $H_1^+(\Lambda)$ by its embedding into $H_{n,1}^+$. The form ω^k appearing in the proposition is then nothing more than the Liouville form on the space of geodesics on Λ.

4. Proof of the main theorem

We start by remarking that to prove the Crofton formula it suffices to show that if D is a domain in a k-flat, then

$$vol(D) = \frac{1}{\epsilon_k}\int_{\zeta\cap D\neq\emptyset}\Phi_{n-k}.$$

Because of the results in the previous section, it is more convenient to work with the forms Ω_{n-k} than with the densities Φ_{n-k}. Note that if D is a domain in an oriented k-flat Π, then the integral of Φ_{n-k} over all $(n-k)$-flats intersecting D equals the absolute value of the integral of Ω_{n-k} over all oriented $(n-k)$-flats which intersect D and whose intersection number with Π is positive. It suffices then to show that

$$vol(D) = \frac{1}{\epsilon_k}\left|\int_{\zeta\cap D>0}\Omega_{n-k}\right|.$$

Let us begin with the simplest case.

Lemma 4.1. *If $D\subset\mathbb{R}^n$ is a domain in an oriented $(n-1)$-flat, then*

$$vol(D) = \frac{1}{\epsilon_{n-1}}\left|\int_{l\cap D>0}\Omega_1\right|,$$

where ϵ_{n-1} is the volume of the Euclidean unit ball of dimension $n-1$.

Proof. By Theorem 3.1 the volume of N equals

$$\frac{1}{2\epsilon_{n-1}}\int_{l\cap D>0}|\omega^{n-1}|,$$

which in turn equals ϵ_{n-1}^{-1} times the absolute value of the integral of ω^{n-1} over the set of all oriented lines which intersect D positively. Since according to Theorem 3.4 $\Omega_1 = \omega^{n-1}$, this finishes the proof. □

The idea of the rest of the proof is to reduce everything to this simplest case by the following procedure:

Let D be a domain in an oriented k-flat and let Λ be an oriented $(k+1)$-flat containing it. Note that Λ inherits a projective Finsler metric from that of $\mathbb{R}^n$ and that D is a hypersurface in Λ. Theorem 3.1 then tells us that

$$vol(D) = \frac{1}{\epsilon_k}\left|\int_{l\cap D>0}\omega^k\right|.$$

To go from an integral over oriented lines in Λ to an integral over oriented $(n-k)$-flats in $\mathbb{R}^n$ we make use of the last construction of the previous section.

If $p : H^+_{n,n-k} \setminus \Delta_\Lambda \to H^+_1(\Lambda)$ is the fibration in Section 3, then by Proposition 3.1

$$\int_{l \cap D > 0} \omega^k = \int_{l \cap D > 0} p_* \Omega_{n-k}.$$

Since the set of oriented lines in Λ which intersect D positively is exactly the image under p of the set of all oriented $(n-k)$-flats which intersect D positively, we have that

$$\int_{l \cap D > 0} p_* \Omega_{n-k} = \int_{\zeta \cap D > 0} \Omega_{n-k}.$$

To summarize:

$$vol(D) = \frac{1}{\epsilon_k} \left| \int_{l \cap D > 0} \omega^k \right| = \frac{1}{\epsilon_k} \left| \int_{l \cap D > 0} p_* \Omega_{n-k} \right| = \frac{1}{\epsilon_k} \left| \int_{\zeta \cap D > 0} \Omega_{n-k} \right|.$$

This finishes the proof of the Crofton formulas. □

References

[1] J.C. Álvarez Paiva, *Integral geometry on Finsler manifolds,* preprint 1998.

[2] J.C. Álvarez Paiva, *Anti-self-dual symplectic forms and integral geometry,* preprint 1998.

[3] J.C. Álvarez Paiva, I.M. Gelfand, and M. Smirnov, *Crofton densities, symplectic geometry, and Hilbert's fourth problem,* in "Arnold-Gelfand Mathematical Seminars, Geometry and Singularity Theory", V.I. Arnold, I.M. Gelfand, and V.S. Retakh (eds.), Birkhäuser, Boston, 1997, pp. 77–92. MR **98a:**52005

[4] V.I. Arnold and A.B. Givental, *Symplectic geometry,* in "Dynamical Systems IV", Encyclopaedia of Mathematical Sciences, Vol. 4, V.I. Arnold and S.P. Novikov (Eds.), Springer-Verlag, Berlin, Heidelberg, New York, 1990. MR **88b:**58044

[5] A. Besse, "Manifolds All of Whose Geodesics are Closed", Springer-Verlag, Berlin, Heidelberg, New York, 1978. MR **80c:**53044

[6] H. Busemann, *Problem IV: Desarguesian spaces,* in "Mathematical developments arising from Hilbert problems", Proceedings of Symposia in Pure Math., Vol. XXVII, Part 1, F. Browder (Ed.), AMS, Providence, Rhode Island, 1974. MR **55:**3940

[7] H. Busemann, *Areas in affine spaces III. The integral geometry of affine area,* Rend. Circ. Mat. Palermo (2) **9** (1960), 226–242. MR **25:**171

[8] H. Busemann, *Geometries in which the planes minimize area,* Ann. Mat. Pura Appl. (4) **55** (1961), 171–190. MR **26:**715

[9] I.M. Gelfand and M. Smirnov, *Lagrangians satisfying Crofton formulas, Radon transforms, and nonlocal differentials,* Adv. in Math. **109** (1994), 188–227. MR **96a:**53092

[10] D. Hilbert, *Mathematical problems,* translation in "Mathematical developments arising from Hilbert problems", Proceedings of Symposia in Pure Math., Vol. XXVII, Part 1, F. Browder (Ed.), AMS, Providence, Rhode Island, 1974. MR **54:**7158

[11] R.D. Holmes and A.C. Thompson, *N-dimensional area and content in Minkowski spaces,* Pacific J. Math. **85** (1979), 77–110. MR **81k:**52023

[12] A.V. Pogorelov, "Hilbert's Fourth Problem", Scripta Series in Mathematics, Winston and Sons, 1979. MR **80j:**53066

[13] R. Schneider and J.A. Wieacker, *Integral geometry in Minkowski spaces,* Adv. in Math. **129** (1997), 222–260. MR **98k:**52015

[14] Z.I. Szabo, *Hilbert's fourth problem, I,* Adv. in Math. **59** (1986), 185–301. MR **88f:**53113

[15] A.C. Thompson, "Minkowski Geometry", Encyclopedia of Math. and Its Applications, Vol. 63, Cambridge Univ. Press, Cambridge, 1996. MR **97f:**52001

Université Catholique de Louvain, Institut de Mathématique Pure et Appl., Chemin du Cyclotron 2, B-1348 Louvain-la-Neuve, Belgium
E-mail address: alvarez@agel.ucl.ac.be

Université Catholique de Louvain, Institut de Mathématique Pure et Appl., Chemin du Cyclotron 2, B-1348 Louvain-la-Neuve, Belgium
E-mail address: fernandes@agel.ucl.ac.be