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AN UPPER BOUND FOR POSITIVE SOLUTIONS OF THE EQUATION $\Delta u = u^\alpha$

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Abstract. In 2002 Mselati proved that every positive solution of the equation $\Delta u = u^2$ in a bounded domain of class C^4 is the limit of an increasing sequence of moderate solutions. (A solution is called moderate if it is dominated by a harmonic function.) As a part of his proof, he established an upper bound (in terms of the capacity of K) for solutions vanishing off a compact subset K of ∂E. We use a different kind of capacity (we call it the Poisson capacity) and we establish in terms of this capacity an upper bound for solutions of $\Delta u = u^\alpha$ with $1 < \alpha \le 2$. This is a part of the program: to classify all positive solutions of this equation.

1. Introduction

1.1. **Main result.** Let $E \subset \mathbb{R}^d$ be a bounded smooth domain of class C^4 in $\mathbb{R}^d$. For $x \in E$, we denote by $\rho(x)$ the distance to the boundary ∂E and by $k(x, y)$ the Poisson kernel in E for the Laplacian Δ.

Let $\mathcal{M}(S)$ stand for the set of all finite measures on a measurable space S. For every $\nu \in \mathcal{M}(\partial E)$, we denote by h_ν the harmonic function $h_\nu(x) = \int_{\partial E} k(x, y)\nu(dy)$.

For every $\alpha > 1$ and every Radon measure m on E, there exists a Choquet capacity given on compact subsets of ∂E by the formula

$$\mathrm{Cap}(K) = \sup_{\nu \in \mathcal{P}(K)} \mathcal{E}(\nu)^{-1} \tag{1.1}$$

where $\mathcal{P}(K)$ is the set of all probability measures on K and

$$\mathcal{E}(\nu) = \int_E h_\nu(x)^\alpha m(dx). \tag{1.2}$$

We call Cap the Poisson capacity.

Our goal is to establish the following theorem.

Theorem 1.1. *Suppose* Cap *is the Poisson capacity corresponding to* $1 < \alpha \le 2$ *and the measure* $m(dx) = \rho(x)dx$. *Let* K *be a compact subset of* ∂E. *There exists a constant* $C(E)$ *depending only on* E, *such that, for every compact* $K \subset \partial E$ *and*

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every solution u *of the boundary value problem*

$$\begin{cases} \Delta u = u^\alpha & \text{in } E, \\ u = 0 & \text{on } \partial E \setminus K, \end{cases} \tag{1.3}$$

we have

$$u(x) \le C(E)\rho(x)\,\mathrm{dist}(x,K)^{-d}\,\mathrm{Cap}(K)^{1/(\alpha-1)}. \tag{1.4}$$

1.2. **Equivalent definitions of the Poisson capacity.** Put

$$\hat{K}f(y) = \int_E f(x)m(dx)k(x,y). \tag{1.5}$$

The following definitions of the Poisson capacity are equivalent to (1.1):

$$\mathrm{Cap}(K)^{1/\alpha} = \sup\{\nu(K) : \nu \in \mathcal{M}(K), \mathcal{E}(\nu) \le 1\} \tag{1.6}$$

and

$$\mathrm{Cap}(K)^{1/\alpha} = \inf\{\|f\|_{\alpha'} : \hat{K}f \ge 1 \text{ on } K\} \tag{1.7}$$

where $\alpha' = \alpha/(\alpha-1)$ and $\|f\|_{\alpha'}$ stands for the norm in $L_{\alpha'}(m)$.

The equivalence of (1.6) and (1.7) is proved, for instance, in [1] (see Theorem 5.1 in Chapter 13). To prove the equivalence of (1.1) and (1.5), we note that $\nu \in \mathcal{M}(K)$ is equal to $t\mu$ where $t = \nu(K)$ and $\mu = \nu/t \in \mathcal{P}(K)$ and

$$\begin{aligned} \sup_{\nu\in\mathcal{M}(K)} \{\nu(K) : \mathcal{E}(\nu) \le 1\} &= \sup_{\mu\in\mathcal{P}(K)} \sup_{t\ge 0}\{t : t^\alpha \mathcal{E}(\mu) \le 1\} \\ &= \sup_{\mu\in\mathcal{P}(K)} \mathcal{E}(\mu)^{-1/\alpha} = (\mathrm{Cap}(K))^{1/\alpha}. \end{aligned}$$

1.3. **Notation.** We denote by $B_r(x)$ a ball of radius r centered at x. Let H be a compact subset of $\partial E \cap B_r(x)$ and let ϕ be a C^∞ function on E such that $0 \le \phi \le 1$. We call ϕ an (R,x)-*truncating function for* H if $\phi = 0$ in a neighborhood of H and $\phi(y) = 1$ if $\mathrm{dist}(x,y) \ge R$. We call ϕ an R-*localizing function* if $\phi = 0$ in a neighborhood of H and $\phi(y) = 1$ if $\mathrm{dist}(y,H) \ge R$.

2. Bounds in a halfspace

2.1. First, we establish some bounds in the case when $E = (0,\infty) \times \mathbb{R}^{d-1}$. A generic element of E is denoted by z or $(s,x), s \in (0,\infty), x \in \mathbb{R}^{d-1}$. We use the notation $f(s,x)$ and $f^s(x)$ for functions on E.

Denote by $\mathbb{E}$ an infinite strip $\{(s,x) : 0 \le s < 1\}$. The measure $m(dz) = 1_{[0,1]}(s)s\,ds\,dx$ is concentrated on $\mathbb{E}$. We denote by $\|f\|_\alpha$ the norm in $L_\alpha(m)$. The Poisson kernel k can be represented by the formula

$$k((s,x),y) = Cq^s(x-y)$$

where C is a constant depending only on the dimension, and

$$q^s(x) = \frac{s}{(|x|^2+s^2)^{d/2}}. \tag{2.1}$$

Lemma 2.1. *Suppose that* $z_0 \in \partial E$ *and* H *is a compact subset of* $\partial E \cap B_1(z_0)$. *If* $\mathrm{Cap}(H) > 0$, *then there exists a* $(3/2, z_0)$-*truncating function* β *for* H *such that* $\beta^s(x) = 1$ *for* $s \ge 1$ *and*

$$\|\nabla^2\beta\|_{\alpha'}^{\alpha'} + \||\nabla\beta|^2\|_{\alpha'}^{\alpha'} + \|\nabla\beta\|_{\alpha'}^{\alpha'} + \left\|\frac{1}{s}\frac{\partial\beta}{\partial s}\right\|_{\alpha'}^{\alpha'} \le C(d)\,\mathrm{Cap}(H)^{1/(\alpha-1)} \tag{2.2}$$

where the constant $C(d)$ depends only on d. If $\mathrm{Cap}(H) = 0$*, then, for every* $\epsilon > 0$*, there exists a* $(3/2, z_0)$*-truncating function* β *for* H *such that* $\beta^r(x) = 1$ *for* $r \geq 1$ *and*

$$\|\nabla^2\beta\|_{\alpha'}^{\alpha'} + \||\nabla\beta|^2\|_{\alpha'}^{\alpha'} + \|\nabla\beta\|_{\alpha'}^{\alpha'} + \left\|\frac{1}{s}\frac{\partial\beta}{\partial s}\right\|_{\alpha'}^{\alpha'} < \epsilon. \tag{2.3}$$

Proof. Let $\mathrm{Cap}(H) > 0$. By (1.7), there exists a function f on E such that $\|f\|_{\alpha'}^{\alpha'} \leq 2\,\mathrm{Cap}(H)^{\alpha'/\alpha}$ and $\hat{K}f \geq 1$ on H. We may assume that $f \geq 0$ (otherwise we just replace f with f^+).

Let $A(t), 0 \leq t < \infty$, be an increasing C^2 function such that $A(t) = 0$ for $t \leq 1$ and $A(t) = 1$ for $t \geq \sqrt{2}$. We set

$$(\mathcal{T}f)^t(y) = \int_0^1 s\,ds A(\sqrt{s/t}) \int_{\mathbb{R}^{d-1}} f^s(x)q^s(y-x)\,dx \quad \text{for } t > 0 \tag{2.4}$$

and

$$(\mathcal{T}f)^0(y) = \lim_{t\downarrow 0}(\mathcal{T}f)^t(y) = \hat{K}f(y) \tag{2.5}$$

(cf. [1, formula (6.3), p. 175]).

By [1, Theorem 13.6.1], we have

$$\|\mathcal{T}f\|_{\alpha'} + \|\nabla\mathcal{T}f\|_{\alpha'} + \|\nabla^2\mathcal{T}f\|_{\alpha'} + \left\|\frac{1}{s}\frac{\partial\mathcal{T}f}{\partial s}\right\|_{\alpha'} \leq C\|f\|_{\alpha'}.$$

Let $g(s,y) = a(s)b(y)$ be such that $0 \leq a, b \leq 1$, a, b are C^2 functions, $a = 1$ in a neighborhood of 0, $b = 1$ in a neighborhood of $\partial E \cap B_1(z_0)$ and $g = 0$ outside $B_{3/2}(z_0)$. Let $h(t)$ be an increasing C^2 function on $[0,\infty)$ such that $h(t) = 0$ if $t \leq 1/4$ and $h(t) = 1$ if $t \geq 3/4$. As in the proof of Lemma 13.6.5 from [1], we put

$$u = \mathcal{T}f, \quad v = gu, \quad \phi = h(v)$$

and, finally,

$$\beta = 1 - \phi = 1 - h(g\mathcal{T}f).$$

By (2.5), $\mathcal{T}f = \hat{K}f \geq 1$ on H, and $\beta = 0$ in a neighborhood of H by the choice of g and h. By direct computation,[1] we get

$$|v| + |\nabla v| + |\nabla^2 v| \leq C(|u| + |\nabla u| + |\nabla^2 u|), \tag{2.6}$$

$$\left|\frac{1}{s}\frac{\partial v}{\partial s}\right| = \left(\left|\frac{a}{s}\frac{\partial u}{\partial s} + \frac{a'}{s}u\right|\right) b \leq C\left(|u| + \frac{1}{s}\left|\frac{\partial u}{\partial s}\right|\right). \tag{2.7}$$

More computation yields

$$|\nabla\phi| \leq C|\nabla v|, \tag{2.8}$$

$$|\nabla^2\phi| \leq C\left(|\nabla^2 v| + \frac{|\nabla v|^2}{v}\right), \tag{2.9}$$

$$|\nabla\phi|^2 \leq C\left(\frac{|\nabla v|^2}{v}\right). \tag{2.10}$$

Therefore

$$\|\nabla^2\beta\|_{\alpha'} + \||\nabla\beta|^2\|_{\alpha'} + \|\nabla\beta\|_{\alpha'} + \left\|\frac{1}{s}\frac{\partial\beta}{\partial s}\right\|_{\alpha'} \leq C\|f\|_{\alpha'} \leq C\,\mathrm{Cap}(H)^{1/(\alpha-1)}.$$

[1] See [1, pp. 181–182], or [2, Section 3].

If $\mathrm{Cap}(H) = 0$, then $\|f\|_{\alpha'}$ can be made arbitrary small, and the same construction yields (2.3). $\square$

2.2. For a set $H \subset R^{d-1}$, we put $\lambda H = \{\lambda x : x \in H\}$.

Lemma 2.2. *For every compact set $H \subset R^{d-1}$ and every $0 < \lambda < 1$,*

$$\mathrm{Cap}(\lambda H)^{1/(\alpha-1)} \leq \lambda^{d-2\alpha'+1}\,\mathrm{Cap}(H)^{1/(\alpha-1)}. \tag{2.11}$$

Proof. Let $\lambda > 0$ and $\nu \in \mathcal{P}(H)$. Then $\nu_\lambda(A) = \nu(A/\lambda)$ is concentrated on λH. Note that $q^{\lambda s}(\lambda x) = \lambda^{d-1} q^s(x)$ and therefore

$$h_\nu(s,x) = \lambda^{d-1} h_{\nu_\lambda}(\lambda s, \lambda x). \tag{2.12}$$

Formula (2.12) and change of variables $t = \lambda s, y = \lambda x$ yield

$$\begin{aligned}
\mathcal{E}(\nu) &= \int_0^1 \int_{\mathbb{R}^{d-1}} h_\nu^\alpha(s,x) s ds dx \\
&= \int_0^1 \int_{\mathbb{R}^{d-1}} \lambda^{(d-1)\alpha} h_{\nu_\lambda}^\alpha(\lambda s, \lambda x) s ds dx \\
&= \int_0^\lambda \int_{\mathbb{R}^{d-1}} \lambda^{(d-1)\alpha} \lambda^{-(d+1)} h_{\nu_\lambda}^\alpha(t,y) t dt dy \\
&\leq \lambda^{(d-1)\alpha} \lambda^{-(d+1)} \mathcal{E}(\nu_\lambda)
\end{aligned}$$

and (1.1) implies

$$\mathrm{Cap}(H) \geq \lambda^{d+1-(d-1)\alpha}\,\mathrm{Cap}(\lambda H). \tag{2.13}$$

Formula (2.11) follows from (1.1) because $d-2\alpha'+1 = -[d+1-(d-1)\alpha]/(\alpha-1)$. $\square$

Lemma 2.3. *Let $z_0 \in \partial E, 0 < \delta < 1$ and let Γ be a compact subset of $\partial E \cap B_\delta(z_0)$. Suppose $\mathrm{Cap}(\Gamma) > 0$. There exists a $(3\delta/2, z_0)$-truncating function $\gamma = \gamma_{\Gamma,\delta}$ for Γ such that*

$$\|\nabla^2\gamma\|_{\alpha'}^{\alpha'} + \||\nabla\gamma|^2\|_{\alpha'}^{\alpha'} + \left\|\frac{1}{\delta}\nabla\gamma\right\|_{\alpha'}^{\alpha'} + \left\|\frac{1}{s}\frac{\partial\gamma}{\partial s}\right\|_{\alpha'}^{\alpha'} \leq C(d)\,\mathrm{Cap}(\Gamma)^{1/\alpha-1} \tag{2.14}$$

where the constant $C(d)$ depends only on d. If $\mathrm{Cap}(\Gamma) = 0$, then the left side of (2.14) can be made smaller than any $\varepsilon > 0$.

Proof. Let $H = \Gamma/\delta$ and let $\beta(s,x)$ be the function constructed in Lemma 2.1 applied to H and z_0/δ. Put $\gamma(s,x) = \beta(r/\delta, x/\delta)$. Since $\beta(s,x) = 1$ if $s \geq 1$, $\gamma(s,x) = 1$ if $s \geq \delta$. Also,

$$\nabla\gamma(s,x) = (1/\delta)\nabla\beta(s/\delta, x/\delta), \qquad \nabla^2\gamma(s,x) = (1/\delta^2)\nabla^2\beta(s/\delta, x/\delta)$$

and therefore

$$\begin{aligned}
\|\nabla\gamma\|_{\alpha'}^{\alpha'} &= \int_0^\delta \int_{\mathbb{R}^{d-1}} |\nabla\gamma(s,x)|^{\alpha'}\, s\, ds\, dx \\
&= \int_0^\delta \int_{\mathbb{R}^{d-1}} \delta^{-2\alpha'} |\nabla\beta(s/\delta, x/\delta)|^{\alpha'}\, s\, ds\, dx \\
&= \int_0^1 \int_{\mathbb{R}^{d-1}} \delta^{d+1-2\alpha'} |\nabla\beta(s,x)|^{\alpha'}\, s\, ds\, dx = \delta^{d+1-2\alpha'} \|\beta\|_{\alpha'}^{\alpha'}.
\end{aligned}$$

In a similar way,

$$\||\nabla\gamma|^2\|_{\alpha'}^{\alpha'} = \delta^{d+1-2\alpha'}\||\nabla\beta|^2\|_{\alpha'}^{\alpha'}, \qquad \left\|\frac{1}{\delta}\nabla\gamma\right\|_{\alpha'}^{\alpha'} = \delta^{d+1-2\alpha'}\|\nabla\beta\|_{\alpha'}^{\alpha'}$$

and

$$\left\|\frac{1}{s}\frac{\partial\gamma}{\partial s}\right\|_{\alpha'}^{\alpha'} = \left\|\frac{1}{s}\frac{\partial\beta}{\partial s}\right\|_{\alpha'}^{\alpha'}.$$

Therefore (2.14) follows from (2.2) and Lemma 2.2. □

3. Bounds in a unit ball

3.1. Now let E be a ball of radius 1 in $\mathbb{R}^d$ centered at a point z_0 with coordinates $s = 1, x = 0$. As before, let $\mathbb{E} = \{(s,x) : 0 \le s < 1\}$. For a point $z = (s,x) \in \mathbb{E}$, we denote by

$$\phi(z) = x/(1-s)$$

a projection of z to $\mathbb{R}^{d-1}$ with center at z_0. For a point $z \in E \cap \mathbb{E}$, we put

$$\psi(z) = (1 - |z - z_0|, \phi(z))$$

(cf. [3], Section 3.1.1). The mapping ψ defines a 1-1 correspondence between $E \cap \mathbb{E}$ and $\mathbb{E}$.

For a set $H \subset \partial E$, denote by $\mathrm{Cap}_E(H)$ the Poisson capacity of H with respect to the domain E and the measure $m(dz) = \mathrm{dist}(z, \partial E)\,dz$. For a set $K \subset \mathbb{R}^{d-1}$, we denote by $\mathrm{Cap}_{\mathbb{E}}(K)$ the Poisson capacity with respect to the halfspace and the measure $m(ds,dx) = 1_{[0,1)}(s)\,s\,ds\,dx$.

Lemma 3.1. *Let H be a compact subset of ∂E that is contained in a ball of radius $1/4$ centered at zero, and let $K = \psi(H)$. There exists a constant C depending only on the dimension, such that*

$$C^{-1}\,\mathrm{Cap}_{\mathbb{E}}(K) \le \mathrm{Cap}_E(H) \le C\,\mathrm{Cap}_{\mathbb{E}}(K).$$

Proof. Let μ be a probability measure on H and let ν be a measure on K defined by the formula $\nu(\Gamma) = \mu(\psi^{-1}(\Gamma))$. It is enough to show that

$$C^{-1}\mathcal{E}_E(\mu) < \mathcal{E}_{\mathbb{E}}(\nu) < C\mathcal{E}_E(\mu) \tag{3.1}$$

for some constant C depending only on the dimension.

Denote by D a ball of radius 1/2 centered at zero. Let $D' = \psi(D)$. Put

$$I_E = \int_{E\setminus D}\rho(z)h_\mu^\alpha(z)\,dz, \qquad J_E = \int_{E\cap D}\rho(x)h_\mu^\alpha(x)\,dx$$

and

$$I_{\mathbb{E}} = \int_{\mathbb{E}\setminus D'}\rho(z)h_\nu^\alpha(z)\,dz, \qquad J_{\mathbb{E}} = \int_{\mathbb{E}\cap D'}\rho(z)h_\nu^\alpha(z)\,dz.$$

Note that

$$C^{-1}\rho(z)/(|z|+1/4)^d < h_\mu(z) \le C\rho(z)/(|z|-1/4)^d$$

on $E \setminus D$ and therefore $C^{-1} < I_E < C$. For the same reason, $C^{-1} < I_{\mathbb{E}} < C$ and therefore

$$C^{-1}I_E < I_{\mathbb{E}} < CI_E. \tag{3.2}$$

On the other hand,

$$C^{-1}h_\nu(\psi(z)) < h_\mu(z) < Ch_\nu(\psi(z))$$

on D (this follows from a similar relation for the Poisson kernels). Since the derivatives of ψ and ψ^{-1} are bounded on D and D', we conclude that

$$C^{-1} J_E < J_{\mathbb{E}} < C J_E. \tag{3.3}$$

Since $\mathcal{E}_E(\mu) = I_E + J_E$ and $\mathcal{E}_{\mathbb{E}}(\nu) = I_{\mathbb{E}} + J_{\mathbb{E}}$, (3.1) follows from (3.2) and (3.3). $\square$

Lemma 3.2. *Let H be a compact subset of ∂E such that $H \subset B_{\delta/8}(0)$, where $\delta \le 2$. Suppose $\operatorname{Cap}_E(H) > 0$. There exists a $(3\delta/8, 0)$-truncating function γ such that*

$$\|\nabla^2\gamma\|_{\alpha'}^{\alpha'} + \||\nabla\gamma|^2\|_{\alpha'}^{\alpha'} + \left\|\frac{1}{\delta}\nabla\gamma\right\|_{\alpha'}^{\alpha'} + \left\|\frac{1}{\rho}\frac{\partial\gamma}{\partial\rho}\right\|_{\alpha'}^{\alpha'} \le C(d)\operatorname{Cap}_E(H)^{1/(\alpha-1)}, \tag{3.4}$$

where the constant $C(d)$ depends only on d. If $\operatorname{Cap}_E(H) = 0$, then the left side of (3.4) can be made smaller than any $\varepsilon > 0$.

Proof. We apply Lemma 2.3 to the set $K = \psi(H)$. Let γ_K be the function constructed in Lemma 2.3. We put $\gamma(s,x) = \gamma_K(\psi(s,x))$ if $s < 1$, and $\gamma(s,x) = 0$ otherwise. Similarly to the proof of [3, Sublemma 3.1.2], we show that

$$\begin{aligned}
&\|\nabla^2\gamma\|_{E,\alpha'}^{\alpha'} + \||\nabla\gamma|^2\|_{E,\alpha'}^{\alpha'} + \left\|\frac{1}{\delta}\nabla\gamma\right\|_{E,\alpha'}^{\alpha'} + \left\|\frac{1}{\rho}\frac{\partial\gamma}{\partial\rho}\right\|_{E,\alpha'}^{\alpha'} \\
&\qquad \le \|\nabla^2\gamma_K\|_{\mathbb{E},\alpha'}^{\alpha'} + \||\nabla\gamma_K|^2\|_{\mathbb{E},\alpha'}^{\alpha'} + \left\|\frac{1}{\delta}\nabla\gamma_K\right\|_{\mathbb{E},\alpha'}^{\alpha'} + \left\|\frac{1}{s}\frac{\partial\gamma_K}{\partial s}\right\|_{\mathbb{E},\alpha'}^{\alpha'}
\end{aligned} \tag{3.5}$$

where $\|\cdot\|_E$ and $\|\cdot\|_{\mathbb{E}}$ stand for $L_{\alpha'}$-norms in E and $\mathbb{E}$. Finally, we apply Lemma 3.1. $\square$

3.2. **Localizing functions.** Let H be a subset of ∂E and let γ be a C^2-function on E. We call γ an ε-localizing function for H if $0 \le \gamma \le 1$, $\gamma = 1$ in a neighborhood of H and $\gamma(z) = 0$ if $\operatorname{dist}(z, \mathcal{H}) > \varepsilon$.

Lemma 3.3. *There exists a constant $C(d)$ such that, for every compact subset K of ∂E with $\operatorname{Cap}(K) > 0$ and $\operatorname{diam}(K) \le 4\delta$, there exists a $\delta/2$-localizing function $\gamma = \gamma_{\delta,K}$ for K such that*

$$\|\nabla^2\gamma\|_{\alpha'}^{\alpha'} + \||\nabla\gamma|^2\|_{\alpha'}^{\alpha'} + \left\|\frac{1}{\delta}\nabla\gamma\right\|_{\alpha'}^{\alpha'} + \left\|\frac{1}{\rho}\frac{\partial\gamma}{\partial\rho}\right\|_{\alpha'}^{\alpha'} \le C(d)\operatorname{Cap}(K)^{1/(\alpha-1)}. \tag{3.6}$$

Proof. As in [3, Lemma 3.1.2], we cover the set K by finitely many balls $B_{\delta/8}(y_k)$ (the number n of the balls depends only on the dimension d). We apply Lemma 3.2 to each of the sets $H_k = K \cap B_{\delta/8}(y_k)$. Denote by γ_k the corresponding truncating function constructed in Lemma 3.2 (we choose $\epsilon = \operatorname{Cap}(\Gamma)^{1/(\alpha-1)}$ if $\operatorname{Cap}(H_k) = 0$ for some k). We set

$$\gamma = \gamma_1 \cdots \gamma_n.$$

Note that

$$|\nabla\gamma| \le \sum_k |\nabla\gamma_k|$$

and

$$|\nabla^2\gamma| \le \sum_k |\nabla^2\gamma_k| + \sum_{k\ne l} |\nabla\gamma_k|\,|\nabla\gamma_l| \le \sum_k |\nabla^2\gamma_k| + \frac{(n-1)}{2}\sum_k |\nabla\gamma_k|^2.$$

By applying Minkowski inequality, we get

$$\begin{aligned}\|\nabla^2\gamma\|_{\alpha'}^{\alpha'} + \||\nabla\gamma|^2\|_{\alpha'}^{\alpha'} + \left\|\frac{1}{\delta}\nabla\gamma\right\|_{\alpha'}^{\alpha'} + \left\|\frac{1}{\rho}\frac{\partial\gamma}{\partial\rho}\right\|_{\alpha'}^{\alpha'} \\ \le C(d,n)\sum_k \operatorname{Cap}(H_k)^{1/(\alpha-1)} + nC(n)\epsilon \\ \le n(C(d,n)+C(n))\operatorname{Cap}(\Gamma)^{1/(\alpha-1)}.\end{aligned} \tag{3.7}$$

□

Lemma 3.4. *Let K be a compact subset of ∂E such that* $\operatorname{diam}(K) \le 4\delta$ *and* $\operatorname{Cap}(K) > 0$. *Let γ be the $\delta/2$-localizing function constructed in Lemma 3.3. Then*

$$\begin{aligned}\int_E u\gamma^{2\alpha'-1}|\Delta\gamma|\rho &\le C\left(\int_E u^\alpha\gamma^{2\alpha'}\rho\right)^{1/\alpha}\operatorname{Cap}(K)^{1/\alpha},\\ \int_E u\gamma^{2\alpha'-2}|\nabla\gamma|^2\rho &\le C\left(\int_E u^\alpha\gamma^{2\alpha'}\rho\right)^{1/\alpha}\operatorname{Cap}(K)^{1/\alpha},\\ \int_E u\gamma^{2\alpha'-1}|\nabla\gamma|\rho &\le C\left(\int_E u^\alpha\gamma^{2\alpha'}\rho\right)^{1/\alpha}\delta\operatorname{Cap}(K)^{1/\alpha},\\ \int_E u\gamma^{2\alpha'-1}\left|\frac{\partial\gamma}{\partial\rho}\right| &\le C\left(\int_E u^\alpha\gamma^{2\alpha'}\rho\right)^{1/\alpha}\operatorname{Cap}(K)^{1/\alpha}\end{aligned} \tag{3.8}$$

whenever u satisfies (1.3).

Proof. The assertion follows from the Hölder inequality, Lemma 3.3, the identity $(\alpha-1)\alpha' = \alpha$ and the inequality $\gamma^{(2\alpha'-1)\alpha} \le \gamma^{(2\alpha'-2)\alpha} = \gamma^{2\alpha'}$. For instance, for the last line in (3.8), we have

$$\begin{aligned}\int_E u\gamma^{2\alpha'-1}\left|\frac{\partial\gamma}{\partial\rho}\right| &= \int_E u\gamma^{2\alpha'-1}\left|\frac{1}{\rho}\frac{\partial\gamma}{\partial\rho}\right|\rho\\ &\le \left(\int_E u^\alpha\gamma^{(2\alpha'-1)\alpha}\rho\right)^{1/\alpha}\left(\int_E\left|\frac{1}{\rho}\frac{\partial\gamma}{\partial\rho}\right|^{\alpha'}\rho\right)^{1/\alpha'}\\ &\le \left(\int_E u^\alpha\gamma^{2\alpha'}\rho\right)^{1/\alpha} C(d)\operatorname{Cap}(K)^{1/[\alpha'(\alpha-1)]}.\end{aligned}$$

□

Lemma 3.5. *Let K, γ, u be as in Lemma 3.4. There exists a constant $C(d)$ such that*

$$\int_E u^\alpha\gamma^{2\alpha'}\rho \le C(d)\operatorname{Cap}(K)^{1/(\alpha-1)} \tag{3.9}$$

whenever u satisfies (1.3).

Proof. This is an adaptation of Lemma 3.1.3 in [3]. Let $E_s = B(s, z_0)$ and $r = |z - z_0|$. By replacing u^2 and γ^4 with u^α and $\gamma^{2\alpha'}$ in the arguments of [3], we get a bound

$$\begin{aligned}\int_E u^\alpha \gamma^{2\alpha'} \rho \le \int_E u\Delta(\gamma^{2\alpha'}(1-r^2)) - 4\int_E u\frac{\partial(\gamma^{2\alpha'})}{\partial r} r \\ + \liminf_{s\to 1-} \int_{\partial E_s} \frac{\partial}{\partial r}(u\gamma^{2\alpha'}(1-r^2)).\end{aligned} \tag{3.10}$$

As in [3, Sublemma 3.1.3], one can show that the last term in (3.10) is negative and can be dropped.

Finally, we note that $1 - r^2 \le 2\rho$ and therefore

$$\begin{aligned}&\left|\int_E u\Delta(\gamma^{2\alpha'}(1-r^2))\right| \\ &\quad\le 2\alpha' \int_E u\gamma^{2\alpha'-1}|\Delta\gamma|(1-r^2) + 2\alpha'(2\alpha'-1)\int_E u\gamma^{2\alpha'-2}|\nabla\gamma|^2\rho \\ &\quad\le 4\alpha' \int_E u\gamma^{2\alpha'-1}|\Delta\gamma|(1-r^2) + 4\alpha'(2\alpha'-1)\int_E u\gamma^{2\alpha'-2}|\nabla\gamma|^2\rho \\ &\quad\le C\left(\int_E u^\alpha\gamma^{2\alpha'}\rho\right)^{1/\alpha} \operatorname{Cap}(K)^{1/\alpha}.\end{aligned} \tag{3.11}$$

In a similar way,

$$\begin{aligned}\left|\int_E u\frac{\partial(\gamma^{2\alpha'})}{\partial r} r\right| = 2\alpha'\left|\int_E u\gamma^{2\alpha'-1}\frac{\partial\gamma}{\partial r} r\right| \le 2\alpha'\left|\int_E u\gamma^{2\alpha'-1}\frac{\partial\gamma}{\partial r}\right| \\ \le C\left(\int_E u^\alpha\gamma^{2\alpha'}\rho\right)^{1/\alpha}\operatorname{Cap}(K)^{1/\alpha}.\end{aligned} \tag{3.12}$$

From (3.10), (3.11) and (3.12), we get

$$\int_E u^\alpha\gamma^{2\alpha'}\rho \le C\left(\int_E u^\alpha\gamma^{2\alpha'}\rho\right)^{1/\alpha}\operatorname{Cap}(K)^{1/\alpha},$$

which implies (3.9). □

Combining (3.8) with Lemma 3.5, we get:

Lemma 3.6. *Let K, γ, u be as in Lemma 3.4. Then*

$$\begin{aligned}\int_E u\gamma^{2\alpha'-1}|\Delta\gamma|\rho &\le C\operatorname{Cap}(K)^{1/(\alpha-1)}, \\ \int_E u\gamma^{2\alpha'-2}|\nabla\gamma|^2\rho &\le C\operatorname{Cap}(K)^{1/(\alpha-1)}, \\ \int_E u\gamma^{2\alpha'-1}|\nabla\gamma|\rho &\le C\delta\operatorname{Cap}(K)^{1/(\alpha-1)}, \\ \int_E u\gamma^{2\alpha'-1}\left|\frac{\partial\gamma}{\partial\rho}\right| &\le C\operatorname{Cap}(K)^{1/(\alpha-1)}.\end{aligned} \tag{3.13}$$

3.3. The rest of the proof of Theorem 1.1 is very close to the corresponding part of the proof of [3, Theorem 3.1.1]. We begin with

Lemma 3.7. *Let K, γ, u be as in Lemma 3.4, and let G_E be the Green operator of E. For every $y \in E, \beta > 0$,*

$$\gamma^\beta(y)u(y) \le \frac{1}{2} G_E\left(\gamma^\beta \Delta u - \Delta(\gamma^\beta u)\right)(y).$$

This is a version of [3, Lemma 3.1.6] (in [3], $\beta = 4$). Bounds from [3, Sublemma 3.1.5] must be replaced with those of [1, Theorem 7.1]. Other modifications are obvious.

As a first step, we establish

Lemma 3.8. *There exists a constant $C(d)$ such that the inequality* (1.4) *holds whenever*

$$\operatorname{dist}(x, K) \ge \frac{\operatorname{diam}(K)}{4}.$$

Proof. The proof is an appropriate modification of the proof of [3, Lemma 3.1.5]. Let $\delta = \operatorname{dist}(x, K)$ and let γ be the $\delta/2$-localizing function constructed in Lemma 3.3. Clearly, $\gamma = 1$ in a neighborhood of x and therefore

$$u(x) = \gamma^{2\alpha'}(x)u(x) \le \frac{1}{2} G_E(\gamma^{2\alpha'}\Delta u - \Delta(\gamma^{2\alpha'}u))(x) \tag{3.14}$$

by Lemma 3.7. As in [3], the right side can be evaluated by means of Green's formula applied to the domain $B(z_0, r) \setminus B(x, \varepsilon)$ and passage to the limit as $\varepsilon \to 0, r \to 1$. Namely, we get

$$\begin{aligned} &G_E(\gamma^{2\alpha'}\Delta u - \Delta(\gamma^{2\alpha'}u))(x) \\ &\qquad = \int_E (2\nabla_y g_E(x,y)\nabla(\gamma^{2\alpha'}(y) - g_E(x,y)\Delta(\gamma^{2\alpha'})(y))u(y)\,dy). \end{aligned}$$

Together with (3.14), this implies

$$\begin{aligned} u(x) \le 2\alpha' &\int_E u(y)\gamma^{2\alpha'-1}(y)\nabla\gamma(y)\nabla_y g_E(x,y)\,dy \\ &+ \alpha' \int_E u(y)\gamma^{2\alpha'-1}(y)\Delta\gamma(y) g_E(x,y)\,dy \\ &+ \alpha'(2\alpha'-1)\int_E u(y)\gamma^{2\alpha'-2}(y)|\nabla\gamma(y)|^2 g_E(x,y)\,dy \end{aligned} \tag{3.15}$$

(cf. [3, (3.13)]). Now, $\gamma(y) = 1$ for all y such that $\operatorname{dist}(y, K) > \delta/2$, in particular for all y such that $|x-y| < \delta/2$, and therefore the integrands are equal to 0 for such y. Following [3], from the bounds for the Green's function and its gradient, we get bounds for the integrals on the right side of (3.15) in terms of integrals (3.13). For instance,

$$g_E(x, y) \le C\rho(x)\rho(y)|x-y|^{-d},$$

and therefore

$$\begin{aligned}\int_E & u(y)\gamma^{2\alpha'-2}(y)|\nabla\gamma(y)|^2 g_E(x,y)\,dy \\ &\le C\rho(x)\int_E u(y)\gamma^{2\alpha'-2}(y)|\nabla\gamma(y)|^2\rho(y)|x-y|^{-d}\,dy \\ &= \int_{E\setminus B(x,\delta/2)} u(y)\gamma^{2\alpha'-2}(y)|\nabla\gamma(y)|^2\rho(y)|x-y|^{-d}\,dy \\ &\le C\rho(x)\delta^{-d}\int_E u\gamma^{2\alpha'-2}(y)|\nabla\gamma|^2(y)\rho(y)\,dy.\end{aligned}$$

In a similar way, we get

$$\begin{aligned}\int_E & u(y)\gamma^{2\alpha'-1}(y)\nabla\gamma(y)\nabla_y g_E(x,y)\,dy \\ &\le C\rho(x)\delta^{-d}\int_E u\gamma^{2\alpha'-1}\left|\frac{\partial\gamma}{\partial\rho}\right| + C\rho(x)\delta^{-d-1}\int_E u\gamma^{2\alpha'-1}|\nabla\gamma|\rho, \qquad (3.16)\\ \int_E & u(y)\gamma^{2\alpha'-1}(y)\Delta\gamma(y)g_E(x,y)\,dy \le C\rho(x)\delta^{-d}\int_E u\gamma^{2\alpha'-1}|\Delta\gamma|\rho.\end{aligned}$$

It remains to use the bounds of Lemma 3.6. □

Theorem 1.1 can be derived from this by using the same construction as in [3]. Let $x \in E$ and K, u be as in Theorem 1.1. Let $\delta = \operatorname{dist}(x, K)$. We set

$$K_1 = K \cap \overline{B(x, 2\delta)},$$

and

$$K_n = K \cap \left(\overline{B(x, 2^n\delta)} \setminus B(x, 2^{n-1}\delta)\right), \quad n \ge 2.$$

Since $K = \bigcap K_n$, we have

$$u \le u_K \le \sum u_{K_n}$$

where u_K stands for the maximal solution of problem (1.3). By construction, we have $\operatorname{dist}(x, K_n) \ge 2^{n-1}\delta$ and $\operatorname{diam}(K_n) \le 2^{n+1}\delta$. Therefore Lemma 3.8 is applicable to every K_n and we get

$$u_{K_n} \le C\rho(x)2^{-nd}\delta^{-d}\operatorname{Cap}(K_n)^{1/(\alpha-1)} \le C\rho(x)2^{-nd}\delta^{-d}\operatorname{Cap}(K)^{1/(\alpha-1)},$$

which implies

$$u(x) \le \sum u_{K_n} \le C\rho(x)\delta^{-d}\operatorname{Cap}(K)^{1/(\alpha-1)}\sum 2^{-nd}.$$

The extension of the theorem to arbitrary C^4 domains is a simple modification of the arguments by Mselati [3].

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