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COUNTEREXAMPLES TO THE NEGGERS-STANLEY CONJECTURE

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ABSTRACT. The Neggers-Stanley conjecture asserts that the polynomial counting the linear extensions of a labeled finite partially ordered set by the number of descents has real zeros only. We provide counterexamples to this conjecture.

A finite partially ordered set (*poset*) P of cardinality p is said to be *labeled* if its elements are identified with the integers $1, 2, \dots, p$. We will use the symbol $\prec$ to denote the partial order on P and $<$ to denote the usual order on the integers. The *Jordan-Hölder set* $\mathcal{L}(P)$ is the set of permutations $\pi = (\pi_1, \dots, \pi_p)$ of $[p] \stackrel{\text{def}}{=} \{1, 2, \dots, p\}$ which encode the linear extensions of P. More precisely, $\pi \in \mathcal{L}(P)$ if $\pi_i \prec \pi_j$ implies $i < j$.

A *descent* in a permutation π is an index i such that $\pi_i > \pi_{i+1}$. Let $\operatorname{des}(\pi)$ denote the number of descents in π. The *W-polynomial* of a labeled poset P is defined by

$$W(P,t) = \sum_{\pi \in \mathcal{L}(P)} t^{\operatorname{des}(\pi)}.$$

W-polynomials appear naturally in many combinatorial contexts [2, 7, 8], and are connected to Hilbert series of the Stanley-Reisner rings of simplicial complexes [10, Section III.7] and algebras with straightening laws [9, Theorem 5.2.].

Example 1. Let $P_{2,2}$ be the labeled poset shown in Figure 1. Then

$$\mathcal{L}(P_{2,2}) = \{(1,3,2,4), (1,3,4,2), (3,1,2,4), (3,1,4,2), (3,4,1,2)\},$$

so $W(P_{2,2}, t) = 4t + t^2$.

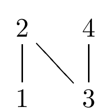
$P_{2,2} =$ 

FIGURE 1. The poset $P_{2,2}$.

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When P is a p-element antichain, then $\mathcal{L}(P)$ consists of all permutations of $[p]$, and $W(P,t)$ is the pth *Eulerian polynomial*. The Eulerian polynomials are known [3] to have only real zeros. In this instance, the *Neggers-Stanley conjecture* holds:

Conjecture 1 (Neggers-Stanley). *For any finite labeled poset P, all zeros of the polynomial $W(P,t)$ are real.*

A poset P is *naturally labeled* if $i \prec j$ implies $i < j$. Conjecture 1 was made by J. Neggers [4] in 1978 for naturally labeled posets, and extended by R. P. Stanley in 1986 to arbitrary labelings. It has been proved in some special cases (see [2, 11]). A weaker *unimodality* property of W-polynomials was recently proved [6] (see also [1]) for *graded* naturally labeled posets.

In this note, we construct counterexamples to Conjecture 1 utilizing the following construction. Let $\mathbf{m} \sqcup \mathbf{n}$ denote the disjoint union of the chains $1 \prec 2 \prec \cdots \prec m$ and $m+1 \prec m+2 \prec \cdots \prec m+n$. Let $P_{m,n}$ be the labeled poset obtained by adding the relation $m+1 \prec m$ to the relations in $\mathbf{m} \sqcup \mathbf{n}$; see Figure 2.

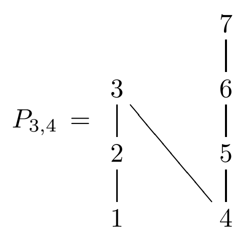

FIGURE 2. The poset $P_{3,4}$.

Theorem 1. *Let M be a positive integer. The polynomial $W(P_{m,n},t)$ has more than M non-real zeros provided* $\min(m,n)$ *is sufficiently large.*

The posets $P_{m,n}$ are not naturally labeled, so the original conjecture of Neggers remains open.

The rest of the paper is devoted to the proof of Theorem 1. At the end, we discuss specific (minimal) counterexamples obtained from Theorem 1.

Lemma 1. $W(P_{m,n},t) = \sum_{k=1}^{\min(n,m)} \binom{m}{k}\binom{n}{k} t^k$.

Proof. Let $\pi = (\pi_1, \pi_2, \ldots)$ be a permutation. If i is a descent in π, we say that π_i is a *descent top* and π_{i+1} is a *descent bottom*. Any $\pi \in \mathcal{L}(\mathbf{m} \sqcup \mathbf{n})$ is uniquely determined by its descent tops (which are necessarily elements of $[m+n] \setminus [m]$) and descent bottoms (which are elements of $[m]$). It follows that the number of permutations in $\mathcal{L}(\mathbf{m} \sqcup \mathbf{n})$ with exactly k descents is $\binom{m}{k}\binom{n}{k}$, implying that $W(\mathbf{m} \sqcup \mathbf{n}, t) = \sum_{k=0}^{\min(n,m)} \binom{m}{k}\binom{n}{k} t^k$. Since the only element of $\mathcal{L}(\mathbf{m} \sqcup \mathbf{n}) \setminus \mathcal{L}(P_{m,n})$ is $(1,2,\ldots,m+n)$, we have $W(\mathbf{m} \sqcup \mathbf{n}, t) = 1 + W(P_{m,n},t)$, and Lemma 1 follows. □

We note that all zeros of $W(\mathbf{m} \sqcup \mathbf{n}, t)$ are real and simple (R. Simion [7]).

Proof of Theorem 1. Recall that the *Bessel function* of order 0 is given by

$$J_0(z) = \frac{2}{\pi} \int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}} dt = \sum_{k=0}^{\infty} \frac{1}{k!k!} \left(\frac{-z^2}{2} \right)^k . \tag{2}$$

It is known that $J_0(z)$ has infinitely many zeros, all of which are real and simple. It follows from (2) that $|J_0(\theta)| \leq 1$ for all real θ, with equality only if $\theta = 0$. Hence the function

$$F(z) = \sum_{k=0}^{\infty} \frac{1}{k!k!} z^k$$

has infinitely many zeros, all of them negative and simple. Also, $|F(\theta)| < 1$ for $\theta < 0$.

Let $f_{m,n}(z) = W(P_{m,n}, z/mn)$. Then

$$f_{m,n}(z) = \sum_{k=1}^{\min(m,n)} \frac{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n})}{k!} \frac{(1-\frac{1}{m})(1-\frac{2}{m})\cdots(1-\frac{k-1}{m})}{k!} z^k.$$

Let $m_1, n_1, m_2, n_2, \ldots$ be positive integers such that $\lim_{j\to\infty} \min(m_j, n_j) = \infty$. Then

$$\lim_{j\to\infty} f_{m_j,n_j}(z) + 1 = F(z),$$

where the convergence is uniform on any compact subset of $\mathbb{C}$. Let $(-a, 0)$ be an interval containing more than M zeros of $F(z)$. It follows from Hurwitz's theorem [5, Theorem 1.3.8] that the polynomial $f_{m_j,n_j}(z) + 1$ has more than M zeros in $(-a, 0)$ for sufficiently large j. By continuity we also have $|f_{m_j,n_j}(z) + 1| < 1$ for $z \in (-a, 0)$ and j large. Thus by subtracting 1 from $f_{m_j,n_j}(z) + 1$, we will lose at least M real zeros. □

By applying Sturm's Theorem [5, Section 10.5], one can find specific counterexamples. The polynomial $W(P_{11,11}, t)$ has two non-real zeros which are approximately

$$z = -0.10902 \pm 0.01308i.$$

A counterexample with a polynomial of lower degree is

$$W(P_{36,6}, t) = 216t + 9450t^2 + 142800t^3 + 883575t^4 + 2261952t^5 + 1947792t^6.$$

This polynomial has two non-real zeros.

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