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COMPLETING LIE ALGEBRA ACTIONS TO LIE GROUP ACTIONS

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Abstract. For a finite-dimensional Lie algebra $\mathfrak{g}$ of vector fields on a manifold M we show that M can be completed to a G-space in a universal way, which however is neither Hausdorff nor T_1 in general. Here G is a connected Lie group with Lie-algebra $\mathfrak{g}$. For a transitive $\mathfrak{g}$-action the completion is of the form G/H for a Lie subgroup H which need not be closed. In general the completion can be constructed by completing each $\mathfrak{g}$-orbit.

1. Introduction. In [7], Palais investigated when one could extend a local Lie group action to a global one. He did this in the realm of non-Hausdorff manifolds, since he showed that completing a vector field X on a Hausdorff manifold M may already lead to a non-Hausdorff manifold on which the additive group $\mathbb{R}$ acts. We reproved this result in [3], being unaware of Palais' result. In [4] this result was extended to infinite dimensions and applied to partial differential equations like Burgers' equation: solutions of the PDE were continued beyond the shocks and the universal completion was identified.

Here we give a detailed description of the universal completion of a Hausdorff $\mathfrak{g}$-manifold to a G-manifold. For a homogeneous $\mathfrak{g}$-manifold (where the finite-dimensional Lie algebra $\mathfrak{g}$ acts infinitesimally transitive) we show that the G-completion (for a Lie group G with Lie algebra $\mathfrak{g}$) is a homogeneous space G/H for a possibly non-closed Lie subgroup H (Theorem 7). In Example 8 we show that each such situation can indeed be realized. For general $\mathfrak{g}$-manifolds we show that one can complete each $\mathfrak{g}$-orbit separately and replace the $\mathfrak{g}$-orbits in M by the resulting G-orbits to obtain the universal completion ${}_GM$ (Theorem 9). All $\mathfrak{g}$-invariant structures on M 'extend' to G-invariant structures on ${}_GM$. The relation between our results and those of Palais are described in Section 10.

2. $\mathfrak{g}$-manifolds. Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-manifold is a (finite-dimensional Hausdorff) connected manifold M together with a homomorphism of Lie algebras $\zeta = \zeta^M : \mathfrak{g} \to \mathfrak{X}(M)$ into the Lie algebra of vector fields on M. We may assume without loss of generality that it is injective; if not, replace $\mathfrak{g}$ by $\mathfrak{g}/\ker(\zeta)$. We shall also say that $\mathfrak{g}$ acts on M.

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The image of ζ spans an integrable distribution on M, which need not be of constant rank. So through each point of M there is a unique maximal leaf of that distribution; we also call it the $\mathfrak{g}$-orbit through that point. It is an *initial submanifold* of M in the sense that a mapping from a manifold into the orbit is smooth if and only if it is smooth into M; see [5, 2.14ff.].

Let $\ell : G \times M \to M$ be a left action of a Lie group with Lie algebra $\mathfrak{g}$. Let $\ell_a : M \to M$ and $\ell^x : G \to M$ be given by $\ell_a(x) = \ell^x(a) = \ell(a, x) = a.x$ for $a \in G$ and $x \in M$. For $X \in \mathfrak{g}$ the *fundamental vector field* $\zeta_X = \zeta_X^M \in \mathfrak{X}(M)$ is given by $\zeta_X(x) = -T_e(\ell^x).X = -T_{(e,x)}\ell.(X, 0_x) = -\partial_t|_0 \exp(tX).x$. The minus sign is necessary so that $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ becomes a Lie algebra homomorphism. For a right action the fundamental vector field mapping without minus would be a Lie algebra homomorphism. Since left actions are more common, we stick to them.

3. The graph of the pseudogroup. Let M be a $\mathfrak{g}$-manifold, effective and connected, so that the action $\zeta = \zeta^M : \mathfrak{g} \to \mathfrak{X}(M)$ is injective. Recall from [1, 2.3] that the pseudogroup $\Gamma(\mathfrak{g})$ consists of all diffeomorphisms of the form

$$\mathrm{Fl}^{\zeta_{X_n}}_{t_n} \circ \ldots \circ \mathrm{Fl}^{\zeta_{X_2}}_{t_2} \circ \mathrm{Fl}^{\zeta_{X_1}}_{t_1} |U,$$

where $X_i \in \mathfrak{g}$, $t_i \in \mathbb{R}$, and $U \subset M$ are such that $\mathrm{Fl}^{\zeta_{X_1}}_{t_1}$ is defined on U, $\mathrm{Fl}^{\zeta_{X_2}}_{t_2}$ is defined on $\mathrm{Fl}^{\zeta_{X_1}}_{t_1}(U)$, and so on.

Now we choose a connected Lie group G with Lie algebra $\mathfrak{g}$, and we consider the integrable distribution of constant rank $d = \dim(\mathfrak{g})$ on $G \times M$ which is given by

$$\{(L_X(g), \zeta^M_X(x)) : (g, x) \in G \times M, X \in \mathfrak{g}\} \subset TG \times TM, \tag{3.1}$$

where L_X is the left invariant vector field on G generated by $X \in \mathfrak{g}$. This gives rise to the foliation $\mathcal{F}_\zeta$ on $G \times M$, which we call the *graph foliation* of the $\mathfrak{g}$-manifold M.

Consider the following diagram, where $L(e, x)$ is the leaf through (e, x) in $G \times M$, $\mathcal{O}_{\mathfrak{g}}(x)$ is the $\mathfrak{g}$-orbit through x in M, and $W_x \subset G$ is the image of the leaf $L(e, x)$ in G. Note that $\mathrm{pr}_1 : L(e, x) \to W_x$ is a local diffeomorphism for the smooth structure of $L(e, x)$.

$$\begin{tikzcd}
 & L(e,x) \arrow[rr, two heads, "\mathrm{pr}_2"] \arrow[dr] \arrow[dd] & & \mathcal{O}_{\mathfrak{g}}(x) \arrow[d, hook] \\
 & & G \times M \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & M \\
{[0,1]} \arrow[r, "c"] \arrow[uur, dashed, "\tilde{c}"] & W_x \arrow[r, hook, "\text{open}"] & G &
\end{tikzcd} \tag{3.2}$$

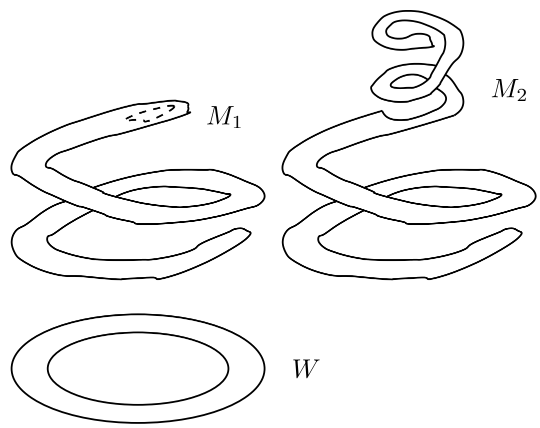
Moreover we consider a piecewise smooth curve $c : [0, 1] \to W_x$ with $c(0) = e$ and we assume that it is liftable to a smooth curve $\tilde{c} : [0, 1] \to L(e, x)$ with $\tilde{c}(0) = (e, x)$. Its endpoint $\tilde{c}(1) \in L(e, x)$ does not depend on small (i.e. liftable to $L(e, x)$) homotopies of c which respect the ends. This lifting depends smoothly on the choice of the initial point x and gives rise to a local diffeomorphism $\gamma_x(c) : U \to \{e\} \times U \to \{c(1)\} \times U' \to U'$, a typical element of the pseudogroup $\Gamma(\mathfrak{g})$ which is defined near x. See [1, 2.3] for more information and Example 4 below. Note that the leaf $L(g, x)$ through (g, x) is given by

$$L(g, x) = \{(gh, y) : (h, y) \in L(e, x)\} = (\mu_g \times \mathrm{Id})(L(e, x)), \tag{3.3}$$

where $\mu : G \times G \to G$ is the multiplication and $\mu_g(h) = gh = \mu^h(g)$.

4. Examples. It is helpful to keep the following examples in mind, which elaborate upon [1, 5.3]. Let $G = \mathfrak{g} = \mathbb{R}^2$, let W be an annulus in $\mathbb{R}^2$ containing 0, and let M_1 be a simply connected piece of finite or infinite length of the universal cover of W. Then the Lie algebra $\mathfrak{g} = \mathbb{R}^2$ acts on M but not the group. Let $p : M_1 \to W$ be the restriction of the covering map, a local diffeomorphism.

Here $G \times_{\mathfrak{g}} M_1 \cong G = \mathbb{R}^2$. Namely, the graph distribution is then also transversal to the fiber of $\mathrm{pr}_2 : G \times M_1 \to M_1$ (since the action is transitive and free on M_1), thus describes a principal G-connection on the bundle $\mathrm{pr}_2 : G \times M_1 \to M_1$. Each leaf is a covering of M_1 and hence diffeomorphic to M_1 since M_1 is simply connected. For $g \in \mathbb{R}^2$ consider $j_g : M_1 \xrightarrow{\mathrm{ins}_g} \{g\} \times M_1 \subset G \times M_1 \xrightarrow{\pi} G \times_{\mathfrak{g}} M_1$ and two points $x \neq y \in M_1$. We may choose a smooth curve γ in M_1 from x to y, lift it into the leaf $L(g,x)$ and project it to a curve c in $g + W$ from g to $c(1) = g + p(y) - p(x) \in g + W$. Then (g,x) and $(c(1),y)$ are on the same leaf. So $j_g(x) = j_g(y)$ if and only if $p(x) = p(y)$. So we see that $j_g(x) = g + p(x)$, and thus $G \times_{\mathfrak{g}} M_1 = \mathbb{R}^2$. This will also follow from 7.



Let us further complicate the situation by now omitting a small disk in M_1 so that it becomes non-simply connected but still projects onto W, and let M_2 be a simply connected component of the universal cover of M_1 with the disk omitted. What happens now is that homotopic curves which act equally on M_1 act differently on M_2.

It is easy to see with the methods described below that the completion ${}_GM_i = \mathbb{R}^2$ in both cases.

5. Enlarging to group actions. In the situation of Section 3 let us denote by ${}_GM = G \times_{\mathfrak{g}} M = G \times M / \mathcal{F}_\zeta$ the space of leaves of the foliation $\mathcal{F}_\zeta$ on $G \times M$, with the quotient topology. For each $g \in G$ we consider the mapping

$$(5.1) \qquad j_g : M \xrightarrow{\mathrm{ins}_g} \{g\} \times M \subset G \times M \xrightarrow{\pi} {}_GM = G \times_{\mathfrak{g}} M.$$

Note that the submanifolds $\{g\} \times M \subset G \times M$ are transversal to the graph foliation $\mathcal{F}_\zeta$. The leaf space ${}_GM$ of $G \times M$ admits a unique smooth structure, possibly singular and non-Hausdorff, such that a mapping $f : {}_GM \to N$ into a smooth manifold N is smooth if and only if the compositions $f \circ j_g : M \to N$ are smooth. For example we may use the structure of a *Frölicher space* or *smooth space* induced by the mappings j_g in the sense of [6, Section 23] on ${}_GM = G \times_{\mathfrak g} M$. The canonical open maps $j_g : M \to {}_GM$ for $g \in G$ are called the charts of ${}_GM$. By construction, for each $x \in M$ and for $g'g^{-1}$ near enough to e in G there exists a curve $c : [0,1] \to W_x$ with $c(0) = e$ and $c(1) = g'g^{-1}$ and an open neighborhood U of x in M such that for the smooth transformation $\gamma_x(c)$ in the pseudogroup $\Gamma(\mathfrak g)$ we have

$$j_{g'}|U = j_g \circ \gamma_x(c). \tag{5.2}$$

Thus the mappings j_g may serve as a replacement for charts in the description of the smooth structure on ${}_GM$. Note that the mappings j_g are not injective in general. Even if $g = g'$, there might be liftable smooth loops c in W_x such that (5.2) holds. Note also some similarity of the system of 'charts' j_g with the notion of an *orbifold* where one uses finite groups instead of pseudogroup transformations.

The leaf space ${}_GM = G \times_{\mathfrak g} M$ is a smooth G-space where the G-action is induced by $(g', x) \mapsto (gg', x)$ in $G \times M$.

Theorem. *The G-completion ${}_GM$ has the following universal properties:*

(5.3) *Given any Hausdorff G-manifold N and $\mathfrak g$-equivariant mapping $f : M \to N$ there exists a unique G-equivariant continuous mapping $\tilde f : {}_GM \to N$ with $\tilde f \circ j_e = f$. Namely, the mapping $\bar f : G \times M \to N$ given by $\bar f(g, x) = g.f(x)$ is smooth and factors to $\tilde f : {}_GM \to N$.*

(5.4) *In the setting of (5.3), the universal property holds also for the T_1-quotient of ${}_GM$, which is given as the quotient $G \times M/\overline{\mathcal F}_\zeta$ of $G \times M$ by the equivalence relation generated by the closure of leaves.*

(5.5) *If M carries a symplectic or Poisson structure or a Riemannian metric such that the $\mathfrak g$-action preserves this structure or is even a Hamiltonian action, then the structure 'can be extended to ${}_GM$ so that the enlarged G-action preserves these structures or is even Hamiltonian'.*

Proof. (5.3) Consider the mapping $\bar f = \ell^N \circ (\mathrm{Id}_G \times f) : G \times M \to N$ which is given by $\bar f(g, x) = g.f(x)$. Then by (3.1) and (3.2) we have for $X \in G$

$$\begin{aligned} T\bar f.(L_X(g), \zeta_X^M(x)) &= T\ell.(L_X(g), T_xf.\zeta_X^M(x)) \\ &= T\ell.(R_{\mathrm{Ad}(g)X}(g), 0_{f(x)}) + T\ell(0_g, \zeta_X^N(f(x))) \\ &= -\zeta_{\mathrm{Ad}(g)X}(g.f(x)) + T\ell_g.\zeta_X^N(f(x)) = 0. \end{aligned}$$

Thus $\bar f$ is constant on the leaves of the graph foliation on $G \times M$ and thus factors to $\tilde f : {}_GM \to N$. Since $\bar f(g.g_1, x) = g.g_1.f(x) = g.\bar f(g, x)$, the mapping $\tilde f$ is G-equivariant. Since N is Hausdorff, $\tilde f$ is even constant on the closure of each leaf, thus (5.4) holds also.

(5.5) Let us treat Poisson structure P on M. For symplectic structures or Riemannian metrics the argument is similar and simpler. Since the Lie derivative along fundamental vector fields of P vanishes, the pseudogroup transformation $\gamma_x(c)$ in (5.2) preserves P. Since ${}_GM$ is the quotient of the disjoint union of all spaces $\{g\} \times M$ for $g \in G$ under the equivalence relation described by (5.2), P 'passes down to this quotient'. Note that we refrain from putting too much meaning on this statement. $\square$

The universal property (5.3) holds also for smooth G-spaces N which need not be Hausdorff, nor T_1, but should have tangent spaces and foliations so that it is meaningful to talk about $\mathfrak{g}$-equivariant mappings. We will not go into this, but see [6, Section 23], for some concepts which point in this direction.

As an application of the universal property of the G-completion ${}_GM$, we see that ${}_GM$ depends on the choice of G in the following way. We write $G = \Gamma\backslash\widetilde{G}$, where $\widetilde{G}$ is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and $\Gamma \subset \widetilde{G}$ is the discrete central subgroup such that $\Gamma \cong \pi_1(G)$. Then we have ${}_GM \cong \Gamma\backslash{}_{\widetilde{G}}M$ as G-spaces, so that ${}_{\widetilde{G}}M$ is potentially less singular than ${}_GM$.

6. Example. Let $\mathfrak{g} = \mathbb{R}^2$ with basis X, Y, let $M = \mathbb{R}^3 \setminus \{(0,0,z) : z \in \mathbb{R}\}$, and let $\zeta^\alpha : \mathfrak{g} \to \mathfrak{X}(M)$ be given by

$$\zeta^\alpha_X = \partial_x + \alpha\,\frac{yz}{x^2+y^2}\,\partial_z, \quad \zeta^\alpha_Y = \partial_y - \alpha\,\frac{xz}{x^2+y^2}\,\partial_z, \qquad \alpha > 0, \tag{6.1}$$

which satisfy $[\zeta^\alpha_X, \zeta^\alpha_Y] = 0$. By construction of the graph foliation $\mathcal{F}_{\zeta^\alpha}$ in (3.1) and the procedure summarized in diagram (3.2), the leaves of $\mathcal{F}_{\zeta^\alpha}$ are determined explicitly as follows. For any smooth curve $c(t) = (\xi(t), \eta(t)) \in G$ starting at (ξ_0, η_0) we have $\dot{c}(t) = \dot{\xi}(t)\, X + \dot{\eta}(t)\, Y \in \mathfrak{g}$ and the lifted curve $(c(t), \mathbf{y}(t))$ is in the leaf $L((\xi_0,\eta_0), \mathbf{y}_0)$ if and only if it satisfies the first order ODE

$$(\mathbf{y}(t), \dot{\mathbf{y}}(t)) = \dot{\xi}(t)\,\zeta^\alpha_X(\mathbf{y}(t)) + \dot{\eta}(t)\,\zeta^\alpha_Y(\mathbf{y}(t)) \tag{6.2}$$

with initial value $\mathbf{y}(0) = \mathbf{y}_0 = (x_0, y_0, u = z_0) \in M$. Substituting (6.1) into (6.2), we see that this ODE is linear, that is, $\dot{x} = \dot{\xi}$, $\dot{y} = \dot{\eta}$ and $\dot{z} = -\alpha\, z\, \frac{x\dot{\eta} - y\dot{\xi}}{r^2} = -\alpha\, z\, \frac{x\dot{y} - y\dot{x}}{r^2}$, where $r^2 = x^2 + y^2$. Thus the projection $\mathbf{x}(t)$ of $\mathbf{y}(t)$ to the (x,y)-plane is given by $\mathbf{x}(t) = c(t) - ((\xi_0, \eta_0) - \mathbf{x}_0) = c(t) - (\xi_0 - x_0, \eta_0 - y_0)$, whereas the third equation leads to

$$z(t) = u\, e^{-\alpha \int_0^t d\theta} = u\, e^{-\alpha(\theta(t) - \theta_0)} = u e^{\alpha\theta_0}\, e^{-\alpha\theta(t)}\,, \tag{6.3}$$

where θ is the angle function in the (x,y)-plane. This depends only on the endpoints $\mathbf{x}_0$, $\mathbf{x}(t)$ and the winding number of the curve $\mathbf{x}$ and is otherwise independent of $\mathbf{x}$. Incompleteness occurs whenever the curve $\mathbf{x}$ goes to $(0,0) \in \mathbb{R}^2$ in finite time $\bar{t} < \infty$, that is, $\mathbf{x}(t) \to (0,0)$, $t \uparrow \bar{t}$, or equivalently $c(t) \to (\xi_0, \eta_0) - \mathbf{x}_0$, $t \uparrow \bar{t}$. It follows that the leaf $L((\xi_0, \eta_0), \mathbf{y}_0)$ is parametrized by $(r, \theta) \in \mathbb{R}_+ \times \mathbb{R}$ with $z = z(\theta)$ being independent of $r > 0$ and that

$$\mathrm{pr}_1 : L((\xi_0,\eta_0), \mathbf{y}_0) \to W_{(\xi_0,\eta_0),\mathbf{y}_0} = \mathbb{R}^2 \setminus \{(\xi_0,\eta_0) - \mathbf{x}_0\} \tag{6.4}$$

in (3.2) is a universal covering. This is visibly consistent with (3.3). In order to parametrize the space of leaves ${}_GM$, we observe that the parameter $\mathbf{x}_0$ can be eliminated. In fact, from the previous formulas we see that

$$L((\xi_0', \eta_0'), (\mathbf{x}_0', u')) = L((\xi_0, \eta_0), (\mathbf{x}_0, u)) \tag{6.5}$$

if and only if $(\xi_0', \eta_0') - \mathbf{x}_0' = (\xi_0, \eta_0) - \mathbf{x}_0$ and $u' = u e^{\alpha(\theta_0 - \theta_0')}$, so that we have $z'(\theta) = u' e^{\alpha\theta_0'}\, e^{-\alpha\theta(t)} = u e^{\alpha\theta_0}\, e^{-\alpha\theta(t)} = z(\theta)$. In particular, it follows that

$$L((\xi_0, \eta_0), \mathbf{y}_0) = L((\xi_0' + 1, \eta_0'), (1, 0, u')), \tag{6.6}$$

where $(\xi_0', \eta_0') = (\xi_0, \eta_0) - \mathbf{x}_0$, $u' = u e^{\alpha\theta_0}$, $\theta_0' = 0$, projecting to $\mathbb{R}^2 \setminus \{(\xi_0', \eta_0')\}$. Therefore the leaves of the form $L((\xi_0 + 1, \eta_0), (1, 0, u))$ are distinct for different values of (ξ_0, η_0) and fixed value of u and from the relation (3.3) we conclude that

$$L((\xi_0 + 1, \eta_0), (1, 0, u)) = (\xi_0, \eta_0) + L((1, 0), (1, 0, u)), \tag{6.7}$$

that is, $G = \mathbb{R}^2$ acts without isotropy on ${}_GM$. We also need to determine the range for the parameter u. Obviously, we have $L((1,0), (1,0,u')) = L((1,0), (1,0,u))$

if and only if $u' = e^{2\pi\alpha n}u$ for $n \in \mathbb{Z}$. Thus these leaves are parametrized by $[u]$, taking values in the quotient of the additive group $\mathbb{R}$ under the multiplicative group $\{e^{2\pi\alpha n} : n \in \mathbb{Z}\}$, that is,

$$\{0\} \cup \mathbb{S}^1_+ \cup \mathbb{S}^1_- \cong \{0\} \cup \mathbb{R}^\times_+/\{e^{2\pi\alpha n} : n \in \mathbb{Z}\} \cup \mathbb{R}^\times_-/\{e^{2\pi\alpha n} : n \in \mathbb{Z}\}. \tag{6.8}$$

The topology on the above space is determined by the leaf closures, respectively the orbit closures. First we have $\overline{L((\xi_0+1,\eta_0),(1,0,u))} = (\xi_0,\eta_0) + \overline{L((1,0),(1,0,u))}$ in $G \times M$, and it is sufficient to determine the closures of $L((1,0),(1,0,u))$. For $(1,0,u) \in M$ with $u \neq 0$ we consider the curve $c(\theta) = e^{i\theta} \in G = \mathbb{R}^2$. It is liftable to $G \times M$ and determines on M the curve $\mathbf{y}(t) = (\cos\theta, \sin\theta, ue^{-\alpha\theta})$. Thus the curve $(c(\theta), \mathbf{y}(\theta))$ in the leaf through $(1,0;1,0,u) \in G \times M \subset \mathbb{R}^5$ has a limit cycle for $\theta \to \infty$ which lies in the different leaf through $(1,0;1,0,0)$, which is closed, given by the (x,y)-plane $(\mathbb{R}^2 \times 0) \setminus 0$ at level $(1,0) \in G$. Thus we have

$$\overline{L((1,0),(1,0,u))} = L((1,0),(1,0,u)) \cup L((1,0),(1,0,0)). \tag{6.9}$$

Hence the leaf $L((1,0),(1,0,u))$ is not closed, and the topological space ${}_GM$ is not T_1 and not a manifold. The orbits of the $\mathfrak{g}$-action are determined by the leaf structure via pr_2 in diagram (3.2), and they look here as follows. The (x,y)-plane $(\mathbb{R}^2 \times 0) \setminus 0$ is a closed orbit. Orbits above this plane are helicoidal staircases leading down and accumulating exponentially at the (x,y)-plane. Orbits below this plane are helicoidal staircases leading up and again accumulating exponentially. Thus the orbit space $M/\mathfrak{g}$ of the $\mathfrak{g}$-action is given by (6.8), with the point 0 being closed. By (6.9), the closure of any orbit represented by a point $[u]$ on one of the circles is given by $\{[u], 0\}$. From (6.6) and (6.7), we see that the G-completion ${}_GM$ has a section over the orbit space ${}_GM/G \cong M/\mathfrak{g}$ given by $[u] \mapsto L((1,0),(1,0,u))$. Therefore ${}_GM \cong G \times M/\mathfrak{g} = \mathbb{R}^2 \times \{\,\{0\} \cup \mathbb{S}^1_+ \cup \mathbb{S}^1_-\,\}$.

The structure of the completion and the orbit spaces are independent of the deformation parameter $\alpha > 0$ in (6.1). However, for $\alpha \downarrow 0$, the completion just means adding in the z-axis, that is, we get ${}_GM \cong \mathbb{R}^3$ with $G = \mathbb{R}^2$ acting by parallel translation on the affine planes $z = c$, and $M/\mathfrak{g} \cong {}_GM/G \cong \mathbb{R}$ as it should be.

It was pointed out to us [2] that one can make this example still more pathological. Consider the above example only in a cylinder over the annulus $0 < x^2+y^2 < 1$. Add an open handle to the disk and continue the $\mathbb{R}^2$-action on the cylinder over the disk with an open handle added in such a way that there is a shift in the z-direction when one traverses the handle. Then one of the helicoidal staircases is connected to the the disk itself, so it accumulates onto itself. This is called a 'resilient leaf' in foliation theory.

7. Theorem. *Let M be a connected transitive effective $\mathfrak{g}$-manifold. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then we have:*

- (7.1) *There exists a subgroup $H \subset G$ such that the G-completion ${}_GM$ is diffeomorphic to G/H.*
- (7.2) *The Hausdorff quotient of ${}_GM$ is the homogeneous manifold $G/\overline{H}$. It has the following universal property: For each smooth $\mathfrak{g}$-equivariant mapping $f : M \to N$ into a Hausdorff G-manifold N there exists a unique smooth G-equivariant mapping $\tilde{f} : G/\overline{H} \to N$ with $f = \tilde{f} \circ \pi \circ j_e : M \to G/H \xrightarrow{\pi} G/\overline{H} \to N$.*

(7.3) *For each leaf* $L(g,x_0) \subset G \times M$ *the projection* $\mathrm{pr}_2 : L(g,x_0) \to M$ *is a smooth fiber bundle with typical fiber* H.

Proof. (7.1) We choose a base point $x_0 \in M$. The G-completion is given by ${}_G M = G \times_{\mathfrak{g}} M$, the orbit space of the $\mathfrak{g}$-action on $G \times M$ which is given by $\mathfrak{g} \ni X \mapsto L_X \times \zeta_X^M$, and the G-action on the completion is given by multiplication from the left. The submanifold $G \times \{x_0\}$ meets each $\mathfrak{g}$-orbit in $G \times M$ transversely, since

$$\begin{aligned} T_{(g,x_0)}(G \times \{x_0\}) + T_{(g,x_0)} L(g,x_0) &= \{L_X(g) \times 0_{x_0} + L_Y(g) \times \zeta_Y(x_0) : X, Y \in \mathfrak{g}\} \\ &= T_{(g,x_0)}(G \times M). \end{aligned}$$

By (3.3) we have $L(g,x) = g.L(e,x)$ so that the isotropy Lie algebra $\mathfrak{h} = \mathfrak{g}_{x_0} = \{X \in \mathfrak{g} : \zeta_X(x_0) = 0\}$ is also given by

$$\begin{aligned} X \in \mathfrak{h} \quad &\iff \quad X \times 0_{x_0} \in T_{(e,x_0)}(G \times \{x_0\}) \cap T_{(e,x_0)} L(e,x_0) \\ &\iff \quad L_X(g) \times 0_{x_0} \in T_{(g,x_0)}(G \times \{x_0\}) \cap T_{(g,x_0)} L(g,x_0). \end{aligned}$$

Since $G \times \{x_0\}$ is a leaf of a foliation and the $L(e,x)$ also form a foliation, $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Let H_0 be the connected Lie subgroup of G which corresponds to $\mathfrak{h}$. Then clearly $H_0 \times \{x_0\} \subset G \times \{x_0\} \cap L(e,x_0)$. Let the subgroup $H \subset G$ be given by

$$H = \{g \in G : (g,x_0) \in L(e,x_0)\} = \{g \in G : L(g,x_0) = L(e,x_0)\};$$

then the C^∞-curve component of H containing e is just H_0. So H consists of at most countably many H_0-cosets. Thus H is a Lie subgroup of G (with a finer topology, perhaps). By construction the orbit space $G \times_{\mathfrak{g}} M$ equals the quotient of the transversal $G \times \{x_0\}$ by the relation induced by intersecting with each leaf $L(g,x_0)$ separately, i.e., $G \times_{\mathfrak{g}} M = G/H$.

(7.2) Obviously the T_1-quotient of G/H equals the Hausdorff quotient $G/\overline{H}$, which is a smooth manifold. The universal property is easily seen.

(7.3) Let $x \in M$ and $(g,x) \in L(e,x_0) = L(g,x) = g.L(e,x)$. So it suffices to treat the leaf $L(e,x)$. We choose $X_1, \dots, X_n \in \mathfrak{g}$ such that $\zeta_{X_1}(x), \dots, \zeta_{X_n}(x)$ form a basis of the tangent space $T_x M$. Let $u : U \to \mathbb{R}^n$ be a chart on M centered at x such that $u(U)$ is an open ball in $\mathbb{R}^n$ and such that $\zeta_{X_1}(y), \dots, \zeta_{X_n}(y)$ are still linearly independent for all $y \in U$. For $y \in U$ consider the smooth curve $c_y : [0,1] \to U$ given by $c_y(t) = u^{-1}(t.u(y))$. We consider

$$\begin{aligned} \partial_t c_y(t) = c_y'(t) &= \textstyle\sum_{i=1}^n f_y^i(t)\, \zeta_{X_i}(c_y(t)), \quad f_y^i \in C^\infty([0,1],\mathbb{R}), \\ X_y(t) &= \textstyle\sum_{i=1}^n f_y^i(t)\, X_i \in \mathfrak{g}, \qquad X \in C^\infty([0,1],\mathfrak{g}), \\ g_y \in C^\infty([0,1],G), &\quad T(\mu_{g_y(t)}) \partial_t g_y(t) = X_y(t), \quad g_y(0) = e, \end{aligned}$$

and everything is also smooth in $y \in U$. Then for $h \in H$ we have $(h.g_y(t), c_y(t)) \in L(e,x)$ since

$$\partial_t (h.g_y(t), c_y(t)) = (L_{X_y(t)}(h.g_y(t)), \zeta_{X_y(t)}(c_y(t))).$$

Thus $U \times H \ni (y,h) \mapsto \mathrm{pr}_2^{-1}(U) \cap L(e,x)$ is the required fiber bundle parameterization. $\square$

8. Example. Let G be a simply connected Lie group and let H be a connected Lie subgroup of G which is not closed. For example, let $G = Spin(5)$, which is compact of rank 2, and let H be a dense 1-parameter subgroup in its 2-dimensional maximal torus. Let $\mathrm{Lie}(G) = \mathfrak{g}$ and $\mathrm{Lie}(H) = \mathfrak{h}$. We consider the foliation of G into right H-cosets gH which is generated by $\{L_X : X \in \mathfrak{h}\}$ and is left invariant under G. Let U be a chart centered at e on G which is adapted to this foliation, i.e. $u : U \to u(U) = V_1 \times V_2 \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ such that the sets $u^{-1}(V_1 \times \{x\})$ are the leaves intersected with U. We assume that V_1 and V_2 are open balls, and that U is so small that $\exp : W \to U$ is a diffeomorphism for a suitable convex open set $W \subset \mathfrak{g}$. Of course, $\mathfrak{g}$ acts on U and respects the foliation, so this $\mathfrak{g}$-action descends to the leaf space M of the foliation on U which is diffeomorphic to V_2.

Lemma. *In this situation, for the G-completion we have $G \times_{\mathfrak{g}} M = G/H$.*

Proof. We use the method described in the end of the proof of Theorem 7: ${}_GM = G \times_{\mathfrak{g}} M$ is the quotient of the transversal $G \times \{x_0\}$ by the relation induced by intersecting with each leaf $L(g, x_0)$ separately. Thus we have to determine the subgroup $H_1 = \{g \in G : (g, x_0) \in L(e, g)\}$.

Obviously any smooth curve $c_1 : [0, 1] \to H$ starting at e is liftable to $L(e, x_0)$ since it does not move $x_0 \in M$. So $H \subseteq H_1$, and moreover H is the C^∞-path component of the identity in H_1.

Conversely, if $c = (c_1, c_2) : [0, 1] \to L(e, x_0) \subset G \times M$ is a smooth curve from (e, x_0) to (g, x_0), then c_2 is a smooth loop through x_0 in M and there exists a smooth homotopy h in M which contracts c_2 to x_0, fixing the ends. Since $\mathrm{pr}_2 : L(e, x_0) \to M$ is a fiber bundle by (7.3), we can lift the homotopy h from M to $L(e, x_0)$ with starting curve c, fixing the ends, and deforming c to a curve c' in $L(e, x_0) \cap \mathrm{pr}_2^{-1}(x_0)$. Then $\mathrm{pr}_1 \circ c'$ is a smooth curve in H_1 connecting e and g.

Thus $H_1 = H$, and consequently ${}_GM = G/H$. $\square$

9. Theorem. *Let M be a connected $\mathfrak{g}$-manifold. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then the G-completion ${}_GM$ can be described in the following way:*

(9.1) *Form the leaf space $M/\mathfrak{g}$, a quotient of M which may be non-Hausdorff and not T_1 etc.*

(9.2) *For each point $z \in M/\mathfrak{g}$, replace the orbit $\pi^{-1}(z) \subset M$ by the homogeneous space G/H_x described in Theorem 7, where x is some point in the orbit $\pi^{-1}(z) \subset M$. One can use transversals to the $\mathfrak{g}$-orbits in M to describe this in more detail.*

(9.3) *For each point $z \in M/\mathfrak{g}$, one can also replace the orbit $\pi^{-1}(z) \subset M$ by the homogeneous space $G/\overline{H_x}$ described in Theorem 7, where x is some point in the orbit $\pi^{-1}(z) \subset M$. The resulting G-space has then Hausdorff orbits which are smooth manifolds, but the same orbit space as $M/\mathfrak{g}$.*

See Example 6 above.

Proof. Let $\mathcal{O}(x) \subset M$ be the $\mathfrak{g}$-orbit through x, i.e., the leaf through x of the singular foliation (with non-constant leaf dimension) on M which is induced by the $\mathfrak{g}$-action. Then the G-completion of the orbit $\mathcal{O}(x)$ is ${}_G\mathcal{O}(x) = G/H_x$ for the Lie subgroup $H_x \subset G$ described in Theorem (7.1). By the universal property of the G-completion we get a G-equivariant mapping ${}_G\mathcal{O}(x) \to {}_GM$ which is injective and a homeomorphism onto its image, since we can repeat the construction of Theorem

(7.1) on M. Clearly the mapping $j_e : M \to {}_GM$ induces a homeomorphism between the orbit spaces $M/\mathfrak{g} \to {}_GM/G$.

Now let $s : V \to M$ be an embedding of a submanifold which is a transversal to the $\mathfrak{g}$-foliation at $s(v_0)$. We have $Ts \cdot T_{v_0}V \oplus \zeta_{s(v_0)}(\mathfrak{g}) = T_{s(v_0)}M$. Then s induces a mapping $V \to G \times M$ and $V \to {}_GM$ and we may use the point $s(v)$ in replacing $\mathcal{O}(s(v))$ by $G/H_{s(v)}$ for v near v_0. $\square$

The following diagram summarizes the relation between the preceding constructions.

(9.4)
$$\begin{tikzcd}
M \arrow[r] \arrow[d, leftrightarrow, "="] & \bigcup_{[x]\in M/\mathfrak{g}} G/H_x \arrow[r, two heads] \arrow[d, "\cong"] & \bigcup_{[x]\in M/\mathfrak{g}} G/\overline{H_x} \arrow[d, dashed] \arrow[dll, dashed] \\
M \arrow[r, "j_e"] \arrow[d, "\pi"] & {}_GM = G \times_{\mathfrak{g}} M \arrow[r, two heads] \arrow[d, "\pi_G"] & G \times M/\overline{\mathcal{F}}_\zeta \arrow[d, "\bar\pi_G"] \\
M/\mathfrak{g} \arrow[r, "\cong"] & {}_GM/G \arrow[r, two heads] & (G \times M/\overline{\mathcal{F}}_\zeta)/G
\end{tikzcd}$$

Note that taking the T_1-quotient $G \times M/\overline{\mathcal{F}}_\zeta$ of the leaf space ${}_GM$ may be a very severe reduction. In Example 6 the isotropy groups H_x are trivial and we have $G \times M/\overline{\mathcal{F}}_\zeta = \mathbb{R}^2 \times \{0\}$ and $(G \times M/\overline{\mathcal{F}}_\zeta)/G = \{0\}$.

10. Palais' treatment of $\mathfrak{g}$-manifolds. In [7], Palais considered $\mathfrak{g}$-actions on finite-dimensional manifolds M in the following way. He assumed from the beginning that M may be a non-Hausdorff manifold, since the completion may be non-Hausdorff. Then he introduced notions which we can express as follows in the terms introduced here:

(10.1) (M, ζ) is called *generating* if it generates a local G-transformation group. See [7, II, 2, Def. V and II, 7, Thm. XI]. This holds if and only if the leaves of the graph foliation on $G \times M$ described in Section 3 are Hausdorff. For Hausdorff $\mathfrak{g}$-manifolds this is always the case.

(10.2) (M, ζ) is called *uniform* if $\mathrm{pr}_1 : L(e, x) \to G$ in (3.2) is a covering map for each $x \in M$. See [7, III, 6, Def. VIII and III, 6, Thm. XVII, Cor., Cor. 2]. In the Hausdorff case the $\mathfrak{g}$-action is then complete and it may be integrated to a $\widetilde{G}$-action, where $\widetilde{G}$ is a simply connected Lie group with Lie algebra $\mathfrak{g}$, so that ${}_{\widetilde{G}}M \cong M$.

(10.3) (M, ζ) is called *univalent* if $\mathrm{pr}_1 : L(e, x) \to G$ in (3.2) is injective for all x. See [7, III, 2, Def. VI and III, 4, Thm. X].

(10.4) (M, ζ) is called *globalizable* if there exists a (non-Hausdorff) G-manifold N which contains M equivariantly as an open submanifold. See [7, III, 1, Def. II and III, 4, Thm. X]. This is a severe condition which is not satisfied in examples 4 and 6 above.

Palais' main result on (non-Hausdorff) manifolds with a vector field says that (10.1), (10.3), and (10.4) are equivalent. See [7, III, 7, Thm. XX].

On (non-Hausdorff) $\mathfrak{g}$-manifolds his main result is that (10.3) and (10.4) are equivalent. See [7, III, 1, Def. II and III, 4, Thm. X], and also [7, III, 2, Def. VI and III, 4, Thm. X].

11. Concluding remarks. (11.1) A suitable setting for further development might be the class of *discrete* $\mathfrak{g}$-manifolds, that is, $\mathfrak{g}$-manifolds for which the $\widetilde{G}$-space ${}_{\widetilde{G}}M$ is T_1, or equivalently the leaves of the graph foliation $\mathcal{F}_\zeta$ on $\widetilde{G} \times M$ are closed. In this case, the charts $j_g : M \to {}_{\widetilde{G}}M$ in (5.1) are local diffeomorphisms with respect to the unique smooth structure on ${}_{\widetilde{G}}M$, and ${}_{\widetilde{G}}M$ is a smooth manifold, albeit not necessarily Hausdorff.

(11.2) In the context of (11.1), there are several definitions of *proper* $\mathfrak{g}$-actions, all of which are equivalent to saying that the $\widetilde{G}$-action on ${}_{\widetilde{G}}M$ is proper. Many properties of proper actions will carry over to this case.

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