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ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE DIRICHLET PROBLEM IN HALF-SPACE FOR LINEAR AND QUASI-LINEAR ELLIPTIC EQUATIONS

VASILY DENISOV AND ANDREY MURAVNIK

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ABSTRACT. We study the Dirichlet problem in half-space for the equation $\Delta u + g(u)|\nabla u|^2 = 0$, where g is continuous or has a power singularity (in the latter case positive solutions are considered). The results presented give necessary and sufficient conditions for the existence of (pointwise or uniform) limit of the solution as $y \to \infty$, where y denotes the spatial variable, orthogonal to the hyperplane of boundary-value data. These conditions are given in terms of integral means of the boundary-value function.

INTRODUCTION

The phenomenon called *stabilization* is well known for *parabolic* equations both in linear (see e.g. [1] and references therein) and non-linear (see e.g. [2] and references therein) cases; it means the existence of a finite limit of the solution as $t \to \infty$. However, there are well-posed non-isotropic *elliptic* boundary-value problems in unbounded domains (see e.g. [3]) for which we can talk about stabilization in the following sense: the solution has a finite limit as a selected *spatial* variable tends to infinity.

This paper is devoted to the Dirichlet problem in half-space for elliptic equations. We present necessary and sufficient conditions for the stabilization of its solution; here the spatial variable, orthogonal to the hyperplane of boundary-value data, plays the role of time. In Section 1, the linear case is presented; Sections 2 and 3 are devoted to quasi-linear equations with the so-called Burgers-Kardar-Parisi-Zhang non-linearity type (see e.g. [4], [5]). Equations with such non-linearities arise, for example, in modeling of directed polymers and interface growth. They also present an independent theoretical interest because they contain second powers of the first derivatives (see e.g. [6] and references therein).

Note that we deal with the stabilization problem in cylindrical domains with an *unbounded* base (in particular, here the base of the cylinder is the whole E^N). As in the parabolic case, this problem is principally different (this refers both to the results and to the methods of research) from the stabilization problem in cylindrical domains with a *bounded* base. The latter problem has been investigated

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for a rather broad class of non-linear elliptic equations by means of methods of dynamical systems (see e.g. [7] and references therein).

1. Linear equations

In the half-space $\{y \geq 0\}$ we consider the Dirichlet problem

$$p(x)\frac{\partial^2 u}{\partial y^2} + \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,y)\frac{\partial u}{\partial x_j}\Big) = 0, \quad (x,y) \in \{y > 0\}, \tag{1}$$

$$u(x,0) = \varphi(x), \quad x \in E^N, \tag{2}$$

where the coefficients of equation (1) are measurable and bounded and the uniform ellipticity condition is satisfied, i.e., there exists a positive constant λ such that for any $x, y \in \{y > 0\}$,

$$\lambda^{-1} \leq p(x) \leq \lambda, \quad \lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x,y)\xi_i\xi_j \leq \lambda|\xi|^2; \tag{3}$$

the boundary-value function $\varphi(x)$ is supposed to be continuous and bounded. We say that $u(x,y)$ is a solution of (1) if u is bounded on $\{y > 0\}$, belongs to W_2^1 in any strictly interior subdomain of $\{y \geq 0\}$, and satisfies the integral identity

$$\int_0^{+\infty}\int_{E^N}\Big[\sum_{i,j=1}^{N} a_{ij}(x,y)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + p(x)\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\Big]dxdy = 0$$

for any $v(x,y) \in C_0^\infty\{y > 0\}$.

It is known (see e.g. [3]) that, under the above assumptions, a solution $u(x,y)$ of (1), (2) exists and is unique; it is bounded and continuous on $\{y > 0\}$ and coincides at $y = 0$ with a continuous boundary-value function $\varphi(x)$ on E^N. Moreover, it follows from [8] and [9] that $u(x,y)$ satisfies Hölder's condition with a constant $\alpha > 0$ depending merely on λ and N.

We discuss the question of finding necessary and sufficient conditions for the existence of

$$\lim_{y\to+\infty} u(x,y) = l \tag{4}$$

for any $x \in E^N$ (or uniformly with respect to $x \in E^N$).

First of all we introduce a few definitions related to the means of multi-variable functions.

Let $g(x)$ be bounded on E^N, let K be a compact subset of E^N, and let R be a large positive parameter. We consider the following family of functions:

$$\{g(Rx)\} = \{g(Rx_1, Rx_2, \ldots, Rx_N)\},\ R > 0.$$

Suppose $g(x)$ has a weak limit at infinity in L^2_{loc}, i.e.,

$$g(Rx) \rightharpoonup \overline{g}(x), \tag{5}$$

where the symbol $\rightharpoonup$ denotes weak convergence in $L^2(K)$. It follows from the boundedness of $g(x)$ that (5) is equivalent to the following two assertions:

$$\lim_{R\to\infty}\int_{E^N} g(Rx)\psi(x)\,dx = \int_{E^N}\overline{g}(x)\psi(x)\,dx \text{ for any } \psi \in C_0^\infty(E^N), \tag{6}$$

$$\lim_{R\to\infty}\int_K g(Rx)\,dx=\int_K \overline{g}(x)\,dx. \tag{7}$$

We say that $g(x)$ has a mean value, if it has a weak limit (5) at infinity in L^2_{loc} and that limit is a constant. Relation (7) yields that the above definition of the mean value is equivalent to the following one:

$$\overline{g}=\lim_{R\to\infty}\frac{1}{|K_R|}\int_{K_R} g(y)\,dy, \tag{8}$$

where $K_R \stackrel{\mathrm{def}}{=} \{y\in E^N \big| y=Rx,\ x\in K\}$, $|K_R|$ denotes the Lebesgue measure of K_R.

We say that $g(x)$ has a uniform mean value $\overline{g}$ on E^N if for any $\psi\in C_0^\infty(E^N)$ there exists

$$\lim_{R\to\infty}\int_{E^N} g(\beta+Rx)\psi(x)\,dx=\overline{g}\int_{E^N}\psi(x)\,dx, \tag{9}$$

and this limit is uniform with respect to $\beta\in E^N$. The latter definition is equivalent to the following one:

$$\lim_{R\to\infty}\frac{1}{|K_R|}\int_{K_R} g(\beta+y)\,dy=\overline{g} \tag{10}$$

uniformly with respect to $\beta\in E^N$.

Let $f(x,y)\stackrel{\mathrm{def}}{=} f(x_1,\dots,x_N;y)$ be bounded on the half-space $\{y\geq 0\}$. We say that it satisfies condition A if there exists a constant $\overline{f}$ such that

$$\lim_{R\to\infty}\int_{|x|\leq 1}\int_0^1 \left[f(Rx,Ry)-\overline{f}\right]^2 dy\,dx=0. \tag{11}$$

Note that an A-type condition was introduced in [10] and [11]. It was developed further in [12].

Now we are ready to formulate our main results concerning linear equations.

Theorem 1. *Suppose $p(x)$ has a mean value and the elliptic operator in (1) is the Laplacian. Then limit (4) exists at any point $x\in E^N$ if and only if*

$$\lim_{R\to\infty}\frac{\int_{|x|\leq R} p(x)\varphi(x)\,dx}{\int_{|x|\leq R} p(x)\,dx}=l. \tag{12}$$

Theorem 2. *Assume $p(x)$ has a mean value and the coefficients $a_{ij}(x,y)$ satisfy condition A with a positive definite matrix $\{\overline{a}_{ij}\}_{i,j=1}^n$. Then limit (4) exists at any point $x\in E^N$ if and only if*

$$\lim_{R\to\infty}\frac{\int_{(Bx,x)\leq R^2}\varphi(x)p(x)\,dx}{\int_{(Bx,x)\leq R^2} p(x)\,dx}=l, \tag{13}$$

where B denotes the inverse matrix of $\{\overline{a}_{ij}\}_{i,j=1}^n$.

Theorem 3. *Limit (4) exists uniformly with respect to $x\in E^N$ if and only if*

$$\lim_{R\to\infty}\frac{\int_{|y|\leq R} p(x+y)\varphi(x+y)\,dy}{\int_{|y|\leq R} p(x+y)\,dy}=l \tag{14}$$

uniformly with respect to $x\in E^N$.

Remark 1. In the case of Cauchy problem for *parabolic* equations, assertions similar to Theorems 1–3 were proved in [10]–[12].

2. Quasi-linear equations with regular coefficients at non-linearities

Hereinafter the point $x = (x_1, \ldots, x_N, x_{N+1})$ is denoted by (x', x_{N+1}), and the half-space $E^N \times (0, +\infty)$ is denoted by E_+^{N+1}.

In this section we consider the following problem:

$$\Delta u + g(u)|\nabla u|^2 = 0,\ x \in E_+^{N+1}; \tag{15}$$

$$u(x', 0) = \varphi(x'),\ x' \in E^N; \tag{16}$$

where g is continuous in $(-\infty, +\infty)$, and φ is continuous and bounded in E^N.

Following e.g. [13], we introduce the function

$$f(s) \overset{\text{def}}{=} \int_0^s e^{\int_0^\tau g(\sigma)d\sigma} d\tau, \tag{17}$$

and prove the following assertions.

Theorem 4. *There exists a unique classical bounded solution of problem* (15), (16).

Theorem 5. *Let $x' \in E^N$, $l \in (-\infty, +\infty)$, and let $u(x)$ be the classical bounded solution of problem* (15), (16). *Then*

$$\lim_{x_{N+1}\to+\infty} u(x) = l \ \textit{ if and only if } \ \lim_{R\to+\infty} \frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}} R^N} \int_{|y|\le R} f[\varphi(y)]dy = f(l).$$

Theorem 6. *Let $l \in (-\infty, +\infty)$, and let $u(x)$ be the classical bounded solution of problem* (15), (16). *Then*

$$u(x) \overset{x_{N+1}\to+\infty}{\longrightarrow} l \ \textit{ uniformly with respect to } \ x' \in E^N$$

if and only if

$$\frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}} R^N} \int_{|y|\le R} f[\varphi(x'+y)]dy \overset{R\to+\infty}{\longrightarrow} f(l) \ \textit{ uniformly with respect to } \ x' \in E^N.$$

Remark 2. The results of this section are valid also for the equation

$$\frac{\partial^2 u}{\partial y^2} + \sum_{i,j=1}^N \frac{\partial}{\partial x_i}\Big(a_{ij}(x,y)\frac{\partial u}{\partial x_j}\Big) + g(u)\left[\sum_{i,j=1}^N a_{ij}(x,y)\Big(\frac{\partial u}{\partial x_j}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2\right] = 0,$$

where, besides (3) and condition A, we have $a_{ij} \in C^{1,q}(\overline{E_+^N})$ with a positive q $(i, j = 1, \ldots, N)$.

3. The case of singular coefficients at non-linearities

In this section, we study the equation

$$\Delta u + \alpha u^\beta |\nabla u|^2 = 0, \tag{18}$$

where $\beta \in [-1, 0)$ and $\alpha > -1$ for $\beta = -1$.

We will also assume that $\varphi(x') \ge 0$ (apart from its continuity and boundedness), and consider positive solutions of (18), (16).

For $\beta \in (-1, 0)$ we use (17) again, but for $\beta = -1$ the function $f(s)$ is defined as $s^{\alpha+1}$. Then the following assertions are valid.

Theorem 7. *If φ is different from the identical zero, then there exists a unique classical positive bounded solution of problem* (18), (16).

Theorem 8. *Let $\beta = -1$ and let $u(x)$ be the classical positive bounded solution of problem* (18), (16), *$x' \in E^N$, $l \geq 0$. Then*

$$\lim_{x_{N+1}\to+\infty} u(x) = l \text{ if and only if } \lim_{R\to+\infty} \frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}R^N} \int_{|y|\leq R} \varphi^{\alpha+1}(y)dy = l^{\alpha+1}.$$

Theorem 9. *Let $\beta \in (-1, 0)$ and let $u(x)$ be the classical positive bounded solution of problem* (18), (16), *$x' \in E^N$, $l \geq 0$. Then*

$$\lim_{x_{N+1}\to+\infty} u(x) = l \text{ if and only if } \lim_{R\to+\infty} \frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}R^N} \int_{|y|\leq R} \tilde{f}[\varphi(y)]dy = f(l),$$

where

$$\tilde{f}(s) = \int_0^s e^{\frac{\alpha}{1-\beta}\tau^{1-\beta}} d\tau. \tag{19}$$

Theorem 10. *Let $\beta = -1$ and let $u(x)$ be the classical positive bounded solution of problem* (18), (16), *$l \geq 0$. Then*

$$u(x) \overset{x_{N+1}\to+\infty}{\longrightarrow} l \text{ uniformly with respect to } x' \in E^N$$

if and only if

$$\frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}R^N} \int_{|y|\leq R} \varphi^{\alpha+1}(x'+y)dy \overset{R\to+\infty}{\longrightarrow} l^{\alpha+1} \text{ uniformly with respect to } x' \in E^N.$$

Theorem 11. *Let $\beta \in (-1, 0)$ and let $u(x)$ be the classical positive bounded solution of problem* (18), (16), *$l \geq 0$. Then*

$$u(x) \overset{x_{N+1}\to+\infty}{\longrightarrow} l \text{ uniformly with respect to } x' \in E^N$$

if and only if

$$\frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}R^N} \int_{|y|\leq R} \tilde{f}[\varphi(x'+y)]dy \overset{R\to+\infty}{\longrightarrow} \tilde{f}(l) \text{ uniformly with respect to } x' \in E^N.$$

Remark 3. In the case of Cauchy problem for *parabolic* equations, assertions similar to Theorems 5–11 were proved in [2].

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Moscow State University, Faculty of Computational Mathematics and Cybernetics, Leninskie gory, Moscow 119899, Russia
E-mail address: `V.Denisov@g23.relcom.ru`

Department of Differential Equations, Moscow State Aviation Institute, Volokolamskoe shosse 4, Moscow, A-80, GSP-3, 125993, Russia
E-mail address: `abm@mailru.com`