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SOME NONEXISTENCE RESULTS FOR HIGHER-ORDER EVOLUTION INEQUALITIES IN CONE-LIKE DOMAINS

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Abstract. Nonexistence of global (positive) solutions of semilinear higher-order evolution inequalities

$$\frac{\partial^k u}{\partial t^k} - \Delta u^m \geq |u|^q, \quad \frac{\partial^k u}{\partial t^k} - \Delta u \geq |x|^\sigma u^q, \quad \frac{\partial^k u}{\partial t^k} - \operatorname{div}(|x|^\alpha Du) \geq u^q$$

with $k = 1, 2, \dots,$ in cone-like domains is studied. The critical exponents q^* are found and the nonexistence results are proved for $1 < q \leq q^*$. Remark that the corresponding result for $k = 1$ (parabolic problem) is sharp.

1. Introduction

This paper is devoted to nonexistence of global (nontrivial) solutions for some semilinear evolution differential inequalities in unbounded cone-like domains. We can formulate the main goal of our investigation as follows: "There are many well-known results on nonexistence in cone-like domains for parabolic equations. Some results can be obtained also for hyperbolic problems. What are the common properties of the evolution equations that imply the nonexistence of global solutions?"

For our proof we use the test function method. We refer the interested readers to the book by Mitidieri and Pohozaev [14] and the references therein, and the papers [13, 18, 17, 15, 7].

Details of the proof can be found in the preprint [11]. The corresponding elliptic and parabolic problems in a cone were investigated (using the test function method) in [8, 14] and [10]. Some types of higher-order evolution equations in a ball were considered in [9].

Let S^{N-1} be the unit sphere in $\mathbf{R}^N$, $N \geq 3$, and (r, ω) polar coordinates in $\mathbf{R}^N$. Let K_ω be a domain of S^{N-1} with smooth boundary ∂K_ω. We denote by K the cone

$$K = \{(r, \omega) : 0 < r < +\infty, \omega \in K_\omega\}.$$

The lateral surface of the cone K is ∂K. "Cone-like domain" K_R, $R > 0$ denotes the set $\{x \in K : |x| > R\}$ with full surface ∂K_R.

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Recall that the Laplace operator Δ in polar coordinates (r,ω) has the form

$$\Delta = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_\omega = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\omega,$$

where Δ_ω denotes the Laplace–Beltrami operator on the unit sphere $S^{N-1}\subset\mathbf{R}^N$.

We shall use the first Helmholtz eigenvalue $\lambda_\omega \equiv \lambda_1(K_\omega) > 0$ and corresponding eigenfunction $\Phi(\omega)$ for the Dirichlet problem of Δ_ω in K_ω,

$$\begin{cases} \Delta_\omega\Phi + \lambda\Phi = 0 \quad \text{in } K_\omega, \\ \Phi|_{\partial K_\omega} = 0. \end{cases} \tag{1.1}$$

It is well known that $\Phi(\omega) > 0$ for $\omega\in K_\omega$. We assume $\Phi(\omega)\le 1$.

Let Ω be an unbounded domain in $\mathbf{R}^{N+1}$ with piecewise smooth boundary. We shall use the anisotropic Sobolev spaces $W_q^{2,k}(\Omega)$, and the local space $L_{q,\mathrm{loc}}(\Omega)$, whose elements belong to $L_q(\Omega')$ for any compact subset Ω': $\overline{\Omega'}\subset\Omega$. Denote the space of continuous functions by $C(\overline{\Omega})$.

The expression $\int_{\partial K}\frac{\partial u}{\partial n}\,dx$ denotes the integral of the directional derivative of u with respect to the outward normal n to the cone lateral surface ∂K.

2. Main results

Let $k\in\mathbf{N}$. Our model problem has the form

$$\begin{cases} \frac{\partial^k u}{\partial t^k} - \Delta u \ge |u|^q \quad \text{in } K\times(0,\infty), \\ u|_{\partial K\times[0,\infty)} \ge 0, \\ \frac{\partial^{k-1}u}{\partial t^{k-1}}|_{t=0} \ge 0, \qquad u\not\equiv 0. \end{cases} \tag{2.1}$$

Definition 2.1. Let $u(x,t)\in C(\overline{K}\times[0,\infty))$ and suppose the locally summable traces $\frac{\partial^i u}{\partial t^i}$, $i=1,\dots,k-1$, as $t=0$, are well defined. The function $u(x,t)$ is called a weak solution of (2.1) if for any nonnegative test function $\varphi(x,t)\in W_\infty^{2;k}(K\times(0,\infty))$ with compact support, such that $\varphi|_{\partial K\times(0,\infty)}=0$, the following inequality holds:

$$\begin{aligned} &\int_0^\infty\int_{\partial K} u\frac{\partial\varphi}{\partial n}\,dxdt + \int_0^\infty\int_K u\left((-1)^k\frac{\partial^k\varphi}{\partial t^k} - \Delta\varphi\right)dxdt \\ &\qquad \ge \int_0^\infty\int_K |u|^q\varphi\,dxdt + \sum_{i=1}^{k-1}(-1)^i\int_K \frac{\partial^{k-1-i}u}{\partial t^{k-1-i}}(x,0)\frac{\partial^i\varphi}{\partial t^i}(x,0)\,dx \\ &\qquad\qquad + \int_K \frac{\partial^{k-1}u}{\partial t^{k-1}}(x,0)\varphi(x,0)\,dx. \end{aligned}$$

Let us introduce the parameters

$$s^* = \frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda_\omega}, \qquad s_* = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda_\omega}; \tag{2.2}$$

here λ_ω is the first eigenvalue of the problem (1.1) introduced above. It is evident that $s^* - s_* = N-2$. These parameters go back to Kondrat'ev [6].

Theorem 2.2. *Let*

$$1 < q \le q_k^* = \frac{s^* + 2/k + 2}{s^* + 2/k} = 1 + \frac{2}{s^* + 2/k},$$

where s^* is defined in (2.2)*. Then the problem* (2.1) *has no nontrivial global solution.*

The following theorem includes sharp results for a parabolic equation and inequality (i.e., $k = 1$):

Theorem 2.3. *Let*

$$1 < q \le q_1^* = 1 + \frac{2}{s^* + 2},$$

where s^* is defined in (2.2)*. Then the problem*

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u \ge |u|^q & \text{in } K \times (0,\infty), \\ u|_{\partial K \times [0,\infty)} \ge 0, & \\ u|_{t=0} \ge 0, & u \not\equiv 0, \end{cases} \tag{2.3}$$

has no nontrivial global solution in the sense of Definition 2.1.

Results of this type were mentioned in the surveys [12] and [2]. Remark that we do not impose any additional constraints on the solution.

For $k = 2$ (hyperbolic problem) we have

Theorem 2.4. *Let*

$$1 < q \le q_2^* = 1 + \frac{2}{s^* + 1},$$

where s^* is defined in (2.2)*. Then the problem*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u \ge |u|^q & \text{in } K \times (0,\infty), \\ u|_{\partial K \times [0,\infty)} \ge 0, & \\ \frac{\partial u}{\partial t}|_{t=0} \ge 0, & u \not\equiv 0, \end{cases} \tag{2.4}$$

has no nontrivial global solution in the sense of Definition 2.1.

We do not know whether this result is sharp. But for hyperbolic problems in the whole $\mathbf{R}^N$ we can compare this critical exponent q_2^* with the classical Kato's result in $\mathbf{R}^N$. Namely, putting $s^* = N - 2$, we obtain Kato's exponent

$$q^* = \frac{N+1}{N-1},$$

but without any assumptions concerning the support of eventual solution. Such results for the general hyperbolic inequalities were obtained, among others, in [1, 4] (using Kato's technique) and [17] (with the test function approach). As shown in [17], this exponent is sharp in the weak sense.

It is interesting that formally passing to the limit as $k \to \infty$, we arrive at the sharp elliptic critical exponent (see, for example, [7, 8, 14] and the references therein)

$$q_\infty^* = 1 + \frac{2}{s^*}.$$

Let us consider the system

$$\begin{cases} \frac{\partial^k u}{\partial t^k} - \Delta u \ge |v|^{q_1} & \text{in } K \times (0,\infty), \\ \frac{\partial^k v}{\partial t^k} - \Delta v \ge |u|^{q_2} & \text{in } K \times (0,\infty), \\ u|_{\partial K \times [0,\infty)} \ge 0, \quad v|_{\partial K \times [0,\infty)} \ge 0, & \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}|_{t=0} \ge 0, \quad \frac{\partial^{k-1} v}{\partial t^{k-1}}|_{t=0} \ge 0, & u \not\equiv 0, \quad v \not\equiv 0. \end{cases} \tag{2.5}$$

Theorem 2.5. *Let* $q_1 > 1$, $q_2 > 1$ *and*

$$\max\{\gamma_1, \gamma_2\} \geq \frac{s^* + 2/k}{2}, \qquad \textit{where} \quad \gamma_1 = \frac{q_1 + 1}{q_1 q_2 - 1}, \quad \gamma_2 = \frac{q_2 + 1}{q_1 q_2 - 1}.$$

Then (2.5) *has no nontrivial global solution.*

In this theorem the concept of solution is understood in the weak sense of Definition 2.1 (with some evident corrections; see for example [8, 9]).

For the parabolic system ($k = 1$)

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u \geq |v|^{q_1} & \text{in } K \times (0,\infty), \\ \frac{\partial v}{\partial t} - \Delta v \geq |u|^{q_2} & \text{in } K \times (0,\infty), \\ u|_{\partial K \times [0,\infty)} \geq 0, \quad v|_{\partial K \times [0,\infty)} \geq 0, & \\ u|_{t=0} \geq 0, \quad v|_{t=0} \geq 0, & u \not\equiv 0, \quad v \not\equiv 0. \end{cases}$$

we obtain from Theorem 2.5 the well-known condition [12, 2]

$$\max\left\{\frac{q_1 + 1}{q_1 q_2 - 1}, \frac{q_2 + 1}{q_1 q_2 - 1}\right\} \geq \frac{s^* + 2}{2}.$$

Let us consider the inequality of porous medium type,

$$\begin{cases} \frac{\partial^k u}{\partial t^k} - \Delta u^m \geq |u|^q \quad \text{in } K \times (0,\infty), \qquad m \geq 1, \quad q > m, \\ u|_{\partial K \times [0,\infty)} \geq 0, \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}|_{t=0} \geq 0, \qquad u \not\equiv 0. \end{cases} \tag{2.6}$$

Definition 2.6. Let $u(x,t) \in C(\overline{K} \times [0,\infty))$ and suppose the locally summable traces $\frac{\partial^i u}{\partial t^i}$, $i = 1, \ldots, k-1$, as $t = 0$, are well defined. The function $u(x,t)$ is called a weak solution of (2.6) if for any nonnegative test function $\varphi(x,t) \in W^{2;k}_\infty(K \times (0,\infty))$ with compact support, such that $\varphi|_{\partial K \times (0,\infty)} = 0$, the following inequality holds:

$$\begin{aligned} \int_0^\infty \int_{\partial K} u^m \frac{\partial \varphi}{\partial n}\, dx dt + (-1)^k \int_0^\infty \int_K u \frac{\partial^k \varphi}{\partial t^k}\, dx dt - \int_0^\infty \int_K u^m \Delta \varphi\, dx dt \\ \geq \int_0^\infty \int_K |u|^q \varphi\, dx dt + \sum_{i=1}^{k-1} (-1)^i \int_K \frac{\partial^{k-1-i} u}{\partial t^{k-1-i}}(x,0) \frac{\partial^i \varphi}{\partial t^i}(x,0)\, dx \\ + \int_K \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \varphi(x,0)\, dx. \end{aligned}$$

Theorem 2.7. *Let*

$$1 \leq m < q \leq q^* = \frac{m(s^* + 2) + 2/k}{s^* + 2/k} = m + 2\frac{m - m/k + 1/k}{s^* + 2/k},$$

where s^* *is defined in* (2.2). *Then the problem* (2.6) *has no nontrivial global solution in the sense of Definition 2.6.*

In the "parabolic" case $k = 1$ we obtain

Theorem 2.8. *Let*

$$1 \leq m < q \leq q^* = m + \frac{2}{s^* + 2},$$

where s^* is defined in (2.2)*. Then the problem*

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u^m \geq |u|^q & \text{in } K \times (0,\infty), \qquad m \geq 1, \quad q > m, \\ u|_{\partial K \times [0,\infty)} \geq 0, & \\ u|_{t=0} \geq 0, & u \not\equiv 0. \end{cases}$$

has no nontrivial global solution in the sense of Definition 2.6.

Now we turn our attention to the singular problem in cone-like domain K_R, $R > 0$,

$$\begin{cases} \frac{\partial^k u}{\partial t^k} - \operatorname{div}(|x|^\alpha Du) \geq u^q & \text{in } K_R \times (0,\infty), \\ u \geq 0, & \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}|_{t=0} \geq 0, & u \not\equiv 0, \end{cases} \tag{2.7}$$

where $-\infty < \alpha < 2$.

Remark that we deal with nonnegative solutions.

Definition 2.9. Let $u(x,t) \in C(\overline{K}_R \times [0,\infty))$ and suppose the locally summable traces $\frac{\partial^i u}{\partial t^i}$, $i = 1, \ldots, k-1$, as $t = 0$, are well defined. The function $u(x,t)$ is called a weak solution of (2.7) if for any nonnegative test function $\varphi(x,t) \in W^{2,k}_\infty(K_R \times (0,\infty))$ with compact support, such that $\varphi|_{\partial K_R \times (0,\infty)} = 0$, the following inequality holds:

$$\begin{aligned} \int_0^\infty \int_{\partial K_R} u|x|^\alpha \frac{\partial \varphi}{\partial n}\, dxdt + \int_0^\infty \int_{K_R} u \left((-1)^k \frac{\partial^k \varphi}{\partial t^k} - \operatorname{div}(|x|^\alpha D\varphi) \right) dxdt \\ \geq \int_0^\infty \int_{K_R} u^q \varphi\, dxdt + \sum_{i=1}^{k-1} (-1)^i \int_{K_R} \frac{\partial^{k-1-i} u}{\partial t^{k-1-i}}(x,0) \frac{\partial^i \varphi}{\partial t^i}(x,0)\, dx \\ + \int_{K_R} \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \varphi(x,0)\, dx. \end{aligned}$$

Let us introduce the parameters

$$s^*_\alpha = \frac{\alpha + N - 2}{2} + \sqrt{\left(\frac{\alpha + N - 2}{2}\right)^2 + \lambda_\omega},$$

$$s_{*\alpha} = -\frac{\alpha + N - 2}{2} + \sqrt{\left(\frac{\alpha + N - 2}{2}\right)^2 + \lambda_\omega}.$$

Theorem 2.10. *Let $-\infty < \alpha < 2$. If*

$$1 < q \leq q^* = \frac{s^*_\alpha + (2-\alpha)/k + 2 - \alpha}{s^*_\alpha + (2-\alpha)/k} = 1 + \frac{2-\alpha}{s^*_\alpha + (2-\alpha)/k},$$

then the problem (2.7) *has no nontrivial nonnegative global solution.*

In the parabolic case we have

Theorem 2.11. *Let $-\infty < \alpha < 2$. If*

$$1 < q \leq q^* = 1 + \frac{2-\alpha}{s^*_\alpha + 2 - \alpha},$$

then the problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(|x|^\alpha Du) \geq u^q & \text{in } K_R \times (0,\infty), \\ u \geq 0, & u \not\equiv 0, \end{cases}$$

has no nontrivial nonnegative global solution.

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