ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 5, Pages 102–107 (July 14, 1999)
S 1079-6762(99)00067-0

A PIERI-CHEVALLEY FORMULA IN THE K-THEORY OF A G/B-BUNDLE

HARSH PITTIE AND ARUN RAM

(Communicated by Efim Zelmanov)

ABSTRACT. Let G be a semisimple complex Lie group, B a Borel subgroup, and $T \subseteq B$ a maximal torus of G. The projective variety G/B is a generalization of the classical flag variety. The structure sheaves of the Schubert subvarieties form a basis of the K-theory $K(G/B)$ and every character of T gives rise to a line bundle on G/B. This note gives a formula for the product of a dominant line bundle and a Schubert class in $K(G/B)$. This result generalizes a formula of Chevalley which computes an analogous product in cohomology. The new formula applies to the relative case, the K-theory of a G/B-bundle over a smooth base X, and is presented in this generality. In this setting the new formula is a generalization of recent $G = GL_n(\mathbb{C})$ results of Fulton and Lascoux.

Let G be a complex, semisimple, simply connected algebraic group and $B \subseteq G$ a Borel subgroup. We fix a smooth closed complex projective variety X and a principal algebraic B-bundle over it: $B \longrightarrow E \stackrel{\pi}{\longrightarrow} X$. For any complex algebraic variety F with a left algebraic B-action, we denote by $E(F)$ the total space of the associated fibre bundle with fibre F. Thus $E(F) = E \times_B F$ and the projection to X is obtained from projection on the first factor.

Fix a maximal torus $T \subseteq B$ and let W be its Weyl group. For each $w \in W$ the Bruhat cell $Y_w^\circ = BwB \subseteq G/B$ and the Schubert variety $Y_w = \overline{BwB} \subseteq G/B$ are B-stable subsets of G/B so we have inclusions of bundles $E(Y_w^\circ) \subseteq E(Y_w) \subseteq E(G/B)$. The closed subvarieties $\Omega_w = E(Y_w)$ determine classes $[\mathcal{O}_{\Omega_w}]$ in $K(E(G/B))$. [1] In fact, by a well-known result of Grothendieck, these classes form a $K(X)$-basis for $K(E(G/B))$. On $E(G/B))$ we also have "homogeneous" line bundles associated to irreducible representations of B (see below). The main result of this announcement is a formula for the tensor product of the class of a homogeneous line bundle with a Schubert class, expressed as a $K(X)$-linear combination of Schubert classes.

We believe that this formula is the most general uniform result in the intersection theory of Schubert classes: it is related to a recent result of Fulton and Lascoux [FL], who presented a similar formula for a $GL_n(\mathbb{C})/B$-bundle. Indeed, in this case, their formula and ours coincide once one knows how to translate between their combinatorics with tableaux and ours with Littelmann paths. O. Mathieu has also proved the positivity which is implied by our formula; see [FP, p. 101].

Received by the editors February 9, 1999 and, in revised form, April 29, 1999.
1991 *Mathematics Subject Classification.* Primary 14M15; Secondary 14C35, 19E08.
Research supported in part by National Science Foundation grant DMS-9622985.

[1] For any smooth variety V, $K(V)$ is the Grothendieck ring of coherent $\mathcal{O}_V$-modules.

Applying the Chern character to our formula, and equating the lowest order terms we obtain a relative version of the result of Chevalley [Ch] alluded to in the title of this paper.

The ring $K(E(G/B))$ is a $K(X)$-module via the map $\pi^*\colon K(X) \to K(E(G/B))$. Since G/B has a unique fixed point for the B-action, there is a canonical section $\sigma\colon X \to E(G/B)$ of the bundle $E(G/B)$. Consider the diagram

$$\begin{array}{ccccc} G & \to & E(G) & \to & X \\ \downarrow & & \downarrow \rho & & \| \\ G/B & \to & E(G/B) & \to & X \end{array}$$

where the vertical maps are quotients by the right action of B on G; precisely, $E(G/B) \simeq (E \times_B G)/B$. Thus ρ is the projection map of a principal B-bundle over $E(G/B)$.

There are two vector bundles naturally associated to each B-module V:

$$E(V) \longrightarrow X, \qquad \text{and} \qquad E_G(V) = E(G) \times_B V \longrightarrow E(G/B),$$

where the projection map for the latter of these is via ρ. This assignment $V \mapsto E_G(V)$ of B-modules to vector bundles over $E(G/B)$ preserves direct sums and tensor products, and hence induces a ring homomorphism $R(B) \xrightarrow{\phi} K(E(G/B))$, where $R(B)$ is the representation ring of B. By construction $\sigma^*(E_G(V)) = E(V)$ as vector bundles on X. One also checks that if V is the restriction of a G-module, then $E_G(V) = \pi^*(\sigma^*(E_G(V)))$. Thus we have a commutative diagram

$$\begin{array}{ccc} R(G) & - - \to & K(X) \\ \downarrow \text{res} & & \downarrow \pi^* \\ R(B) & \longrightarrow & K(E(G/B)) \end{array}$$

and a map

$$K(X) \otimes_{R(G)} R(B) \quad \xrightarrow{\pi^* \otimes \phi} \quad K(E(G/B)),$$

where $R(G)$ is the representation ring of G and the $R(G)$-action on $K(X)$ is given by the map $V \mapsto E(V)$.

Let P be the weight lattice of $\mathfrak{g} = \mathrm{Lie}(G)$. Then $R(B) = R(T) \cong \mathbb{Z}[P]$, the group algebra of P, and $R(G) = R(T)^W$. If $\lambda \in P$, let e^λ be the corresponding element of $R(T)$ and define

$$x^\lambda = E(e^\lambda) \in K(X) \qquad \text{and} \qquad y^\lambda = E_G(e^\lambda) \in K(E(G/B)). \tag{1}$$

The statement that $E_G(V) = \pi^*(\sigma^*(E_G(V)))$ if V is a G-module is equivalent to the statement that, in $K(E(G/B))$,

$$\chi(x) = E(\chi) \quad \text{is equal to} \quad \chi(y) = E_G(\chi), \quad \text{for all } \chi \in R(T)^W.$$

We recall from [P] that $R(T)$ is a free $R(G)$-module of rank $|W|$, and $R(T) \otimes_{R(G)} \mathbb{Z} \longrightarrow K(G/B)$ is an isomorphism.[2] According to Steinberg, [S] there is an $R(G)$-basis of $R(T)$ of the form $\{e^{\varepsilon_w} \mid w \in W\}$, where the ε_w are certain specific elements of P. Since the set $\{y^{\varepsilon_w} \mid w \in W\}$ is a set of globally defined elements in $K(E(G/B))$ which behaves properly under restriction, and which forms a basis

[2]The discussion in [P] is entirely in terms of compact groups and the K-theory of C^∞ vector bundles; with trivial modifications the results hold in the present context also.

locally, it follows from standard yoga that it is also a $K(X)$-basis for $K(E(G/B))$. Thus the map $K(X) \otimes_{R(G)} R(T) \longrightarrow K(E(G/B))$ is an isomorphism and

$$K(E(G/B)) \cong \frac{K(X) \otimes R(T)}{\mathcal{I}}, \tag{2}$$

where $\mathcal{I}$ is the ideal in $K(X) \otimes R(T)$ generated by the set $\{\chi(x) \otimes 1 - 1 \otimes \chi \mid \chi \in R(T)^W\}$.

Define a W-action on $K(X) \otimes R(T)$ as the $K(X)$-linear extension of the action given by

$$wy^\lambda = y^{w\lambda}, \qquad \text{for } w \in W,\, \lambda \in P.$$

This action descends to an action on $K(E(G/B))$, since the generators of the ideal $\mathcal{I}$ are W-invariants for this action. Using this W-action on $K(E(G/B))$, we can define the analogues of BGG-operators in this context. Such operators were defined in the "absolute case" ($X = \text{pt}$) by Demazure, in $K_T(G/B)$ by Kostant and Kumar [KK], and finally by Fulton and Lascoux [FL] when $G = SL(n, \mathbb{C})$. To make the definition, let α be a positive root with respect to the pair (B, T) and let $s_\alpha \in W$ be the corresponding reflection. Define $T_\alpha \colon R(T) \to R(T)$ by setting

$$T_\alpha(e^\lambda) = (e^{\lambda+\alpha} - s_\alpha(e^\lambda))/(e^\alpha - 1)$$

and extending $\mathbb{Z}$-linearly. Since T_α fixes elements of $R(T)^W$, this operation can be extended $K(X)$-linearly to a well-defined operator on $K(E(G/B))$.

Now fix a simple system of roots $\alpha_1, \ldots, \alpha_\ell$ for (B, T) and let P_j be the minimal parabolic subgroup corresponding to α_j; this is the closed connected subgroup of G whose Lie algebra $\mathfrak{p}_j$ is spanned by the Lie algebra $\mathfrak{b}$ of B and the root space $\mathfrak{g}_{-\alpha_j}$. Let $f_j : E(G/B) \longrightarrow E(G/P_j)$ be the projection induced from the B-equivariant $\mathbb{P}^1$-bundle $G/B \longrightarrow G/P_j$ (the canonical projection). The following result explains the geometric significance of the operators T_{α_j} (henceforth abbreviated as T_j). P. Deligne pointed out an error in the proof of (a) below in an earlier version of this preprint. We are grateful to him for pointing this out and have corrected the argument.

Proposition. *With the notation as above,*

(a) $(f_j)^! \circ (f_j)_!([\mathcal{O}_{\Omega_w}]) = \begin{cases} [\mathcal{O}_{\Omega_{ws_j}}] & \text{if } \ell(ws_j) > \ell(w), \\ [\mathcal{O}_{\Omega_w}] & \text{if } \ell(ws_j) < \ell(w). \end{cases}$

(b) *For any element* $x \in K(E(G/B))$, $\ (f_j)^! \circ (f_j)_!(x) = T_j(x)$.

Proof. (a) Let $\bar{w} = \{w, ws_j\}$ be the coset of w relative to $\langle s_j \rangle$. The essential point is to prove the following two equations:

$$f_![\mathcal{O}_{\Omega_v}] = [\mathcal{O}_{\Omega_{\bar{w}}}], \qquad \text{for } v \in \bar{w},$$

where $\Omega_{\bar{w}} \subseteq E(G/P_j)$ is the relative Schubert variety constructed from $Y_{\bar{w}} \subseteq E(G/P_j)$. In turn, these equations will follow from the isomorphisms

$$\text{(i)} \quad f_*(\mathcal{O}_{\Omega_v}) = \mathcal{O}_{\Omega_{\bar{w}}}, \qquad \text{(ii)} \quad R^q f_*(\mathcal{O}_{\Omega_v}) = 0, \quad \text{for } q > 0,\, v \in \bar{w}.$$

To prove (i) and (ii) relabel the elements of $\bar{w}$ as w' and w'' where $w' < w''$. Then $f \colon \Omega_{w'} \to \Omega_{\bar{w}}$ is birational, and since the varieties in question have at worst rational singularities, (i) and (ii) for w' follow from known arguments. Secondly, $f : \Omega_{w''} \to \Omega_{\bar{w}}$ is a $\mathbb{P}^1$-bundle, so (i) and (ii) are standard. Finally, $f^![\mathcal{O}_{\Omega_{\bar{w}}}] = [\mathcal{O}_{\Omega_{w''}}]$ follows because, in this case, f is the projection of a $\mathbb{P}^1$-bundle.

(b) There is a 2-dimensional algebraic vector bundle $E_j \longrightarrow E(G/P_j)$ associated to a 2-dimensional representation of P_j. Its projectivization is $E(G/B)$, i.e., $\mathbb{P}(E_j) \simeq E(G/B)$ as bundles over $E(G/P_j)$. It follows that $K(E(G/B))$ is a free module over $K(E(G/P_j))$ on two generators, 1 and L_{ω_j}, where ω_j is the j^{th} fundamental weight. Since both sides are $K(X)$-linear, it suffices to check the assertion for 1 and L_{ω_j}, and this reduces to the "same" computation as in the absolute case. □

The operators T_i, $1 \le i \le \ell$, satisfy $T_i^2 = T_i$ and the generalized braid relations. For each $w \in W$, let $w = s_{i_1} \cdots s_{i_p}$ be a reduced word for w and define $T_w = T_{i_1} \cdots T_{i_p}$. Since the T_i satisfy the braid relations, the operators T_w are well defined and, by the above Proposition,

$$[\mathcal{O}_{\Omega_w}] = T_{w^{-1}}[\mathcal{O}_{\Omega_1}], \qquad \text{for } w \in W. \tag{3}$$

For each $\lambda \in P$ let Y^λ be the "left multiplication" operator on $K(E(G/B))$ defined by $Y^\lambda(x) = y^\lambda x$. Since $[\mathcal{O}_{\Omega_1}] = \sigma(X)$,

$$Y^\lambda[\mathcal{O}_{\Omega_1}] = x^\lambda[\mathcal{O}_{\Omega_1}], \tag{4}$$

where x^λ is as in (1). As operators on $K(E(G/B))$,

$$Y^\lambda T_i = T_i Y^{s_i\lambda} + \frac{Y^\lambda - Y^{s_i\lambda}}{1 - Y^{-\alpha_i}}, \tag{5}$$

where the second term is always viewed as a linear combination of Y^μ, $\mu \in P$. We will iterate this formula to obtain an expansion of the product $e^\lambda[\mathcal{O}_{X_w}]$ in $K(G/B)$ in terms of the $K(X)$-basis $\{[\mathcal{O}_{X_v}] \mid v \in W\}$ of $K(E(G/B))$. The path model of P. Littelmann [Li] is exactly what is needed for controlling the resulting expansion.

Let $\mathfrak{h}^* = \mathbb{R} \otimes P$ be the real span of the weight lattice. A path in $\mathfrak{h}^*$ is a piecewise linear map $\pi\colon [0,1] \to \mathfrak{h}^*$ such that $\pi(0) = 0$. P. Littelmann [Li] defined *root operators* $f_1, \ldots, f_\ell$ which act on the paths. The action of a root operator f_i on a path π either produces another path or returns 0.

Let λ be a dominant integral weight and let W_λ be the stabilizer of λ. The cosets in W/W_λ are partially ordered by the Bruhat-Chevalley order. Let π_λ be the path given by

$$\pi_\lambda(t) = t\lambda, \;\; 0 \le t \le 1, \quad \text{and let} \quad \mathcal{T}^\lambda = \{f_{i_1} f_{i_2} \cdots f_{i_l} \pi_\lambda\}$$

be the set of all paths obtained by applying sequences of root operators $f_i = f_{\alpha_i}$, $1 \le i \le \ell$ to π_λ. Each path $\pi \in \mathcal{T}^\lambda$ can be encoded with a pair of sequences

$$\begin{aligned} \vec{\tau} &= (\tau_1 > \tau_2 > \cdots > \tau_r), && \tau_i \in W/W_\lambda, \qquad \text{and} \\ \vec{a} &= (0 = a_0 < a_1 < a_2 < \cdots < a_r = 1), && a_i \in \mathbb{Q}, \end{aligned}$$

so that π is given by

$$\pi(t) = (t - a_{j-1})\tau_j\lambda + \sum_{i=1}^{j-1} (a_i - a_{i-1})\tau_i\lambda, \qquad \text{for } a_{j-1} \le t \le a_j.$$

The *initial direction* of π is $\iota(\pi) = \tau_1$ and the *endpoint* of π is $\pi(1) \in \mathfrak{h}^*$.

Fix $w \in W$, let $\bar{w} = wW_\lambda \in W/W_\lambda$ and assume that π is a path in the set

$$\mathcal{T}^\lambda_{\le w} = \{\pi \in \mathcal{T}^\lambda \mid \iota(\pi) \le \bar{w}\}.$$

A *maximal lift of $\vec{\tau}$ with respect to w* is a choice of representatives $t_i \in W$ of the cosets τ_i such that $w \geq t_1 > \cdots > t_r$ and each t_i is maximal in Bruhat order such that $t_{i-1} > t_i$. The *final direction* of π with respect to w is

$$v(\pi, w) = t_r,$$

where $w \geq t_1 > \cdots > t_r$ is a maximal lift of $\tau_1 > \cdots > \tau_r$ with respect to w.

Theorem. *Let λ be a dominant integral weight and let $w \in W$. Then*

$$Y^\lambda T_{w^{-1}} = \sum_{\eta \in \mathcal{T}^\lambda_{\leq w}} T_{v(\eta,w)^{-1}} Y^{\eta(1)}$$

as operators on $K(E(G/B))$.

Sketch of proof. Fix a simple root α_i. Every path is in a unique *α_i-string* of paths

$$S_{\alpha_i}(\pi) = \{f_i^m \pi, \ldots, f_i^2\pi, f_i\pi, \pi\},$$

where $f_i^m\pi = 0$ and there does not exist any path η such that $f_i\eta = \pi$. In a manner similar to that of [Li, Lemma 5.3] one shows that, for any α_i-string $S_{\alpha_i}(\pi)$,

$$\sum_{\eta \in S_{\alpha_i}(\pi)} T_{v(\eta,w)^{-1}} Y^{\eta(1)} = T_{v(\pi,w)^{-1}} Y^{\pi(1)} T_i.$$

Given these facts, the proof of the Theorem follows the same lines as the proof of the Demazure character formula given in [Li, 5.5]. □

By applying the formula in the Theorem to the element $[\mathcal{O}_{\Omega_1}] \in K(E(G/B))$ and using (3) and (4) we obtain the following.

Corollary. *Let λ be a dominant integral weight and let $w \in W$. In $K(E(G/B))$,*

$$y^\lambda [\mathcal{O}_{\Omega_w}] = \sum_{\eta \in \mathcal{T}^\lambda_w} [\mathcal{O}_{\Omega_{v(\eta,w)}}] x^{\eta(1)}.$$

Acknowledgements

We are grateful to many people for comments, suggestions and encouragement: we would particularly like to thank Jim Carlson, Mark Green, Shrawan Kumar, Bob MacPherson, Ted Shifrin and Al Vasquez.

References

[Ch] C. Chevalley, *Sur les decompositions cellulaires des espaces G/B*, in *Algebraic Groups and their Generalizations: Classical Methods*, W. Haboush and B. Parshall eds., Proc. Symp. Pure Math., Vol. **56** Pt. 1, Amer. Math. Soc. (1994), 1–23. MR **95e:**14041

[FL] W. Fulton and A. Lascoux, *A Pieri formula in the Grothendieck ring of a flag bundle*, Duke Math. J. **76** (1994), 711–729. MR **96j:**14036

[FP] W. Fulton and P. Pragacz, *Schubert varieties and degeneracy loci*, Lecture Notes in Math. **1689**, Springer-Verlag, Berlin 1998. CMP 98:17

[KK] B. Kostant and S. Kumar, *T-equivariant K-theory of generalized flag varieties*, J. Differential Geom. **32** (1990), 549–603. MR **92c:**19006

[Li] P. Littelmann, *A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras*, Invent. Math. **116** (1994), 329–346. MR **95f:**17023

[P] H. Pittie, *Homogeneous vector bundles over homogeneous spaces*, Topology **11** (1972), 199–203. MR **44:**7583

[S] R. Steinberg, *On a theorem of Pittie*, Topology **14** (1975), 173–177. MR **51:**9101

Department of Mathematics, Graduate Center, City University of New York, New York, NY 10036

Department of Mathematics, Princeton University, Princeton, NJ 08544
E-mail address: `rama@math.princeton.edu`