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A NEW INEQUALITY FOR SUPERDIFFUSIONS AND ITS APPLICATIONS TO NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. Our motivation is the following problem: to describe all positive solutions of a semilinear elliptic equation $Lu = u^\alpha$ with $\alpha > 1$ in a bounded smooth domain $E \subset \mathbb{R}^d$. In 1998 Dynkin and Kuznetsov solved this problem for a class of solutions which they called σ-moderate. The question if all solutions belong to this class remained open. In 2002 Mselati proved that this is true for the equation $\Delta u = u^2$ in a domain of class C^4. His principal tool—the Brownian snake—is not applicable to the case $\alpha \neq 2$. In 2003 Dynkin and Kuznetsov modified most of Mselati's arguments by using superdiffusions instead of the snake. However a critical gap remained. A new inequality established in the present paper allows us to close this gap.

1. Introduction

1.1. **Diffusions and superdiffusions.** We denote by $\mathcal{M}(S)$ the set of all finite measures, and by $\mathcal{P}(S)$ the set of all probability measures on a measurable space S. $\mathcal{B}(E)$ stands for the set of all positive Borel functions on E. We use notation $\langle u, \mu \rangle$ for the integral of u with respect to a measure μ, and notation $P\{A, Y\}$ for the integral $\int_A Y\, dP$.

Let L be an elliptic differential operator of the second order in $\mathbb{R}^d$. Under mild assumptions on the coefficients of L, there exists a continuous Markov process $\xi = (\xi_t, \Pi_x)$ in $\mathbb{R}^d$ whose transition density is a fundamental solution of the parabolic equation $\partial u/\partial t = Lu$. We call this process a diffusion. For every open set D we denote by τ_D the first exit time of ξ from D.

Let ψ be a positive Borel function on $\mathbb{R}_+ = [0, \infty)$. Suppose that to every open set D and every $\mu \in \mathcal{M}(\mathbb{R}^d)$ there corresponds a random measure (X_D, P_μ) on $\mathbb{R}^d$ such that, for every $f \in \mathcal{B}(\mathbb{R}^d)$,

$$P_\mu e^{-\langle f, X_D \rangle} = e^{-\langle u, \mu \rangle} \tag{1.1}$$

where u satisfies the equation

$$u(x) + \Pi_x \int_0^{\tau_D} \psi[u(\xi_t)]dt = \Pi_x f(\xi_{\tau_D}). \tag{1.2}$$

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We call the family $X = (X_D, P_\mu)$ a superdiffusion. [Heuristically, we have here a model of a random evolution of the cloud of particles, and X_D is a mass distribution on ∂D if each particle is frozen at the first exit from D.]

The existence of a superdiffusion is proved for a convex class of positive convex functions which contains the functions

$$\psi(u) = u^\alpha, \quad 0 < \alpha \le 2. \tag{1.3}$$

[See, e.g., Chapter 4 in [Dy02].]

By restricting the family (X_D, P_μ) to $D \subset E$ and $\mu \in \mathcal{M}(E)$ we define a superdiffusion in an open set E.

A new tool—a family of measures $\mathbb{N}_x, x \in E$ (defined on the same space $\mathcal{O}O$ as measures P_μ)—was introduced in [DK04]. Our inspiration was the role played by an analog of these measures in Le Gall's theory of the Brownian snake.[1]

The range $\mathcal{R}_E$ of a superdiffusion in a domain E is a minimal closed set which supports, P_x-a.s. and $\mathbb{N}_x$-a.s., an exit measure X_O for an arbitrary open set $O \subset E$ and for every $x \in E$.

1.2. **Stochastic boundary values of harmonic functions.** We say that a function h in E is harmonic in E if $Lh = 0$ in E and we use the notation $\mathcal{H}(E)$ for the set of all positive harmonic functions. If E is smooth,[2] then there exists a 1-1 correspondence between $\mathcal{H}(E)$ and $\mathcal{M}(\partial E)$. The harmonic function h_ν corresponding to $\nu \in \mathcal{M}(\partial E)$ is given by the formula

$$h_\nu(x) = \int_{\partial E} k_E(x, y)\nu(dy) \tag{1.4}$$

where $k_E(x, y)$ is the Poisson kernel for L in E. For every $\nu \in \mathcal{M}(\partial E)$, there exists a random variable Z_ν such that

$$Z_\nu = \lim \langle h_\nu, X_{D_n}\rangle \quad P_x\text{-a.s. and} \quad \mathbb{N}_x\text{-a.s.} \tag{1.5}$$

for every $x \in E$ and for every sequence D_n exhausting E.[3] We call Z_ν the stochastic boundary value of h_ν.

The energy function for $\nu \in \mathcal{M}(\partial E)$ is defined by the formula

$$\mathcal{E}_x(\nu) = \Pi_x \int_0^{\tau_E} \psi[h_\nu(\xi_t)]dt. \tag{1.6}$$

1.3. **Principal result.**

Theorem 1.1. *Suppose that D is a smooth open subset of a smooth domain E. If ν is a finite measure concentrated on $\partial D \cap \partial E$ and if $\mathcal{E}_x(\nu) < \infty$, then*

$$\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu \neq 0\} \ge C(\alpha)[\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\}]^{\alpha/(\alpha-1)}\mathcal{E}_x(\nu)^{-1/(\alpha-1)} \tag{1.7}$$

where $C(\alpha) = (\alpha - 1)^{-1}\Gamma(\alpha - 1)$.[4]

We prove Theorem 1.1 in Section 5 after the necessary tools have been prepared in Sections 2–4.

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[1] Definitions of $\mathbb{N}_x$ and other tools mentioned in the Introduction will be given in Section 2.

[2] We use the term smooth for open sets of class $C^{2,\lambda}$ unless another class is indicated explicitly.

[3] Domains D_n exhaust E if $\bar{D}_n \subset D_{n+1}$ and if the union of D_n is equal to E.

[4] Here Γ is Euler's gamma-function.

1.4. **Applications to differential equations.** We denote by $\mathcal{U}(E)$ the set of all positive solutions of the equation

$$Lu = \psi(u) \quad \text{in } E. \tag{1.8}$$

We say that an element u of $\mathcal{U}(E)$ is moderate if $u \le h$ for some $h \in \mathcal{H}(E)$. There exists a 1-1 correspondence between the set $\mathcal{U}_1(E)$ of all moderate solutions and a subset $\mathcal{H}_1(E)$ of $\mathcal{H}(E)$: $h \in \mathcal{H}_1(E)$ is the minimal harmonic function dominating $u \in \mathcal{U}_1(E)$, and u is the maximal solution dominated by h. We put $\nu \in \mathcal{N}_1^E$ if $h_\nu \in \mathcal{H}_1(E)$. We denote by u_ν the element of $\mathcal{U}_1(E)$ corresponding to h_ν. These elements are related by the formula

$$u_\nu(x) + \mathcal{E}_x(\nu) = h_\nu(x). \tag{1.9}$$

If $\mathcal{E}_x(\nu) < \infty$ for some $x \in E$, then $\nu \in \mathcal{N}_1^E$.[5]

To every closed subset K of ∂E there correspond two elements of $\mathcal{U}(E)$:[6]

$$w_K(x) = -\log P_x\{\mathcal{R}_E \cap K = \emptyset\} = \mathbb{N}_x\{\mathcal{R}_E \cap K \neq \emptyset\} \tag{1.10}$$

and

$$u_K(x) = \sup u_\nu(x) \tag{1.11}$$

where the supremum is taken over all $\nu \in \mathcal{N}_1^E$ concentrated on K.

We say that $u \in \mathcal{U}(E)$ is σ-moderate if there exist moderate solutions u_n such that $u_n \uparrow u$. All solutions u_K are σ-moderate.

Theorem 1.1 in combination with the results presented in Chapter 11, Section 7.1 of [Dy02] and in [Dy04a], [Dy04c], [DK03], [DK04], [Ku04] makes it possible to prove the following two theorems:[7]

Theorem 1.2. *If E is a domain of class C^4 and if L is the Laplacian Δ, then*

$$u_K = w_K \quad \textit{for all closed } K \subset \partial E. \tag{1.12}$$

Theorem 1.3. *Under the conditions of Theorem 1.2 all elements of $\mathcal{U}(E)$ are σ-moderate.*

[Marcus and Véron proved in [MV04] that the equation (1.12) can be established by a purely analytical method applicable to all $\alpha > 1$.]

2. Tools

2.1. ***h*-transform and conditional diffusion.** Suppose ξ is a diffusion in a domain E with the transition function $p_t(x, dy)$, and let $h \in \mathcal{H}(E)$. Then

$$p_t^h(x, dy) = \frac{1}{h(x)} p_t(x, dy) h(y) \tag{2.1}$$

is the transition function of a continuous Markov process $(\xi_t, \hat{\Pi}_x^h)$ in E called the h-transform of ξ. We prefer to deal with measures $\Pi_x^h = h(x)\hat{\Pi}_x^h$ which depend linearly on h. Put $\Pi_x^\nu = \Pi_x^{h_\nu}$ and $\hat{\Pi}_x^y = \hat{\Pi}_x^{\delta_y}$ where δ_y is the unit mass at a point y. The process $(\xi_t, \hat{\Pi}_x^y)$ can be interpreted as a diffusion starting from $x \in E$ and conditioned to exit from E at y.

The following lemma is proved, for instance, in [Dy02], page 103:

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[5]This follows, for instance, from Theorem 3.2 of Chapter 8 in [Dy02].

[6]w_K can be characterized as the maximal element of $\mathcal{U}(E)$ vanishing on $\partial E \setminus K$.

[7]Proofs of these theorems are sketched in [Dy04b]. The complete proofs are contained in the forthcoming book [Dy04d].

Lemma 2.1. *For every stopping time τ and every pre-τ positive Y,*

$$\Pi^h_x Y 1_{\tau<\tau_E} = \Pi_x Y h(\xi_\tau) 1_{\tau<\tau_E}. \tag{2.2}$$

2.2. **Measures $\mathbb{N}_x$.** Denote by $\mathcal{Z}_x$ the class of all functions of the form

$$Z = \sum_1^n \langle f_i, X_{O_i}\rangle \tag{2.3}$$

where $O_1, \dots, O_n$ is a finite family of neighborhoods of x and $f_1, \dots, f_n \in \mathcal{B}(\mathbb{R}^d)$. By Theorem 1.1 in [DK04], for every $x \in E$, there exists a unique measure $\mathbb{N}_x$ with the properties:

(i) For every $Z \in \mathcal{Z}_x$,

$$\mathbb{N}_x(1 - e^{-Z}) = -\log P_x e^{-Z}. \tag{2.4}$$

(ii) If $\bar{\Omega}$ is the intersection of $\{X_O = 0\}$ over all neighborhoods O of x, then $\mathbb{N}_x(\bar{\Omega}) = 0$.

2.3. **Stochastic boundary values and range.** Suppose that $u \in \mathcal{B}(E)$. A random variable Z_u is called a stochastic boundary value of u [we write $Z_u = \mathrm{SBV}(u)$] if

$$Z_u = \lim \langle u, X_{D_n}\rangle \quad P_x\text{-a.s. and} \quad \mathbb{N}_x\text{-a.s.} \tag{2.5}$$

for every $x \in E$ and every sequence D_n exhausting E.

For every $u \in \mathcal{B}(\mathbb{R}^d)$, we put

$$V_D(u)(x) = -\log P_x e^{-\langle u, X_D\rangle}.$$

Denote by $\mathcal{U}^-(E)$ the set of u such that $V_D(u) \le u$ for all $D \subset E$. This condition holds for all $u \ge 0$ such that $Lu \le \psi(u)$ in E. In particular, it holds for $u \in \mathcal{H}(E)$. Since $V_D(u_1 + u_2) \le V_D(u_1) + V_D(u_2)$ (Theorem 2.1 of Chapter 8 in [Dy02]), the sum of two elements of $\mathcal{U}^-(E)$ belongs to $\mathcal{U}^-(E)$.

By Theorem 1.2 in [DK04], a stochastic boundary value Z_u exist for every $u \in \mathcal{U}^-(E)$ and

$$\mathbb{N}_x(1 - e^{-Z_u}) = -\log P_x e^{-Z_u}. \tag{2.6}$$

Formula (1.5) means that $Z_\nu = \mathrm{SBV}(h_\nu)$. If $\nu \in \mathcal{N}_1^E$, then Z_ν is also $\mathrm{SBV}(u_\nu)$.

By Theorem 1.3 in [DK04], for every domain E, there exists a random closed set $\mathcal{R}_E$ with the properties:

(a) For every open $O \subset E$ and every $x \in E$, the measure X_O is concentrated, P_x-a.s. and $\mathbb{N}_x$-a.s., on $\mathcal{R}_E$.

(b) If (a) holds for a random closed set F, then, for every $x \in E$, $\mathcal{R} \subset F$ P_x-a.s. and $\mathbb{N}_x$-a.s.

We call $\mathcal{R}_E$ the range of X in E. We denote by $\mathcal{R}$ the range of X in $\mathbb{R}^d$.

2.4. **More relations between measures P_x and $\mathbb{N}_x$.** By Theorem 1.4 in [DK04], for every $u \in \mathcal{U}^-(E)$ and every Borel set $\Gamma \subset \partial E$,

$$-\log P_x\{\mathcal{R}_E \cap \Gamma = \emptyset, e^{-Z_u}\} = \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma \ne \emptyset\} + \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma = \emptyset, 1 - e^{-Z_u}\}. \tag{2.7}$$

This function is the maximal element of $\mathcal{U}(E)$ dominated by $w_\Gamma + u$.

By taking $Z = 0$, we get

$$-\log P_x\{\mathcal{R}_E \cap \Gamma = \emptyset\} = \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma \ne \emptyset\}. \tag{2.8}$$

It follows from (2.7) and (2.8) that, if

$$P_x\{\mathcal{R}_E \cap \Gamma = \emptyset\} > 0,$$

then

$$\mathbb{N}_x\{\mathcal{R}_E \cap \Gamma = \emptyset, 1 - e^{-Z}\} = -\log P_x\{e^{-Z} \mid \mathcal{R}_E \cap \Gamma = \emptyset\}. \tag{2.9}$$

By applying (2.7) to λZ and passing to the limit as $\lambda \to +\infty$, we get

$$-\log P_x\{\mathcal{R}_E \cap \Gamma = \emptyset, Z = 0\} = \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma \neq \emptyset\} + \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma = \emptyset, Z \neq 0\}. \tag{2.10}$$

By Proposition 1.1 in [DK04],

$$\mathbb{N}_x Z_\nu = P_x Z_\nu \quad \text{if } P_x Z_\nu < \infty. \tag{2.11}$$

On the other hand, for every $f \in \mathcal{B}(\bar{D})$,

$$P_x\langle f, X_D\rangle = \Pi_x f(\xi_{\tau_D}) \tag{2.12}$$

(see, e.g., [Dy02], Chapter 4, Lemma 4.1). It follows from (2.5), (2.11), (2.12), Fatou's lemma and the mean value property of harmonic functions, that

$$\mathbb{N}_x Z_\nu = P_x Z_\nu \le h_\nu(x) < \infty \quad \text{for every } \nu \in \mathcal{M}(\partial E). \tag{2.13}$$

Proposition 2.1. *Suppose $x \in D$, Λ is a Borel subset of ∂D and $\mathcal{A} = \{\mathcal{R} \cap \Lambda = \emptyset\}$. We have $P_x\mathcal{A} > 0$ and, for all $Z', Z'' \in \mathcal{Z}_x$,*

$$\begin{aligned} \mathbb{N}_x\{\mathcal{A}, (e^{-Z'} - e^{-Z''})^2\} \\ = -2\log P_x\{e^{-Z'-Z''} \mid \mathcal{A}\} + \log P_x\{e^{-2Z'} \mid \mathcal{A}\} + \log P_x\{e^{-2Z''} \mid \mathcal{A}\}. \end{aligned} \tag{2.14}$$

If $Z' = Z''$ P_x-a.s. on $\mathcal{A}$ and if $P_x\{\mathcal{A}, Z' < \infty\} > 0$, then $Z' = Z''$ $\mathbb{N}_x$-a.s. on $\mathcal{A}$.

Proof. First, $P_x\mathcal{A} > 0$ because $P_x\mathcal{A} = e^{-w_\Lambda(x)}$. Next

$$(e^{-Z'} - e^{-Z''})^2 = 2(1 - e^{-Z'-Z''}) - (1 - e^{-2Z'}) - (1 - e^{-2Z''}).$$

Therefore (2.14) follows from (2.9). The second part of the proposition is an obvious implication of (2.14). ☐

2.5. **Properties of superdiffusions.** The following properties are often used in the theory of superdiffusions. [They are a part of the definition of branching exit Markov systems, and superdiffusions are a special case of such systems (see [Dy02], Chapters 3 and 4).]

2.5.A. (Markov property) If $Y \ge 0$ is measurable with respect to the σ-algebra generated by $X_{D'}, D' \subset D$ and $Z \ge 0$ is measurable with respect to the σ-algebra generated by $X_{D''}, D'' \supset D$, then

$$P_\mu(YZ) = P_\mu(Y P_{X_D} Z). \tag{2.15}$$

2.5.B. If $\mu(E) = 0$, then $P_\mu\{X_E = \mu\} = 1$.

We use 2.5.A, 2.5.B and Proposition 2.1 to prove the next proposition.

Proposition 2.2. *Let $D \subset E$ be two open sets. Then, for every $x \in D$, X_D and X_E coincide P_x-a.s. and $\mathbb{N}_x$-a.s. on the set $\mathcal{A} = \{\mathcal{R}_D \subset D^*\}$.*

[Note that

$$D^* = \{x \in \bar{D} : d(x, \Lambda) > 0\} \tag{2.16}$$

where $\Lambda = \partial D \cap E$.]

3. Relations between superdiffusions and conditional diffusions in two open sets

3.1. Now we consider two bounded smooth open sets $D \subset E$. We denote by $\tilde{Z}_\nu$ the stochastic boundary value of $\tilde{h}_\nu(x) = \int_{\partial D} k_D(x,y)\nu(dy)$ in D; $\tilde{\Pi}_x^y$ refers to the diffusion in D conditioned to exit at $y \in \partial D$.

Theorem 3.1. *Put $\mathcal{A} = \{\mathcal{R}_D \subset D^*\}$. For every $x \in D$,*

$$\mathcal{R}_E = \mathcal{R}_D \quad P_x\text{-a.s. and} \quad \mathbb{N}_x\text{-a.s.} \tag{3.1}$$

and

$$Z_\nu = \tilde{Z}_\nu \quad P_x\text{-a.s. and} \quad \mathbb{N}_x\text{-a.s. on } \mathcal{A} \tag{3.2}$$

for all $\nu \in \mathcal{N}_1^E$ concentrated on $\partial D \cap \partial E$.

Proof. 1°. First, we prove (3.1). Clearly, $\mathcal{R}_D \subset \mathcal{R}_E$ P_x-a.s. and $\mathbb{N}_x$-a.s. for all $x \in D$. We get (3.1) if we show that, if O is an open subset of E, then, for every $x \in D$, $X_O = X_{O\cap D}$ P_x-a.s. on $\mathcal{A}$ and, for every $x \in O \cap D$, $X_O = X_{O\cap D}$ $\mathbb{N}_x$-a.s. on $\mathcal{A}$. For $x \in O \cap D$ this follows from Proposition 2.2 applied to $O \cap D \subset O$ because $\{\mathcal{R}_D \subset D^*\} \subset \{\mathcal{R}_{O\cap D} \subset (O\cap D)^*\}$. For $x \in D \setminus O$, $P_x\{X_O = X_{D\cap O} = \delta_x\} = 1$.

2°. Put

$$D_m^* = \{x \in \bar{D} : d(x, E \setminus D) > 1/m\}. \tag{3.3}$$

To prove (3.2), it is sufficient to prove that it holds on $\mathcal{A}_m = \{\mathcal{R}_D \subset D_m^*\}$ for all sufficiently large m. First we prove that, for all $x \in D$,

$$Z_\nu = \tilde{Z}_\nu \quad P_x\text{-a.s. on } \mathcal{A}_m. \tag{3.4}$$

We get (3.4) by proving that both Z_ν and $\tilde{Z}_\nu$ coincide P_x-a.s. on $\mathcal{A}_m$ with the stochastic boundary value Z^* of h_ν in D.

Let

$$E_n = \{x \in E : d(x, \partial E) > 1/n\}, \quad D_n = \{x \in D : d(x, \partial D) > 1/n\}.$$

If $n > m$, then

$$\mathcal{A}_m \subset \mathcal{A}_n \subset \{\mathcal{R}_D \subset D_n^*\} \subset \{\mathcal{R}_{D_n} \subset D_n^*\}.$$

We apply Proposition 2.2 to $D_n \subset E_n$ and we get that, P_x-a.s. on $\{\mathcal{R}_{D_n} \subset D_n^*\} \supset \mathcal{A}_m$, $X_{D_n} = X_{E_n}$ for all $n > m$, which implies $Z^* = Z_\nu$.

3°. Now we prove that

$$Z^* = \tilde{Z}_\nu \quad P_x\text{-a.s. on } \mathcal{A}_m. \tag{3.5}$$

Consider $h^0 = h_\nu - \tilde{h}_\nu$ and $Z^0 = Z_\nu^* - \tilde{Z}_\nu$. If $y \in \partial D \cap \partial E$, then

$$k_E(x,y) = k_D(x,y) + \Pi_x\{\tau_D < \tau_E, k_E(\xi_{\tau_D}, y)\}. \tag{3.6}$$

Therefore

$$h^0(x) = \Pi_x\{\xi_{\tau_D} \in \partial D \cap E, h_\nu(\xi_{\tau_D})\}. \tag{3.7}$$

This is a harmonic function in D. It vanishes on $\Gamma_m = \partial D \cap D_m^* = \partial E \cap D_m^*$.

We claim that, for every $\varepsilon > 0$ and every m, $h^0 < \varepsilon$ on $\Gamma_{m,n} = \partial E_n \cap D_m^*$ for all sufficiently large n. [If this is not true, then there exists a sequence $n_i \to \infty$ such that $z_{n_i} \in \Gamma_{m,n_i}$ and $h^0(z_{n_i}) \geq \varepsilon$. If z is a limit point of z_{n_i}, then $z \in \Gamma_m$ and $h^0(z) \geq \varepsilon$.]

All measures X_{D_n} are concentrated, P_x-a.s., on $\mathcal{R}_D$. Therefore $\mathcal{A}_m$ implies that they are concentrated, P_x-a.s., on D^*_m. Since $\Gamma_{m,n} \subset D^*_m$, we conclude that, for all sufficiently large n, $\langle h^0, X_{D_n}\rangle < \varepsilon\langle 1, X_{D_n}\rangle$ P_x-a.s. on $\mathcal{A}_m$. This implies (3.5).

4°. If $\nu \in \mathcal{M}(\partial E)$ and $Z_\nu = \mathrm{SBV}(h_\nu)$, then

$$\mathbb{N}_x Z_\nu = P_x Z_\nu \le h_\nu(x) < \infty. \tag{3.8}$$

Note that $P_x\mathcal{A} > 0$. It follows from (3.8) that $Z_\nu < \infty$ P_x-a.s. and therefore $P_x\{\mathcal{A}, Z_\nu < \infty\} > 0$. By Proposition 2.1, (3.2) follows from (3.4) . □

3.2. We also need the following result (see [Dy04a], Lemma 3.2).

Theorem 3.2. *Suppose that $D \subset E$ are smooth open sets. Denote by $\tilde{\mathcal{F}}$ the σ-algebra in Ω generated by the sets $\{s < \tau_D, \xi_s \in B\}$ where $s \ge 0, B \in \mathcal{B}(E)$. We have*

$$\tilde{\Pi}^y_x Y = \Pi^y_x\{\tau_D = \tau_E, Y\} \tag{3.9}$$

for all $x \in D, y \in \partial E \cap \partial D$ and for all $Y \in \tilde{\mathcal{F}}$.

Corollary 3.1. *If*

$$F_t = \exp\Big[-\int_0^t a(\xi_s)\, ds\Big] \tag{3.10}$$

where a is a positive continuous function on $[0,\infty)$, then, for $y \in \partial D \cap \partial E$,

$$\tilde{\Pi}^y_x F_{\tau_D} = \Pi^y_x\{\tau_D = \tau_E, F_{\tau_E}\}. \tag{3.11}$$

Indeed, it is easy to see that $F_{\tilde{\tau}} \in \tilde{\mathcal{F}}$.

4. Equations connecting P_x and $\mathbb{N}_x$ with Π^ν_x

4.1.

Theorem 4.1. *Let $Z_\nu = \mathrm{SBV}(h_\nu), Z_u = \mathrm{SBV}(u)$ where $\nu \in \mathcal{N}^E_1$ and $u \in \mathcal{U}(E)$. Then*

$$P_x Z_\nu e^{-Z_u} = e^{-u(x)}\Pi^\nu_x e^{-\Phi(u)} \tag{4.1}$$

and

$$\mathbb{N}_x Z_\nu e^{-Z_u} = \Pi^\nu_x e^{-\Phi(u)} \tag{4.2}$$

where

$$\Phi(u) = \int_0^{\tau_E} \psi'[u(\xi_t)]dt. \tag{4.3}$$

Proof. Formula (4.1) follows from Theorem 3.1 in Chapter 9 of [Dy02]. To prove (4.2), we observe that, for every $\lambda \ge 0$, $\lambda Z_\nu + Z_u = \mathrm{SBV}(v)$ where $v = \lambda h_\nu + u \in \mathcal{U}^-(E)$ and therefore, by (2.6),

$$\mathbb{N}_x(1 - e^{-\lambda Z_\nu - Z_u}) = -\log P_x e^{-\lambda Z_\nu - Z_u}. \tag{4.4}$$

By taking the derivatives with respect to λ at $\lambda = 0$,[8] we get

$$\mathbb{N}_x Z_\nu e^{-Z_u} = P_x Z_\nu e^{-Z_u}/P_x e^{-Z_u}. \tag{4.5}$$

By Theorem 1.1 of Chapter 9 in [Dy02],

$$P_x e^{-Z_u} = e^{-u(x)}. \tag{4.6}$$

[8]The differentiation under the integral signs is justified by (2.13).

Therefore (4.4) follows from (4.1), (4.5) and (4.6). □

Theorem 4.2. *Suppose that $D \subset E$ are bounded smooth open sets and Λ, L, D^* are the sets introduced in Theorem 1.1. Let ν be a finite measure on $\partial D \cap \partial E$, $x \in E$ and $\mathcal{E}_x(\nu) < \infty$. Put*

$$w_\Lambda(x) = \mathbb{N}_x\{\mathcal{R}_D \cap \Lambda \neq \emptyset\},$$
$$v_s(x) = w_\Lambda(x) + \mathbb{N}_x\{\mathcal{R}_D \cap \Lambda = \emptyset, 1 - e^{-sZ_\nu}\} \tag{4.7}$$

for $x \in D$ and let $w_\Lambda(x) = v_s(x) = 0$ for $x \in E \setminus D$. For every $x \in E$, we have

$$\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\} = \Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\}, \tag{4.8}$$

$$\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu \neq 0\} = \int_0^\infty \Pi_x^\nu\{A, e^{-\Phi(v_s)}\}ds \tag{4.9}$$

where Φ is defined by (4.3) and

$$A = \{\tau_E = \tau_D\} = \{\xi_t \in D \text{ for all } t < \tau_E\}. \tag{4.10}$$

Remark. Since $\mathcal{E}_x(\nu) < \infty$, ν belongs to $\mathcal{N}_x^E$ and to $\mathcal{N}_x^D$.

Proof. 1°. If $x \in E \setminus D$, then, $\mathbb{N}_x$-a.s., $\mathcal{R}_E$ is not a subset of D^* because $\mathcal{R}_E$ contains supports of X_O for all neighborhoods O of x and we can choose O such that $\bar{O} \cap D^* = \emptyset$. On the other hand, $\Pi_x^\nu(A) = 0$. Therefore (4.8) and (4.9) hold independently of values of w_Λ and v_s.

2°. Now we assume that $x \in D$. Put $\mathcal{A} = \{\mathcal{R}_D \subset D^*\}$. We claim that

$$\mathcal{A} = \{\mathcal{R}_E \subset D^*\} \quad \mathbb{N}_x\text{-a.s.}$$

Indeed, $\{\mathcal{R}_E \subset D^*\} \subset \mathcal{A}$ because $\mathcal{R}_D \subset \mathcal{R}_E$. By Theorem 3.1, $\mathcal{A} \subset \{\mathcal{R}_D = \mathcal{R}_E\}$ $\mathbb{N}_x$-a.s. Hence, $\mathcal{A} \subset \{\mathcal{R}_E \subset D^*\}$.

By Theorem 3.1, $\mathcal{R}_D = \mathcal{R}_E$ and $Z_\nu = \tilde{Z}_\nu$ $\mathbb{N}_x$-a.s. on $\mathcal{A}$. Therefore

$$\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\} = \mathbb{N}_x\{\mathcal{A}, Z_\nu\} = \mathbb{N}_x\{\mathcal{A}, \tilde{Z}_\nu\},$$
$$\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu e^{-sZ_\nu}\} = \mathbb{N}_x\{\mathcal{A}, Z_\nu e^{-sZ_\nu}\} = \mathbb{N}_x\{\mathcal{A}, \tilde{Z}_\nu e^{-s\tilde{Z}_\nu}\}. \tag{4.11}$$

Formula (4.7) defines two elements of $\mathcal{U}(D)$. The stochastic boundary value Z_Λ of w_Λ in D is equal to $\infty 1_{\mathcal{A}^c}$ (Remark 1.2 on p. 133 in [Dy02]) and therefore

$$e^{-Z_\Lambda} = 1_{\mathcal{A}}. \tag{4.12}$$

By (2.7) and (2.8), $v_s(x) = -\log P_x\{\mathcal{R}_D \cap \Lambda = \emptyset, e^{-sZ_\nu}\}$ and, by Remark 2.1 on p. 137 in [Dy02], the stochastic boundary value Z^s of v_s in D is equal to $Z_\Lambda + s\tilde{Z}_\nu$. Hence,

$$e^{-Z^s} = 1_{\mathcal{A}} e^{-s\tilde{Z}_\nu}. \tag{4.13}$$

By (4.11), (4.12) and (4.13),

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu\} = \mathbb{N}_x\{\tilde{Z}_\nu e^{-Z_\Lambda}\} \tag{4.14}$$

and

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu e^{-sZ_\nu}\} = \mathbb{N}_x\{\tilde{Z}_\nu e^{-Z^s}\}. \tag{4.15}$$

By applying formula (4.2) to $\tilde{Z}_\nu$ and to the restriction of w_Λ to D, we conclude from (4.14) that

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu\} = \tilde{\Pi}_x^\nu \exp\big[-\int_0^{\tau_D} \psi'[w_\Lambda(\xi_s)]ds\big] \tag{4.16}$$

and, by Corollary 3.1,

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu\} = \Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\}. \tag{4.17}$$

Analogously, (4.2) applied to the restriction of v_s to D, in combination with (4.15) and (3.11), yields

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu e^{-sZ_\nu}\} = \Pi_x^\nu\{A, e^{-\Phi(v_s)}\}. \tag{4.18}$$

Formula (4.8) follows from (4.17) and formula (4.9) follows from (4.18) because

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu \neq 0\} = \lim_{t\to\infty} \mathbb{N}_x\{\mathcal{A}, 1 - e^{-tZ_\nu}\} \tag{4.19}$$

and

$$1 - e^{-tZ_\nu} = \int_0^t Z_\nu e^{-sZ_\nu} ds. \tag{4.20}$$

□

5. Proof of Theorem 1.1

We use the following two elementary inequalities:

5.A. For all $a, b \geq 0$ and $0 < \beta < 1$,

$$(a + b)^\beta \leq a^\beta + b^\beta. \tag{5.1}$$

Proof. It is sufficient to prove (5.1) for $a = 1$. Put $f(t) = (1 + t)^\beta - t^\beta$. Note that $f(0) = 1$ and $f'(t) \leq 0$ for $t > 0$. Hence $f(t) \leq 1$ for $t \geq 0$. □

5.B. For every finite measure M, every positive measurable function Y and every $\beta > 0$,

$$M(Y^{-\beta}) \geq M(1)^{1+\beta}(MY)^{-\beta}.$$

Indeed $f(y) = y^{-\beta}$ is a convex function on $\mathbb{R}_+$, and we get 5.B by applying Jensen's inequality to the probability measure $M/M(1)$.

Proof of Theorem 1.1. 1°. If $x \in E \setminus D$, then, $\mathbb{N}_x$-a.s., $\mathcal{R}_E$ is not a subset of D^* (see the proof of Theorem 4.2). Hence, both sides of (1.7) vanish.

2°. Suppose $x \in D$. By (2.6), $\mathbb{N}_x(1 - e^{-sZ_\nu}) = u_{s\nu}$. Thus (4.7) implies $v_s \leq w_\Lambda + u_{s\nu}$. Therefore, by 5.A, $v_s^{\alpha-1} \leq w_\Lambda^{\alpha-1} + u_{s\nu}^{\alpha-1}$ and, since $u_{s\nu} \leq h_{s\nu} = sh_\nu$, $\Phi(v_s) \leq \Phi(w_\Lambda) + s^{\alpha-1}\Phi(h_\nu)$.

Put $\mathcal{A} = \{\mathcal{R}_E \subset D^*\}$. It follows from (4.9) that

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu \neq 0\} \geq \Pi_x^\nu\{A, \int_0^\infty e^{-\Phi(w_\Lambda) - s^{\alpha-1}\Phi(h_\nu)} ds\}. \tag{5.2}$$

Note that $\int_0^\infty e^{-as^\beta} ds = Ca^{-1/\beta}$ where $C = \int_0^\infty e^{-t^\beta} dt$. Therefore (5.2) implies

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu \neq 0\} \geq C\Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\Phi(h_\nu)^{-1/(\alpha-1)}\}. \tag{5.3}$$

The right side in (5.3) is equal to $CM(Y^{-\beta})$ where $\beta = 1/(\alpha-1), Y = \Phi(h_\nu)$ and M is the measure with the density $1_A e^{-\Phi(w_\Lambda)}$ with respect to Π_x^ν. We get from (5.3) and 5.B that

$$\begin{aligned}\mathbb{N}_x\{\mathcal{A}, Z_\nu \neq 0\} &\geq CM(1)^{1+\beta}(MY)^{-\beta} \\ &= C[\Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\}]^{\alpha/(\alpha-1)}[\Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\Phi(h_\nu)\}]^{-1/(\alpha-1)}.\end{aligned}$$

By (4.8), $\Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\} = \mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\}$ and, since $\Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\Phi(h_\nu)\} \leq \Pi_x^\nu\Phi(h_\nu)$, we have

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu \neq 0\} \geq C[\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\}]^{\alpha/(\alpha-1)}[\Pi_x^\nu\Phi(h_\nu)]^{-1/(\alpha-1)}. \tag{5.4}$$

3°. By the definition of h-transform, for every $f \in \mathcal{B}(E)$ and every $h \in \mathcal{H}(E)$,

$$\Pi_x^h \int_0^{\tau_E} f(\xi_t)dt = \int_0^\infty \Pi_x^h\{t < \tau_E, f(\xi_t)\}dt = \int_0^\infty \Pi_x\{t < \tau_E, f(\xi_t)h(\xi_t)\}dt.$$

By taking $f = \alpha h_\nu^{\alpha-1}$ and $h = h_\nu$ we get

$$\Pi_x^\nu\Phi(h_\nu) = \alpha\mathcal{E}_x(\nu). \tag{5.5}$$

Formula (1.7) follows from (5.4) and (5.5). $\square$

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