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THICKNESS MEASURES FOR CANTOR SETS

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Abstract. For a fixed $k \geq 1$ let $C_1, \ldots, C_k$ be generalized Cantor sets. We examine various criteria under which $C_1 + \cdots + C_k$ contains an interval. When these criteria do not hold, we give a lower bound for the Hausdorff dimension of $C_1 + \cdots + C_k$. Our work will involve the development of two different types of thickness measures.

1. Introduction

We define a *generalized Cantor set* (henceforth known as a *Cantor set*) to be any set C of real numbers of the form

$$C = I \setminus \bigcup_{i \geq 1} O_i$$

where I is a finite closed interval and $\{O_i\ ;\ i \geq 1\}$ is a countable (finite or infinite) collection of disjoint open intervals contained in I. We may inductively define a tree $\mathcal{D}$ that will represent C. Let the root of the tree be the interval I. We say that $\{I\}$ is the *zeroth level* of the tree. Now suppose we have defined our tree up to the n^{th} level. We define the $(n+1)^{th}$ level of the tree as follows. Let I^w be an n^{th} level vertex of our tree. Assume first that

$$I^w \cap \Big(\bigcup_{i \geq 1} O_i \Big) \neq \emptyset.$$

Let O_{I^w} be the interval in the set $\{O_i\ ;\ i \geq 1\}$ of least index which is contained in I^w, and let I^{w0} and I^{w1} be closed intervals with

$$I^w = I^{w0} \cup O_{I^w} \cup I^{w1}.$$

We let I^{w0} and I^{w1} be subvertices of I^w in $\mathcal{D}$. If

$$I^w \cap \Big(\bigcup_{i \geq 1} O_i \Big) = \emptyset,$$

then we set $I^{w0} = I^w$ and let I^{w0} be a subvertex of I^w in $\mathcal{D}$. We repeat this process for every vertex I^w in the n^{th} level of $\mathcal{D}$. The $(n+1)^{th}$ level of the tree is the set

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of vertices I^v in $\mathcal{D}$ with $|v| = n+1$, where $|v|$ denotes the length of the word v. We continue this process inductively, creating the infinite tree $\mathcal{D}$. Note that

$$\{O_{I^w}\ ;\ I^w \text{ is a bridge of } \mathcal{D}\} = \{O_i\ ;\ i \geq 1\},$$

hence

$$C = \bigcap_{n=0}^{\infty} \Big(\bigcup_{|w|=n} I^w \Big).$$

Any tree with this property is said to be a *derivation* of the Cantor set C from I. The intervals $I, I^0, \ldots$ are called *bridges* of the derivation, while the open intervals $O_I, O_{I^0}, \ldots$ are called *gaps* of C.

Cantor sets arise naturally in many areas of mathematical inquiry, including the examination of the Markoff spectrum and the study of the chaotic behavior of certain families of functions (see, for example, [2] and [4]). Of interest to us here is the following problem. Define the sum of sets $E_1, \ldots, E_n$ to be the set

$$E_1 + \cdots + E_n = \{e_1 + \cdots + e_n\ ;\ e_i \in E_i \text{ for } 1 \leq i \leq n\}.$$

For $k \geq 2$ let $C_1, \ldots, C_k$ be Cantor sets derived from $I_1, \ldots, I_k$ respectively. In this paper we discuss conditions under which $C_1 + \cdots + C_k$ contains an interval. We also give bounds for the Hausdorff dimension of $C_1 + \cdots + C_k$.

2. Thickness

Let C be a Cantor set. We define the *thickness* of C, $\tau(C)$, to be infinity if $\{O_i\ ;\ i \geq 1\}$ is empty. Otherwise we put

$$\tau(C) = \sup_{\mathcal{D}} \inf_{A \in \mathcal{D}} \min \left\{ \frac{|A^0|}{|O_A|}, \frac{|A^1|}{|O_A|} \right\}$$

where the supremum is over all derivations $\mathcal{D}$ of C and the infimum is over all bridges A of $\mathcal{D}$. It is not difficult to show that the supremum is always attained (see, for example, [1], Lemma 3.1).

In 1979 Sheldon Newhouse [3] proved the following result.

Theorem 2.1. *Let C_1 and C_2 be Cantor sets derived from I_1 and I_2 respectively, with $\tau(C_1)\tau(C_2) > 1$. Then either $I_1 \cap I_2 = \emptyset$, C_1 is contained in a gap of C_2, C_2 is contained in a gap of C_1 or $C_1 \cap C_2 \neq \emptyset$.*

In fact, if Newhouse's proof is slightly altered, then we may replace the condition "$\tau(C_1)\tau(C_2) > 1$" in Theorem 2.1 with the weaker condition "$\tau(C_1)\tau(C_2) \geq 1$". This strengthened version of Theorem 2.1 has the following corollary.

Theorem 2.2. *For $j = 1$ or $j = 2$ let C_j be a Cantor set derived from I_j, with O_j a gap of maximal size in C_j. Assume that*

$$|O_1| \leq |I_2| \quad \textit{and} \quad |O_2| \leq |I_1|.$$

If $\tau(C_1)\tau(C_2) \geq 1$, then $C_1 + C_2 = I_1 + I_2$.

If $\tau(C_1)\tau(C_2) < 1$, then the work of Newhouse does not yield any non-trivial results. This case was the main focus of the author in [1], where a best-possible lower bound for the thickness of a finite sum of Cantor sets was found.

For a Cantor set C we define the *normalized thickness* of C, $\gamma(C)$, to be

$$\gamma(C) = \frac{\tau(C)}{\tau(C) + 1}.$$

Theorem 2.3. *Let k be a positive integer and for $j = 1, 2, \ldots, k$ let C_j be a Cantor set derived from I_j, with O_j a gap of maximal size in C_j. Let $S_\gamma = \gamma(C_1) + \cdots + \gamma(C_k)$.*

1. *If $S_\gamma \geq 1$, then $C_1 + \cdots + C_k$ contains an interval. Otherwise $C_1 + \cdots + C_k$ contains a Cantor set of thickness at least*
$$\frac{S_\gamma}{1 - S_\gamma}.$$
2. *If*

$$|I_{r+1}| \geq |O_j| \quad \text{for } r = 1, \ldots, k-1 \text{ and } j = 1, \ldots, r, \tag{1}$$
$$|I_1| + \cdots + |I_r| \geq |O_{r+1}| \quad \text{for } r = 1, \ldots, k-1 \tag{2}$$

and $S_\gamma \geq 1$, then
$$C_1 + \cdots + C_k = I_1 + \cdots + I_k.$$

3. *If (1) and (2) hold and $S_\gamma < 1$, then*
$$\tau(C_1 + \cdots + C_k) \geq \frac{S_\gamma}{1 - S_\gamma}.$$

Proof. See [1], Theorem 2.4. □

Note that in the case $k = 2$ the condition "$\gamma(C_1) + \gamma(C_2) \geq 1$" is equivalent to the condition "$\tau(C_1)\tau(C_2) \geq 1$", hence Theorem 2.3 implies Theorem 2.2.

Let $\dim_H(E)$ denote the Hausdorff dimension of the set E. There is a connection between thickness and Hausdorff dimension, as illustrated by the next theorem.

Theorem 2.4. *If C is a Cantor set, then*
$$\dim_H(C) \geq \frac{\log 2}{\log\left(2 + \frac{1}{\tau(C)}\right)}.$$

Proof. See [4], p. 77. □

Using Theorem 2.3 and Theorem 2.4 we can establish the following lower bound for the Hausdorff dimension of a sum of Cantor sets.

Theorem 2.5. *For $k \in \mathbb{Z}^+$ let $C_1, \ldots, C_k$ be Cantor sets. Then*
$$\dim_H(C_1 + \cdots + C_k) \geq \frac{\log 2}{\log\left(1 + \frac{1}{\min\{\gamma(C_1) + \cdots + \gamma(C_k), 1\}}\right)}.$$

It may be the case that $C_1 + \cdots + C_k$ contains an interval yet $\gamma(C_1) + \cdots + \gamma(C_k) < 1$. In this case better results may be gained by employing a concept known as *maximal thickness.*

3. Maximal thickness

We define the *maximal thickness* and *normalized maximal thickness* of a Cantor set C to be
$$\tau_M(C) = \sup_{C' \subseteq C} \tau(C') \quad \text{and} \quad \gamma_M(C) = \frac{\tau_M(C)}{\tau_M(C) + 1}$$
respectively, where the supremum is taken over all Cantor sets C' contained in C. Note that for any Cantor set C it follows trivially that $\tau(C) \leq \tau_M(C)$. Using Theorems 2.3 and 2.5 we may establish the following result.

Theorem 3.1. *Let $C_1, \dots, C_k$ be Cantor sets. If*

$$\gamma_M(C_1) + \cdots + \gamma_M(C_k) > 1,$$

then $C_1 + \cdots + C_k$ contains an interval. Otherwise

$$\gamma_M(C_1 + \cdots + C_k) \geq \gamma_M(C_1) + \cdots + \gamma_M(C_k)$$

and

$$\dim_H(C_1 + \cdots + C_k) \geq \frac{\log 2}{\log\left(1 + \frac{1}{\gamma_M(C_1) + \cdots + \gamma_M(C_k)}\right)}.$$

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