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ESTIMATES ON THE DIMENSION OF AN ATTRACTOR FOR A NONCLASSICAL HYPERBOLIC EQUATION

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Abstract. In this paper, we estimate the dimension of a global attractor for a nonclassical hyperbolic equation with a viscoelastic damping term in Hilbert spaces $H_0^2 \times L^2$ and $D(A) \times H_0^2$, where $D(A) = \{v \in H_0^2 \mid Av \in L^2\}$ and $A = \Delta^2$. We obtain an explicit formula of the upper bound of the dimension of the attractor. The obtained dimension decreases as damping grows and is uniformly bounded for large damping, which conforms to physical intuition.

1. Introduction

We are concerned with the estimates on the dimension of a global attractor for the following initial value problem for a nonclassical hyperbolic equation with a viscoelastic damping term:

$$\begin{cases} u_{tt} + \Delta^2 u + \delta u_t + ku^+ + g(u) = h, \quad (x,t) \in \Omega \times R^+, \\ u(x,0) = u_1(x) \quad \text{and} \quad u_t(x,0) = u_2(x), \\ u(x,t) = \nabla u(x,t) = \Delta u(x,t) = 0, \quad x \in \Gamma, \quad t \in R^+, \end{cases} \tag{1.1}$$

where $\Omega \subset R^2$ is an open bounded set of R^2 with a smooth boundary Γ sufficiently regular, and the initial data $u_1(x)$, $u_2(x)$ are in appropriate function spaces, $k > 0$ is a constant, and $\delta > 0$ is a damping. The force $u^+ = \max\{u, 0\}$ is the positive part of u. Further details can be found in [1] and [9].

Throughout the paper we use the notation $H = L^2(\Omega)$, $V = H_0^2(\Omega)$, with the scalar product $(\cdot,\cdot)$ and the norm $|\cdot|$, respectively $((\cdot,\cdot))$ and $\|\cdot\|$, and the Sobolev spaces $H^s(\Omega) = W^{s,2}(\Omega)$. We define $(u,v) = \int_\Omega u(x)v(x)dx$, $((u,v)) = \int_\Omega \Delta u(x)\Delta v(x)dx$, and $D(A) = \{v \in V \mid Av \in H\}$, where $Au = \Delta^2 u$. We have

$$D(A) \subset V \subset H = H^* \subset V^*, \tag{1.2}$$

where all embeddings are compact, and H^*, V^* are the dual spaces of H, V, respectively. We can define the *power* A^s for $s \in R$, which is also an operator on the space $D(A^s)$. We have $D(A^0) = H$, $D(A^{1/2}) = V$, $D(A^{-1/2}) = V^*$, and write

$$V_{2s} = D(A^s), \qquad \forall\, s \in R. \tag{1.3}$$

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Applying norms on $D(A^{\frac{s}{4}})$ equivalent to the $H^s(\Omega)$ norms and interpolation inequality, we have

$$H_0^s(\Omega) \subset D(A^{\frac{s}{4}}) \subset H^s(\Omega) \subset H^{s_1}(\Omega) \qquad \text{for} \quad s > s_1 \geq 0. \tag{1.4}$$

We assume that $h \in H,\ g \in C^2(\Omega)$, and the function $g : R \to R$ satisfies the following conditions:

$$\liminf_{|s|\to\infty} \frac{P(s)}{s^2} \geq 0, \quad \text{where } P(s) = \int_0^s g(\tau)d\tau, \tag{1.5}$$

$$\limsup_{|s|\to\infty} \frac{|g'(s)|}{|s|^\sigma} = 0, \quad \text{for } 0 \leq \sigma < \infty, \tag{1.6}$$

and there exists a positive constant c_0 such that

$$\liminf_{|s|\to\infty} \frac{sg(s) - c_0 P(s)}{s^2} \geq 0. \tag{1.7}$$

Furthermore, we need the following assumptions on g':

(i) there exist $\beta \in (0,1)$ and c_1 such that

$$|g'(s_1) - g'(s_2)| \leq c_1 |s_1 - s_2|^\beta; \tag{1.8}$$

(ii) g' is a bounded continuous mapping from V into $\mathcal{L}(V,H)$ and a bounded mapping from $D(A)$ into $\mathcal{L}(V_\alpha, H)$ for some $\alpha \in [0,1)$.

Results on the existence of a solution and attractor in a bounded domain can be found in [10, 11, 7], and so on. In the book [11], the notion of dimension of a global attractor was considered. Later, V. V. Chepyzhov and A. A. Ilyin presented an approach that is well suited for studying the dimension of the global attractor arising in [3]. The construction of the Hausdorff dimension and fractal dimension of global attractors admits the same upper bound under quite general assumptions. Further references are [2, 4, 5, 6, 8], and [12]. However, there is not yet a formula for the upper bound of the dimension of a global attractor for a nonclassical hyperbolic equation. In this paper, we obtain a rather strict upper bound for the dimension a global attractor by careful estimations and give the optimal value for the parameter α that leads to the minimal estimate of the dimension of the attractor so that the value $\gamma = \gamma(k, \|g'(\varphi)\|_{\mathcal{L}(V_\alpha,H)})$ in the main estimates for the dimension is determined explicitly. The obtained dimension decreases as the damping coefficient δ grows and is uniformly bounded for large δ, which conforms to physical intuition. Meanwhile, this gives us a relatively easy way to estimate the upper bound of the dimension of the attractor.

The main result is the following theorem.

Theorem 1.1. *Let the nonclassical hyperbolic equation be given by equations* (1.1). *Assume that $g(u)$ satisfies conditions* (1.5) – (1.8). *Then for any $\delta \geq \delta_0 > 0$, the Hausdorff and fractal dimensions of the global attractor $\mathcal{A}$ for the system* (1.1) *satisfy*

$$\begin{aligned}\dim(\mathcal{A}) &\leq \min\{m \mid \frac{1}{m}\sum_{j=1}^m \lambda_j^{\alpha-1} \leq \frac{2\lambda_1\delta^2}{\gamma^2\sqrt{\delta^2+4\lambda_1}(\delta+\sqrt{\delta^2+4\lambda_1})}\} \\ &\leq \min\{m \mid \frac{1}{m}\sum_{j=1}^m \lambda_j^{\alpha-1} \leq \frac{2\lambda_1\delta_0^2}{\gamma^2\sqrt{\delta_0^2+4\lambda_1}(\delta_0+\sqrt{\delta_0^2+4\lambda_1})}\},\end{aligned}$$

where $\{\lambda_j\}_{j\in N}$, $0 < \lambda_1 \le \lambda_2 \le \cdots$, *are the eigenvalues of the operator* Δ^2 *with the Dirichlet boundary condition on* Ω, $\gamma = \gamma(k, \|g'(\varphi)\|_{\mathcal{L}(V_\alpha,H)})$ *is a positive constant, and* $\alpha = 2\sigma/p$ *with* $0 \le 2\sigma < p < \infty$, $p \ge 1$.

From Theorem 1.1, we see that the upper bound of the dimension is decreasing with respect to δ and remains small when damping coefficient δ is very large because

$$\phi(\delta) = \frac{2\lambda_1\delta^2}{\gamma^2\sqrt{\delta^2+4\lambda_1}(\delta+\sqrt{\delta^2+4\lambda_1})}$$

increases as δ grows and

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{\alpha-1} = 0, \quad \lim_{\delta\to+\infty}\phi(\delta) = \frac{\lambda_1}{\gamma^2}.$$

2. Preliminary results

We use (1.2) to give the following definition of a weak solution to system (1.1).

Definition 2.1. Let $h \in H$, $E_0 = V \times H$, $E_1 = D(A) \times V$, and $\{u_1, u_2\} \in E_0$. A weak solution (1.1) is a function $u(x,t)$ such that

(i)

$$u \in L^2((0,T),V), \quad u_t \in L^2((0,T),H), \quad u_{tt} \in L^2((0,T),V^*); \tag{2.1}$$

(ii) for all $v \in C_0^\infty([0,T]\times\Omega)$, the generalized formula holds:

$$\frac{d}{dt}(u_t,v) + \delta(u_t,v) + (\Delta u,\Delta v) + (ku^+,v) + (g(u),v) = (h,v). \tag{2.2}$$

Now we recall some basic results in [10, 11, 7].

Lemma 2.2. *Suppose that the constants* $T > 0$, $k > 0$ *and the initial conditions* $u_1(x) \in V$ *and* $u_2(x) \in H$ *are given. Also,* $g(u)$ *satisfies* (1.5) – (1.7). *Then for system* (1.1) *there exists a unique (weak) solution such that*

$$u \in C([0,T],V) \quad \text{and} \quad u_t \in C((0,T],H). \tag{2.3}$$

The next two lemmas are results on the existence of a global attractor of the dynamical system $S(t)$, $t \ge 0$.

Lemma 2.3. *Let* g *satisfy* (1.5) – (1.7), $h \in H$. *Then the dynamical system associated to the system* (1.1) *possesses a global attractor* $\mathcal{A}$, *which is compact, connected, and maximal among the functional invariant sets in* E_0.

For the proof of the main result, we use the following result, which gives additional information about the global attractor $\mathcal{A}$.

Lemma 2.4. *Assume that* g *satisfies conditions* (1.5) – (1.7) *and* $h \in H$. *Then the global attractor* $\mathcal{A}$ *is included and is bounded in the space* E_1.

3. The Hausdorff and fractal dimensions of the attractor

We shall prove in this section that the global attractor given in Lemma 2.3 is finite dimensional. For every $t \in R$, we define the mapping

$$S(t) : \{u_1, u_2\} \to \{u(t), u_t(t)\}, \tag{3.1}$$

which maps $E_0 = V \times H$ and $E_1 = D(A) \times V$ into themselves and they enjoy the group properties. Therefore, for every t, $S(t)$ is a homeomorphism from E_0 onto E_0

(or from E_1 onto E_1) and is continuous in E_0 (or E_1). We define the inner product in E_0 by $(\{p,q\},\{\tilde{p},\tilde{q}\})_{E_0} = (p,\tilde{p})_V + (q,\tilde{q})_H$ for $p,\ \tilde{p} \in V$ and $q,\ \tilde{q} \in H$.

There exists a complete orthonormal family in H, $\{\omega_j\}_{j\in N}$, consisting of the eigenvectors of A:

$$\begin{gathered} A\omega_j = \lambda_j \omega_j, \ \ \forall j \in N, \\ 0 < \lambda_1 \le \lambda_2 \le \cdots, \ \ \lambda_j \to \infty \ \text{ as } \ j \to \infty. \end{gathered} \tag{3.2}$$

Let $\varepsilon_0 = \min(\delta/4, \lambda_1/2\delta)$; for any $\varepsilon \in (0,\varepsilon_0)$, we consider the semigroup of operators $S_\varepsilon(t) := R_\varepsilon S(t) R_{-\varepsilon}$ defined by

$$S_\varepsilon(t) : \{u_1, u_2 + \varepsilon u_1\} \to \{u(t), u_t(t) + \varepsilon u(t)\}.$$

The operator R_ε, $\varepsilon \in R$, is an isomorphism of E_0, given by the formula $R_\varepsilon : \{x,y\} \to \{x, y + \varepsilon x\}$, for any $x, y \in E_0$. If $\mathcal{A}$ is the maximal attractor defined by Lemma 2.3 for $S(t)$, then $R_\varepsilon \mathcal{A}$ is the maximal attractor for $S_\varepsilon(t)$.

We first check assumption (ii) on g' for the operators $S(t)$ to give the optimal value for the parameter α.

Lemma 3.1. *There exists $\alpha \ge 0$ such that for every $\varphi \in D(A)$, the differential $g'(\varphi)$ belongs to $\mathcal{L}(V_\alpha, H)$ and, for every $R > 0$,*

$$K = \sup_{\|\varphi\|_{D(A)} \le R} \|g'(\varphi)\|_{\mathcal{L}(V_\alpha, H)} < \infty. \tag{3.3}$$

Proof. From (1.6) we have

$$|g'(\varphi)| \le c_2(1 + |\varphi|^\sigma) \quad \text{with} \quad 0 \le \sigma < \infty, \quad \Omega \subset R^2.$$

The case $\sigma = 0$ is obvious.

For $\sigma > 0$, if $\varphi \in D(A)$ with $\|\varphi\|_{D(A)} \le R$, then by (1.4) and the Sobolev imbedding inequality

$$\|g'(\varphi)\|_{L^{p/\sigma}} \le c_3(R) \quad \text{with} \quad 0 < 2\sigma < p < \infty, \ \ p \ge 1.$$

Let $\psi \in V_\alpha$ and $\phi \in H$; we apply Hölder's inequality with exponents p/σ, $2p/(p-2\sigma)$, 2, and find

$$\Big| \int_\Omega g'(\varphi)\psi\phi dx \Big| \le \|g'(\varphi)\|_{L^{p/\sigma}} \|\psi\|_{L^{2p/(p-2\sigma)}} \|\phi\|_H.$$

For $n = 2$ due to (1.4) and the Sobolev imbedding theorem we have

$$\|\psi\|_{L^{2p/(p-2\sigma)}} \le c_4 \|\psi\|_\alpha \quad \text{if} \ \ \alpha = \frac{2\sigma}{p}.$$

Therefore,

$$\Big| \int_\Omega g'(\varphi)\psi\phi dx \Big| \le c_3(R) c_5 \|\psi\|_\alpha \|\phi\|_H. \tag{3.4}$$

This shows that $g'(\varphi)\psi$ is in the space $H (= H^*)$ and its norm in H is bounded by $c_3(R)c_5\|\psi\|_\alpha$, i.e., $K \le c_3(R)c_5$. □

From Lemma 3.1 we obtain the optimal value for the parameter $\alpha = 2\sigma/p$ with $0 \le \alpha < 1$ that leads to the minimal estimate of the dimension of the attractor.

Proposition 3.2. *Assume that g satisfies* (1.5)−(1.7). *Then the linearized problem of* (1.1) *around the solution u,*

$$V_{tt} + \Delta^2 V + \delta V_t + f'(u)V + g'(u)V = 0, \tag{3.5}$$

where $f(u) = ku^+$, with initial data

$$V(x,0) = y \in V, \quad V_t(x,0) = z \in H, \tag{3.6}$$

has a unique (weak) solution.

Proof. Using Lemma 2.2, we are able to prove that the linearized problem (3.5) − (3.6) possesses a unique solution $V(t)$ such that

$$V(t) \in C([0,T],V) \qquad \text{and} \qquad V_t(t) \in C([0,T],H). \qquad \square$$

We can then define a linear map $L(t,\varphi_0) : E_0 \to E_0$ by setting

$$L(t,\varphi_0) : \{y,z\} \to \{V(t), V_t(t)\}. \tag{3.7}$$

It can also be proved that $L(t,\varphi_0)$ is bounded and that $\{S(t)\}_{t\geq 0}$ is uniformly differentiable on $\mathcal{A}$, i.e.,

$$\frac{\|S(t)(\varphi_0+\eta) - S(t)(\varphi_0) - L(t,\varphi_0)\eta\|^2_{E_0}}{\|\eta\|^2_{E_0}} \to 0 \ \text{ as } \ \eta = \{y,z\} \to 0.$$

Setting $\theta = R_\varepsilon \varphi = \{u,\ u_t + \varepsilon u\}$ and choosing

$$\varepsilon = \frac{\lambda_1 \delta}{\delta^2 + 4\lambda_1},$$

we may rewrite system (1.1) as a first evolution equation of the form

$$\theta_t = B(\theta) = -A_\varepsilon \theta - b(\theta) + \bar{h},$$

where $b(\theta) = \{0, G(u)\}$, $\bar{h} = \{0, h\}$, and

$$A_\varepsilon = \begin{pmatrix} \varepsilon I & -I \\ \Delta^2 - \varepsilon(\delta-\varepsilon)I & (\delta-\varepsilon)I \end{pmatrix}.$$

Here I denotes the identity mapping. Also for simplicity of the presentation, we denote by $\{u,v\}^T$ the transposed form of $\{u,v\}$. In the above notation, the first variation equation (3.5) has the form

$$U_t = B'(\theta)U = -A_\varepsilon U - b'(\theta)U, \tag{3.8}$$

$$U(0) = \xi, \tag{3.9}$$

where $U = \{V, V_t + \varepsilon V\}$, $b'(\theta) = \{0, G'(u)V\}$, $\xi = \{y,z\} \in E_0$. We consider m solutions $U(t) = U_1(t), \ldots, U_m(t)$ of (3.5) − (3.6) corresponding to initial data $\xi = \xi_1, \ldots, \xi_m$, $\xi_k \in E_0$, $k = 1, 2, \ldots, m$.

For the proof of the main result we need the following lemma, which can be found in [12].

Lemma 3.3. *For any $\varphi = (y,z) \in E_0$,*

$$(A_\varepsilon \varphi, \varphi)_{E_0} \geq \rho \|\varphi\|^2_{E_0} + \frac{\delta}{2}|z|^2, \tag{3.10}$$

where

$$\rho = \frac{\lambda_1 \delta}{\sqrt{\delta^2 + 4\lambda_1}(\delta + \sqrt{\delta^2 + 4\lambda_1})}. \tag{3.11}$$

Recalling that in the generalized Liouville formula

$$
\begin{aligned}
&|U_1(t)\wedge\cdots\wedge U_m(t)|_{\wedge^m E_0}\\
&\qquad = |\xi_1(t)\wedge\cdots\wedge \xi_m(t)|_{\wedge^m E_0}\exp\int_0^t Tr(B'(S_\varepsilon(\tau)\theta_0)\circ Q_m(\tau))d\tau,
\end{aligned}
\tag{3.12}
$$

we have that the m-trace $Tr(B'(S_\varepsilon(\tau)\theta_0)\circ Q_m(\tau))$ provides information for the evolution of the m-dimensional volumes, transported along $S_\varepsilon(t)\theta_0$, by the first variation equation. We denote by $Q_m(t)$ the orthogonal projector in E_0 onto the subspace spanned by $U_1(t),\dots,U_m(t)$. We also denote by

$$
\Phi_j(t)=\{y_j,z_j\},\quad j=1,\dots,m,
$$

the orthonormal basis of $\mathrm{span}\{U_1(t),\dots,U_m(t)\}=Q_m(t)E_0$. We have that

$$
\begin{aligned}
Tr(B'(S_\varepsilon(\tau)\theta_0)\circ Q_m(\tau)) &= \sum_{j=1}^{\infty}[B'(S_\varepsilon(\tau)\theta_0)\circ Q_m(\tau)\Phi_j(\tau),\Phi_j(\tau)]_{E_0}\\
&= \sum_{j=1}^{m}(B'(\theta(\tau))\Phi_j(\tau),\Phi_j(\tau))_{E_0}.
\end{aligned}
\tag{3.13}
$$

Under the above notation, we prove our main result.

Proof of Theorem 1.1. From Lemma 2.4 we have that $R_\varepsilon\mathcal{A}\subset E_1$. Using this fact and conditions (1.5) – (1.8), we write

$$
\begin{aligned}
(B'(\theta(s))\Phi_j,\Phi_j)_{E_0} &= (-A_\varepsilon\Phi_j,\Phi_j)_{E_0}+(-b'(\theta)\Phi_j,\Phi_j)_{E_0}\\
&\le -\rho\|\Phi_j\|_{E_0}^2-\frac{\delta}{2}|z_j|^2+(\{0,G'(u)y_j\},\{y_j,z_j\})_{E_0}.
\end{aligned}
\tag{3.14}
$$

Applying Lemma 3.1, we take

$$
\gamma=\sup_{|Au|\le R}|G'(u)|_{\mathcal{L}(V_\alpha,H)}=k+K<\infty.
$$

This allows us to majorize $|(G'(u)y_j,z_j)|$ by

$$
|G'(u)y_j||z_j|\le\gamma\|y_j\|_\alpha|z_j|.
$$

Hence, applying Lemma 3.3 and Young's inequality to relation (3.14), we deduce

$$
\begin{aligned}
&(B'(\theta(s))\Phi_j,\Phi_j)_{E_0}\\
&\qquad\le -\rho\|\Phi_j\|_{E_0}^2-\frac{\delta}{2}|z_j|^2+\gamma\int_\Omega\|y_j\|_\alpha|z_j|dx\\
&\qquad\le -\rho\|\Phi_j\|_{E_0}^2-\frac{\delta}{2}|z_j|^2+\frac{\delta}{2}|z_j|^2+\frac{\gamma^2}{2\delta}\|y_j\|_\alpha^2\\
&\qquad\le -\rho\|\Phi_j\|_{E_0}^2+\frac{\gamma^2}{2\delta}\|y_j\|_\alpha^2.
\end{aligned}
\tag{3.15}
$$

Since $\{\Phi_j\}$ is an orthonormal basis of Q_mE_0, we have from (3.15) the inequality

$$
\sum_{j=1}^{m}(B'(\theta(s))\Phi_j(s),\Phi_j(s))_{E_0}\le -m\rho+\frac{\gamma^2}{2\delta}\sum_{j=1}^{m}\|y_j\|_\alpha^2.
\tag{3.16}
$$

Therefore, applying the result of Lemma VI.6.3 in [11] to any orthogonal family of elements $\{y_j, z_j\},\ j = 1, \dots, m$, of E_0, we obtain that

$$\sum_{j=1}^{m} \|y_j\|_\alpha^2 \le \sum_{j=1}^{m} \lambda_j^{\alpha-1} \ \text{ with } \ \alpha = 2\sigma/p\,, \ \ 0 \le 2\sigma < p < \infty, \ \ p \ge 1. \tag{3.17}$$

Thus, by (3.16) and (3.17), we have the following estimate:

$$Tr(B'(S_\varepsilon(\tau)\theta_0) \circ Q_m(\tau)) \le -m\rho + \frac{\gamma^2}{2\delta} \sum_{j=1}^{m} \lambda_j^{\alpha-1}. \tag{3.18}$$

We integrate (3.18) with respect to time t to obtain the relation

$$q_m(t) = \sup_{\varphi_0 \in R_\varepsilon \mathcal{A}} \sup_{\|\xi_i\| \le 1} \frac{1}{t} \int_0^t Tr(B'(S_\varepsilon(\tau)\theta_0) \circ Q_m(\tau)) d\tau$$

for $\xi_i \in E_0,\ i = 1, \dots, m,$ where $q_m = \limsup_{t\to\infty} q_m(t)$. Hence,

$$q_m \le -m\rho + \frac{\gamma^2}{2\delta} \sum_{j=1}^{m} \lambda_j^{\alpha-1}. \tag{3.19}$$

We infer from (3.19) and (2.49) in [11, Chapter V] the following bound on the uniform Lyapunov exponents $\mu_j,\ j \in N$, of $\mathcal{A}$ (or $R_\varepsilon\mathcal{A}$):

$$\mu_1 + \dots + \mu_j \le -m\rho + \frac{\gamma^2}{2\delta} \sum_{j=1}^{m} \lambda_j^{\alpha-1}. \tag{3.20}$$

From (3.2), we obtain that

$$\frac{1}{m} \sum_{j=1}^{m} \lambda_j^{\alpha-1} \to 0 \ \text{ as } \ m \to \infty.$$

Hence, there exists m (≥ 1) such that

$$\frac{1}{m} \sum_{j=1}^{m} \lambda_j^{\alpha-1} \le \frac{2\delta\rho}{\gamma^2}. \tag{3.21}$$

Consequently, we have that $q_m \le 0$. □

Therefore, by Theorem V.3.3 in [11] and Theorem 2.1 in [3], the dimensions of the attractor $\mathcal{A}$ are

$$\begin{aligned} &\dim_H(\mathcal{A}) \le \min\Big\{m \,\Big|\, \frac{1}{m} \sum_{j=1}^{m} \lambda_j^{\alpha-1} \le \frac{2\lambda_1\delta^2}{\gamma^2\sqrt{\delta^2 + 4\lambda_1}(\delta + \sqrt{\delta^2 + 4\lambda_1})}\Big\}, \\ &\dim_f(\mathcal{A}) \le \min\Big\{m \,\Big|\, \frac{1}{m} \sum_{j=1}^{m} \lambda_j^{\alpha-1} \le \frac{2\lambda_1\delta^2}{\gamma^2\sqrt{\delta^2 + 4\lambda_1}(\delta + \sqrt{\delta^2 + 4\lambda_1})}\Big\}. \end{aligned} \tag{3.22}$$

Remark 3.4. The estimate from Theorem 1.1 for the dimension of the attractor depends on the parameter p, which seems to be quite arbitrary. For convenience, we assume that the value of p is optimal.

From (3.2) and (3.22) we deduce that $\alpha = 0$, i.e., $p = +\infty$ is the optimal value for the minimal estimate of the dimension of the attractor.

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