ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 6, Pages 52–63 (July 19, 2000)
S 1079-6762(00)00081-0

ON THE CONNECTEDNESS OF THE SPACE OF INITIAL DATA FOR THE EINSTEIN EQUATIONS

BRIAN SMITH AND GILBERT WEINSTEIN

(Communicated by Richard Schoen)

Abstract. Is the space of initial data for the Einstein vacuum equations connected? As a partial answer to this question, we prove the following result: Let $\mathcal{M}$ be the space of asymptotically flat metrics of non-negative scalar curvature on $\mathbb{R}^3$ which admit a global foliation outside a point by 2-spheres of positive mean and Gauss curvatures. Then $\mathcal{M}$ is connected.

Introduction

The Einstein vacuum equations of general relativity read:

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}_{\mu\nu} = 0, \tag{1}$$

where $\bar{R}_{\mu\nu}$ is the Ricci curvature tensor of a Lorentzian 4-manifold, and $\bar{R}$ the scalar curvature. The basic problem for these equations is the Cauchy problem: *given data on a time-slice M, consisting of a Riemannian metric g and a second fundamental form k on M, find the evolution of space-time according to* (1). Not all the equations in (1) are evolution equations. Using the twice-contracted Gauss equation and the Codazzi equations of the Riemannian submanifold M, one finds that the normal-normal and normal-tangential components of (1) are:

$$R - |k|^2 + (\operatorname{tr} k)^2 = 0, \tag{2}$$

$$\nabla^j k_{ij} - \nabla_i \operatorname{tr} k = 0, \tag{3}$$

where R is the scalar curvature of M, and k its second fundamental form. These equations, called the *Vacuum Constraint Equations*, involve no time derivatives and hence are to be considered as restrictions on the data g and k; see [14]. We will only consider *asymptotically flat* (AF) solutions of these equations, i.e., solutions satisfying the decay:

$$\begin{aligned} g_{ij} - \delta_{ij} &= O(r^{-1}), \\ k_{ij} &= O(r^{-2}), \\ R &\in L^1(M). \end{aligned}$$

Received by the editors May 27, 1999.
2000 *Mathematics Subject Classification.* Primary 83C05; Secondary 58G11.
This research was supported in part by NSF grant DMS 9704760.

It is standard to choose the *maximal gauge* $\operatorname{tr} k = 0$ in (2)–(3), which, as shown by Bartnik [1], involves no loss of generality. In this case, we get the *Maximal Gauge Vacuum Constraint Equations*:

$$R = |k|^2\,, \tag{4}$$

$$\operatorname{tr} k = 0, \tag{5}$$

$$\operatorname{div} k = 0. \tag{6}$$

These form an underdetermined system of elliptic equations on M for g and k.

Much work has been devoted to finding solutions of (4)–(6); see for example [3, 4, 6, 7, 9, 10, 15] and the references therein. However, certain fundamental questions remain unanswered. For example, it is not known whether the space of AF initial data on a given 3-manifold M is connected, not even in the case $M = \mathbb{R}^3$. Since the evolution equations trace a continuous path in the phase space of initial data, either answer to this question would be of considerable significance for the dynamics of the Einstein equations.

The standard method for solving (4)–(6) has been the *conformal method.* In this method the free data is the conformal class of an asymptotically flat Riemannian metric g, and a trace-free divergence-free symmetric 2-tensor k on M. Since the trace-free and divergence-free conditions on k are invariant under the transformation $g \mapsto \phi^4 g$, $k \mapsto \phi^{-2}k$, it suffices to find ϕ so that (4) is satisfied. This will be so provided that the Lichnerowicz equation is satisfied:

$$\Delta\phi - \frac{1}{8}R\phi + |k|^2\,\phi^{-7} = 0.$$

A solution of this equation can be found if the negative part of the scalar curvature is small enough in the $L^{3/2}$ norm; see [9]. In particular, the question above can be reduced to the following purely geometric problem: is the space of AF metrics of non-negative scalar curvature on a 3-manifold M connected?

In this paper, we announce, and sketch the proof of a result which gives a partial answer in the affirmative to the question posed above; details will appear in [13]. We say that a topological 2-sphere S in M is *quasiconvex* if both the Gauss and the mean curvature of S are positive [8]. Let $\mathcal{M}$ be the space of $C^{2,\alpha}_{-1}$ metrics g on $\mathbb{R}^3$ with non-negative scalar curvature $R \in L^1$ which admit a global coordinate system whose coordinate spheres are quasiconvex, and which satisfy in this coordinate system:

$$\begin{aligned} g_{ij} - \delta_{ij} &= O(r^{-1}) \qquad \text{as } r \to \infty,\\ g_{ij} - \delta_{ij} &= 0 \qquad\qquad\ \ \text{at } r = 0. \end{aligned}$$

The $C^{2,\alpha}_{-1}$ topology on $\mathcal{M}$ is generated by the following system of neighborhoods of any metric $g \in \mathcal{M}$:

$$\left\{ g' \in \mathcal{M}\colon\ \sum_{\alpha=0}^{2} \sup \left|(1+r)^{m+1}\partial^m(g_{ij} - g'_{ij})\right| + [\partial^2 g_{ij} - \partial^2 g'_{ij}]_{\alpha,-3} < \epsilon \right\},$$

where

$$[f]_{\alpha,-k} = \sup_r \left((1+r)^{k+\alpha} \sup_{x,y\in B_r} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} \right)$$

is the weighted Hölder norm with exponent α of f on $\mathbb{R}^3$. In fact, in view of the general covariance of the Einstein Equations, we are only interested in the quotient of $\mathcal{M}$ by the group $\mathcal{G}$ of diffeomorphisms of $\mathbb{R}^3$.

Main Theorem. *The quotient of $\mathcal{M}/\mathcal{G}$ is path connected in the quotient topology induced by $C^{2;\alpha}_{-1}$ on $\mathcal{M}$.*

Of course, this raises the following question: when does an AF metric g of non-negative scalar curvature belong to $\mathcal{M}$? Clearly, a necessary condition is that g possesses no compact minimal surfaces. However we do not even know whether the absence of compact minimal surfaces suffices to guarantee the existence of a global foliation with positive mean curvature.

To prove our Main Theorem, we generalize a method introduced by Bartnik [2] to construct quasispherical metrics of prescribed scalar curvature. A metric is *quasispherical* if it can be foliated by *round spheres*, spheres of constant curvature. Bartnik observed that prescribing scalar curvature for this type of metric could be viewed as a parabolic equation on the sphere for one of the metric coefficients, $u = |\nabla r|^{-1}$, where r is the foliating function, provided that the mean curvature was also positive. We combine this with the Poincaré Uniformization as in [8] to get a general method to prescribe scalar curvature for metrics in $\mathcal{M}$. As an application, we prove the Main Theorem.

Denote by r the foliating function normalized so that the area of the spheres is $4\pi r^2$, and by γ the induced metric on the spheres. Any smooth enough metric $g \in \mathcal{M}$ can be written as:

$$g = u^2 dr^2 + e^{2v}\bar{\gamma}_{AB}(\hat{\beta}^A dr + r d\theta^A)(\hat{\beta}^B dr + r d\theta^B), \tag{7}$$

where (θ^1, θ^2) are local coordinates on $\mathbb{S}^2$, $\bar{\gamma}_{AB}$ is a fixed (independent of r) round metric of area 4π, and $\hat{\beta} = \hat{\beta}^A \partial_A$ is the *shift vector*. Here, and throughout, we use the summation convention: repeated indices are summed over their range, $0, 1, 2, 3$ for Greek indices, $1, 2, 3$ for lower case Latin indices, and $1, 2$ for upper case Latin indices. Let χ be the second fundamental form, $H = \operatorname{tr}_\gamma \chi$ be the mean curvature of the spheres, and $\Pi = \mathcal{L}_{\hat{\beta}}\gamma$ be the deformation tensor of $\hat{\beta}$ on the spheres; then it can be checked that

$$\bar{\chi} = ru\chi = ((1 + rv_r)\gamma - \Pi/2)\,, \tag{8}$$

$$\bar{H} = ruH = \left(2 + 2rv_r - e^{-2v}\operatorname{div}_{\bar{\gamma}}\beta\right), \tag{9}$$

where $\beta = e^{2v}\hat{\beta}$. It is important to note that both $|\bar{\chi}|^2_\gamma$ and $\bar{H}$ can be calculated in terms of only β, v, r, and the round metric $\bar{\gamma}$ on $\mathbb{S}^2$. Let N be the outer unit normal to the foliation spheres, let $\bar{N} = ruN = r\partial_r - \hat{\beta}$, let $\not{\Delta}_\gamma$ be the Laplacian on the spheres with respect to γ, and let

$$\kappa = r^{-2}e^{-2v}(1 - \not{\Delta} v) \tag{10}$$

be the Gauss curvature of the spheres. Then the equation for the scalar curvature R of g can be written as

$$\bar{H}\partial_{\bar{N}} u = r^2 u^2 \not{\Delta}_\gamma u + \bar{A}u - \bar{B}u^3, \tag{11}$$

where

$$\bar{A} = \partial_{\bar{N}}\bar{H} - \bar{H} + \frac{1}{2}|\bar{\chi}|^2_{\gamma} + \frac{1}{2}\bar{H}^2,$$
$$\bar{B} = r^2(\kappa - \frac{1}{2}R) = e^{-2v}(1 - \not\Delta v) - \frac{1}{2}r^2 R.$$

Noting that the Laplacian with respect to $\bar{\gamma}$ is $\not\Delta = r^2 e^{2v}\not\Delta_{\gamma}$, we obtain, provided that $H > 0$, the following *Bernoulli-type* parabolic equation for u on the unit sphere:

$$r\partial_r u - \beta \cdot \not\nabla u = \Gamma u^2 \not\Delta u + Au - Bu^3, \tag{12}$$

where $\not\nabla u$ is the tangential component of the gradient of u , $\Gamma = e^{-2v}/\bar{H}$, $A = \bar{A}/\bar{H}$ and $B = \bar{B}/\bar{H}$. It follows from the comment following equations (8)–(9) that the coefficients Γ, A and B can be calculated in terms of only β, v, r, the round metric $\bar{\gamma}$ on $\mathbb{S}^2$, and R. The quasispherical case can be recovered by setting $v = 0$, and $\kappa = 1$, see [2].

The proof of the Main Theorem is based on the study of equation (12). The deformation to a flat metric is accomplished in several steps. First, the metric is smoothed out with the scalar curvature R truncated to be compactly supported. Next, we deform the metric to one satisfying $2\kappa > R$. Then, we deform to a metric with compactly supported β and v. Finally, we deform to a flat metric. The last three steps are all based on the following strategy. The deformation g_λ is defined explicitly on a ball B_{r_0}. In the exterior of B_{r_0} we consider β_λ, v_λ, and R_λ as free data, and solve equation (12) on $[r_0, \infty) \times \mathbb{S}^2$ for u_λ with initial conditions $u_\lambda|_{S_{r_0}}$. In order for this to be feasible, and for the resulting metric g_λ to yield a continuous path, we must ensure that β_λ, v_λ, and R_λ are continuous in the appropriate spaces, that $\bar{H}_\lambda$ is positive, and that R_λ is non-negative. In addition, one must verify conditions that guarantee the global existence of the solution u_λ, its appropriate decay as $r \to \infty$, and continuity with respect to λ. The regularity of u_λ across S_{r_0} is obtained by solving equation (12) on $[r', r_0 + \epsilon) \times \mathbb{S}^2$, $0 < r' < r_0$, with initial data $u_\lambda|_{S_{r'}}$ and by using the uniqueness and regularity of solutions.

The plan of the paper is as follows. In the next section, we derive equation (11). In Section 2, we collect the analytical results we need on existence, uniqueness, asymptotic behavior, and continuous dependence on parameters of solutions of (12). Then in Section 3, we sketch the proof of the Main Theorem by deforming any metric in $\mathcal{M}$ to the Euclidean metric.

1. The scalar curvature of a 3-manifold foliated by spheres

In this section, we derive equation (11). First, note that $\nabla r = u^{-1}N$, and consequently $u^{-1} = N^i \nabla_i r$. It follows that

$$\nabla_N N = -u^{-1}\not\nabla u, \tag{13}$$

where $\not\nabla u$ is the tangential part of ∇u. Furthermore, using $N^j \nabla_i N_j = 0$, we get

$$\begin{aligned}\nabla_i\nabla_j u^{-1} &= N^k\nabla_i\nabla_j\nabla_k r + (\nabla_j\nabla_k r)(\nabla_i N^k)\\ &= N^k\nabla_i\nabla_k\nabla_j r + \nabla_j(u^{-1}N_k)(\nabla_i N^k)\\ &= N^k\nabla_k\nabla_i\nabla_j r + u^{-1}R_{jlik}N^k N^l + u^{-1}(\nabla_j N_k)(\nabla_i N^k),\end{aligned}$$

which after tracing with respect to g leads to

$$\Delta u^{-1} = \nabla_N \Delta r + u^{-1}\operatorname{Ric}(N,N) + u^{-1}|\nabla N|^2, \tag{14}$$

where Ric is the Ricci tensor of g. We now need to calculate the first and third terms on the right hand side of (14). Since the tangential part of ∇N is the second fundamental form χ, and the normal part is given by (13), it is easy to see that:

$$|\nabla N|^2 = |\chi|^2_\gamma + u^{-2}|\nabla\!\!\!/ u|^2_\gamma.$$

Next, using equation (13) again, we find:

$$\begin{aligned}\nabla_N \Delta r &= N^k N^i \nabla_k \nabla_i u^{-1} + (\nabla_N N^i)(\nabla_i u^{-1}) + \nabla_N(u^{-1}\nabla^i N_i)\\ &= \nabla^2 u^{-1}(N,N) + u^{-3}|\nabla\!\!\!/ u|^2_\gamma + \nabla_N(u^{-1}H).\end{aligned}$$

Substituting the last two equations into (14) gives

$$\Delta u^{-1} = \nabla^2 u^{-1}(N,N) + 2u^{-3}|\nabla\!\!\!/ u|^2_\gamma + u^{-1}\operatorname{Ric}(N,N) + \nabla_N(u^{-1}H) + u^{-1}|\chi|^2_\gamma,$$

where $\nabla^2 u^{-1}$ denotes the Hessian of the function u^{-1}. On the other hand:

$$\Delta u^{-1} = \nabla^2 u^{-1}(N,N) + H\,\nabla_N u^{-1} + \Delta\!\!\!/_\gamma u^{-1}.$$

Combining the last two equations we obtain:

$$\Delta\!\!\!/_\gamma u = 2u^{-1}|\nabla\!\!\!/ u|^2 - u^2\Delta\!\!\!/ u^{-1} = -u\operatorname{Ric}(N,N) - u\nabla_N H - u|\chi|^2.$$

Substituting $\operatorname{Ric}(N,N) = \frac{1}{2}(H^2 - |\chi|^2 + R - R\!\!\!/)$ from the Gauss equation, and the definitions $\bar H = ruH$, $\bar\chi = ru\chi$, and $\bar N = ruN$ from the introduction, we get (11).

2. Bernoulli-type parabolic PDEs on $\mathbb{S}^2$

In this section we collect the analytical results we need to prove the Main Theorem: (conditions for) global existence, uniqueness, asymptotic behavior and continuous dependence on parameters for solutions of (12). All these rely on a simple pointwise a priori bound whose proof we present here. Many of the results presented in this section are adapted from [2]. In order to ensure the uniform parabolicity of (12) we assume throughout this section that $\Gamma = e^{-2v}/\bar H$ is bounded above and below by positive constants.

First, we define parabolic Banach spaces to be used in our study of (12). Let $0 < r_0 < r_1 \le \infty$, $I = [r_0, r_1] \subset \mathbb{R}^+$, and let $A_I = I \times \mathbb{S}^2$. Given a function f on A_I, define:

$$[f]_{\alpha;I} = \sup_{\substack{(r_1,\theta_1),(r_2,\theta_2)\in A_I\\ \operatorname{dist}(\theta_1,\theta_2)<\pi}} \left[\frac{|f(r_2,\theta_2) - f(r_1,\theta_1)|}{|1 - r_2/r_1|^{\alpha/2} + \operatorname{dist}(\theta_2,\theta_1)^\alpha}\right],$$

$$\|f\|_{0;I} = \sup_{A_I}|f|, \qquad \|f\|_{0,\alpha;I} = \|f\|_{0;I} + [f]_{\alpha;I}.$$

Here $\operatorname{dist}(\cdot,\cdot)$ denotes the geodesic distance on $\mathbb{S}^2$. If $f(r,\cdot)$ is a tensor field on $\mathbb{S}^2$, then $f(r_2,\theta_2)$ is understood to mean the parallel translate of $f(r_2,\theta_2)$ back to θ_1 along the unique geodesic from θ_1 to θ_2. With these conventions, we can now define:

$$\|f\|_{k,\alpha;I} = \sum_{i+2j\le k}\left\|\nabla\!\!\!/^i (r\partial_r)^j f\right\|_{0;I} + \sum_{i+2j=k}[\nabla\!\!\!/^i (r\partial_r)^j f]_{\alpha;I},$$

where $\not\nabla^i f$ is to be interpreted as the i-th covariant derivative of f in the standard metric on $\mathbb{S}^2$. Now define $H_I^{k,\alpha}$ to be the space of functions f on A_I for which $\|f\|_{k,\alpha;I}$ is defined and finite. Equipped with the norm $\|\cdot\|_{k,\alpha;I}$, the space $H_I^{k,\alpha}$ is a Banach space. Following Bartnik in [2], we also use the notation:

$$f^*(r) = \sup_{\theta\in\mathbb{S}^2} f(r,\theta), \qquad f_*(r) = \inf_{\theta\in\mathbb{S}^2} f(r,\theta).$$

2.1. **Conditions for global existence and uniqueness.** Our first observation is that equation (12) is uniformly parabolic with r as the 'time' variable. Therefore, given any initial data $u(r_0,\theta) = u_0(\theta)$, it is standard to obtain the existence of a unique solution on a short time interval $[r_0, r_0+\epsilon)$ for some $\epsilon > 0$. Furthermore, it is well known that, for some choices of coefficients and initial data, a classical solution can blow up in finite time. Thus, our main objective here is to derive conditions which guarantee the existence of a global positive solution on the time interval $[r_0,\infty]$.

The principal ingredient in this and future subsections is a simple a priori bound on solutions of (12). To derive this bound, we use the familiar substitution $w = u^{-2}$ well-known from the elementary method used to solve the corresponding Bernoulli ordinary differential equation. If $u > 0$ satisfies (12) on $[r_0, r_1]$, then w satisfies

$$r\partial_r w - \beta\cdot\not\nabla u = 2(-\Gamma u^{-1}\not\Delta u - Aw + B). \tag{15}$$

Since this equation is only used to derive pointwise a priori bounds, and since u has a maximum where w has a minimum and vice versa, there is no need to transform the gradient and Laplacian terms. For example, it follows from (15) that

$$r\partial_r w_* + 2A^* w_* \geq 2B_*, \tag{16}$$

which upon integration yields the lower bound:

$$rw_* \geq \left[r_0 w_*(r_0) + \int_{r_0}^{r} 2B_* \exp\left(\int_{r_0}^{r'}(2A^*-1)\frac{dt}{t}\right)dr'\right]\exp\left(\int_{r_0}^{r}(1-2A^*)\frac{dt}{t}\right). \tag{17}$$

Similarly, one obtains the upper bound:

$$rw^* \leq \left[r_0 w^*(r_0) + \int_{r_0}^{r} 2B^* \exp\left(\int_{r_0}^{r'}(2A_*-1)\frac{dt}{t}\right)dr'\right]\exp\left(\int_{r_0}^{r}(1-2A_*)\frac{dt}{t}\right). \tag{18}$$

In particular it follows immediately that w is bounded above, and hence $u \geq c > 0$, where c depends on r_1. Suppose furthermore that

$$K = \frac{1}{r_0}\left(\sup_{r_0<r<\infty}\left(-\int_{r_0}^{r} 2B_* \exp\left(\int_{r_0}^{s}(2A^*-1)\frac{dt}{t}\right)ds\right)\right)^{-1/2} > 0. \tag{19}$$

If the initial data $u_0 < K$, then w is bounded below, and hence $u < \infty$. We note that if $B \geq 0$ then $K = \infty$, which gives $u < \infty$ for any positive initial data. The above considerations lead to the following result:

Theorem 1. *Let $I = [r_0,\infty)$, suppose $\beta, \Gamma, A, B \in C^\alpha(I\times\mathbb{S}^2)$, and suppose that the constant K defined in* (19) *is positive. Then, for any positive function $u_0 \in C(\mathbb{S}^2)$ satisfying $\sup_{\mathbb{S}^2} u_0 < K$, the equation* (12) *has a unique positive global classical solution u on $I\times\mathbb{S}^2$ with initial data $u(r_0,\theta) = u_0(\theta)$.*

Now, suppose that the coefficients in (12) satisfy

$$\|\beta\|_{2,\alpha;I'} + \|\Gamma\|_{2,\alpha;I'} + \|A\|_{2,\alpha;I'} + \|B\|_{2,\alpha;I'} \leq C,$$

for some $C > 0$, and that u is a solution which is bounded above and below, $C^{-1} \leq u \leq C$ on A_I, where $I \subset I'$. Then standard parabolic Schauder theory gives

$$\|u\|_{4,\alpha;I} \leq C', \tag{20}$$

where C' depends on C and the length of I. Using the scaling properties of the $H_I^{k,\alpha}$-norms and of equation (12) we can also derive (20) for $I_\lambda = [\lambda r_0, \lambda r_1]$ with C' independent of λ provided $I_\lambda \subset I'$. Furthermore, if in addition $|2A - 1| < C/r$ on $[r_0, \infty) \times \mathbb{S}^2$ and $u(r_0, \theta) < K$, we may use the bounds (17) and (18) to obtain uniform infimum and supremum bounds on every subinterval of $[r_0, \infty)$. Thus, we obtain the following result.

Theorem 2. *Let $I = [r_0, \infty)$, and suppose that there is a constant $C > 0$ such that $\beta, \Gamma, A, B \in H_I^{2,\alpha}$ satisfy*

$$\|\beta\|_{2,\alpha;I}, \|\Gamma\|_{2,\alpha;I}, \|B\|_{2,\alpha;I}, \|r(2A-1)\|_{2,\alpha;I} \leq C,$$
$$C^{-1} \leq \Gamma \leq C,$$

and $u_0 \in C^{4,\alpha}(\mathbb{S}^2)$ satisfies $0 < u_0 \leq K - C^{-1}$. Then there is a unique positive solution $u \in H_I^{4,\alpha}$ of (12) with initial condition $u(r_0, \theta) = u_0(\theta)$. Furthermore, we have

$$\|u\|_{4,\alpha;I} \leq C',$$

where C' depends only on C.

Our final result concerning global existence is a simple consequence of the maximum principle.

Theorem 3. *Let $I = [r_0, \infty)$, and suppose that $u > 0$ is a classical solution of equation (12) on $I \times \mathbb{S}^2$ with coefficients $\beta, \Gamma, A, B \in H_I^{2,\alpha}$ and with initial data $u_0 \in C^{4,\alpha}(\mathbb{S}^2)$. Let $\tilde{B} \in H_I^{2,\alpha}$ satisfy $\tilde{B} \geq B$. Then the equation*

$$r\partial_r u - \beta \cdot \nabla\!\!\!/ u = \Gamma u^2 \Delta\!\!\!/ u + Au - \tilde{B} u^3$$

has a unique solution $\tilde{u} \in H_I^{4,\alpha}$ with the same initial data $\tilde{u}(r_0, \theta) = u_0(\theta)$. Furthermore, we have $0 < \tilde{u} \leq u$.

Proof. It suffices to prove a supremum a priori bound for $\tilde{u}$ on any interval $[r_0, r_1)$ where the solution $\tilde{u} > 0$ exists. Subtracting the equation for u from the equation for $\tilde{u}$, we get an equation for $v = \tilde{u} - u$:

$$r\partial_r v - \beta \cdot \nabla\!\!\!/ v = \Gamma \tilde{u}^2 \Delta\!\!\!/ v + \tilde{A} v - (\tilde{B} - B)\tilde{u}^3, \tag{21}$$

where $\tilde{A} = A + \Gamma(\tilde{u} + u)\Delta\!\!\!/ u - B(\tilde{u}^2 + \tilde{u}u + \tilde{u}^2)$. Since $(\tilde{B} - B)\tilde{u}^3 > 0$, the maximum principle applies to give $v \leq 0$. It follows that $\tilde{u} \leq u$ on $[r_0, r_1)$. $\square$

For the remainder of Section 2 we set $I = [r_0, \infty)$.

2.2. **Asymptotic behavior.** To study the asymptotic behavior of solutions of equation (12) we define:

$$m = \frac{r}{2}\,(1 - u^{-2}).$$

If u is a global solution of equation (12) on $I \times \mathbb{S}^2$, then m satisfies

$$r\partial_r m - \beta \cdot \nabla\!\!\!/\, m = r\Gamma \frac{\Delta\!\!\!/\, u}{u} - (2A-1)m + r(A-B). \tag{22}$$

We note that if $r\,|2A-1| \le C$, and

$$|A-B|^* \in L^1(I), \tag{23}$$

then using the maximum principle, it follows that $|m|$ is bounded. Applying Schauder theory to equation (22) we then obtain:

Theorem 4. *Let $I = [r_0, \infty)$, let $\beta, \Gamma, A, B \in H_I^{2,\alpha}$, suppose that (23) is satisfied, and let $u \in H_I^{2,\alpha}$ be a positive solution of equation (12). Suppose that there is a constant $C > 0$ such that*

$$\|u\|_{2,\alpha;I}\,, \|r(2A-1)\|_{2,\alpha;I}\,, \|B\|_{2,\alpha;I}, \||A-B|^*\|_{L^1(I)} \le C,$$
$$C^{-1} \le \Gamma \le C.$$

Then $m = r(1-u^{-2})/2$ satisfies

$$\|m\|_{4,\alpha;I} < C', \tag{24}$$

where C' depends only on C.

We will also need the bound (24) in some cases where the condition (23) cannot be verified. To this end, we state a comparison result that establishes a bound on m. Let u_i, $i = 1, 2$, satisfy

$$r\partial_r u_i - \beta \cdot \nabla\!\!\!/\, u_i = \Gamma u_i^2 \Delta\!\!\!/\, u_i + A u_i - B_i u_i^3. \tag{25}$$

Setting $m_i = r(1-u_i^{-2})/2$, $i = 1, 2$, and $\tilde{m} = m_2 - m_1$, we obtain, in view of (22),

$$r\partial_r \tilde{m} - \tilde{\beta} \cdot \nabla\!\!\!/\, \tilde{m} = \Gamma u_2^2 \Delta\!\!\!/\, \tilde{m} + (2\tilde{A} - 1)\tilde{m} + r(B_1 - B_2),$$

where

$$\tilde{\beta} = \beta + 3\Gamma u_2^4 \nabla\!\!\!/ (m_1 + m_2),$$
$$\tilde{A} = A - \Gamma u_1^2 u_2^2 \left(r^{-1} \Delta\!\!\!/\, m_1 + 3r^{-2}(u_1^2 + u_2^2)\left|\nabla\!\!\!/\, m_1\right|^2\right).$$

If we also assume that m_1, $\nabla\!\!\!/\, m_1$, $\Delta\!\!\!/\, m_1$, u_1, and u_2 are bounded, then $r|2\tilde{A} - 1|$ is bounded, and provided that also $|B_1 - B_2|^* \in L^1(I)$, we may apply the previous argument to derive a bound on $|\tilde{m}|$, and consequently also on $|m_2|$. This leads to the following theorem.

Theorem 5. *Let $I = [r_0, \infty)$, let $\beta, \Gamma, A, B_1, B_2 \in H_I^{2,\alpha}$ and suppose that $u_i > 0$, $i = 1, 2$, are bounded classical solutions of equation (25) on $I \times \mathbb{S}^2$. Let $B = B_1 - B_2$, and suppose also that $|B|^* \in L^1(I)$, that m_1 is bounded, and that there is a constant $C > 0$ such that*

$$\|u_1\|_{2,\alpha;I}\,, \|u_2\|_{2,\alpha;I}\,, \|m_1\|_{2,\alpha;I}\,, \|r(2A-1)\|_{2,\alpha;I}\,, \|B\|_{2,\alpha;I}, \||B|^*\|_{L^1(I)} \le C,$$
$$C^{-1} \le \Gamma \le C.$$

Then $\tilde{m} = r(1 - \tilde{u}^{-2})/2$ *satisfies*

$$\|\tilde{m}\|_{4,\alpha;I} \le C',$$

where C' *depends only on* C.

2.3. **Continuous dependence on parameters.**

Theorem 6. *Let* $I = [r_0, \infty)$*, and suppose that* $u_\lambda \in H_I^{k+2,\alpha}$*,* $a \le \lambda \le b$*, is a family of solutions of* (12) *with* $\beta_\lambda, \Gamma_\lambda, A_\lambda, B_\lambda$ *satisfying* $r(2A_\lambda - 1), r(2B_\lambda - 1), \beta_\lambda, \Gamma_\lambda \in C^0([a,b], H_I^{k,\alpha})$ *and with the initial data* $u_\lambda(r_0) \in C^0\big([a,b], C^{k+2,\alpha}(\mathbb{S}^2)\big)$*. Suppose also that one of the following conditions is satisfied:*

1. $|A_\lambda - B_\lambda|^* \in C^0\big([a,b], L^1(I)\big)$*;*
2. *A continuous family of solutions* $m'_\lambda \in C^0\big([a,b], H_I^{k+2,\alpha}\big)$ *of* (22) *exist.*

Then $u_\lambda, m_\lambda \in C^0\big([a,b], H_I^{k,\alpha}\big)$.

3. Deformation of metrics in $\mathcal{M}$

We are now in a position to sketch the proof of the Main Theorem. Recall that any metric $g \in \mathcal{M}$ can be written as in (7). We define a nested sequence of subsets $\mathcal{M} = \mathcal{M}_0 \supset \cdots \supset \mathcal{M}_4$:

$\mathcal{M}_1 = \{g \in \mathcal{M}_0 \colon\ r(1-u) \in H^{4,\alpha}_{[r_0,\infty)};\ r\beta, rv \in H^{8,\alpha}_{[r_0,\infty)}, \forall r_0 > 0, \operatorname{supp} R \text{ is compact}\},$

$\mathcal{M}_2 = \{g \in \mathcal{M}_1 \colon\ 2\kappa - R > 0\},$

$\mathcal{M}_3 = \{g \in \mathcal{M}_2 \colon\ \beta, v \text{ are compactly supported}\},$

$\mathcal{M}_4 = \{g \in \mathcal{M}_3 \colon\ g \text{ is flat}\}.$

Let us say that $\mathcal{M}_i$ *is connected to* $\mathcal{M}_{i+1}$ if for each $g \in \mathcal{M}_i$ there is a path Γ in $\mathcal{M}_i$, continuous in the topology of $C^{2,\alpha}_{-1}$, with $\Gamma(0) = g$ and $\Gamma(1) \in \mathcal{M}_{i+1}$. We will show that $\mathcal{M}_i$ is connected to $\mathcal{M}_{i+1}$ for each $i = 0, \ldots, 3$. The Main Theorem follows by joining these paths.

Lemma 7. $\mathcal{M}_0$ *is connected to* $\mathcal{M}_1$.

Proof. Let $g = g_0 \in \mathcal{M}_0$. It is not difficult, using a truncation followed by a standard smoothing, to construct a deformation g_λ, continuous in $C^{2,\alpha}_{-1}$, from g_0 to a smooth metric g_1 which is flat outside a large enough ball, with scalar curvature $R_\lambda \in L^1$, and such that $g_\lambda - g_0$ is small in $C^{2,\alpha}_{-1/2}$ for all λ. Since g_λ is close to g, the coordinate spheres are still quasiconvex in g_λ, and the negative part R^-_λ of the scalar curvature of g_λ is small in $L^{3/2}$. It follows that the operator $-8\Delta_{g_\lambda} + R_\lambda$ is injective [4], and hence also a bijection; see [5, 12]. We can now choose a smooth positive function of compact support S_λ which is close to R_λ in $C^\alpha_{-5/2}$, and solve the equation

$$(-8\Delta + R_\lambda)\psi_\lambda = R_\lambda - S_\lambda.$$

It follows from the above that ψ_λ is small in $C^{2,\alpha}_{-1/2}$. Taking $\phi_\lambda = 1 + \psi_\lambda$, we see that the metrics $\phi_\lambda^4 g_\lambda$ have positive scalar curvature, quasiconvex coordinate spheres, and form a continuous path from g_0 to a smooth metric $\tilde{g}_1 = \phi_1^4 g_1$. Since $R_1 - S_1$ is of compact support, and g_1 is flat outside a compact set, it follows that $\tilde{g}_1 \in C^{k,\alpha}_{-1}$ for all k. On each coordinate sphere equipped with the metric γ induced by $\tilde{g}_1$, it is possible, using the techniques of [8, Chapter 2], to find a uniformization factor $r^2 e^{2v}$, with bounds as required in $\mathcal{M}_1$, so that $\bar{\gamma} = r^{-2} e^{-2v} \gamma$ is a round metric with surface area $4\pi r^2$. We conclude that the continuous path $\phi_\lambda^4 g_\lambda$ joins $g \in \mathcal{M}_0$ to a metric $\tilde{g}_1 \in \mathcal{M}_1$. □

It is important to note that since the round metric $\bar{\gamma}$ on the coordinate spheres will in general vary with r, it is most likely necessary to change the background flat metric when writing $\tilde{g}_1$ as in (7). Nevertheless, these two flat metrics are asymptotic as $r \to \infty$; see [13] for details.

As outlined in the Introduction, the deformation is obtained in the next three steps by deforming g_λ explicitly inside a ball B_{r_0}, while solving (12) outside B_{r_0} with the deformation of β_λ, v_λ and R_λ defined so that $\kappa_\lambda, \bar{H}_\lambda > 0$, $R_\lambda \geq 0$, and so that theorems from Section 2 guarantee global existence, asymptotic behavior as $r \to \infty$, and continuity of u_λ in $H^{4,\alpha}_{[r_0,\infty)}$. Note that in order to ensure continuity at the end point of the deformation, it is necessary to have a $\bar{H}_\lambda$ uniformly bounded below by a positive constant. Now, if g_λ is a path in $\mathcal{M}_i$, $i = 1, 2, 3$, such that for some $0 < r' < r_0$, $g_\lambda|_{B_{r_0}}$ is continuous in C^2, and $r(1 - u_\lambda)$, $r\beta_\lambda$, and rv_λ are continuous in $H^{4,\alpha}_{[r',\infty)}$, then g_λ is continuous in $C^{2,\alpha}_{-1}$.

Lemma 8. $\mathcal{M}_1$ *is connected to* $\mathcal{M}_2$*.*

Proof. Let $g \in \mathcal{M}_1$, choose r_1 such that $2\kappa - R > 0$ for $r < r_1$, and r_0 with $r_0 < r_1$. For each $\lambda \in [0, 1]$, define $g_\lambda = g$ inside B_{r_0}, $\beta_\lambda = \beta$, and $v_\lambda = v$ everywhere. We have $\kappa_\lambda, \bar{H}_\lambda > 0$. Let $\varphi(r)$ be a smooth cut-off function on $[0, \infty)$, satisfying $0 \leq \varphi \leq 1$, $\varphi = 1$ on $[0, r_0]$, and $\varphi = 0$ on $[r_1, \infty)$. Define $\varphi_\lambda(r) = (1 - \lambda) + \lambda\varphi(r)$ and define $R_\lambda = \varphi_\lambda R$. Then R_λ is monotonically decreasing in λ, $R_\lambda = R$ on B_{r_0}, and $\operatorname{supp}(R_1) \subset B_{r_1}$. Thus, Theorems 3 and 5 can be used to solve equation (12) on $[r_0, \infty) \times \mathbb{S}^2$ for $u_\lambda \in H^{4,\alpha}$. The continuity of u_λ and m_λ with respect to λ is obtained from Theorem 6. Clearly, $g_1 \in \mathcal{M}_2$ and the lemma follows. □

Lemma 9. $\mathcal{M}_2$ *is connected to* $\mathcal{M}_3$*.*

Proof. Let $g \in \mathcal{M}_2$, and put $R_\lambda = R$. For $\lambda \in [1, \infty)$ define $\tilde{\beta}_\lambda = (\phi_\lambda)^*\beta$, $\tilde{v}_\lambda = (\phi_\lambda)^* v$, where $\phi_\lambda(r, \theta) = (\lambda r, \theta)$. Note that $r\tilde{\beta}_\lambda$ and $r\tilde{v}_\lambda$ are continuous in $H^{6,\alpha}_I$ since $r\tilde{\beta}_\lambda$ and $r\tilde{v}_\lambda$ are uniformly bounded in $H^{8,\alpha}_I$. Now, let $\beta_\lambda = \varphi\beta + (1 - \varphi)\tilde{\beta}_\lambda$, and define v_λ by $e^{2v_\lambda} = \varphi e^{2v} + (1 - \varphi)e^{2\tilde{v}_\lambda}$, where $\varphi(r)$ is a cut-off function as in the proof of the previous lemma. It follows from (9) that

$$e^{2v_\lambda}\bar{H}_\lambda = \varphi e^{2v}\bar{H} + (1 - \varphi)e^{2\tilde{v}_\lambda}\tilde{H}_\lambda + (e^{2v} - e^{2\tilde{v}_\lambda})r\varphi',$$

where $\tilde{H}_\lambda = (\phi_\lambda)^*\bar{H}$. Thus, since v and $\tilde{v}_\lambda$ tend to zero as $r \to \infty$, it follows that if r_0 and r_1/r_0 are large enough, then $\bar{H}_\lambda > 0$ for $r > r_0$. Furthermore, in view of (10), the Gauss curvature κ_λ is given by

$$\begin{aligned} r^2 e^{2v_\lambda}\kappa_\lambda = 1 - \not{\Delta} v_\lambda = r^2(\varphi e^{2v}\kappa + (1 - \varphi)e^{2\tilde{v}_\lambda}\tilde{\kappa}_\lambda) + \left|\not{\nabla} v\right|^2 \\ - 2e^{-2v_\lambda}(\varphi e^{2v}\left|\not{\nabla} v\right|^2 + (1 - \varphi)e^{2\tilde{v}_\lambda}\left|\not{\nabla}\tilde{v}_\lambda\right|^2), \end{aligned}$$

where $\tilde{\kappa}_\lambda = (\phi_\lambda)^*\kappa$. Hence, since also $\left|\not{\nabla} v\right|$ and $\left|\not{\nabla}\tilde{v}_\lambda\right|$ tend to zero as $r \to \infty$, we see that if r_0 is large enough, then $\kappa_\lambda > 0$ for $r > r_0$. By choosing r_0 large enough, we can also ensure that $R = 0$ outside B_{r_0}. As in the proof of the previous lemma, we define $g_\lambda = g$ inside B_{r_0}, and solve equation (12) for u_λ outside B_{r_0}. The existence of u_λ for all $r \geq r_0$ is now guaranteed by Theorem 1. Note that outside B_{r_1}, $\beta_\lambda = \tilde{\beta}_\lambda$, $v_\lambda = \tilde{v}_\lambda$, hence we have a uniformly bounded solution $\lambda^{-1}(\phi_\lambda)^* m$ of equation (22), and therefore Theorem 5 applies to give the asymptotic behavior of u_λ for $r \to \infty$. It is easy to see that the path g_λ can be extended continuously to $[1, \infty]$, and since β_λ and v_λ tend to zero as $\lambda \to \infty$ for $r > r_1$, it follows that $g_\infty \in \mathcal{M}_3$. □

Lemma 10. *$\mathcal{M}_3$ is connected to $\mathcal{M}_4$.*

Proof. Let $g \in \mathcal{M}_3$, choose $r_0 > 0$ so that R, β, and v are supported in B_{r_0}, and let $\varphi(r)$ be a cut-off function as above. For $\lambda \in [1, \infty)$, define $\beta_\lambda = \varphi \left(\phi_{1/\lambda}\right)^* \beta$, and $\tilde{v}_\lambda = \left(\phi_{1/\lambda}\right)^* v$. Let $r_0 < r_1 < r_2$, and let $\zeta(r)$ be a smooth function satisfying $0 \le \zeta \le 1$, $\zeta = 1$ on $[0, r_1]$, $\zeta = 0$ on $[r_2, \infty)$. Let $\tilde{H}_\lambda = \left(\phi_{1/\lambda}\right)^* \bar{H}$, and note that $\inf \tilde{H}_\lambda = \inf \bar{H} > 0$, hence $h = \inf \tilde{H}_\lambda$ is independent of λ. Let $f(r)$ be a smooth non-negative function supported on $[r_0, r_2]$, satisfying on $[r_0, r_1]$ the inequality:

$$f > -r^{-a-1}(\varphi h + r\varphi'),$$

where $a = \max\{-(2 + 2r\partial_r \tilde{v}_\lambda), 0\}$. Let $\xi(r) = r^a \int_{r_0}^r f(s)\, ds$, $\psi = \xi + \varphi$, and $v_\lambda = \zeta(\tilde{v}_\lambda + \frac{1}{2} \log \psi)$. Since $\xi \ge 0$, it now follows from (9) that we have for $r_0 < r < r_1$:

$$\psi \bar{H}_\lambda = \xi(2 + 2r\partial_r \tilde{v}_\lambda) + r\xi' + \varphi \tilde{H}_\lambda + r\varphi' > -a\xi + r\xi' - r^{a+1} f = 0.$$

Furthermore, since $\beta_\lambda = 0$ in $B_{r_2} \setminus B_{r_1}$, we can also choose ζ so as to ensure that $\bar{H}_\lambda > 0$ there, provided that r_2/r_1 is large enough. Since the deformation of v is radial, it is clear that $\kappa_\lambda > 0$. Define $g_\lambda = \lambda^2 \left(\phi_{1/\lambda}\right)^* g$ in B_{r_0}, and as before, solve for u_λ in (12) on $[r_0, \infty) \times \mathbb{S}^2$ with initial data $u_\lambda|_{S_{r_0}}$. Global existence and asymptotic behavior as $r \to \infty$ is obtained from Theorems 1 and 4. The path g_λ can be extended continuously to $[1, \infty]$, and since β_λ, v_λ, and R_λ tend to zero as $\lambda \to \infty$, it follows that u_λ tends to 1. Consequently g_∞ is flat, and the continuous path g_λ joins g_1 to a flat metric $g_\infty \in \mathcal{M}_4$. However, note that $g_1 \ne g$, since clearly $v_1 \ne v$. In order to complete the proof of the lemma, we now define a continuous path g_λ, $\lambda \in [0, 1]$, between g and g_1. Define $g_\lambda = g$ in B_{r_0}, $\beta_\lambda = \beta$, $R_\lambda = R$, and $v_\lambda = \zeta\big(v + (1/2)\log(1 - \lambda + \lambda\psi)\big)$. Then from (9) we get $e^{2v_\lambda} \bar{H}_\lambda = (1 - \lambda) e^{2v_0} \bar{H}_0 + \lambda e^{2v_1} \bar{H}_1 > 0$ in $[r_0, r_1]$, and as before $\bar{H}_\lambda > 0$ also in $[r_1, r_2]$ provided r_2/r_1 is large enough. Clearly, as before, we have $\kappa_\lambda > 0$. Thus, we can solve for u_λ in (12) as above. □

Note that due to the uniformization step in Lemma 7, the final flat metric g_∞ in Lemma 10 may be different from the background flat metric originally given in $\mathcal{M}_0$.

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University of Alabama at Birmingham, Birmingham, AL 35205
E-mail address: `smith@math.uab.edu`

University of Alabama at Birmingham, Birmingham, AL 35205
E-mail address: `weinstei@math.uab.edu`