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A NOTE ON THE CONSTRUCTION OF NONSEPARABLE WAVELET BASES AND MULTIWAVELET MATRIX FILTERS OF $L^2(\mathbb{R}^n)$, WHERE $n \geq 2$

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Abstract. In this note, we announce a general method for the construction of nonseparable orthogonal wavelet bases of $L^2(\mathbb{R}^n)$, where $n \geq 2$. Hence, we prove the existence of such type of wavelet bases for any integer $n \geq 2$. Moreover, we show that this construction method can be extended to the construction of n-D multiwavelet matrix filters.

1. Introduction

In this note, we are interested in the construction of dyadic nonseparable compactly supported wavelet bases of $L^2(\mathbb{R}^n)$. This type of wavelet bases is defined as follows.

Definition 1. Consider a discrete set of functions

$$\{\Psi^i_{j,\mathbf{k}}(\boldsymbol{x}) = 2^{-j/2}\Psi^i(2^{-j}\boldsymbol{x} - \mathbf{k}),\ j \in \mathbb{Z},\ \mathbf{k} \in \mathbb{Z}^n,\ i = 1, \dots, 2^n - 1\},$$

obtained by translations and dilations of $2^n - 1$ mother wavelets Ψ^i, $i = 1, \dots, 2^n - 1$. If the set $\{\Psi^i_{j,\mathbf{k}}\}_{i,j,\mathbf{k}}$ satisfies the following three conditions:

(c_1) $\{\Psi^i_{j,\mathbf{k}}\}_{i,j,\mathbf{k}}$ is orthogonal with respect to the usual L^2 inner product

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(\boldsymbol{x})\overline{g}(\boldsymbol{x})\, d\boldsymbol{x},$$

(c_2) $\forall f \in L^2(\mathbb{R}^n)$, $f(\boldsymbol{x}) = \sum_{i=1}^{2^n-1} \sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} \langle f, \Psi^i_{j,\mathbf{k}}\rangle \Psi^i_{j,\mathbf{k}}(\boldsymbol{x})$, where the equality holds a.e., and

(c_3) $\{\Psi^i_{j,\mathbf{k}}\}_{i,j,\mathbf{k}}$ satisfies the stability condition, that is, $\exists c_2 > c_1 > 0$ such that for any $f \in L^2(\mathbb{R}^n)$, we have

$$c_1\|f\|_2^2 \leq \sum_{i=1}^{2^n-1} \sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n} |\langle f, \Psi^i_{j,\mathbf{k}}\rangle|^2 \leq c_2\|f\|_2^2,$$

then $\{\Psi^i_{j,\mathbf{k}}\}_{i,j,\mathbf{k}}$ is called a dyadic orthogonal wavelet basis of $L^2(\mathbb{R}^n)$.

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Note that in general, the set of mother wavelets Ψ^i, $i = 1, \ldots, 2^n - 1$ is related to a mother scaling function Φ which is an appropriate solution of the following refinement equation:

$$\Phi(\boldsymbol{x}) = 2^n \sum_{\mathbf{k} \in \mathbb{Z}^n} \alpha_k \Phi(2\boldsymbol{x} - k), \tag{1}$$

with $\int \Phi(\boldsymbol{x})\, d\boldsymbol{x} = 1$. Note that if in (1), $(\alpha_{\mathbf{k}})_{\mathbf{k}}$ is a finite sequence satisfying $\sum_{\mathbf{k}} \alpha_{\mathbf{k}} = 1$ and if $\mathcal{H}_0$ denotes the n-variate trigonometric polynomial given by

$$\mathcal{H}_0(\omega_1, \ldots \omega_n) = \sum_{\mathbf{k}} \alpha_{\mathbf{k}} e^{-i\mathbf{k}\cdot(\omega_1, \ldots, \omega_n)},$$

then

$$\widehat{\Phi}(\omega_1, \ldots, \omega_n) = \prod_{j=1}^{\infty} \mathcal{H}_0\left(\frac{\omega_1}{2^j}, \ldots, \frac{\omega_n}{2^j}\right).$$

In this case, the different mother wavelets $\Psi^i, i = 1, \ldots, 2^n - 1$ are given by

$$\widehat{\Psi}^i(\omega_1, \ldots, \omega_n) = \mathcal{H}_i\left(\frac{\omega_1}{2}, \ldots, \frac{\omega_n}{2}\right) \widehat{\Phi}\left(\frac{\omega_1}{2}, \ldots, \frac{\omega_n}{2}\right), \quad \forall i = 1, \ldots, 2^n - 1,$$

for some appropriate n-variate trigonometric polynomials, also called high-pass wavelet filters.

Finally, we should mention that unlike the 1-D case where an extensive work has been done, the construction of nonseparable orthogonal wavelet bases is still a challenging problem. To this date, only very few references have dealt with the construction of such multivariate orthogonal wavelet bases in some special cases; see [1]. Nonetheless, many references have dealt with the construction of multidimensional biorthogonal wavelet bases.

Finally, this note is organized as follows. In section 2, we construct the different orthogonal wavelet filters, candidates for generating nonseparable wavelet bases of $L^2(\mathbb{R}^n)$, and study the stability of the associated wavelets. In section 3, we show how to extend the previous construction to the design of multidimensional multiwavelet filters.

Notation. In this note, we denote $\boldsymbol{\omega}_k = (\omega_1, \ldots, \omega_k) \in \mathbb{R}^k, \boldsymbol{\eta}_i = (\eta_i^1, \ldots, \eta_i^n) \in E_n = \{0, \pi\}^n, \pi_k = (\pi, \ldots, \pi) \in \mathbb{R}^k$.

2. Construction of n-D compactly supported and orthogonal wavelet bases

2.1. **Construction of multivariate orthogonal wavelet filters.** We first construct a family of 2-bands orthogonal and compactly supported low-pass wavelet filters (n-variate trigonometric polynomials) $H_n(\omega_1, \ldots, \omega_n)$. This family is given by the following proposition.

Proposition 1. *Let $H_1(\omega)$ and $G_1(\omega) = e^{-i\omega}\overline{H}_1(\omega + \pi)$ be any 1-D low-pass and the corresponding high-pass orthogonal wavelet filters of $L^2(\mathbb{R})$, respectively. Define* *an n-variate low-pass wavelet filter $H_n(\omega_1,\dots,\omega_n)$ by the following iterative process:*

$$\begin{array}{rcl} \forall\ 2\le k\le n & & \textit{choose an integer } 1\le l_k\le k-1,\\ P_{k-1}(\omega_1,\dots,\omega_k) &=& H_{k-1}(2\omega_1,\dots,2\omega_{k-1}),\\ Q_{k-1}(\omega_1,\dots,\omega_k) &=& H_{k-1}(2\omega_1+\pi,\dots,2\omega_{k-1}+\pi),\\ H_k(\omega_1,\dots,\omega_k) &=& P_{k-1}(\omega_1,\dots,\omega_{k-1})H_{l_k}(\omega_{k-l_k+1},\dots,\omega_k)\\ & & +\,Q_{k-1}(\omega_1,\dots,\omega_{k-1})G_{l_k}(\omega_{k-l_k+1},\dots,\omega_k), \end{array}$$

where $G_l(\omega_1,\dots,\omega_l)=e^{-i\omega_l}\overline{H_l}(\omega_1+\pi,\dots,\omega_l+\pi)$. Then $H_n(0,\dots,0)=1$. Moreover, H_n satisfies the following orthogonality condition:

$$|H_n(\omega_1,\dots,\omega_n)|^2+|H_n(\omega_1+\pi,\dots,\omega_n+\pi)|^2=1,\quad \forall(\omega_1,\dots,\omega_n)\in\mathbb{R}^n. \tag{2}$$

Proof. See [6]. $\square$

It is well known that the construction of dyadic, orthogonal and compactly supported wavelet bases of $L^2(\mathbb{R}^n)$ requires the construction of one low-pass filter $\mathcal{H}_0$ and 2^n-1 high-pass filters $\mathcal{H}_i, i=1,\dots,2^n-1$. These different wavelet filters must satisfy the following matrix equation (see [4, 8]):

$$\begin{bmatrix} \overline{\mathcal{H}_0(\boldsymbol{\omega})} & \overline{\mathcal{H}_0(\boldsymbol{\omega}+\boldsymbol{\eta}_1)} & \cdots & \overline{\mathcal{H}_0(\boldsymbol{\omega}+\boldsymbol{\eta}_{2^n-1})}\\ \overline{\mathcal{H}_1(\boldsymbol{\omega})} & \overline{\mathcal{H}_1(\boldsymbol{\omega}+\boldsymbol{\eta}_1)} & \cdots & \overline{\mathcal{H}_1(\boldsymbol{\omega}+\boldsymbol{\eta}_{2^n-1})}\\ \vdots & \vdots & & \vdots\\ \overline{\mathcal{H}_{2^n-1}(\boldsymbol{\omega})} & \overline{\mathcal{H}_{2^n-1}(\boldsymbol{\omega}+\boldsymbol{\eta}_1)} & \cdots & \overline{\mathcal{H}_{2^n-1}(\boldsymbol{\omega}+\boldsymbol{\eta}_{2^n-1})} \end{bmatrix}$$

$$\times\begin{bmatrix} \mathcal{H}_0(\boldsymbol{\omega}) & \mathcal{H}_1(\boldsymbol{\omega}) & \cdots & \mathcal{H}_{2^n-1}(\boldsymbol{\omega})\\ \mathcal{H}_0(\boldsymbol{\omega}+\boldsymbol{\eta}_1) & \mathcal{H}_1(\boldsymbol{\omega}+\boldsymbol{\eta}_1) & \cdots & \mathcal{H}_{2^n-1}(\boldsymbol{\omega}+\boldsymbol{\eta}_1)\\ \vdots & \vdots & & \vdots\\ \mathcal{H}_0(\boldsymbol{\omega}+\boldsymbol{\eta}_{2^n-1}) & \mathcal{H}_1(\boldsymbol{\omega}+\boldsymbol{\eta}_{2^n-1}) & \cdots & \mathcal{H}_{2^n-1}(\boldsymbol{\omega}+\boldsymbol{\eta}_{2^n-1}) \end{bmatrix}=I_{2^n},$$

where the $\boldsymbol{\eta}_i$ are the different points of the set $E_n=\{0,\pi\}^n$. A solution to the above matrix equation is obtained by the use of a family of $n-1$ matrices $D_i\in\mathbb{Z}^{n\times n}$ satisfying the following three properties:

- (P_1) $\forall\boldsymbol{\eta}_j\in E_n$, $D_i(\boldsymbol{\eta}_j)=D_i(\boldsymbol{\eta}_j^s) \bmod (2\pi\mathbb{Z}^n)$, where $\boldsymbol{\eta}_j^s=\pi_n-\boldsymbol{\eta}_j$.
- (P_2) For all $\boldsymbol{\eta}_{j'}\neq\boldsymbol{\eta}_j$ and all $\boldsymbol{\eta}_{j'}\neq\boldsymbol{\eta}_j^s$, we have $D_i(\boldsymbol{\eta}_j)\neq D_i(\boldsymbol{\eta}_{j'}) \bmod (2\pi\mathbb{Z}^n)$.
- (P_3) If $F_i=D_iD_{i-1}\cdots D_1(E_n)/_{2\pi\mathbb{Z}^n}$, then $|F_i|=\frac{|E_n|}{2^i}$ and F_i is a symmetric subset of E_n, i.e. $\forall\boldsymbol{\eta}\in F_i,\ \boldsymbol{\eta}^s\in F_i$.

Remark 1. If $\{e_k\}_{k=1,\dots,n}$ denotes the usual basis of $\mathbb{R}^n$, and if $\forall 1\le i\le n$, we denote D_i the matrix defined by

$$D_i(e_k)=e_k \text{ if } k\neq i,i+1,\quad D_i(e_i)=e_i+e_{i+1},\quad D_i(e_{i+1})=\sum_{k=1,k\neq i}^{n}e_k-e_i,$$

then the family $\{D_i;\ i=1,\dots,n-1\}$ satisfies the above three properties.

The construction of $\mathcal{H}_0$ is then given by the following proposition.

Proposition 2. *Let $H_0(\omega_1,\dots,\omega_n)$ be the n-D filter of Proposition 1. Define $\mathcal{H}_0$ by*

$$\mathcal{H}_0(\omega_1,\dots,\omega_n)=\prod_{k=0}^{n-1}H_0(D_k\cdots D_0(\omega_1,\dots,\omega_n)).$$

Then $\mathcal{H}_0(0,\ldots,0)=1$. *Moreover,* $\mathcal{H}_0$ *satisfies the following orthogonality condition:*

$$\sum_{i=0}^{2^n-1} |\mathcal{H}_0(\boldsymbol{\omega}_n+\boldsymbol{\eta}_i)|^2 = 1, \quad \forall \boldsymbol{\omega}_n \in \mathbb{R}^n. \tag{3}$$

Proof. Since $H_0(0,\ldots,0)=1$, we have

$$\mathcal{H}_0(0,\ldots,0) = \prod_{k=0}^{n-1} H_0(D_k\cdots D_0(0,\ldots,0)) = 1.$$

Moreover, since

$$D_1(\boldsymbol{\eta}_i) = D_1(\boldsymbol{\eta}_i^s) \bmod (2\pi\mathbb{Z}^n), \quad \forall \boldsymbol{\eta}_i \in E_n,$$

it follows that

$$\begin{aligned}
&\sum_{i=0}^{2^n-1} |\mathcal{H}_0(\boldsymbol{\omega}+\boldsymbol{\eta}_i)|^2 \\
&= \sum_{\boldsymbol{\eta}_i\in F_1} \left[|H_0|^2(\boldsymbol{\omega}+\boldsymbol{\eta}_i) + |H_0|^2(\boldsymbol{\omega}+\boldsymbol{\eta}_i^s)\right]\cdot \prod_{k=1}^{n-1} |H_0|^2(D_k\cdots(D_1(\boldsymbol{\omega})+\boldsymbol{\eta}_i)) \\
&= \sum_{\boldsymbol{\eta}_i\in F_1} \prod_{k=1}^{n-1} |H_0|^2(D_k\cdots(D_1(\boldsymbol{\omega})+\boldsymbol{\eta}_i)) \\
&= \sum_{\boldsymbol{\eta}_i\in F_2} \left[|H_0|^2(D_1(\boldsymbol{\omega})+\boldsymbol{\eta}_i) + |H_0|^2(D_1(\boldsymbol{\omega})+\boldsymbol{\eta}_i^s)\right]\cdot \prod_{k=2}^{n-1} |H_0|^2(D_k\cdots(D_1(\boldsymbol{\omega})+\boldsymbol{\eta}_i)) \\
&= \sum_{\boldsymbol{\eta}_i\in F_2} \prod_{k=2}^{n-1} |H_0|^2(D_k\cdots(D_1(\boldsymbol{\omega})+\boldsymbol{\eta}_i)) \\
&\;\vdots \\
&= \sum_{\boldsymbol{\eta}_i\in F_{n-1}} |H_0|^2(D_{n-1}\cdots D_1(\boldsymbol{\omega})+\boldsymbol{\eta}_i) \\
&= |H_0|^2(D_{n-1}\cdots D_1(\boldsymbol{\omega})+\boldsymbol{\eta}_0) + |H_0|^2(D_{n-1}\cdots D_1(\boldsymbol{\omega})+\boldsymbol{\eta}_0^s) = 1.
\end{aligned}$$

□

Once $\mathcal{H}_0$ is constructed, one is faced with the problem of finding the remaining 2^n-1 high-pass wavelet filters $\mathcal{H}_i, i=1,\ldots,2^n-1$, satisfying the following equations:

$$\sum_{\boldsymbol{\eta}_i\in E_n} \mathcal{H}_j(\boldsymbol{\omega}_n+\boldsymbol{\eta}_j)\overline{\mathcal{H}_{j'}}(\boldsymbol{\omega}_n+\boldsymbol{\eta}_i) = \delta_{jj'}, \quad \forall\, 0\le j,j'\le 2^n-1. \tag{4}$$

In general, a solution of (4) is hard to obtain. Nevertheless and thanks to the particular structure of $\mathcal{H}_0$, a solution to the above equations is given by the following theorem. Note that the biorthogonal version of this theorem is given in [5].

Theorem 1. *Let* H_0 *be the n-D wavelet filter of Proposition 1. Let* $D_1, D_2, \ldots, D_{n-1}$ *be a set of matrices with integer coefficients and satisfying the three properties* (P_1), (P_2) *and* (P_3). *Define a filter* H_1 *by*

$$H_1(\boldsymbol{\omega}_n) = e^{-i\omega_1}\overline{H_0}(\boldsymbol{\omega}_n+\pi_n).$$

For $i = 1, \dots, 2^n - 1$, define $\mathcal{H}_i$ by

$$\mathcal{H}_i(\boldsymbol{\omega}_n) = \prod_{j=0}^{n-1} [\epsilon_j^i H_0(D_j D_{j-1} \cdots D_0 \boldsymbol{\omega}_n) + (1 - \epsilon_j^i) H_1(D_j D_{j-1} \cdots D_0 \boldsymbol{\omega}_n)], \tag{5}$$

where $(\epsilon_0^i, \epsilon_1^i, \dots, \epsilon_{n-1}^i)_{i=1,2^n-1}$ are the different points of $\{0,1\}^n \backslash (0,0,\dots,0)$. Then $\mathcal{H}_i, i = 0, 2^n - 1$, is a solution of (4).

Proof. See [6]. □

2.2. **Stability of the orthogonal wavelet bases.** It is well known that a solution of (4) does not ensure the construction of wavelet bases of $L^2(\mathbb{R}^n)$. In fact, the translates of the scaling function Φ derived from $\mathcal{H}_0$ must satisfy the stability condition, that is,

$$0 < c_1 \le \sum_{\mathbf{k} \in \mathbb{Z}^n} |\widehat{\Phi}(\boldsymbol{\omega}_n + 2\pi \mathbf{k})|^2 \le c_2 < +\infty, \tag{6}$$

for some constants c_1, c_2.

It is known (see [3, 7, 9]) that if $T_{\mathcal{H}_0}$ denote the transition operator associated with $\mathcal{H}_0$ and defined by

$$T_{\mathcal{H}_0}(f)(\boldsymbol{\omega}_n) = \sum_{\boldsymbol{\eta}_j \in E_n} |\mathcal{H}_0(\frac{\boldsymbol{\omega}_n}{2} + \boldsymbol{\eta}_j)|^2 f(\frac{\boldsymbol{\omega}_n}{2} + \boldsymbol{\eta}_j)$$

and if V is a finite-dimensional subspace of n-variate trigonometric polynomials and V is invariant under the action of $T_{\mathcal{H}_0}$, then the stability and the orthogonality of our wavelet basis can be deduced from the spectral radius of $T_{\mathcal{H}_0}/_V$, the restriction of $T_{\mathcal{H}_0}$ to V. More precisely, if 1 is a simple eigenvalue of $T_{\mathcal{H}_0}/_V$ and if the other eigenvalues are inside the unit circle, then $\mathcal{H}_0$ gives rise to an orthogonal wavelet basis of $L^2(\mathbb{R}^n)$. The following theorem provides us with an explicit invariant subspace V under $T_{\mathcal{H}_0}$ as well as the different coefficients of the matrix representing $T_{\mathcal{H}_0}/_V$.

Theorem 2. *Let $\mathcal{H}_0$ be the filter of Proposition 2 and*

$$|\mathcal{H}_0(\omega_1, \dots, \omega_n)|^2 = \sum_{m_1, \dots, m_n = -M}^{M} \beta_{m_1, \dots, m_n} \cos \left[\sum_{i=1}^{n} m_i \omega_i \right].$$

Let $V = \operatorname{span} \Omega$, where $\Omega = \{U_{\mathbf{k}} = \cos[\sum_{i=1}^n k_i \omega_i],\ \mathbf{k} = (k_1, \dots, k_n) \in \operatorname{supp} |\mathcal{H}_0|^2\}$. Define a matrix $B_{\mathcal{H}_0}$ by

$$B_{\mathcal{H}_0}(U_{\mathbf{k}}) = 2^{n-1} \sum_{\mathbf{l}, \mathbf{k}, \mathbf{l}+2\mathbf{k} \in \operatorname{supp} |\mathcal{H}_0|^2} \beta_{\mathbf{k}+2\mathbf{l}} [U_{\mathbf{l}+\mathbf{k}} + U_{\mathbf{l}}].$$

Then, $B_{\mathcal{H}_0}$ represents the restricted operator $T_{\mathcal{H}_0}/_V$. In particular, if 1 is a simple eigenvalue of $B_{\mathcal{H}_0}$ and if the other eigenvalues are inside the unit circle, then $\mathcal{H}_0$ generates an orthogonal wavelet basis of $L^2(\mathbb{R}^n)$.

Proof. See [6]. □

3. Construction of n-D multiwavelet filters

In this section we show that the techniques of the previous section can be used for the construction of orthogonal multiwavelet bases of $L^2(\mathbb{R}^n)$. Note that a 1-D orthogonal multiwavelet basis of multiplicity $r \geq 2, r \in \mathbb{N}$ is an orthogonal and stable set generated by translations and dilations of a vector-valued mother multiwavelet function $\Psi = (\psi^1, \dots, \psi^r)^T$. The associated vector-valued multiscaling function $\Phi = (\phi^1, \dots, \phi^r)^T$ satisfies the functional equation

$$\Phi(x) = \sum_{\alpha \in \mathbb{Z}} P_\alpha \Phi(2x - \alpha), \tag{7}$$

where P_α is an $r \times r$ matrix with real coefficients. If we define the trigonometric matrix function $P(\omega)$ by

$$P(\omega) = \frac{1}{2} \sum_{\alpha \in \mathbb{Z}} P_\alpha \exp(-i\alpha\omega), \tag{8}$$

then by applying Fourier transform to (7), the latter can be written as

$$\widehat{\Phi}(\omega) = P\left(\frac{\omega}{2}\right) \cdot \widehat{\Phi}\left(\frac{\omega}{2}\right). \tag{9}$$

Unlike the one-dimensional case, where an extensive work has been done in the theory and the design of 1-D multiwavelet bases, only very few references have dealt with the theory and the design of multidimensional multiwavelet bases [2]. For the sake of simplicity, we will be concerned with the construction of n-D multiwavelet matrix filters in the special case $r = 2$. Moreover, we will use very often the following basic condition B on the low-pass matrix filter $P(\boldsymbol{\omega})$, which is a necessary condition for the existence of multiwavelet basis and it is due to [9]. This condition is given as follows:

Basic condition B. We say that the matrix mask P satisfies the basic condition B, if the following two conditions are satisfied:

- (C_1) $P(0)$ has the form $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right]$, where λ is an $(r-1) \times (r-1)$ matrix such that $\rho(\lambda) < 1$. Here $\rho(\cdot)$ denotes the spectral radius of a given matrix.
- (C_2) $\forall \mu \in \{0,1\}^n$, $e_1^T P(\mu\pi) = \delta_\alpha e_1^T$, where $e_1^T = (1, 0, \dots, 0) \in \mathbb{R}^r$.

It is shown in [9] that if $P(\boldsymbol{\omega})$ satisfies condition B, then the infinite product $\prod_{j=1}^{\infty} P\left(\frac{\boldsymbol{\omega}}{2^j}\right)$ converges to the $r \times r$ matrix $\left[\widehat{\Phi}(\boldsymbol{\omega})A, 0, \dots, 0\right]$, where A is an eigenvector of $P(0)$ corresponding to the eigenvalue 1. Also, it is well known that a necessary condition for the orthonormality of the r components of Φ is the following equality:

$$P(\omega)P^*(\omega) + P(\omega + \pi)P^*(\omega + \pi) = I_r, \tag{10}$$

where $P^* = \overline{P}^T$, the transpose of the conjugate of P. Also, it is well known that the construction of n-D dyadic multiwavelet matrix filters with multiplicity $r = 2$ requires the design of one low-pass 2×2 matrix filter $\mathcal{P}_n(\boldsymbol{\omega})$ and $2^n - 1$ high-pass 2×2 matrix filters $\mathcal{Q}_n^i(\boldsymbol{\omega})$ such that the following matrix equation holds:

$$\begin{bmatrix} \mathcal{P}_n(\boldsymbol{\omega} + \mu\pi) \\ \mathcal{Q}_n^1(\boldsymbol{\omega} + \mu\pi) \\ \vdots \\ \mathcal{Q}_n^{2^n-1}(\boldsymbol{\omega} + \mu\pi) \end{bmatrix}_{\mu \in \{0,1\}^n} \cdot \begin{bmatrix} \mathcal{P}_n(\boldsymbol{\omega} + \mu\pi) \\ \mathcal{Q}_n^1(\boldsymbol{\omega} + \mu\pi) \\ \vdots \\ \mathcal{Q}_n^{2^n-1}(\boldsymbol{\omega} + \mu\pi) \end{bmatrix}^*_{\mu \in \{0,1\}^n} = I_{2^{2n}}. \tag{11}$$

One major difficulty in the design of multidimensional multiwavelet matrix filters is the noncommutativity of the matrix product. As we will see, this extra difficulty does not affect very much the construction method of the previous section. To solve (11), we need the result of the following proposition, whose proof is given in [6].

Proposition 3. *Let P_1, Q_1 be a 1-D matrix filters generating 1-D multiscaling function and multiwavelet with multiplicity 2. Let H_k be the p-variate wavelet filter of Proposition 1. For $2 \leq k \leq n$, define two sequences of matrix filters P_k, Q_k by the following iterative process:*

$$\begin{aligned} \forall\ 2 \leq p \leq k, \qquad & \textit{let} \\ L_p(\boldsymbol{\omega}_p) &= H_p(2\boldsymbol{\omega}_p), \quad M_p(\boldsymbol{\omega}_p) = H_p(2\boldsymbol{\omega}_p + \pi_p), \\ P_p(\boldsymbol{\omega}_p) &= L_p(\boldsymbol{\omega}_p) P_{p-1}(\boldsymbol{\omega}_{p-1}) + \overline{M_p}(\boldsymbol{\omega}_p) Q_{p-1}(\boldsymbol{\omega}_{p-1}), \\ Q_p(\boldsymbol{\omega}_p) &= M_p(\boldsymbol{\omega}_p) P_{p-1}(\boldsymbol{\omega}_{p-1}) - \overline{L_p}(\boldsymbol{\omega}_p) Q_{p-1}(\boldsymbol{\omega}_{p-1}). \end{aligned}$$

Then $P_n(\cdot), Q_n(\cdot)$ is a solution of the following equation:

$$\begin{bmatrix} P_n(\boldsymbol{\omega}_n) & P_n(\boldsymbol{\omega}_n + \pi_n) \\ Q_n(\boldsymbol{\omega}_n) & Q_n(\boldsymbol{\omega}_n + \pi_n) \end{bmatrix} \cdot \begin{bmatrix} P_n^*(\boldsymbol{\omega}_n) & Q_n^*(\boldsymbol{\omega}_n) \\ P_n^*(\boldsymbol{\omega}_n + \pi_n) & Q_n^*(\boldsymbol{\omega}_n + \pi_n) \end{bmatrix} = I_4. \tag{12}$$

Next, the following proposition gives us an n-variate matrix filter candidate for generating an n-D multiscaling function.

Proposition 4. *Let P_n be as given by Proposition 3 and let $D_0 = I_n$ and $D_i, i = 1, \ldots, n-1$ be a set of matrices satisfying the three properties (P_1), (P_2) and (P_3). Define an n-variate matrix filter $\mathcal{P}_n(\boldsymbol{\omega}_n)$ by*

$$\mathcal{P}_n(\boldsymbol{\omega}_n) = \prod_{k=0}^{n-1} P_n(D_{n-k-1} \cdots D_0 \boldsymbol{\omega}_n); \tag{13}$$

then $\mathcal{P}_n$ satisfies the basic condition B. Moreover, $\mathcal{P}_n$ is a solution of the following matrix equation:

$$\sum_{\boldsymbol{\eta}_k \in E_n = \{0,\pi\}^n} \mathcal{P}_n(\boldsymbol{\omega}_n + \boldsymbol{\eta}_k) \mathcal{P}_n^*(\boldsymbol{\omega}_n + \boldsymbol{\eta}_k) = I_2, \quad \forall \boldsymbol{\omega}_n \in \mathbb{R}^n. \tag{14}$$

Proof. See [6]. □

Finally, the different $2^n - 1$ matrix filters candidates for generating the different n-D mother multiwavelet functions are given by the following theorem, whose proof is given in [6].

Theorem 3. *Let P_n, Q_n be as given by Proposition 3 and let $\mathcal{P}_n$ be as given by the previous proposition. Let $D_0 = I_n$ and let $D_1, \ldots, D_{n-1}$ be a set of matrices satisfying the three properties P_1, P_2, P_3. Define $2^n - 1$ matrix filters $\mathcal{Q}_n^i, i = 1, \ldots, 2^n - 1$ by*

$$\mathcal{Q}_n^i(\boldsymbol{\omega}) = \prod_{k=0}^{n-1} \left[\epsilon_k^i P_n(D_{n-k-1} \cdots D_0 \boldsymbol{\omega}) + (1 - \epsilon_k^i) Q_n(D_{n-k-1} \cdots D_0 \boldsymbol{\omega}) \right], \tag{15}$$

where $(\epsilon_0^i, \ldots, \epsilon_{n-1}^i)_{i=1,2^n-1}$ are the different points of $\{0,1\}^n \setminus \{\mathbf{0_n}\}$. The set

$$\{\mathcal{P}_n, \mathcal{Q}_n^1, \ldots, \mathcal{Q}_n^{2^n-1}\}$$

is a solution of (11).

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