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PRINCIPAL BUNDLES WITH PARABOLIC STRUCTURE

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Abstract. We define a principal bundle analog of vector bundles with parabolic structure over a normal crossing divisor. Various results on parabolic vector bundles and usual principal bundles are extended to the context of parabolic principal bundles.

1. Introduction

Although parabolic vector bundles have been existing for a long time, a satisfactory definition of parabolic G-bundles is still lacking. In this note we define a G-bundle analog of vector bundles with parabolic structure over a normal crossing divisor and rational parabolic weights.

Let Y be an algebraic variety. Nori showed that a principal G-bundle over Y can equivalently be thought of as a functor from the category of finite-dimensional G-modules to the category of vector bundles over Y which is compatible with the tensor product, direct sum and dualization operation. A G-bundle P sends a G-module V to the associated vector bundle $P \times_G V$. The content of the observation of Nori is that this functor determines P uniquely.

Like in the case of usual vector bundles, the parabolic vector bundles also have *parabolic* tensor product, direct sum and dualization operations. Therefore, the above definition of G-bundles can be extended to parabolic bundles. This is done in Section 2. However, it is desirable to have a parabolic G-bundle as concrete object, a scheme, *representing* this functor.

If we restrict ourselves to the situation where the parabolic divisor is a normal crossing divisor and the parabolic weights are all rational, then a parabolic G-bundle over X can be realized as an equivariant G-bundle over a ramified Galois cover Y over X.

Let Γ denote the Galois group of the cover Y of X. Given an equivariant G-bundle P on Y, we may take the quotient of P by Γ, which is a principal G-bundle over the complement of the parabolic divisor. Using the properties of this quotient that we establish, it is possible to characterize all G-spaces that are of the form P/Γ. This enables us to define a parabolic G-bundle as a G-space with certain properties.

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2. The parabolic analog of principal bundles

Let G be an affine algebraic group over $\mathbb{C}$. We will briefly recall a reformulation of the definition of principal G-bundles constructed by Nori in [10], [11].

Let X be a connected smooth projective variety over $\mathbb{C}$. Denote by $\mathrm{Vect}(X)$ the category of vector bundles over X. The category $\mathrm{Vect}(X)$ is equipped with an algebra structure defined by the tensor product operation

$$\mathrm{Vect}(X) \times \mathrm{Vect}(X) \longrightarrow \mathrm{Vect}(X),$$

which sends any pair (E, F) to $E \otimes F$, and the direct sum operation $\bigoplus$, making it an additive tensor category in the sense of [6, Definition 1.15].

Let $\mathrm{Rep}(G)$ denote the category of all finite-dimensional complex left representations of the group G, or equivalently, left G-modules. By a G-module (or a representation) we shall always mean a left G-module (or a left representation).

Given a principal G-bundle P over X and a left G-module V, the associated fiber bundle $P \times_G V$ has a natural structure of a vector bundle over X. Consider the functor

$$F(P) : \mathrm{Rep}(G) \longrightarrow \mathrm{Vect}(X), \tag{2.1}$$

which sends any V to the vector bundle $P \times_G V$ and sends any homomorphism between two G-modules to the naturally induced homomorphism between the two corresponding vector bundles. The functor $F(P)$ enjoys several natural abstract properties. For example, it is compatible with the algebra structures of $\mathrm{Rep}(G)$ and $\mathrm{Vect}(X)$ defined using direct sum and tensor product operations. Furthermore, $F(P)$ takes an exact sequence of G-modules to an exact sequence of vector bundles, it also takes the trivial G-module $\mathbb{C}$ to the trivial line bundle on X, and the dimension of V coincides with the rank of the vector bundle $F(P)(V)$.

In Proposition 2.9 of [10] (also Proposition 2.9 of [11]) it has been established that the collection of principal G-bundles over X is in bijective correspondence with the collection of functors from $\mathrm{Rep}(G)$ to $\mathrm{Vect}(X)$ satisfying the abstract properties that the functor $F(P)$ in (2.1) enjoys. The four abstract properties are described on page 31 of [10], where they are marked F1–F4. The bijective correspondence sends a principal bundle P to the functor $F(P)$ defined in (2.1).

We will define parabolic G-bundles along the above lines.

Let D be an effective divisor on X. For a coherent sheaf E on X, the image of $E \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$ in E will be denoted by $E(-D)$. The following definition of a parabolic sheaf was introduced in [9].

Definition 2.1. Let E be a torsion free $\mathcal{O}_X$-coherent sheaf on X. A *quasi-parabolic* structure on E over D is a filtration by $\mathcal{O}_X$-coherent subsheaves

$$E = F_1(E) \supset F_2(E) \supset \cdots \supset F_l(E) \supset F_{l+1}(E) = E(-D)\,.$$

The integer l is called the *length of the filtration.* A *parabolic structure* is a quasi-parabolic structure, as above, together with a system of *weights* $\{\alpha_1, \ldots, \alpha_l\}$ such that

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{l-1} < \alpha_l < 1,$$

where the weight α_i corresponds to the subsheaf $F_i(E)$.

We shall denote the parabolic sheaf defined above by (E, F_*, α_*). When there is no possibility of confusion, it will be denoted by E_*.

For a parabolic sheaf (E, F_*, α_*), define the following filtration $\{E_t\}_{t\in\mathbb{R}}$ of coherent sheaves on X parameterized by $\mathbb{R}$:

$$E_t := F_i(E)(-[t]D),$$

where $[t]$ is the integral part of t and $\alpha_{i-1} < t - [t] \leq \alpha_i$, with the convention that $\alpha_0 = \alpha_l - 1$ and $\alpha_{l+1} = 1$.

If the underlying sheaf E is locally free, then E_* will be called a parabolic vector bundle. *Henceforth, all parabolic sheaves will be assumed to be parabolic vector bundles.*

The class of parabolic vector bundles that are dealt with in the present work satisfies certain conditions which will be explained now. The first condition is that all parabolic divisors are assumed to be *divisors with normal crossings.* In other words, any parabolic divisor is assumed to be reduced, each of its irreducible components is smooth, and furthermore the irreducible components intersect transversally. The second condition is that all the parabolic weights are *rational numbers.* Before stating the third condition, we remark that quasi-parabolic filtrations on a vector bundle can be defined by giving filtrations by subsheaves of the restriction of the vector bundle to each component of the parabolic divisor. The third and final condition states that on each component of the parabolic divisor the filtration is given by *subbundles.* The precise formulation of the last condition is given in [4, Assumptions 3.2 (1)]. *Henceforth, all parabolic vector bundles will be assumed to satisfy the above three conditions.* Note that if $\dim X = 1$, then all conditions except the rationality of weights are automatically satisfied.

Let $\mathrm{PVect}(X, D)$ denote the category whose objects are parabolic vector bundles over X with parabolic structure over the divisor D satisfying the above three conditions, and the morphisms of the category are homomorphisms of parabolic vector bundles. For any two parabolic bundles $E_*, V_* \in \mathrm{PVect}(X, D)$, their parabolic tensor product $E_* \otimes V_*$ is also an element of $\mathrm{PVect}(X, D)$. (See [3], [13] for the definition of parabolic tensor product.) The trivial line bundle with the trivial parabolic structure (this means that the length of the parabolic flag is zero) acts as the identity element for the parabolic tensor multiplication. The parabolic tensor product operation on $\mathrm{PVect}(X, D)$ has all the abstract properties enjoyed by the usual tensor product operation of vector bundles.

The direct sum of two vector bundles with parabolic structures has an obvious parabolic structure. Evidently, $\mathrm{PVect}(X, D)$ is closed under the operation of taking direct sum. The category $\mathrm{PVect}(X, D)$ is an additive tensor category with the direct sum and the parabolic tensor product operation. Also the notion of dual of a vector bundle can be extended to the context of parabolic bundles.

For an integer $N \geq 2$, let $\mathrm{PVect}(X, D, N) \subseteq \mathrm{PVect}(X, D)$ denote the subcategory consisting of all parabolic vector bundles all of whose parabolic weights are multiples of $1/N$. It is straightforward to check that $\mathrm{PVect}(X, D, N)$ is closed under all the above operations, namely parabolic tensor product, direct sum and taking the parabolic dual.

Definition 2.2. A *parabolic principal G-bundle* with parabolic structure over D is a *functor F* from the category $\mathrm{Rep}(G)$ to the category $\mathrm{PVect}(X, D)$ satisfying the four conditions of [10] mentioned earlier. The functor is further required to satisfy the condition that there is an integer N, which depends on the functor, such that the image of the functor is contained in $\mathrm{PVect}(X, D, N)$.

A justification of the above definition will be provided by the following proposition:

Proposition 2.3. *The collection of parabolic $GL(n,\mathbb{C})$-bundles on X with parabolic structure over D is identified, in a bijective fashion, with the subclass of* $\mathrm{PVect}(X,D)$ *consisting of parabolic vector bundles of rank n. Under this identification, a parabolic $GL(n,\mathbb{C})$-bundle is identified with the parabolic vector bundle associated to it for the standard representation of $GL(n,\mathbb{C})$ on $\mathbb{C}^n$.*

The proof is given in [1].

3. Semistability for parabolic principal bundles

Fix an ample line bundle L over X. For a coherent sheaf F over X, define the degree $\deg(F)$ of F using L.

Let P be a principal G-bundle over X. A reduction of the structure group of P to a subgroup $Q \subset G$ is defined by giving a section of the fiber bundle $P/Q \to X$ with fiber G/Q. Henceforth, we will assume G to be semisimple.

Definition 3.1 ([12])**.** Let $P(Q)$ denote a reduction of the structure group of P to a maximal parabolic subgroup $Q \subset G$ over an open set $U \subseteq X$ with $\mathrm{codim}(X-U) \geq 2$. The principal G-bundle P is called *semistable* (respectively, *stable*) if for every such situation, the line bundle over U associated to $P(Q)$ for any character of Q dominant with respect to a Borel subgroup contained in Q, is of nonpositive degree (respectively, strictly negative degree). The principal bundle P is called *polystable* if there is a reduction of the structure group of P to M, namely $P(M) \subset P$, where $M \subset G$ is a maximal reductive subgroup of a parabolic subgroup of G, such that $P(M)$ is a stable principal M-bundle and furthermore, for any character of M trivial on the intersection with the center of G, the corresponding line bundle associated to $P(M)$ is of degree zero.

Given a homomorphism $G \to H$ and a principal G-bundle P, the space $P \times_G H$, where G acts as left translations of H, has a natural structure of a principal H-bundle. This construction of a principal H-bundle from the principal G-bundle P is called the *extension of the structure group* of P to H. From [12] we know that if P is a semistable (respectively, polystable) G-bundle, then $P \times_G H$ is a semistable (respectively, polystable) H-bundle.

If H is reductive and P is a semistable G-bundle, then the extension $P \times_G H$ is also semistable ([12]). This indicates how we may define semistability in the context of parabolic G-bundles. The following proposition, which is proved in [1], will be needed for that purpose. The definition of parabolic semistable and parabolic polystable vector bundles is given in [9] and [8].

Proposition 3.2. *Let E_* , $F_* \in$ $\mathrm{PVect}(X,D)$ be two parabolic semistable* (*respectively, parabolic polystable*) *vector bundles on X. Then the parabolic tensor product $E_* \otimes F_*$ is also parabolic semistable* (*respectively, parabolic polystable*)*, and furthermore the parabolic dual of E_* is also parabolic semistable* (*respectively, parabolic polystable*)*.*

Definition 3.3. Let P_* be a functor from the category $\mathrm{Rep}(G)$ to the category $\mathrm{PVect}(X,D)$ defining a parabolic G-bundle as in Definition 2.2. This functor P_* will be called a *parabolic semistable* (*respectively, parabolic polystable*) *principal*

G-bundle if and only if the image of the functor is contained in the category of parabolic semistable (respectively, parabolic polystable) vector bundles.

We observe that Proposition 3.2 implies that the subcategory of $\mathrm{PVect}(X, D)$ consisting of parabolic semistable (respectively, parabolic polystable) vector bundles is closed under tensor product. Furthermore, to check parabolic semistability (respectively, parabolic polystability) it is not necessary to check the criterion for $V_1 \otimes V_2 \in \mathrm{Rep}(G)$ if it has been checked for V_1 and V_2 individually.

The following proposition is proved in [1].

Proposition 3.4. *A parabolic G-bundle P_* is parabolic semistable* (*respectively, parabolic polystable*) *if and only if there is a faithful representation*

$$\rho : G \longrightarrow GL(V)$$

such that the corresponding parabolic vector bundle $P_*(\rho)$ is parabolic semistable (*respectively, parabolic polystable*)*. Consequently, if for one faithful representation ρ the parabolic vector bundle $P_*(\rho)$ is parabolic semistable* (*respectively, parabolic polystable*)*, then for any representation ρ', the parabolic vector bundle $P_*(\rho')$ is parabolic semistable* (*respectively, parabolic polystable*)*.*

The following proposition is immediate.

Proposition 3.5. *If $G \longrightarrow H$ is a homomorphism of groups and if P_* is a parabolic semistable* (*respectively, parabolic polystable*) *G-bundle, then the parabolic H-bundle, obtained by the extension of the structure group of P_*, is also parabolic semistable* (*respectively, parabolic polystable*)*.*

The following theorem is proved in [1]. The definition of Chern classes of parabolic bundles is given in [5].

Theorem 3.6. *A parabolic G-bundle P_* admits a unitary flat connection if and only if the following two conditions hold*:

(1) *P_* is parabolic polystable*;
(2) $c_2^*(P_*(\mathrm{ad})) = 0$*, where c_2^* is the second parabolic Chern class.*

Furthermore, a parabolic G-bundle satisfying the above two conditions admits a unique unitary flat connection.

For parabolic vector bundles over a Riemann surface, this theorem was proved in [8].

4. Orbifold bundles and parabolic bundles

The following is the well-known "covering lemma" of Y. Kawamata.

Given X and a divisor $D = \sum_{i=1}^{c} D_i$ as above, where D_i are irreducible components, and an integer N, there is a Galois cover

$$p : Y \longrightarrow X,$$

where Y is a connected smooth projective manifold, $p^{-1}(D)_{\mathrm{red}}$ is a normal crossing divisor and $p^{-1}(D_i) = k_i N (p^{-1}(D_i))_{\mathrm{red}}$ ([7]). We fix such a cover. Let $\Gamma := \mathrm{Gal}(Y/X)$.

An *orbifold* G-bundle over Y is a G-bundle P together with a lift of the action of Γ on the total space of P as bundle automorphisms. An orbifold G-bundle will be called a (Γ, G)-bundle.

Let P be a (Γ, G)-bundle on Y. It is called Γ-*semistable* (respectively, Γ-*polystable*) if and only if P satisfies the condition of semistability (respectively, polystability) in Definition 3.1 with all reductions of the structure group being Γ-equivariant.

Proposition 4.1. *A* (Γ, G)*-bundle* P *is* Γ*-semistable* (*respectively,* Γ*-polystable*) *if and only if* P *is semistable* (*respectively, polystable*) *according to Definition 3.1.*

If we fix N, the space of all parabolic principal G-bundles satisfying the condition that the image of the corresponding functor in Definition 2.2 is contained in $\mathrm{PVect}(X, D, N)$ will be denoted by $PG(X, D, N)$.

Let $[\Gamma, G, N]$ denote the collection of (Γ, G)-bundles on Y satisfying the following two conditions:

1. for a general point y of an irreducible component of $(p^*D_i)_{\mathrm{red}}$, the action of Γ_y on P_y is of order a divisor of N;
2. for a general point y of an irreducible component of a ramification divisor for p not contained in $(p^*D)_{\mathrm{red}}$, the action of Γ_y on P_y is the trivial action.

In [1] we construct a map from $[\Gamma, G, N]$ to $PG(X, D, N)$ and also a map from $PG(X, D, N)$ to $[\Gamma, G, N]$. Using the result of Nori, the constructions reduce to a construction of maps between parabolic vector bundles and orbifold vector bundles. Such constructions are given in [4]. In [1] we prove

Theorem 4.2. *The above-mentioned maps between* $PG(X, D, N)$ *and* $[\Gamma, G, N]$ *are inverses of each other. Furthermore,* $P_* \in PG(X, D, N)$ *is parabolic semistable* (*respectively, parabolic polystable*) *if and only if the corresponding bundle* $P \in [\Gamma, G, N]$ *is* Γ*-semistable* (*respectively,* Γ*-polystable*).

The constructions in Theorem 4.2 suggest a more concrete definition of a parabolic G-bundle.

5. Alternative definition of parabolic G-bundles

Henceforth we will assume $\dim X = 1$. This is only to simplify the exposition. Everything in this section can be extended to higher dimensions.

Suppose $P \in [\Gamma, G, N]$. Then both G and Γ act on the total space of P. It is easy to see that these two actions commute. Therefore, the quotient space P/Γ, which we will denote by P', has an action of G. Let

$$\phi : P' \longrightarrow P'/G = X \tag{5.1}$$

be the quotient map. For any $y \in Y$, let $\Gamma_y \subset \Gamma$ denote the isotropy of y. If Γ_y is trivial, then the action of G on $\phi^{-1}(p(y))$ is free and transitive. If $x \in X \setminus D$ and $y \in p^{-1}(x)$, then Γ_y may be nontrivial, but by assumption, the action of Γ_y on the fiber P_y is trivial. Therefore, the map ϕ in (5.1) defines a principal G-bundle over $X \setminus D$.

For any $x \in D$, the action of Γ_y on P_y for any $y \in p^{-1}(x)$ is of order a divisor of N. Note that Γ_y is a cyclic group whose order is a multiple of N.

It can be checked that the variety P' is smooth, but the map ϕ is not smooth over any point $x \in D$ that has the property that the action of Γ_y on P_y is nontrivial, where $y \in p^{-1}(x)$ ([2]).

The G-bundle P can be constructed back from P'. Indeed, P is the normalization of the fiber product $P' \times_X Y$ ([2]). This shows that $[\Gamma, G, N]$ can be identified with G-spaces of certain type.

To explain this, let Q be a smooth variety on which G acts with the property that there are only finitely many orbits on which the isotropy is nontrivial. Suppose, moreover that at each of these finite orbits, the isotropy is a cyclic subgroup of order which divides N. Furthermore, assume that $Q/G = X$ and the image of the orbits with nontrivial isotropy is contained in D. The quotient is in the sense of geometric invariant theory.

From the earlier remarks on P' it follows that the space of all such G-spaces, $Q \longrightarrow Q/G = X$, is in bijective correspondence with $[\Gamma, G, N]$.

Definition 5.1. A parabolic G-object is a smooth variety Q with an action of G satisfying the above properties.

Using Theorem 4.2 it follows that parabolic G-objects are in bijective correspondence with parabolic G-bundles.

For a finite-dimensional G-module V, consider the diagonal action of G on $Q \times V$. The invariant sections on G-saturated open sets define a vector bundle over X with a parabolic structure. The parabolic structure is obtained the same way a parabolic bundle is obtained in [4] from an orbifold bundle. Therefore, we get a functor from $\mathrm{Rep}(G)$ to $\mathrm{PVect}(X, D, N)$ that sends any V to the parabolic vector bundle constructed this way. In other words, we get a parabolic G-bundle according to Definition 2.2. For the parabolic G-object P', the parabolic G-bundle obtained this way coincides with the one associated to P by Theorem 4.2.

Fix a maximal torus T and a Borel subgroup B of G, with $T \subset B$. Let H be a closed subgroup of G. A reduction of the structure group of Q to H is a section $\sigma : X \to Q/H$. So, in particular, H acts on $f^{-1}(\sigma(X))$, where $f : Q \to Q/H$ is the projection. If W is a finite-dimensional H-module, then consider $f^{-1}(\sigma(X)) \times_H W$, which is a parabolic vector bundle over X. This parabolic vector bundle will be denoted by $Q(W)_*$.

Definition 5.2. A parabolic G-object Q is semistable (respectively, stable) if for every reduction of structure group of Q to H, where H is any parabolic subgroup containing B, and any nontrivial dominant character χ of P, the parabolic line bundle $Q(\chi)_*$ has nonpositive (respectively, negative) parabolic degree.

Polystability of parabolic G-objects can be defined similarly.

A parabolic G-object Q is semistable (respectively, stable) if and only if the corresponding parabolic G-bundle in the sense of Definition 2.2 is semistable (respectively, stable) [2]. Definition 5.2 can be reformulated in terms of the parabolic degree of the parabolic vector bundle associated to the pullback of the relative tangent bundle.

Using Proposition 4.1, the Harder-Narasimhan reduction of the structure group of an orbifold G-bundle is simply the Harder-Narasimhan reduction of the underlying G-bundle. Now using the identification of orbifold G-bundles with parabolic G-objects the notion of Harder-Narasimhan reduction extends to the context of parabolic G-objects.

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