

GENERATORS AND RELATIONS FOR SCHUR ALGEBRAS

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ABSTRACT. We obtain a presentation of Schur algebras (and q -Schur algebras) by generators and relations, one which is compatible with the usual presentation of the enveloping algebra (quantized enveloping algebra) corresponding to the Lie algebra \mathfrak{gl}_n of $n \times n$ matrices. We also find several new bases of Schur algebras.

1. INTRODUCTION

The classical Schur algebra $S(n, d)$ (over \mathbb{Q}) may be defined as the algebra $\text{End}_{\Sigma_d}(V^{\otimes d})$ of linear endomorphisms on the d th tensor power of an n -dimensional \mathbb{Q} -vector space V commuting with the action of the symmetric group Σ_d , acting by permutation of the tensor places (see [6]). Schur algebras determine the polynomial representation theory of general linear groups, and they form an important class of quasi-hereditary algebras.

We identify V with \mathbb{Q}^n . Then $V_{\mathbb{Z}} = \mathbb{Z}^n$ is a lattice in V . One can define a \mathbb{Z} -order $S_{\mathbb{Z}}(n, d)$ (the integral Schur algebra) in $S(n, d)$ by setting $S_{\mathbb{Z}}(n, d) = \text{End}_{\Sigma_d}(V_{\mathbb{Z}}^{\otimes d})$. For any field K , one then obtains the Schur algebra $S_K(n, d)$ over K by setting $S_K(n, d) = S_{\mathbb{Z}}(n, d) \otimes_{\mathbb{Z}} K$. Moreover, $S_{\mathbb{Q}}(n, d) \cong S(n, d)$.

In the quantum case one can replace \mathbb{Q} by $\mathbb{Q}(v)$ (v an indeterminate), V by an n -dimensional $\mathbb{Q}(v)$ -vector space, and Σ_d by the corresponding Hecke algebra $\mathbf{H}(\Sigma_d)$. Then the resulting commuting algebra, $\mathbf{S}(n, d)$, is known as the q -Schur algebra, or quantum Schur algebra. It appeared first in work of Dipper and James, and, independently, Jimbo. Beilinson, Lusztig, and MacPherson [1] have given a geometric construction of $\mathbf{S}(n, d)$ in terms of orbits of flags in vector spaces. (See also [5].)

In this situation \mathbb{Z} is replaced by the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, and there is a corresponding “integral” form $\mathbf{S}_{\mathcal{A}}(n, d)$ in $\mathbf{S}(n, d)$. The above quantized objects specialize to their classical versions when $v = 1$. We have more detailed information about the Schur algebras in rank 1; see [2] and [3]. Proofs of the main results will appear in [4].

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2. SERRE'S PRESENTATION OF U

Let $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}$ be the root system of type A_{n-1} , with simple roots $\Delta = \{\varepsilon_i - \varepsilon_{i+1} \mid i = 1, \dots, n-1\}$, where $\{\varepsilon_1, \dots, \varepsilon_n\}$ denotes the standard basis of \mathbb{Z}^n . The corresponding set Φ^+ of positive roots is the set $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}$.

The abelian group \mathbb{Z}^n has a bilinear form $(\ , \)$ given by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$ (Kronecker delta). We write $\alpha_j := \varepsilon_j - \varepsilon_{j+1}$.

The enveloping algebra $U = U(\mathfrak{gl}_n)$ is the associative algebra (with 1) on generators e_i, f_i ($i = 1, \dots, n-1$) and H_i ($i = 1, \dots, n$) with relations

$$\begin{aligned} \text{(R1)} \quad & H_i H_j = H_j H_i, \\ \text{(R2)} \quad & e_i f_j - f_j e_i = \delta_{ij} (H_j - H_{j+1}), \\ \text{(R3)} \quad & H_i e_j - e_j H_i = (\varepsilon_i, \alpha_j) e_j, \quad H_i f_j - f_j H_i = -(\varepsilon_i, \alpha_j) f_j, \\ \text{(R4)} \quad & e_i^2 e_j - 2e_i e_j e_i + e_j e_i^2 = 0 \quad (|i - j| = 1), \\ & e_i e_j - e_j e_i = 0 \quad (\text{otherwise}), \\ \text{(R5)} \quad & f_i^2 f_j - 2f_i f_j f_i + f_j f_i^2 = 0 \quad (|i - j| = 1), \\ & f_i f_j - f_j f_i = 0 \quad (\text{otherwise}). \end{aligned}$$

For $\alpha \in \Phi$, let x_α denote the corresponding root vector of \mathfrak{gl}_n viewed as an element of U . We have in particular $e_i = x_{\alpha_i}$ and $f_i = x_{-\alpha_i}$.

3. THE QUANTIZED ENVELOPING ALGEBRA

The Drinfeld-Jimbo quantized enveloping algebra $\mathbf{U} = \mathbf{U}_v(\mathfrak{gl}_n)$, by definition, is the $\mathbb{Q}(v)$ -algebra with generators E_i, F_i ($1 \leq i \leq n-1$), $K_i^{\pm 1}$ ($1 \leq i \leq n$) and relations

$$\begin{aligned} \text{(Q1)} \quad & K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \\ \text{(Q2)} \quad & E_i F_j - F_j E_i = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{v - v^{-1}}, \\ \text{(Q3)} \quad & K_i E_j = v^{(\varepsilon_i, \alpha_j)} E_j K_i, \quad K_i F_j = v^{-(\varepsilon_i, \alpha_j)} F_j K_i, \\ \text{(Q4)} \quad & E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad (|i - j| = 1), \\ & E_i E_j - E_j E_i = 0 \quad (\text{otherwise}), \\ \text{(Q5)} \quad & F_i^2 F_j - (v + v^{-1}) F_i F_j F_i + F_j F_i^2 = 0 \quad (|i - j| = 1), \\ & F_i F_j - F_j F_i = 0 \quad (\text{otherwise}). \end{aligned}$$

For $\alpha \in \Phi^+$, let E_α and F_α be the positive and negative quantum root vectors of \mathbf{U} as defined by Jimbo [8]. We have in particular $E_i = E_{\alpha_i}$ and $F_i = F_{\alpha_i}$.

4. MAIN RESULTS: CLASSICAL CASE

We now give a precise statement of our main results in the classical case. The first result describes a presentation by generators and relations of the Schur algebra over the rational field \mathbb{Q} . This presentation is compatible with the usual presentation (see section 2) of $U = U(\mathfrak{gl}_n)$.

Theorem 1. *Over \mathbb{Q} , the Schur algebra $S(n, d)$ is isomorphic to the associative algebra (with 1) on the same generators as for U subject to the same relations (R1)–(R5) as for U , together with the additional relations:*

$$(R6) \quad H_1 + H_2 + \cdots + H_n = d,$$

$$(R7) \quad H_k(H_k - 1) \cdots (H_k - d) = 0 \quad (k = 1, \dots, n).$$

The next result gives a basis for the Schur algebra which is the analogue of the Poincare-Birkhoff-Witt (PBW) basis of U .

Theorem 2. *The algebra $S(n, d)$ has a “truncated PBW” basis (over \mathbb{Q}) which can be described as follows. Fix an integer k_0 with $1 \leq k_0 \leq n$ and set*

$$G = \{x_\alpha \mid \alpha \in \Phi\} \cup \{H_k \mid k \in \{1, \dots, n\} - \{k_0\}\}$$

and fix some ordering for this set. Then the set of all monomials in G (with specified order) of total degree at most d is a basis for $S(n, d)$.

Our next result constructs the integral Schur algebra $S_{\mathbb{Z}}(n, d)$ in terms of the generators given above. We need some more notation. For B in \mathbb{N}^n , we write

$$H_B = \prod_{k=1}^n \binom{H_k}{b_k}.$$

Let $\Lambda(n, d)$ be the subset of \mathbb{N}^n consisting of those $\lambda \in \mathbb{N}^n$ satisfying $|\lambda| = d$; this is the set of n -part compositions of d . Given $\lambda \in \Lambda(n, d)$ we set $1_\lambda := H_\lambda$. One can show that the collection $\{1_\lambda\}$ as λ varies over $\Lambda(n, d)$ forms a set of pairwise orthogonal idempotents in $S_{\mathbb{Z}}(n, d)$ which sum to the identity element.

For $m \in \mathbb{N}$ and $\alpha \in \Phi$, set $x_\alpha^{(m)} := x_\alpha^m / (m!)$. Any product in U of elements of the form

$$x_\alpha^{(m)}, \quad \binom{H_k}{m} \quad (m \in \mathbb{N}, \alpha \in \Phi, k \in \{1, \dots, n\}),$$

taken in any order, will be called a *Kostant monomial*. Note that the set of Kostant monomials is multiplicatively closed, and spans $U_{\mathbb{Z}}$. We define a function χ (content function) on Kostant monomials by setting

$$\chi(x_\alpha^{(m)}) := m \varepsilon_{\max(i, j)}, \quad \chi\left(\binom{H_k}{m}\right) := 0,$$

where $\alpha = \varepsilon_i - \varepsilon_j$ ($i \neq j$), and by declaring that $\chi(XY) = \chi(X) + \chi(Y)$ whenever X, Y are Kostant monomials.

We write any $A \in \mathbb{N}^{\Phi^+}$ in “multi-index” form $A = (a_\alpha)_{\alpha \in \Phi^+}$ and set $|A| := \sum_{\alpha \in \Phi^+} a_\alpha$. For $A, C \in \mathbb{N}^{\Phi^+}$ we write

$$e_A = \prod_{\alpha \in \Phi^+} x_\alpha^{(a_\alpha)}, \quad f_C = \prod_{\alpha \in \Phi^-} x_\alpha^{(c_\alpha)}$$

where the products in e_A and f_C are taken relative to any fixed orders on Φ^+ and Φ^- .

Theorem 3. *The integral Schur algebra $S_{\mathbb{Z}}(n, d)$ is the subring of $S(n, d)$ generated by all divided powers $e_j^{(m)}, f_j^{(m)}$ ($j \in \{1, \dots, n-1\}, m \in \mathbb{N}$). Moreover, this algebra has a \mathbb{Z} -basis consisting of all*

$$(B1) \quad e_A 1_\lambda f_C \quad (\chi(e_A f_C) \preceq \lambda)$$

and another such basis consisting of all

$$(B2) \quad f_A 1_\lambda e_C \quad (\chi(f_A e_C) \preceq \lambda)$$

where $A, C \in \mathbb{N}^{\Phi^+}$, $\lambda \in \Lambda(n, d)$, and where \preceq denotes the componentwise partial ordering on \mathbb{N}^n .

Finally, we have another presentation of the Schur algebra by generators and relations. This presentation has the advantage that it possesses a quantization of the same form, in which we can set $v = 1$ to recover the classical version.

Theorem 4. *For each $i \in \{1, \dots, n-1\}$ write $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$. The algebra $S(n, d)$ is the associative algebra (with 1) given by generators 1_λ ($\lambda \in \Lambda(n, d)$), e_i , f_i ($i \in \{1, \dots, n-1\}$) subject to the relations*

$$(S1) \quad \begin{aligned} 1_\lambda 1_\mu &= \delta_{\lambda, \mu} 1_\lambda \quad (\lambda, \mu \in \Lambda(n, d)), \\ \sum_{\lambda \in \Lambda(n, d)} 1_\lambda &= 1, \end{aligned}$$

$$(S2) \quad \begin{aligned} e_i 1_\lambda &= \begin{cases} 1_{\lambda + \alpha_i} e_i & \text{if } \lambda + \alpha_i \in \Lambda(n, d), \\ 0 & \text{otherwise,} \end{cases} \\ f_i 1_\lambda &= \begin{cases} 1_{\lambda - \alpha_i} f_i & \text{if } \lambda - \alpha_i \in \Lambda(n, d), \\ 0 & \text{otherwise,} \end{cases} \\ 1_\lambda e_i &= \begin{cases} e_i 1_{\lambda - \alpha_i} & \text{if } \lambda - \alpha_i \in \Lambda(n, d), \\ 0 & \text{otherwise,} \end{cases} \\ 1_\lambda f_i &= \begin{cases} f_i 1_{\lambda + \alpha_i} & \text{if } \lambda + \alpha_i \in \Lambda(n, d), \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

$$(S3) \quad e_i f_j - f_j e_i = \delta_{ij} \sum_{\lambda \in \Lambda(n, d)} (\lambda_j - \lambda_{j+1}) 1_\lambda,$$

along with the Serre relations (R4), (R5), for $i, j \in \{1, \dots, n-1\}$.

5. MAIN RESULTS: QUANTIZED CASE

Our main results in the quantized case are similar in form to those in the classical case. The first result describes a presentation by generators and relations of the quantized Schur algebra over the rational function field $\mathbb{Q}(v)$. This presentation is compatible with the usual presentation (see section 3) of $\mathbf{U} = \mathbf{U}_v(\mathfrak{gl}_n)$.

Theorem 1'. *Over $\mathbb{Q}(v)$, the q -Schur algebra $\mathbf{S}(n, d)$ is isomorphic with the associative algebra (with 1) on the same generators as for \mathbf{U} subject to the same relations (Q1)–(Q5) as for \mathbf{U} , together with the additional relations:*

$$(Q6) \quad K_1 K_2 \cdots K_n = v^d,$$

$$(Q7) \quad (K_j - 1)(K_j - v)(K_j - v^2) \cdots (K_j - v^d) = 0 \quad (j = 1, \dots, n).$$

The next result gives a basis for the q -Schur algebra which is the analogue of the Poincare-Birkhoff-Witt (PBW) type basis of \mathbf{U} , given in Lusztig [9, Prop. 1.13].

Theorem 2'. *The algebra $\mathbf{S}(n, d)$ has a “truncated PBW type” basis which can be described as follows. Fix an integer k_0 with $1 \leq k_0 \leq n$ and set*

$$G' = \{E_\alpha, F_\alpha \mid \alpha \in \Phi^+\} \cup \{K_k \mid k \in \{1, \dots, n\} - \{k_0\}\}$$

and fix some ordering for this set. Then the set of all monomials in G' (with specified order) of total degree at most d is a basis for $\mathbf{S}(n, d)$.

Note that setting $v = 1$ in the basis of Theorem 2' does not yield the basis of Theorem 2 since K_i acts as the identity when $v = 1$.

Our next result constructs the integral q -Schur algebra $\mathbf{S}_A(n, d)$ in terms of the generators given above. For B in \mathbb{N}^n , we write

$$K_B = \prod_{j=1}^n \begin{bmatrix} K_j \\ b_j \end{bmatrix},$$

where for indeterminates X, X^{-1} satisfying $XX^{-1} = X^{-1}X = 1$ and any $t \in \mathbb{N}$ we formally set

$$\begin{bmatrix} X \\ t \end{bmatrix} := \prod_{s=1}^t \frac{Xv^{-s+1} - X^{-1}v^{s-1}}{v^s - v^{-s}},$$

an expression that makes sense if X is replaced by any invertible element of a $\mathbb{Q}(v)$ -algebra.

Given $\lambda \in \Lambda(n, d)$ we set (when we are in the quantum case) $1_\lambda := K_\lambda$. It turns out that, just as in the classical case, the collection $\{1_\lambda\}$ as λ varies over $\Lambda(n, d)$ forms a set of pairwise orthogonal idempotents in $\mathbf{S}_A(n, d)$ which sum to the identity element.

Let $[m]$ denote the quantum integer $[m] := (v^m - v^{-m})/(v - v^{-1})$ and set $[m]! := [m][m-1] \cdots [1]$. Then the q -analogues of the divided powers of root vectors are defined to be $E_\alpha^{(m)} := E_\alpha/[m]!$ and $F_\alpha^{(m)} := F_\alpha/[m]!$. Any product in \mathbf{U} of elements of the form

$$E_\alpha^{(m)}, \quad F_\alpha^{(m)}, \quad K_j^{\pm 1}, \quad \begin{bmatrix} K_j \\ m \end{bmatrix} \quad (m \in \mathbb{N}, \alpha \in \Phi, j \in \{1, \dots, n\}),$$

taken in any order, will be called a *Kostant monomial*. As before, the set of Kostant monomials is multiplicatively closed, and spans \mathbf{U}_A . The definition of content χ of a Kostant monomial is obtained similarly, by setting

$$\chi(E_\alpha^{(m)}) = \chi(F_\alpha^{(m)}) := m \varepsilon_{\max(i, j)}, \quad \chi(K_l^{\pm 1}) = \chi\left(\begin{bmatrix} K_l \\ m \end{bmatrix}\right) := 0$$

where $\alpha = \varepsilon_i - \varepsilon_j \in \Phi^+$, and by declaring that $\chi(XY) = \chi(X) + \chi(Y)$ whenever X, Y are Kostant monomials. For $A, C \in \mathbb{N}^{\Phi^+}$ we write

$$E_A = \prod_{\alpha \in \Phi^+} E_\alpha^{(a_\alpha)}, \quad F_C = \prod_{\alpha \in \Phi^+} F_\alpha^{(c_\alpha)}$$

where the products in E_A and F_C are taken relative to any two specified orderings on Φ^+ .

Theorem 3'. *The integral q -Schur algebra $\mathbf{S}_A(n, d)$ is the subring of $\mathbf{S}(n, d)$ generated by all quantum divided powers $E_j^{(m)}, F_j^{(m)}$ ($j \in \{1, \dots, n-1\}, m \in \mathbb{N}$), along*

with the elements $\begin{bmatrix} K_j \\ m \end{bmatrix}$ ($j \in \{1, \dots, n\}$, $m \in \mathbb{N}$). Moreover, this algebra has a basis over \mathcal{A} consisting of all

$$(B1') \quad E_A 1_\lambda F_C \quad (\chi(e_A f_C) \preceq \lambda)$$

and another such basis consisting of all

$$(B2') \quad F_A 1_\lambda E_C \quad (\chi(f_A e_C) \preceq \lambda)$$

where $A, C \in \mathbb{N}^{\Phi^+}$, $\lambda \in \Lambda(n, d)$, and where \preceq denotes the componentwise partial ordering on \mathbb{N}^n .

Unlike the truncated PBW basis, the bases of Theorem 3' do specialize when $v = 1$ to their classical analogues given in Theorem 3.

Finally, we have another presentation of the q -Schur algebra by generators and relations. This presentation has the advantage that by setting $v = 1$, we recover the classical version given in Theorem 4. The relations of the presentation are similar to relations that hold for Lusztig's modified form $\bar{\mathbf{U}}$ of \mathbf{U} . (See [9, Chap. 23].)

Theorem 4'. For each $i \in \{1, \dots, n-1\}$ write $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$. The algebra $\mathbf{S}(n, d)$ is the associative algebra (with 1) given by generators 1_λ ($\lambda \in \Lambda(n, d)$), E_i , F_i ($i \in \{1, \dots, n-1\}$) subject to the relations

$$(S1') \quad \begin{aligned} 1_\lambda 1_\mu &= \delta_{\lambda, \mu} 1_\lambda \quad (\lambda, \mu \in \Lambda(n, d)), \\ \sum_{\lambda \in \Lambda(n, d)} 1_\lambda &= 1, \end{aligned}$$

$$(S2') \quad \begin{aligned} E_i 1_\lambda &= \begin{cases} 1_{\lambda + \alpha_i} E_i & \text{if } \lambda + \alpha_i \in \Lambda(n, d), \\ 0 & \text{otherwise,} \end{cases} \\ F_i 1_\lambda &= \begin{cases} 1_{\lambda - \alpha_i} F_i & \text{if } \lambda - \alpha_i \in \Lambda(n, d), \\ 0 & \text{otherwise,} \end{cases} \\ 1_\lambda E_i &= \begin{cases} E_i 1_{\lambda - \alpha_i} & \text{if } \lambda - \alpha_i \in \Lambda(n, d), \\ 0 & \text{otherwise,} \end{cases} \\ 1_\lambda F_i &= \begin{cases} F_i 1_{\lambda + \alpha_i} & \text{if } \lambda + \alpha_i \in \Lambda(n, d), \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

$$(S3') \quad E_i F_j - F_j E_i = \delta_{ij} \sum_{\lambda \in \Lambda(n, d)} [\lambda_j - \lambda_{j+1}] 1_\lambda,$$

along with the q -Serre relations (Q4), (Q5), for $i, j \in \{1, \dots, n-1\}$.

6. OTHER RESULTS

6.1. Triangular decomposition. One can define the plus, minus, and zero parts of Schur algebras in terms of the generators, as follows. (These subalgebras have been studied before.) The plus part $S^+(n, d)$ (resp., the minus part $S^-(n, d)$) is the subalgebra of $S(n, d)$ generated by all x_α , $\alpha \in \Phi^+$ (resp., $\alpha \in \Phi^-$). The zero part $S^0(n, d)$ is the subalgebra generated by all H_j , $j = 1, \dots, n$. We also have the Borel Schur algebras $S^{\geq 0}(n, d)$ (resp., $S^{\leq 0}(n, d)$), the subalgebra generated by $S^+(n, d)$ (resp., $S^-(n, d)$) together with $S^0(n, d)$.

The appellations $S_{\mathbb{Z}}^+(n, d)$, $S_{\mathbb{Z}}^-(n, d)$, $S_{\mathbb{Z}}^0(n, d)$, $S_{\mathbb{Z}}^{\geq 0}(n, d)$, $S_{\mathbb{Z}}^{\leq 0}(n, d)$ will denote the intersection of the appropriate algebra from above with the integral form $S_{\mathbb{Z}}(n, d)$.

The algebra $S = S(n, d)$ has a triangular decomposition $S = S^+S^0S^-$. We show that in this decomposition one can permute the three factors in any order. Moreover, the same result holds over \mathbb{Z} .

The zero part $S_{\mathbb{Z}}^0(n, d)$ is the algebra generated by all $\binom{H_j}{m}$ for $j = 1, \dots, n$ and $m \in \mathbb{N}$. We give in [4] a presentation of $S^0(n, d)$ by generators and relations. In particular, we prove that $H_B = 0$ whenever $|B| > d$ ($B \in \mathbb{N}^n$) and that the set of (pairwise orthogonal) idempotents 1_λ , $\lambda \in \Lambda(n, d)$, is a \mathbb{Z} -basis of $S_{\mathbb{Z}}^0(n, d)$.

$S_{\mathbb{Z}}^+(n, d)$ (resp., $S_{\mathbb{Z}}^-(n, d)$) is the algebra generated by all divided powers $x_\alpha^{(m)}$ for $\alpha \in \Phi^+$ (resp., $\alpha \in \Phi^-$) and $m \in \mathbb{N}$. It is an easy consequence of the commutation formulas (S2) that each generator x_α is nilpotent of index $d + 1$; see [4] for details. Moreover, from our main results we see easily that the set of all e_A (resp., f_A) such that $|A| \leq d$ is a \mathbb{Z} -basis for the algebra $S_{\mathbb{Z}}^+(n, d)$ (resp., $S_{\mathbb{Z}}^-(n, d)$).

It also follows immediately from our results that the set of all $e_A 1_\lambda$ (resp., $1_\lambda f_A$) satisfying $\chi(e_A) \preceq \lambda$ is an integral basis for the Borel Schur algebra $S_{\mathbb{Z}}^{\geq 0}(n, d)$ (resp., $S_{\mathbb{Z}}^{\leq 0}(n, d)$).

Similar statements to the above hold in the quantum case. In particular, as a corollary of the commutation formulas (S2') one can give a simple proof of [7, Prop. 2.3].

6.2. Explicit reduction formulas. Fix a positive root α and write $\alpha = \varepsilon_i - \varepsilon_j$ for $i < j$.

Then from [2] we have the following reduction formulas in $S_{\mathbb{Z}}(n, d)$, for any $a, b, c \in \mathbb{N}$:

$$(1) \quad f_\alpha^{(a)} \binom{H_j}{b} e_\alpha^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \binom{k-1}{s-1} \binom{b+k}{k} f_\alpha^{(a-k)} \binom{H_j}{b+k} e_\alpha^{(c-k)},$$

$$(2) \quad e_\alpha^{(a)} \binom{H_i}{b} f_\alpha^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \binom{k-1}{s-1} \binom{b+k}{k} e_\alpha^{(a-k)} \binom{H_i}{b+k} f_\alpha^{(c-k)},$$

where $s = a + b + c - d$ and $s \geq 1$.

We do not have a q -analogue of these formulas.

The results of [3] give another type of reduction formula for $\mathbf{S}_{\mathcal{A}}(n, d)$. If $b_1, b_2 \in \mathbb{N}$ satisfy $b_1 + b_2 = d$, set $\lambda := b_1 \varepsilon_i + b_2 \varepsilon_j \in \Lambda(n, d)$. Then for all $s \geq 1$ we have:

$$(3) \quad E_\alpha^{(a)} 1_\lambda F_\alpha^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \begin{bmatrix} k-1 \\ s-1 \end{bmatrix} \begin{bmatrix} b_1+k \\ k \end{bmatrix} E_\alpha^{(a-k)} 1_{\lambda+k\alpha} F_\alpha^{(c-k)},$$

$$(4) \quad F_\alpha^{(a)} 1_\lambda E_\alpha^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \begin{bmatrix} k-1 \\ s-1 \end{bmatrix} \begin{bmatrix} b_2+k \\ k \end{bmatrix} F_\alpha^{(a-k)} 1_{\lambda-k\alpha} E_\alpha^{(c-k)},$$

where $s = a + b_1 + c - d$ in (3) and $s = a + b_2 + c - d$ in (4).

The classical analogues of formulas (3) and (4) hold in $S(n, d)$.

6.3. Another presentation (for $n = 2$). We have the following result from [2], which presents $S(2, d)$ as a quotient of $U(\mathfrak{sl}_2)$.

Theorem 5. *Over \mathbb{Q} , the Schur algebra $S(2, d)$ is isomorphic with the associative algebra (with 1) generated by e, f, h subject to the relations:*

$$\begin{aligned} he - eh &= 2e; & ef - fe &= h; & hf - fh &= -2f; \\ (h + d)(h + d - 2) \cdots (h - d + 2)(h - d) &= 0. \end{aligned}$$

Moreover, this algebra has a “truncated PBW” basis over \mathbb{Q} consisting of all $f^a h^b e^c$ such that $a + b + c \leq d$.

Note that if we eliminate the last relation, we have the usual presentation of $U(\mathfrak{sl}_2)$ over \mathbb{Q} . The last relation is the minimal polynomial of h in the representation on tensor space. The problem of presenting $S(n, d)$ as a quotient of $U(\mathfrak{sl}_n)$ seems to be more difficult for $n > 2$.

We also have from [3] the following q -version of the above, which presents the q -Schur algebra $\mathbf{S}(2, d)$ as a quotient of the quantized enveloping algebra $\mathbf{U}(\mathfrak{sl}_2)$.

Theorem 5'. *Over $\mathbb{Q}(v)$, the quantum Schur algebra $\mathbf{S}(2, d)$ is isomorphic to the algebra generated by $E, F, K^{\pm 1}$ subject to the relations:*

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, \\ KEK^{-1} &= v^2E, & KFK^{-1} &= v^{-2}F, \\ EF - FE &= \frac{K - K^{-1}}{v - v^{-1}}, \\ (K - v^d)(K - v^{d-2}) \cdots (K - v^{-d+2})(K - v^{-d}) &= 0. \end{aligned}$$

6.4. Hecke algebras. Suppose that $n \geq d$. Let $\omega = (1^d)$. Then the subalgebra $1_\omega \mathbf{S}(n, d) 1_\omega$ is isomorphic with the Hecke algebra $\mathbf{H}(\Sigma_d)$. The nonzero elements of the basis (B1') of the form $1_\omega E_A F_C 1_\omega$ form a basis of the Hecke algebra; similarly for elements of the basis (B2') of the form $1_\omega F_A E_C 1_\omega$.

Moreover, taking $d = n$, we can see that $\mathbf{H} = \mathbf{H}(\Sigma_n)$ is generated by the elements $1_\omega E_i F_i 1_\omega$ ($1 \leq i \leq n-1$). Alternatively, \mathbf{H} is generated by the $1_\omega F_i E_i 1_\omega$ ($1 \leq i \leq n-1$). One can easily describe the relations on these generators, thus obtaining a presentation of \mathbf{H} which is closely related to one in [10].

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