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WELL-APPROXIMABLE ANGLES AND MIXING FOR FLOWS ON $\mathbb{T}^2$ WITH NONSINGULAR FIXED POINTS

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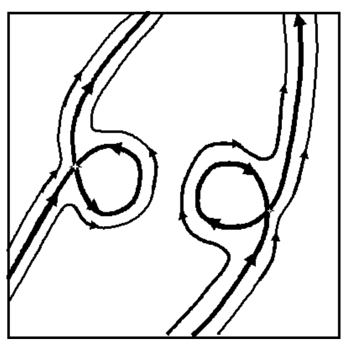
(Communicated by Svetlana Katok)

To the Anniversary of Anatole Katok, my Friend and Teacher.

ABSTRACT. We consider special flows over circle rotations with an asymmetric function having logarithmic singularities. If some expressions containing singularity coefficients are different from any negative integer, then there exists a class of well-approximable angles of rotation such that the special flow over the rotation of this class is mixing.

Examples of smooth flows on a two-dimensional torus with a smooth invariant measure and nonsingular hyperbolic fixed points appear naturally in Arnold's paper [1]. The phase space of such a flow decomposes into cells bounded by closed separatrices of regular fixed points and filled with periodic orbits, and an ergodic component in which orbits on one side of a fixed point visit its neighborhood more frequently than on the other. (See the figure, for example.)



V. I. Arnold has shown that there exists a smooth closed curve transversal to the orbits of the ergodic component. The invariant measure and the flow naturally induce a smooth parameterization on the curve (this procedure was described in detail, e.g., in [3]). The first-return map is determined everywhere on the curve, except for a finite number of points that are the points of the last intersection of the stable separatrices with the curve. In the induced parameterization, this map is a circle rotation. The return time is a smooth function of the parameter everywhere

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except for the same points. In the same work, it was shown that this function has logarithmic singularities at these points since the residence time within a small neighborhood of the nondegenerate saddle point is of the order of $\log\tau$, where τ, roughly speaking, is the distance from the point of intersection of the orbit with the boundary of the neighborhood to the separatrix.

Thus, the ergodic component of such a flow is isomorphic to a special flow S^t over rotation T of the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and under a "roof" function with logarithmic singular points which, in general, is asymmetric. In view of the ergodicity, the angle of rotation is irrational. Each singularity as usual is asymmetric since an orbit, sufficiently close to the separatrix, visits a neighborhood of the fixed point once or twice, depending on whether it passes on the left or on the right of the separatrix. In the general case, the function is asymmetric; however, in some cases, it may be symmetric (see definitions below).

We say that a "roof" function has *logarithmic singularities* if it satisfies the following conditions:

1) f has K singular points $\overline{x}_1, \dots, \overline{x}_K$;
2) $f \in C^1(\mathbb{T}^1 \setminus \bigcup_{i=1}^K \overline{x}_i)$, $f(x) \geq c > 0$;
3) for any $i = 1, \dots, K$,

$$f'(x) = \frac{1}{\{x - \overline{x}_i\}}(-A_i + o(1)) \quad \text{for} \quad x \to \overline{x}_i + 0,$$
$$f'(x) = \frac{1}{\{\overline{x}_i - x\}}(B_i + o(1)) \quad \text{for} \quad x \to \overline{x}_i - 0,$$

where A_i, $B_i > 0$.

Let

$$A = \sum_{i=1}^K A_i, \quad B = \sum_{i=1}^K B_i.$$

The function f is called *symmetric* if

$$A = B,$$

asymmetric if

$$A \neq B,$$

and *strongly asymmetric* if for any i

$$\frac{A_i - B_i}{A - B} > 0.$$

For symmetric functions it is known [3] that if

$$f(x) = f_0(x) + \sum_{i=1}^K \left(A_i \log \frac{1}{\{x - \overline{x}_i\}} + B_i \log \frac{1}{\{\overline{x}_i - x\}} \right),$$

where f_0 has a bounded variation, $A = B$, and ρ admits an approximation by rationals with the rate $\frac{\text{const}}{q^2 \log q}$, then the special flow over the circle rotation by ρ with "roof" function f is not mixing.

The conjecture about the possibility of mixing in special flows with an asymmetric "roof" function was proposed in [3]. Khanin and Sinai proved it for a certain class of Diophantine rotation angles ρ. More precisely, let $\rho = [k_1, \dots, k_n, \dots]$ be the expansion of ρ in a continued fraction, and p_n/q_n the nth convergent to ρ. In [2], the restriction on ρ is

$$k_{n+1} \leq \text{const } n^{1+\gamma}, \quad 0 < \gamma < 1.$$

In [4], the mixing is proved for angles satisfying

$$(*) \qquad \log k_{n+1} = o\left(\log q_n\right).$$

It is easy to show that, if for some $\gamma > 0$, perhaps greater than 1, the inequality $k_{n+1} \leq \text{const } n^{1+\gamma}$ holds for all n, then ρ satisfies $(*)$.

If f is a strongly asymmetric function with logarithmic singularities and ρ is an arbitrary irrational angle, then the special flow over a circle rotation through the angle ρ with the "roof" function f is mixing [5].

In this paper we return to the general nonsymmetric case. We extend the theorem about mixing to some class of well-approximable angles if the coefficients of logarithmic singularities satisfy no negative integer value condition defined in Theorem 1 below.

One more extension of old results is related to the definition of a strictly logarithmic singularity which describes this singularity more precisely.

We say that the singular point $\overline{x}_i$ is *strictly logarithmic* if there exists a function $\Psi \in C^2(0,1]$ such that $\Psi > 0$, $\Psi' > 0$, $\Psi'' \leq 0$, $\Psi(+0) = 0$, and in some right half-neighborhood of $\overline{x}_i$

$$\left| f'(x) + \frac{A_i}{\{x - \overline{x}_i\}} \right| \leq \Psi'(\{x - \overline{x}_i\}),$$

and in some left half-neighborhood of $\overline{x}_i$

$$\left| f'(x) - \frac{B_i}{\{\overline{x}_i - x\}} \right| \leq \Psi'(\{\overline{x}_i - x\}).$$

Note that, as follows from [3], if the invariant measure on $\mathbb{T}^2$ has a smooth positive density, then the singularities of a "roof" function of the special flow are strictly logarithmic.

Given a subset I of $\{1, 2, \ldots, K\}$ we denote

$$D_I = \sum_{i \in I} \frac{B_i - A_i}{B - A}.$$

If $I = \varnothing$ then $D_I = 0$. Let I_{s} be the set of numbers of strictly logarithmic singularities and I_{ns} be the set of numbers of other singularities.

Theorem 1. *Let f be asymmetric and such that for any $I \subset \{1, 2, \ldots, K\}$ and any $l \in \mathbb{N}$*

$$(1) \qquad D_I \neq -l.$$

Assume in addition that $D_I \neq 0$ whenever $I \cap I_{\mathrm{ns}} \neq \varnothing$.

Then for any ρ satisfying

$$(2) \qquad \liminf_{n \to +\infty} \frac{\log k_{n+1}}{\log q_n} > \max_{I : D_I < 0} \frac{[|D_I|] + 1}{\{|D_I|\}},$$

the special flow constructed over a ρ-rotation and under f is mixing. (Here $[x]$ is the integer part of x, and $\{x\}$ is its fractional part.)

The following theorem is an extension of the main theorem in [5].

Theorem 2. *If $(B_i - A_i)/(B - A) > 0$ for any $i \in I_{\mathrm{ns}}$, and $(B_i - A_i)/(B - A) \geq 0$ for strictly logarithmic singular points of f, then for any irrational ρ, the special flow constructed from ρ and f is mixing.*

To give a sketch of the proof we recall some notions and facts from [5].

In special flows over a circle rotation the only possible cause of mixing is the difference in the times that various points take to get from the "floor" to the "roof". This can cause, as time passes, a small rectangle to be strongly stretched and almost uniformly distributed along trajectories and hence over the phase space.

The divergence of adjacent points is described via Birkhoff sums of the "roof" function

$$f^r(x) = \sum_{k=0}^{r-1} f(T^k x).$$

It is obvious from the relation $S^t(x, y) = S^{y+t-f^r(x)}(T^r x, 0)$, where (x, y) denotes a point of the phase space. Strong and almost uniform distribution of a small rectangle over the phase space is ensured by a strong and almost uniform stretching of Birkhoff sums for $r \approx t$.

Formally this is described by the theorem below.

For $x \in \mathbb{T}^1$, let $\mathcal{R}(t, x)$ be the number of "jumps" which the point $(x; 0)$ has done under the action of the special flow S^t over the time t. For any measurable $X \subset \mathbb{T}^1$ we set

$$\mathcal{R}(t, X) = \bigcup_{x \in X} \mathcal{R}(t, x).$$

Theorem (Sufficient condition for mixing)**.** *Let T be an ergodic circle rotation. For some $t_0 > 0$ assume that the following objects are fixed for each $t > t_0$:*

— *a finite partial partition ξ_t into closed intervals: $\xi_t = \{C\}$, where*

$$\lim_{t \to +\infty} \max_{C \in \xi_t} |C| = 0, \qquad \lim_{t \to +\infty} \mu([\xi_t]) = 1$$

($[\xi_t]$ is the union of elements of ξ_t);

— *positive functions $\varepsilon(t)$, $H(t)$, such that $\varepsilon(t) \to 0$, $H(t) \to +\infty$ as $t \to +\infty$.*

If for each $t > t_0$ for any $C \in \xi_t$ and $r \in \mathcal{R}(t, [\xi_t]) \cap (t/\sqrt{2}, \sqrt{2}t)$,

$$(f^r)'(x) = M(r, C)(1 + \gamma(r, x)),$$
$$|C||M(r, C)| \geq H(t), \quad |\gamma(r, x)| < \varepsilon(t),$$

($M(r, C)$, $\gamma(r, x)$ are real functions), *then S^t is mixing.*

At first, we consider the derivatives of "ideal logarithmic functions". Let u, $v \colon \mathbb{R} \to \mathbb{R}$ be the functions with period 1 which are defined as

$$u(x) = 1/x \text{ if } x \in (0, 1],$$
$$v(x) = 1/(1 - x) \text{ if } x \in [0, 1).$$

Then $x_0 = 0$ is their singular point in $\mathbb{T}^1$. One can show that for every x, except singular points of f^r,

$$(f^r)'(x) = \sum_{i \in I_{\mathrm{ns}}} \left(u^r(x - \overline{x}_i)(-A_i + \alpha_i^-(r, x)) + v^r(x - \overline{x}_i)(B_i + \alpha_i^+(r, x)) \right) + \sum_{i \in I_{\mathrm{s}}} \left(-A_i u^r(x - \overline{x}_i) + (\psi_i^-)^r(x) + B_i v^r(x - \overline{x}_i) + (\psi_i^+)^r(x) \right), \tag{3}$$

where $|\psi_i^-(x)| \le \Psi'(x-\overline{x}_i)$, $|\psi_i^+(x)| \le \Psi'(\overline{x}_i - x)$, $|\alpha_i^\pm(r,x)| \le \alpha(r)$, $\alpha(r) \to 0$ as $r \to +\infty$.

Suppose that $q_m \le r < q_{m+1}$, and $r = l_m q_m + \cdots + l_0 q_0$ is the expansion of r by denominators q_n with integer nonnegative coefficients, such that

$$1 \le l_m \le k_{m+1},$$
$$0 \le l_n \le k_{n+1} \quad \text{for} \quad n = 0, 1, \ldots, m-1,$$
$$l_{n-1}q_{n-1} + \cdots + l_0 q_0 < q_n \quad \text{for} \quad n = 1, \ldots, m.$$

Then

$$u^r(x) = u^{l_m q_m}(x) + \cdots + u^{l_n q_n}(T^{r_{n+1}}x) + \cdots + u^{l_0 q_0}(T^{r_1}x),$$

where $r_n = l_m q_m + \cdots + l_n q_n$.

For each n we decompose u in two terms:

$$\widehat{u}_n(x) = u(x), \quad \check{u}_n(x) = 0 \text{ for } x \in (0, 1/q_n),$$
$$\widehat{u}_n(x) = 0, \quad \check{u}_n(x) = u(x) \text{ for } x \notin (0, 1/q_n).$$

Then $u^{l_n q_n}(T^{r_{n+1}}x) = \widehat{u}_n^{l_n q_n}(T^{r_{n+1}}x) + \check{u}_n^{l_n q_n}(T^{r_{n+1}}x)$.

One can show that for any x

$$\check{u}_n^{l_n q_n}(x) = l_n q_n \log q_n + P^-_{l_n q_n}(T^{r_{n+1}}x), \quad |P^-_{l_n q_n}(T^{r_{n+1}}x)| < 4 l_n q_n.$$

Thus, we get

$$u^r(x) = e(r) + \sum_{n=0}^{m} Z_n^-(r,x) + o(e(r)),$$

where

$$e(r) = \sum_{n=1}^{m} l_n q_n \log q_n,$$
$$Z_n^-(r,x) = \widehat{u}_n^{l_n q_n}(T^{r_{n+1}}x).$$

We call the term $e(r)$ the *ergodic* component of u^r, and Z_n^- the *resonant*. The component $e(r)$ does not depend on x, the terms Z_n^- depend on x and r, and their value essentially depend on the approximability of ρ by p_n/q_n.

Fix t large enough and choose m such that $\sqrt{2}q_m \le t < \sqrt{2}q_{m+1}$. We may construct the set $V(t) \subset \mathbb{T}^1$ by deleting in some special way neighborhoods of singular points of u^r, and estimate $\sum_{n=0}^m Z_n^-(r,x)$ on $V(t)$.

We decompose the set $X^{(n)} = X^{(n)}(r)$ of singular points of $u^{l_n q_n}(T^{r_{n+1}}x)$ into subsets

$$X_i^{(n)} = \left\{T^{-r_{n+1}-i-jq_n}x_0,\ j = 0, \ldots, l_n - 1\right\},\ i = 0, \ldots, q_n - 1,$$

which we call *clusters* of rank n. Each cluster consists of l_n points, forming an arithmetic progression with a step $\delta_n = q_n\rho - p_n$. By $[X_i^{(n)}]$ we denote the minimal segment containing $X_i^{(n)}$, $[X^{(n)}] = \bigcup_i [X_i^{(n)}]$, and $\partial[X^{(n)}]$ is the bound of $[X^{(n)}]$. The length of each segment is $|X_i^{(n)}| = (l_n - 1)|\delta_n| \approx \frac{l_n/k_{n+1}}{q_n}$, $\quad \mu([X^{(n)}]) \approx l_n/k_{n+1}$. Segments $[X_i^{(n)}]$ are disjoint, thus $\partial[X^{(n)}]$ is the union of the ends of $[X_i^{(n)}]$.

Let us choose a sequence σ_n, $n \in \mathbb{Z}_+$, depending on ρ and satisfying the conditions

$$\sigma_n \searrow 0; \quad \sigma_n > (\log q_n)^{-1/4} \quad (\text{for} \ \ n > 1), \quad \sigma_n^2 \log q_n \nearrow +\infty.$$

For the set W, we define $U(\varepsilon, W) = \bigcup_{x\in W} U(\varepsilon, x)$, where $U(\varepsilon, x)$ is the ε-neighborhood of x.

Theorem 3 (On the "main resonant term")**.** *For sufficiently large m, the following cases are possible:*

1. *The "main resonant term" is absent: for any $s < m$*

$$q_{s+1} \log k_{s+1} \le \sigma_m t \log q_m$$

and additionally $\log k_{m+1} \le \sigma_m^2 \log q_m$ or $\sqrt{2} q_m \le t < \sqrt{2}\sigma_m q_{m+1}$. Then for any $r \in (t/\sqrt{2}, \sqrt{2}t)$ and $x \in V(t)$

$$0 \le \sum_n Z_n^-(r, x) < 20\sigma_m e(r).$$

2. *The "main resonant term" is of rank m: for any $s < m$, we have*

$$q_{s+1} \log k_{s+1} \le \sigma_m t \log q_m,$$

and

$$\log k_{m+1} > \sigma_m^2 \log q_m, \quad \sqrt{2}\sigma_m q_{m+1} \le t < \sqrt{2} q_{m+1}.$$

Then for any $x \in V(t)$

$$\sum_{n\ne m} Z_n^-(r, x) < 16\sigma_m e(r),$$

and for $Z_m^-(r, x)$, when $r \in (t/\sqrt{2}, \sqrt{2}t)$ and $x \in V(t)\setminus U(\sigma_m/q_m, \partial[X^{(m)}(r)])$, there is an alternative:

— *if $x \notin [X^{(m)}(r)]$, then $Z_m^-(r, x) < \sigma_m e(r)$;*
— *if $x \in [X^{(m)}(r)]$, then $q_{m+1} \ln k_{m+1} - \sigma_m e(r) < Z_m^-(r, x) < q_{m+1} \ln k_{m+1} + \sigma_m e(r)$.*

3. *The "main resonant term" is of rank $s < m$: there exists $s < m$ such that $q_{s+1} \log k_{s+1} > \sigma_m t \log q_m$. Then for any $r \in (t/\sqrt{2}, \sqrt{2}t)$ and $x \in V(t)$*

$$\sum_{n\ne s} Z_n^-(r, x) < \sigma_m e(r);$$

for $Z_s^-(r, x)$, when $r \in (t/\sqrt{2}, \sqrt{2}t)$ and $x \in V(t) \setminus U(\sigma_m/q_s, \partial[X^{(s)}(r)])$, there is an alternative:

— *if $x \notin [X^{(s)}(r)]$, then $Z_s^-(r, x) < \sigma_m e(r)$;*
— *if $x \in [X^{(s)}(r)]$, then $q_{s+1} \log k_{s+1} - \sigma_m e(r) < Z_s^-(r, x) < q_{s+1} \log k_{s+1} + \sigma_m e(r)$.*

Thus, in the case 1, we have: for $x \in V(t)$ and $r \in (t/\sqrt{2}, \sqrt{2}t)$,

$$u^r(x) = e(r) + o(r);$$

in the cases 2 and 3, we have: for $x \in V(t) \setminus U(\sigma_m/q_s, \partial[X^{(s)}(r)])$, $r \in (t/\sqrt{2}, \sqrt{2}t)$

$$u^r(x) = e(r) + q_{s+1} \log k_{s+1} \chi(x) + o(r),$$

where χ is the indicator of $[X^{(s)}(r)]$.

Similar estimates are valid for v^r.

Using $V(t)$ and representations for u^r and v^r, we constructed in [5, § 4] the partial partition ξ_t, functions $H(t) \to +\infty$, $\varepsilon_e(t)$, $\varepsilon_L(t) \to 0$ as $t \to +\infty$, and the decomposition of $(f^r)'$ satisfying the conditions:

If $C \in \xi_t$, then for any $x \in C$ and $r \in \mathcal{R}(t, [\xi_t]) \cap (t/\sqrt{2}, \sqrt{2}t)$

$$(f^r)'(x) = (B - A)\big(e(\overline{r})(1 + \gamma_e(r, x)) + L(C)(1 + \gamma_L(r, x))\big), \tag{4}$$

$$L(C) = q_{s+1} \ln k_{s+1} \sum_{i=1}^{K} \frac{B_i - A_i}{B - A} \chi(x - \overline{x}_i), \tag{5}$$

$$|C| e(r) \geq \frac{H(t)}{|B - A|}, \quad |\gamma_e(r, x)| < \varepsilon_e(t), \quad |\gamma_L(r, x)| < \varepsilon_L(t), \tag{6}$$

where $\overline{r}$ is some fixed number, s is the rank of the "main resonant term" (if it exists; otherwise $L(C) = 0$) and χ is the indicator function of its support; $\chi(x - \overline{x}_i)$ is constant on each C, and if we set $I(C) = \{i \colon \chi(x - \overline{x}_i) = 1 \text{ for } x \in C\}$, then

$$L(C) = D_{I(C)} q_{s+1} \ln k_{s+1}. \tag{7}$$

Using (3) and the fact that $(\Psi^r)'(x) = o(e(r))$ uniformly on $V(t)$, one can show that *this decomposition is valid if $D_I \neq 0$ for any I such that $I \cap I_{\text{ns}} \neq \varnothing$.*

Theorem 2 follows directly from this statement.

The first step to a proof of Theorem 1 is almost obvious.

Lemma 1. *Let $\theta \in (0, 1)$ be a constant such that for any sufficiently large t, any $C \in \xi_t$ and any $r \in (t/\sqrt{2}, \sqrt{2}t)$*

$$|e(r) + L(C)| \geq \theta e(r).$$

Then the special flow under consideration is mixing.

The condition of lemma is equivalent to

$$\left| \frac{e(r)}{L(C)} + 1 \right| \geq \theta' > 0$$

for C such that $L(C) < 0$.

Let $\lambda > 0$ be such that for sufficiently large m

$$\frac{\ln q_m}{\ln k_{m+1}} \leq \frac{1}{\lambda}. \tag{8}$$

It is easy to see that

$$1 < \frac{\ln q_{m+1}}{\ln k_{m+1}} \leq 1 + \frac{1}{\lambda} + \nu_{m+1}, \quad \nu_{m+1} = \frac{q_{m-1}}{k_{m+1} q_m \ln k_{m+1}}.$$

Lemma 2. *For sufficiently large t, for any $r \in (t/\sqrt{2}, \sqrt{2}t)$, if $L(C) < 0$, then*

$$\frac{e(r)}{L(C)} \in \widetilde{\mathcal{D}}(\lambda, m) = \bigcup_{l \in \mathbb{Z}_+} \bigcup_{I : D_I < 0} \left(\frac{1}{D_I}\Big(l + \frac{l+1}{\lambda} + l\nu'_m\Big), \frac{l}{D_I} \right),$$

where $\nu'_m = \max(\nu_{m+1}, \nu_m)$.

Proof. It will suffice to consider only cases 2 and 3 described in Theorem 3.

We begin with the case 3. In this case, there exists a number $s < m$ such that $q_{s+1} \ln k_{s+1} > \sigma_m t \ln q_m$. It is obvious that $s = m - 1$. Indeed, suppose that $s + 1 < m$. Then since $t > q_m$, we have

$$\frac{q_{s+1} \ln k_{s+1}}{t \ln q_m} < \frac{q_{s+1} \ln k_{s+1}}{q_m \ln q_m} < \frac{1}{k_m} \leq \frac{1}{q_m^{\lambda/(\lambda+2)}} < \frac{1}{\ln q_m} < \sigma_m,$$

which contradicts the definition of the case 3.

Thus, in view of (7)

$$L(C) = D_{I(C)} q_m \ln k_m.$$

From the condition $q_m \ln k_m > \sigma_m t \ln q_m$, for sufficiently large m, it follows that

$$t < \frac{q_m \ln k_m}{\sigma_m \ln q_m} < q_m \ln k_m < q_m k_m/2 < q_{m+1}/2,$$

therefore $q_m < r < q_{m+1}$. Suppose that $lq_m \le r < (l+1)q_m$, $1 \le l \le k_{m+1}$. Then

$$lq_m \ln q_m \le e(r) < lq_m \ln q_m + q_m \ln q_{m-1}.$$

At $D_I < 0$, we obtain (for brevity, we write I instead of $I(C)$):

$$\frac{l}{D_I} > \frac{1}{D_I}\frac{l \ln q_m}{\ln k_m} \ge \frac{e(r)}{L(C)} > \frac{1}{D_I}\frac{l \ln q_m + \ln q_{m-1}}{\ln k_m} > \frac{1}{D_I}\left(l\big(1 + \frac{1}{\lambda} + \nu_m\big) + \frac{1}{\lambda}\right),$$

that is, for $r \in [lq_m, (l+1)q_m)$ we have

$$\frac{e(r)}{L(C)} \in \left(\frac{1}{D_I}\big(l + \frac{l+1}{\lambda} + l\nu_m\big), \frac{l}{D_I}\right).$$

The case 2 will take place if $\sqrt{2}\max(q_m, \sigma_m q_{m+1}) \le t < \sqrt{2} q_{m+1}$, and $r \in (\max(q_m, \sigma_m q_{m+1}), 2q_{m+1})$.

First, consider the case of $q_m < r < q_{m+1}$. We have $q_m \ln q_m \le e(r) < q_{m+1} \ln q_m$, $L(C) = D_{I(C)} q_{m+1} \ln k_{m+1}$. If $D_I = D_{I(C)} < 0$, then

$$0 > \frac{1}{D_I}\frac{q_m \ln q_m}{q_{m+1} \ln k_{m+1}} \ge \frac{e(r)}{L(C)} > \frac{1}{D_I}\frac{\ln q_m}{\ln k_{m+1}} > \frac{1}{D_I \lambda},$$

i.e.

$$\frac{e(r)}{L(C)} \in \left(\frac{1}{D_I \lambda}, 0\right).$$

Now, suppose that $q_{m+1} \le r < 2q_{m+1}$ (and hence, $r < q_{m+2}$). Then

$$q_{m+1} \ln q_{m+1} \le e(r) < q_{m+1}(\ln q_{m+1} + \ln q_m),$$

from which

$$\frac{1}{D_I} > \frac{e(r)}{L(C)} > \frac{1}{D_I}\left(1 + \frac{2}{\lambda} + \nu_{m+1}\right)$$

at $D_I < 0$, i.e.,

$$\frac{e(r)}{L(C)} \in \left(\frac{1}{D_I}\big(1 + \frac{2}{\lambda} + \nu_{m+1}\big), \frac{1}{D_I}\right).$$

Combining all the cases, we obtain the conclusion of the lemma. □

Hence, if

$$-1 \notin \mathcal{D}(\lambda) = \bigcup_{l \in \mathbb{Z}_+} \bigcup_{I : D_I < 0} \left[\frac{1}{D_I}\big(l + \frac{l+1}{\lambda}\big), \frac{l}{D_I}\right], \tag{9}$$

then there exists $\theta' > 0$ such that for sufficiently large t, for any $C \in \xi_t$ and $r \in (t/\sqrt{2}, \sqrt{2}t)$

$$\left|\frac{e(r)}{L(C)} + 1\right| \ge \theta'.$$

From this we obtain, that, if for any I and any $l \in \mathbb{N}$, the relation $D_I \ne -l$ is true, then there exists λ for which $-1 \notin \mathcal{D}(\lambda)$, and hence the special flow is mixing.

It is easy to deduce from (2) the existence of λ satisfying (8) and (9).

The full proof will be published in Mat. Sbornik, perhaps, in 2004.

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