ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 12, Pages 44–52 (April 28, 2006)
S 1079-6762(06)00159-4

PERIPHERAL FILLINGS OF RELATIVELY HYPERBOLIC GROUPS

D. V. OSIN

(Communicated by Efim Zelmanov)

Abstract. A group-theoretic version of Dehn surgery is studied. Starting with an arbitrary relatively hyperbolic group G we define a *peripheral filling procedure*, which produces quotients of G by imitating the effect of the Dehn filling of a complete finite-volume hyperbolic 3-manifold M on the fundamental group $\pi_1(M)$. The main result of the paper is an algebraic counterpart of Thurston's hyperbolic Dehn surgery theorem. We also show that peripheral subgroups of G "almost" have the Congruence Extension Property and the group G is approximated (in an algebraic sense) by its quotients obtained by peripheral fillings.

1. Introduction and main results

Let M be a compact orientable 3-manifold with finitely many toric boundary components $T_1, \ldots, T_k$. Topologically distinct ways to attach a solid torus to T_i are parameterized by *slopes* on T_i, i.e., isotopy classes of unoriented essential simple closed curves in T_i. For a collection $\sigma = (\sigma_1, \ldots, \sigma_k)$, where σ_i is a slope on T_i, the *Dehn filling* $M(\sigma)$ of M is the manifold obtained from M by attaching a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ to each boundary component T_i so that the meridian $\partial\mathbb{D}^2$ goes to a simple closed curve of the slope σ_i. The fundamental theorem of Thurston [34] asserts that if $M - \partial M$ admits a complete finite-volume hyperbolic structure, then the resulting closed manifold $M(\sigma)$ is hyperbolic provided σ does not contain slopes from a fixed finite set.

Given a subset S of a group G, we denote by $\langle S\rangle^G$ the normal closure of S in G. Clearly,

$$\pi_1(M(\sigma)) = \pi_1(M)/\langle x_1, \ldots, x_k\rangle^{\pi_1(M)},$$

where $x_i \in \pi_1(T_i) \leq \pi_1(M)$ is the element corresponding to the slope σ_i. Thus Thurston's theorem implies the following group-theoretic result:

Let G be a fundamental group of a complete finite-volume hyperbolic 3*-manifold, and $H_1, \ldots, H_k$ the cusp subgroups of G. Then there exists a finite subset $\mathcal{F}$ of G such that for any collection of (primitive) elements $x_i \in H_i \setminus \mathcal{F}$, the quotient group $G/\langle x_1, \ldots, x_k\rangle^G$ is (word) hyperbolic.*

We generalize this result in two directions. First, instead of the class of fundamental groups of complete finite-volume hyperbolic manifolds, we consider its

Received by the editors November 7, 2005.
2000 *Mathematics Subject Classification.* Primary 20F65, 20F67, 57M27.

far-reaching generalization, the class of *relatively hyperbolic groups*. Second, instead of single elements $x_i \in H_i$, we deal with normal subgroups generated by arbitrary subsets of the cusp subgroups.

Recall that the notion of relative hyperbolicity was introduced in group theory by Gromov in [9], and since then it has been elaborated from different points of view [4, 5, 8, 24]. Here we mention some examples and refer the reader to the next section for the precise definition of relatively hyperbolic groups.

- If M is a complete finite-volume manifold of pinched negative sectional curvature, then $\pi_1(M)$ is hyperbolic with respect to the collection of the cusp subgroups [4, 8].
- Any (word) hyperbolic group G is hyperbolic relative to the trivial subgroup.
- Geometrically finite convergence groups acting on nonempty perfect compact metric spaces are hyperbolic relative to the set of maximal parabolic subgroups [38].
- Free products of groups and their small cancellation quotients, as defined in [19], are hyperbolic relative to the factors [24].
- Fundamental groups of finite graphs of groups with finite edge groups are hyperbolic relative to the vertex groups [4]. In particular, according to the famous Stallings Theorem [32], any group with infinite number of ends carries such a relatively hyperbolic structure.
- Finitely generated groups acting freely on R^n-trees are hyperbolic relative to the maximal noncyclic abelian subgroups [13]. This class of examples includes limit groups studied by Kharlampovich, Myasnikov [18], and independently by Sela [31] in their solutions of the famous Tarskii problem.

The group-theoretic analogue of Dehn filling is defined as follows. Suppose that $\{H_\lambda\}_{\lambda\in\Lambda}$ is a collection of subgroups of a group G. To each collection $\mathfrak{N} = \{N_\lambda\}_{\lambda\in\Lambda}$, where N_λ is a normal subgroup of H_λ, we associate the quotient group

$$G(\mathfrak{N}) = G/\left\langle \bigcup\nolimits_{\lambda\in\Lambda} N_\lambda \right\rangle^G . \tag{1}$$

The purpose of this note is to announce the following result obtained in [27].

Theorem 1.1. *Suppose that a group G is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. Then there exists a finite subset $\mathcal{F}$ of nontrivial elements of G with the following property. Let $\mathfrak{N} = \{N_\lambda\}_{\lambda\in\Lambda}$ be a collection of subgroups $N_\lambda \lhd H_\lambda$ such that $N_\lambda \cap \mathcal{F} = \emptyset$ for all $\lambda \in \Lambda$. Then:*

1) For each $\lambda \in \Lambda$, the natural map $H_\lambda/N_\lambda \to G(\mathfrak{N})$ is injective.

2) The quotient group $G(\mathfrak{N})$ is hyperbolic relative to the collection $\{H_\lambda/N_\lambda\}_{\lambda\in\Lambda}$. Moreover, for any finite subset $S \subseteq G$, there exists a finite subset $\mathcal{F}(S)$ of nontrivial elements of G such that the restriction of the natural homomorphism $G \to G(\mathfrak{N})$ to S is injective whenever $\mathfrak{N}$ satisfies the condition $N_\lambda \cap \mathcal{F}(S) = \emptyset$ for all $\lambda \in \Lambda$.

It is worth noting that the theorem applies to general (not necessarily finitely generated) relatively hyperbolic groups. In case the group G is finitely generated, the condition $N_\lambda \cap \mathcal{F} = \emptyset$ simply means that the subgroups N_λ contain no nontrivial elements of small (word) length. Jason Manning and Daniel Groves recently found ([12]) another (topological) proof of Theorem 1.1 in the particular case when the group G is torsion-free and finitely generated.

2. Applications

Recall that if a finitely generated group G is hyperbolic relative to a collection of hyperbolic subgroups, then G itself is a hyperbolic group [8, 24]. The following corollary may be regarded as a generalization of the group-theoretic version of Thurston's hyperbolic Dehn surgery theorem. Indeed, in case G is a fundamental group of a complete finite-volume hyperbolic 3-manifold, all cusp subgroups are isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$ and for any nontrivial element $x \in \mathbb{Z}\oplus\mathbb{Z}$, the quotient $\mathbb{Z}\oplus\mathbb{Z}/\langle x\rangle$ is hyperbolic.

Corollary 2.1. *Under the assumptions of Theorem* 1.1*, suppose in addition that* G *is finitely generated and* H_λ/N_λ *is hyperbolic for each* $\lambda \in \Lambda$*. Then* $G(\mathfrak{N})$ *is hyperbolic.*

On the other hand, Theorem 1.1 can also be applied to manifolds of higher dimension. Indeed, let M be a complete finite-volume hyperbolic n-manifold with cusp ends $E_1, \dots, E_k$. For simplicity we assume that each cusp end is homeomorphic to $T^{n-1} \times \mathbb{R}^+$, where T^{n-1} is an $(n-1)$-dimensional torus. Let $1 \le l \le k$. For each E_i, $1 \le i \le l$, we fix a torus $T_i^{n-1} \subset E_i$ and a closed simple curve σ_i in T_i^{n-1}. We now perform Dehn filling on the collection of cusps E_i, $1 \le i \le l$, by attaching a solid torus $\mathbb{D}^2 \times T^{n-2}$ onto T_i^{n-1} via a homeomorphism sending $\mathbb{S}^1 = \partial\mathbb{D}^2$ to σ_i. Let $\sigma = (\sigma_1, \dots, \sigma_l)$. The topological type of the resulting manifold $M(\sigma)$ depends only on the homotopy class of unoriented curves σ_i, i.e., on $[\pm\sigma_i] \in \pi_1(T_i^{n-1})$ [30]. If $l = k$, $M(\sigma)$ is a closed manifold; otherwise it has $k - l$ remaining cusps.

The Gromov–Thurston 2π-theorem states that $M(\sigma)$ has a complete metric of nonpositive sectional curvature for "most" choices of σ. (Although the theorem was originally proved in the context of 3-manifolds, the same proof actually holds in any dimension as observed in [2].) This means that the fundamental group of $M(\sigma)$ is *semihyperbolic* in the sense of [1]. The following immediate corollary of Theorem 1.1 shows that, in fact, $\pi_1(M(\sigma))$ is *hyperbolic relative to finitely generated free abelian subgroups*, which is a much stronger property than semihyperbolicity. In particular, $\pi_1(M(\sigma))$ is *biautomatic* [29] in the sense of [7] (and therefore, it is semihyperbolic [1]). We also note that relatively hyperbolic groups of this type are Hopfian [11, 10], are C^*-exact [28], have finite asymptotic dimension [26] (hence they satisfy the Novikov Conjecture [39]), and have many other nice properties. Below we regard $\pi_1(T_i^{n-1})$ as a subgroup of $\pi_1(M)$, and set $x_i = [\sigma_i] \in \pi_1(T_i^{n-1})$, $1 \le i \le l$, and $x_i = 1$ for $l < i \le k$.

Corollary 2.2. *There is a finite subset of nontrivial elements* $\mathcal{F} \subset \pi_1(M)$ *such that if* $x_i \notin \mathcal{F}$ *for all* $1 \le i \le k$*, then the quotient groups* $\pi_1(T_i^{n-1})/\langle x_i\rangle$ *naturally inject into* $M(\sigma)$*, and* $\pi_1(M(\sigma))$ *is hyperbolic relative to the collection of finitely generated free abelian subgroups* $\{\pi_1(T_i^{n-1})/\langle x_i\rangle\}_{1\le i\le k}$.

Let us discuss some algebraic applications of Theorem 1.1. Recall that a subgroup H of a group G has the *Congruence Extension Property* if for any $N \lhd H$, we have $H \cap N^G = N$ (or, equivalently, the natural homomorphism $H/N \to G/N^G$ is injective). An obvious example of the CEP is provided by the pair G, H, where H is a free factor of G. Another example is a cyclic subgroup $H = \langle w\rangle$ generated by an arbitrary element w of a free group F. In this case the CEP for H is equivalent to the assertion that the element represented by w has order n in the one relator group $F/\langle w^n\rangle^F$, which is a part of the well known theorem of Karrass, Magnus, and

Solitar [17]. Olshanskii [22] noted that the free group of rank 2 contains subgroups of arbitrary rank having CEP. This easily implies the Higman–Neumann–Neumann theorem stating that any countable group can be embedded into a 2-generated group. The CEP has also been extensively studied for semigroups and universal algebras (see [3, 33, 37] and references therein). It plays an important role in some constructions of groups with "exotic" properties [23].

We say that a subgroup H of a group G *almost has CEP* if there is a finite set of nontrivial elements $\mathcal{F} \subseteq H$ such that $H \cap N^G = N$ whenever $N \cap \mathcal{F} = \emptyset$. Recall that a subgroup H of a group G is said to be *almost malnormal* if $H^g \cap H$ is finite for all $g \notin H$. Bowditch [4] proved that if G is a hyperbolic group and H is an almost malnormal quasi-convex subgroup of G, then G is hyperbolic relative to H (see also [25]). Thus the following is an immediate corollary of Theorem 1.1.

Corollary 2.3. *Any almost malnormal quasi-convex subgroup of a hyperbolic group almost has CEP.*

If G is a free group, any almost malnormal subgroup $H \leq G$ is malnormal (i.e., it satisfies $H^g \cap H = \{1\}$ for all $g \notin H$). It is also well known that a subgroup of a finitely generated free group is quasi-convex if and only if it is finitely generated. Even the following result seems to be new.

Corollary 2.4. *Any finitely generated malnormal subgroup of a free group almost has CEP.*

If the free group is finitely generated, this is a particular case of the previous corollary. To prove Corollary 2.4 in full generality, it suffices to notice that any finitely generated subgroup H of a free group F belongs to a finitely generated free factor F_0 of F, and F_0 has CEP as a subgroup of F. This easily implies that H almost has CEP in F.

Considering a series of subgroups $K \lhd H \lhd F$ in a free group F, where K is not normal in F, it is easy to see that the word "malnormal" cannot be removed from the corollary. It is less trivial that, in general, malnormal subgroups of free groups do not have CEP. Here we sketch an example suggested by A. Klyachko. Let F be the free group with basis x, y. By using small cancellation arguments, it is not hard to construct a malnormal subgroup H of F generated by x and some word $w \in [F, F]$. Then $\langle x\rangle^F = H \neq \langle x\rangle^H$ since $w \in \langle x\rangle^F$.

Theorem 1.1 also implies that, in an algebraic sense, the group G is approximated by its images obtained by peripheral fillings. To be more precise, we recall that a group G is *fully residually* $\mathcal{C}$, where $\mathcal{C}$ is a class of groups, if for any finite subset $S \subseteq G$, there is a homomorphism of G onto a group from $\mathcal{C}$ that is injective on S. The study of this notion has a long history and is motivated by the following observation: If $\mathcal{C}$ is a class of "nice" groups in a certain sense, then any (fully) residually $\mathcal{C}$ group also enjoys some nice properties.

Using Theorem 1.1, we will obtain some nontrivial examples of fully residually hyperbolic groups. We recall that a group is called *nonelementary* if it does not contain a cyclic subgroup of finite index.

Corollary 2.5. *Suppose that a finitely generated group G is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$, and for each $\lambda \in \Lambda$, the group H_λ is fully residually hyperbolic. Then G is fully residually hyperbolic. Moreover, if G is nonelementary and all subgroups H_λ are proper, then G is fully residually nonelementary hyperbolic.*

For instance, fundamental groups of complete finite-volume Riemannian manifolds of pinched negative curvature are hyperbolic relative to the cusp subgroups [4, 8], which are virtually nilpotent [6]. It is well known that any nilpotent group is residually finite [14], and hence so is any virtually nilpotent group. Finally we recall that finite groups are hyperbolic. Combining this with Corollary 2.5 we obtain

Corollary 2.6. *Fundamental groups of complete finite-volume Riemannian manifolds of pinched negative curvature are fully residually nonelementary hyperbolic.*

Note that any fully residually nonelementary hyperbolic group G has infinite quotients of bounded period and, moreover, $\bigcap_{n=1}^{\infty} G^n = \{1\}$, where $G^n = \{g^n \mid g \in G\}$. This easily follows from the result of Ivanov and Olshanskii [16].

Another application is related to the well-known question of whether all hyperbolic groups are residually finite. Although in many particular cases the answer is known to be positive (see [36] and references therein), in the general case the question is still open. The following obvious consequence of Corollary 2.5 shows that this problem is equivalent to its relative analogue. In particular, in order to construct a nonresidually finite hyperbolic group, it suffices to find a nonresidually finite group that is hyperbolic relative to a collection of residually finite subgroups.

Corollary 2.7. *The following assertions are equivalent.*

(1) *Suppose that a finitely generated group G is hyperbolic relative to a collection of residually finite subgroups. Then G is residually finite.*
(2) *Any hyperbolic group is residually finite.*

3. On the proof

The proof of Theorem 1.1 is based on methods developed by the author in [24, 25]. The starting point is the following isoperimetric characterization of relatively hyperbolic groups suggested in [24].

Let G be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a collection of subgroups of G, and X a subset of G. We say that X is a *relative generating set of G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$* if G is generated by X together with the union of all H_λ. (In what follows we always assume X to be symmetric.) In this situation the group G can be regarded as a quotient group of the free product

$$F = \left(\mathop{\ast}_{\lambda\in\Lambda} H_\lambda \right) \ast F(X), \tag{2}$$

where $F(X)$ is the free group with basis X. If the kernel of the natural homomorphism $F \to G$ is a normal closure of a subset $\mathcal{R}$ in the group F, we say that G has *relative presentation*

$$\langle X,\ H_\lambda, \lambda \in \Lambda \mid \mathcal{R} \rangle. \tag{3}$$

If $\sharp X < \infty$ and $\sharp \mathcal{R} < \infty$, the relative presentation (3) is said to be *finite* and the group G is said to be *finitely presented relative to the collection of subgroups* $\{H_\lambda\}_{\lambda\in\Lambda}$.

Set

$$\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} (H_\lambda \setminus \{1\}). \tag{4}$$

Given a word W in the alphabet $X \cup \mathcal{H}$ such that W represents 1 in G, there exists an expression

$$W =_F \prod_{i=1}^{k} f_i^{-1} R_i^{\pm 1} f_i \tag{5}$$

with the equality in the group F, where $R_i \in \mathcal{R}$ and $f_i \in F$ for $i = 1, \ldots, k$. The smallest possible number k in a representation of the form (5) is called the *relative area* of W and is denoted by $Area^{rel}(W)$.

Definition 3.1. A group G is *hyperbolic relative to a collection of subgroups* $\{H_\lambda\}_{\lambda \in \Lambda}$ if G is finitely presented relative to $\{H_\lambda\}_{\lambda \in \Lambda}$ and there is a constant $C > 0$ such that for any word W in $X \cup \mathcal{H}$ representing the identity in G, we have $Area^{rel}(W) \leq C\|W\|$. The constant C is called an *isoperimetric constant* of the relative presentation (3). In particular, G is an ordinary *hyperbolic group* if G is hyperbolic relative to the trivial subgroup.

The proof of Theorem 1.1 consists of two ingredients. The first is a surgery on van Kampen diagrams; some ideas used here go back to methods developed by Alexander Olshanskii in his geometric solution of the Burnside problem [20, 21].

Recall that a *van Kampen diagram* Δ over a presentation

$$G = \langle \mathcal{A} \mid \mathcal{O} \rangle \tag{6}$$

is a finite oriented connected planar 2-complex endowed with a labelling function $\mathbf{Lab} : E(\Delta) \to \mathcal{A}$, where $E(\Delta)$ denotes the set of oriented edges of Δ, such that $\mathbf{Lab}\,(e^{-1}) \equiv (\mathbf{Lab}\,(e))^{-1}$. Given a cell Π of Δ, we denote by $\partial\Pi$ the boundary of Π; similarly, $\partial\Delta$ denotes the boundary of Δ. An additional requirement is that for any cell Π of Δ, the label $\mathbf{Lab}\,(\partial\Pi)$ is equal to (a cyclic permutation of) a word $P^{\pm 1}$, where $P \in \mathcal{O}$.

The van Kampen lemma states that a word W over an alphabet $\mathcal{A}$ represents the identity in the group given by (6) if and only if there exists a connected simply-connected planar diagram Δ over (6) such that $\mathbf{Lab}\,(\partial\Delta) \equiv W$ [19, Ch. 5, Theorem 1.1].

Van Kampen diagrams come into the proof of Theorem 1.1 via the following observation. Let G be a group given by the relative presentation (3) with respect to a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$. We denote by $\mathcal{S}$ the set of all words in the alphabet $\mathcal{H}$ representing the identity in the groups F defined by (2). Then G has the ordinary (nonrelative) presentation

$$G = \langle X \cup \mathcal{H} \mid \mathcal{S} \cup \mathcal{R} \rangle. \tag{7}$$

A cell in a van Kampen diagram Δ over (7) is called an *$\mathcal{R}$-cell* if its boundary is labeled by a word from $\mathcal{R}$. We denote by $N_{\mathcal{R}}(\Delta)$ the number of $\mathcal{R}$-cells of Δ. Obviously given a word W in $X \cup \mathcal{H}$ that represents 1 in G, we have

$$Area^{rel}(W) = \min_{\mathbf{Lab}\,(\partial\Delta) \equiv W} N_{\mathcal{R}}(\Delta),$$

where the minimum is taken over all disk van Kampen diagrams with boundary label W.

The second ingredient of the proof of Theorem 1.1 is a result concerning geodesic polygons in Cayley graphs of relatively hyperbolic groups (Proposition 3.4), which seems to be of independent interest.

Recall that the *Cayley graph* $\Gamma(G,\mathcal{A})$ of a group G generated by a (symmetric) set $\mathcal{A}$ is an oriented labelled 1-complex with vertex set $V(\Gamma(G,\mathcal{A}))=G$ and edge set $E(\Gamma(G,\mathcal{A}))=G\times\mathcal{A}$. An edge $e=(g,a)$ goes from the vertex g to the vertex ga and has label $\mathbf{Lab}\,(e)\equiv a$. The graph $\Gamma(G,\mathcal{A})$ can be thought of as a metric space if we set the length of each edge to be equal to 1.

Throughout the rest of the section, let G denote a group that is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. Also let X be a finite generating set of G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$, and Ω a subset of G. To state Proposition 3.4, we need some auxiliary terminology introduced in [24].

Definition 3.2. Let q be a path in the Cayley graph $\Gamma(G,X\cup\mathcal{H})$. A (nontrivial) subpath p of q is called an H_λ-*subpath* for some $\lambda\in\Lambda$ if the label of p is a word in the alphabet $H_\lambda\setminus\{1\}$. If p is a maximal H_λ-subpath of q, i.e., it is not contained in a bigger H_λ-subpath, then p is called an H_λ-*component* (or simply a *component*) of q.

Two H_λ-subpaths (or H_λ-components) p_1,p_2 of a path q in $\Gamma(G,X\cup\mathcal{H})$ are called *connected* if there exists a path c in $\Gamma(G,X\cup\mathcal{H})$ that connects some vertex of p_1 to some vertex of p_2 and $\mathbf{Lab}\,(c)$ is a word consisting of letters from $H_\lambda\setminus\{1\}$. In algebraic terms this means that all the vertices of p_1 and p_2 belong to the same coset gH_λ for a certain $g\in G$. Note that we can always assume that c has length at most 1, as every nontrivial element of $H_\lambda\setminus\{1\}$ is included in the set of generators. An H_λ-component p of a path q is called *isolated* (in q) if no distinct H_λ-component of q is connected to p.

To every subset Ω of G, we can associate a (partial) distance function $dist_\Omega$: $G\times G\to[0,\infty]$ as follows. If $g_1,g_2\in G$ and $g_1^{-1}g_2\in\langle\Omega\rangle$, we set $dist_\Omega(g_1,g_2)=|g_1^{-1}g_2|_\Omega$, where $|\cdot|_\Omega$ is the word length with respect to Ω. If $g_1^{-1}g_2\notin\langle\Omega\rangle$, we set $dist_\Omega(g_1,g_2)=\infty$. Finally, for any path p in $\Gamma(G,X\cup\mathcal{H})$, we define its Ω-length as

$$l_\Omega(p)=dist_\Omega(p_-,p_+).$$

Definition 3.3. Let $\mathcal{Q}_n$ denote the set of all pairs $(\mathcal{P},I)$, where $\mathcal{P}=p_1\ldots p_n$ is an n-gon in $\Gamma(G,X\cup\mathcal{H})$ with geodesic sides $p_1,\ldots,p_n$, and I is a subset of $\{p_1,\ldots,p_n\}$ such that each $p_i\in I$ is an isolated component of $\mathcal{P}$. Below we also use the following notation for vertices of $\mathcal{P}$:

$$x_1=(p_n)_+=(p_1)_-,\ x_2=(p_1)_+=(p_2)_-,\ \ldots,\ x_n=(p_{n-1})_+=(p_n)_-.$$

Given $(\mathcal{P},I)\in\mathcal{Q}_n$, we set

$$s_\Omega(\mathcal{P},I)=\sum_{p_i\in I}l_\Omega(p_i),$$

where Ω is a subset of G, and consider the quantity

$$s(n)=\sup_{(\mathcal{P},I)\in\mathcal{Q}_n}s(\mathcal{P},I).$$

Proposition 3.4. *There exists a finite subset Ω of G and a constant $D>0$ such that $s_\Omega(n)\le Dn$ for any $n\in\mathbb{N}$.*

To prove that the group $G(\mathfrak{N})$ is relatively hyperbolic, we consider the following presentation:

$$G(\mathfrak{N})=\langle X\cup\mathcal{H}\mid\mathcal{S}\cup\mathcal{Q}\cup\mathcal{R}\rangle,\tag{8}$$

where $\mathcal{Q} = \bigcup_{\lambda\in\Lambda} \mathcal{Q}_\lambda$ and $\mathcal{Q}_\lambda$ consists of all words (not necessary irreducible) in the alphabet $H_\lambda \setminus \{1\}$ representing elements of N_λ in G. Given an arbitrary word W in $X \cup \mathcal{H}$ representing 1 in $G(\mathfrak{N})$, let Δ_0 be an arbitrary diagram over (8) such that $\mathbf{Lab}\,(\partial\Delta_0) \equiv W$. By cutting all cells in Δ_0 labelled by words from $\mathcal{Q}$, we get a diagram Δ over (7) such that:

(D1) Topologically Δ is a disk with $k \geq 0$ holes. More precisely, the boundary of Δ is decomposed as $\partial\Delta = \partial_{ext}\Delta \sqcup \partial_{int}\Delta$, where $\partial_{ext}\Delta$ is the boundary of the disk and $\partial_{int}\Delta$ consists of disjoint cycles (*components*) $c_1, \ldots, c_k$ that bound the holes.

(D2) For any $i = 1, \ldots, k$, the label $\mathbf{Lab}\,(c_i)$ is a word in the alphabet $H_\lambda \setminus \{1\}$ for some $\lambda \in \Lambda$ and this word represents an element of N_λ in G.

Using the above-mentioned diagram surgery we transform Δ into a new diagram Δ_1 that still satisfies (D1) and (D2) and, in addition, has the following properties:

(1) There is a set of (disjoint) simple paths $\{t_1, \ldots, t_k\}$ in Δ_1 such that after cutting Δ_1 along $t_1, \ldots, t_k$, we get a connected simply connected diagram $\widetilde{\Delta}_1$. Note that the cycle $\mathcal{P}$ in $\Gamma(G, X \cup \mathcal{H})$ corresponding to $\partial\widetilde{\Delta}_1$ can be naturally regarded as an n-gon, $n \leq 4k + \|W\|$.

(2) The labels of holes in Δ_1 lead to sides that are isolated components of $\mathcal{P}$.

Now Proposition 3.4 allows us to derive the linear isoperimetric inequality for $G(\mathfrak{N})$. The other two assertions of the theorem are proved in a similar way. All details can be found in [27].

Acknowledgments

The author is grateful to Koji Fujiwara, Anton Klyachko, Alexander Olshanskii, and John Ratcliffe for useful discussions.

References

1. J. Alonso, M. Bridson, Semihyperbolic groups, *Proc. London Math. Soc.* (3) **70** (1995), no. 1, 56–114. MR1300841 (95j:20033)
2. M. T. Anderson, Dehn filling and Einstein metrics in higher dimensions, Preprint, 2003; available at http://www.arxiv.org/abs/math.DG/0303260.
3. W. J. Blok, D. Pigozzi, On the congruence extension property, *Algebra Universalis* **38** (1997), no. 4, 391–394. MR1626343 (99m:08007)
4. B. H. Bowditch, Relatively hyperbolic groups, Preprint, 1999.
5. C. Druţu, M. Sapir, Tree-graded spaces and asymptotic cones of groups. With an appendix by D. Osin and M. Sapir, *Topology* **44** (2005), no. 5, 959–1058. MR2153979 (2006d:20078)
6. P. Eberlein, Lattices in spaces of nonpositive curvature, *Annals of Math.* **111** (1980), 435–476. MR0577132 (82m:53040)
7. D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, W. Thurston, Word processing in groups, Jones and Bartlett Publishers, Boston, MA, 1992. MR1161694 (93i:20036)
8. B. Farb, Relatively hyperbolic groups, *GAFA* **8** (1998), 810–840. MR1650094 (99j:20043)
9. M. Gromov, Hyperbolic groups, Essays in Group Theory, MSRI Series, Vol. 8 (S. M. Gersten, ed.), Springer, 1987, pp. 75–263. MR0919829 (89e:20070)
10. D. Groves, Limits of (certain) CAT(0) groups, II: The Hopf property and the shortening argument, Preprint, 2004; http://www.arxiv.org/abs/math.GR/0408080.
11. D. Groves, Limits of (certain) CAT(0) groups, I: Compactification, Preprint, 2004; available at http://www.arxiv.org/abs/math.GR/0404440.
12. D. Groves and J. Manning, Dehn filling in relatively hyperbolic groups, arXiv: math.GR/0601311.
13. V. Guirardel, Limit groups and groups acting freely on $\mathbb{R}^n$-trees, *Geometry & Topology* **8** (2004), 1427–1470. MR2119301 (2005m:20060)
14. Ph. Hall, The Edmonton notes on nilpotent groups, Queen Mary College Mathematics Notes, Mathematics Department, Queen Mary College, London, 1969. MR0283083 (44:316)

15. G. Higman, B. H. Neumann, H. Neumann, Embedding theorems for groups, *J. London Math. Soc.* **24** (1949), 247–254. MR0032641 (11:322d)
16. S. V. Ivanov, A. Yu. Olshanskii, Hyperbolic groups and their quotients of bounded exponents, *Trans. Amer. Math. Soc.* **348** (1996), no. 6, 2091–2138. MR1327257 (96m:20057)
17. A. Karrass, W. Magnus, D. Solitar, Elements of finite order in groups with a single defining relation, *Comm. Pure Appl. Math.* **13** (1960), 57–66. MR0124384 (23:A1696)
18. O. Kharlampovich, A. Myasnikov, Description of fully residually free groups and irreducible affine varieties over a free group. Summer School in Group Theory in Banff, 1996, 71–80, CRM Proc. Lecture Notes, vol. 17, Amer. Math. Soc., Providence, RI, 1999. MR1653685 (99j:20032)
19. R. C. Lyndon, P. E. Shupp, Combinatorial Group Theory, Springer-Verlag, 1977. MR0577064 (58:28182)
20. A. Yu. Olshanskii, Geometry of defining relations in groups. Mathematics and its Applications (Soviet Series), vol. 70, Kluwer Academic Publishers Group, Dordrecht, 1991. MR1191619 (93g:20071)
21. A. Yu. Olshanskii, Periodic quotients of hyperbolic groups, *Math. USSR Sbornik* **72** (1992), no. 2, 519–541. MR1119008 (92d:20050)
22. A. Yu. Olshanskii, SQ-universality of hyperbolic groups, *Mat. Sb.* **186** (1995), no. 8, 119–132; English transl., *Sb. Math.* **186** (1995), no. 8, 1199–1211. MR1357360 (97b:20057)
23. A. Yu. Olshanskii, M. V. Sapir, Nonamenable finitely presented torsion-by-cyclic groups, *Publ. Math. Inst. Hautes Études Sci.* **96** (2002), 43–169. MR1985031 (2004f:20061)
24. D. V. Osin, Relatively hyperbolic groups: Intrinsic geometry, algebraic properties, and algorithmic problems, Memoirs Amer. Math. Soc., to appear; available at http://www.arxiv.org/abs/math.GR/0404040.
25. D. V. Osin, Elementary subgroups of hyperbolic groups and bounded generation, *Int. J. Alg. Comp.*, to appear; available at http://www.arxiv.org/abs/math.GR/0404118.
26. D. V. Osin, Asymptotic dimension of relatively hyperbolic groups, *Internat. Math. Res. Notices* **2005**, no. 35, 2143–2162. MR2181790
27. D. Osin, Peripheral fillings of relatively hyperbolic groups, arXiv: math.GR/0510195.
28. N. Ozawa, Boundary amenability of relatively hyperbolic groups, Preprint, 2005; available at http://www.arxiv.org/abs/math.GR/0501555.
29. Y. D. Rebbechi, Algorithmic properties of relatively hyperbolic groups, PhD thesis, Rutgers Univ. (Newark); available at http://www.arxiv.org/abs/math.GR/0302245.
30. D. Rolfsen, Knots and links, Math. Lect. Series **7**, Publish or Perish Inc., Houston, Texas, 1976. MR0515288 (58:24236)
31. Z. Sela, Diophantine geometry over groups I: Makanin–Razborov diagrams, *IHES Publ. Math.* **93** (2001), 31–105. MR1863735 (2002h:20061)
32. J. Stallings, Group theory and three-dimensional manifolds, Yale Mathematical Monographs, vol. 4, Yale University Press, New Haven, Conn.-London, 1971. MR0415622 (54:3705)
33. X. Tang, Semigroups with the congruence extension property, *Semigroup Forum* **56** (1998), no. 2, 228–264. MR1490295 (99i:20079)
34. W. P. Thurston, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc. (N.S.)* **6** (1982), no. 3, 357–381. MR0648524 (83h:57019)
35. P. Tukia, Convergence groups and Gromov's metric hyperbolic spaces, *New Zealand J. Math.* **23** (1994), no. 2, 157–187. MR1313451 (96c:30042)
36. D. Wise, The residual finiteness of negatively curved polygons of finite groups, *Invent. Math.* **149** (2002), no. 3, 579–617. MR1923477 (2003e:20033)
37. M. Wu, The fuzzy congruence extension property in groups, *JP J. Algebra Number Theory Appl.* **2** (2002), no. 2, 153–160. MR1942382 (2003j:20106)
38. A. Yaman, A topological characterization of relatively hyperbolic groups, *J. Reine Angew. Math.* **566** (2004), 41–89. MR2039323 (2005e:20064)
39. G. Yu, The Novikov conjecture for groups with finite asymptotic dimension, *Ann. of Math.* (2) **147** (1998), no. 2, 325–355. MR1626745 (99k:57072)

Department of Mathematics, City College of CUNY, New York, NY 10031
E-mail address: denis.osin@gmail.com