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GLOBAL WEAK SOLUTIONS OF NON-ISOTHERMAL FRONT PROPAGATION PROBLEM

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Abstract. We show the global existence of weak solutions for a free-boundary problem arising in the non-isothermal crystallization of polymers. In particular, the free interface is shown to be of codimension one for every time t in two space dimensions; Hölder continuity of the temperature u is proven.

1. Introduction and main results

We are interested in the global existence of weak solution to the following system of partial differential equations, which is the level-set formulation of the non-isothermal crystallization process of polymers (see [5, 20]):

$$\partial_t(u + \chi(\varphi)) - \Delta u = 0, \tag{1.1}$$

$$\varphi_t + G(u)|\nabla\varphi| = 0, \tag{1.2}$$

$$u(x, 0) = u_0(x), \qquad \varphi(x, 0) = \varphi_0(x), \tag{1.3}$$

where

$$\chi(\varphi) = \operatorname{sign}(\varphi) = \begin{cases} -1 & \text{if } \varphi < 0, \\ [-1, 1] & \text{if } \varphi = 0, \\ 1 & \text{if } \varphi > 0. \end{cases} \tag{1.4}$$

Based upon the physics of the problem, we assume throughout the paper that for all $u, u_1, u_2 \in \mathbb{R}$,

$$0 < C_1 \leq G(u) \leq C_2 < \infty, \tag{1.5}$$

$$|G(u_1) - G(u_2)| \leq L|u_1 - u_2|. \tag{1.6}$$

In the isothermal case—the Stefan problem—the temperature is fixed on the moving interface. Therefore, the weak formulation of the classical Stefan problem is as follows:

$$\partial_t(u + \chi(u)) - \Delta u = 0. \tag{1.7}$$

There is vast literature on the study of Stefan problems (see, e.g., [22, 19, 24, 36]).

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However, the best known regularity of the temperature u for (1.7) is just continuity (see [8, 14, 37, 30]), and the codimension-one property of the moving front is still open.

Because of the change of temperature on the interface in the non-isothermal polymer crystallization, $\mathrm{sign}(u)$ in the Stefan problem (1.7) is replaced by $\mathrm{sign}(\varphi)$ in (1.1), where φ is governed by the Hamilton-Jacobi equation (1.2). Earlier works by Avrami [1], Kolmogorov [21], and Evans [17] in the 1930s and 1940s addressed spatially homogeneous growth of crystals, where the phase could be computed analytically.

Recently the modeling of non-isothermal crystallization has been extensively studied in [27, 31, 34, 29, 15, 16, 3, 5, 6, 7]. The local existence of a single smooth moving front in (1.1)–(1.3) was shown under the assumption that initial interface is a small perturbation of a sphere in Friedman-Velazquez [20]. In practice there are many crystals created by nucleation so that a situation with multiple crystals should be studied instead. As crystallines grow, they definitely infringe each other in finite time as long as there is a positive lower bound on the propagation speed. Consequently, the moving front may not be smooth.

To capture the physical features of front propagation, we study a global weak solution of (1.1)–(1.3). In this paper, we prove the global existence of a weak solution of (1.1)–(1.3) for large initial data. In particular, we show that temperature is Hölder continuous in space and the moving front is a Lipschitz graph in time-space.

The main novel features of our work are:

(i) the moving interface is shown to be of codimension one;

(ii) finite Hölder continuous propagation speed yields an intrinsic estimate of finite codimension-one Hausdorff measure of the moving interface for every time t in two space dimensions;

(iii) based upon (ii), we prove Hölder continuity of the temperature u by a decomposition argument.

Now we need to interpret the weak solution (φ, u) of (1.1) and (1.2). The temperature u satisfies (1.1) in the standard distributional sense. The best available Hölder regularity of the temperature u is weaker than Lipschitz continuity due to the non-smoothness of the moving interface, and so is the normal velocity of the moving interface. In reality, there is also thermal noise in the heat transfer. Then the temperature u may at best be Hölder continuous, and consequently the phase function φ has to be understood in an L^∞ sense (see [10, 11, 12]) since the level-set equation (1.2) allows a discontinuity to develop in finite time even if the initial phase function is smooth (because of the non-uniqueness of ODE solutions given a Hölder continuous right-hand side).

In this paper, we shall use the notion of L^∞ solution of Hamilton-Jacobi equations.

Denote $\{\varphi(\cdot,t)<0\}$ by $\Omega(t)$ and let $\Gamma(t)=\partial\Omega(t)$. Our main theorem is stated as follows.

Theorem 1.1. *Assume that the initial front* $\Gamma(0)$ *is* $\bigcup_{i=1}^m \Gamma_i^0$, *where each* Γ_i^0 *is a curve satisfying the local flattening condition: for any points* A *and* B *on* Γ_i^0 *with* $L(\overline{AB})<1$, $\sup_{x\in\widetilde{AB}} \mathrm{dist}(x,\overline{AB}) \le \xi L(\overline{AB})^{1+\beta}$, *where* $\widetilde{AB}$ *is the connecting part of* Γ_i^0 *between the point* A *and* B. *Moreover, let* $\|u_0\|_{C^{0,1}(\mathbb{R}^2)} \le C$. *Then for any*

$T>0$, the temperature u of (1.1)–(1.3) *satisfies*

$$\|u\|_{L^\infty(\mathbb{R}^2\times[0,T])}\le C, \tag{1.8}$$

$$\|u(\cdot,t_1)-u(\cdot,t_2)\|_{L^p(\mathbb{R}^2\times[0,T])}\le C|t_1-t_2|^\gamma \text{ for any } \gamma\in(0,\frac{1}{2}), \tag{1.9}$$

$$|u(x_1,t)-u(x_2,t)|\le C(\alpha)|x_1-x_2|^\alpha \text{ for any } \alpha\in(0,1), \tag{1.10}$$

for any $x_1,x_2\in\mathbb{R}^2$ and $t_1,t_2>0$. The phase function $\varphi\in L^\infty(R^2\times(0,T])\cap BV_{loc}(R^2\times(0,T])$ satisfies $\partial\{\varphi<0\}=\partial\{\varphi>0\}$. Furthermore, $\partial\{\varphi<0\}$ is a Lipschitz graph in time-space, i.e., $|\gamma(y_1,\varphi)-\gamma(y_2,\varphi)|\le\frac{|y_1-y_2|}{C_1}$. For each time t, $\Gamma(t)$ satisfies

$$\mathcal{H}^1(\Gamma(t)\cap B_r(y_0))\le C(m,\xi,\beta,\alpha,C_1,C_2,r), \tag{1.11}$$

for any $y_0\in\mathbb{R}^2$. As $r\to 0$, $\mathcal{H}^1(\Gamma(t)\cap B_r(y_0))\to 0$ for each $t\in(0,T]$.

2. Outline of proof

To study (1.1)–(1.3), we consider the following approximate equations:

$$\frac{\partial}{\partial t}(u+\chi_\epsilon(x,t,\varphi))-\Delta u=0 \qquad \text{in } R^n\times[0,T], \tag{2.1}$$

$$\frac{\partial\varphi}{\partial t}+G(u)|\nabla\varphi|=0 \qquad \text{in } R^n\times[0,T], \tag{2.2}$$

$$u(x,0)=u_0(x), \qquad \varphi(x,0)=\varphi_0(x). \tag{2.3}$$

Here χ_ϵ is defined by

$$\chi_\epsilon(x,t,\varphi)=\begin{cases}-1 & \text{if } t-\gamma(x,\varphi)>2\epsilon,\\ -1+\frac{\gamma(x,\varphi)-t}{\epsilon} & \text{if } 0\le t-\gamma(x,\varphi)\le 2\epsilon,\\ 1 & \text{if } \varphi(x,t)>0,\end{cases} \tag{2.4}$$

where $\gamma(x,\varphi)=\inf\{s:\varphi(x,s)<0\}$ for a monotone decreasing function φ in time t.

Remark 2.1. Observe that the solution of the level-set equation (2.2) is monotone decreasing in time; thus γ is well defined.

We introduce the following notation. Suppose that $A\subset R^n$ and $B\subset R^n$; then the distance between the sets A and B is given by

$$\operatorname{dist}(A,B)=\sup_{x\in A}\operatorname{dist}(x,B)+\sup_{y\in B}\operatorname{dist}(y,A). \tag{2.5}$$

Let T be any given positive number. Denote by Γ_ϕ^- the boundary of $\{\phi<0\}$ in $R^n\times(0,T]$, and by Γ_ϕ^+ the boundary of $\{\phi>0\}$ in $R^n\times(0,T]$ for a function ϕ monotone decreasing in time t. We need the following assumption on the initial front before we state the approximation lemma:

$$m(\partial\{\varphi_0<0\}\cap\partial\{\varphi_0>0\}\cap\{\varphi_0=0\})=0. \tag{A1}$$

We denote $\{\varphi_0<0\}$ by $\Gamma(0)$ as long as the assumption (A1) holds.

Lemma 2.2. *Suppose that the initial temperature satisfies $u_0\in C^2(R^n)\cap L^\infty(R^n)$, and the initial almost-everywhere-continuous phase function $\varphi_0\in L^\infty(R^n)$ satisfies the assumption* (A1) *with the following property: $\varphi_0<-1$ if $\varphi_0\le 0$, and $\varphi_0>1$ if $\varphi_0>0$. The material function G satisfies assumptions* (1.5) *and* (1.6). *Then for every $\epsilon>0$, there exists a solution $u^\epsilon\in W^{2,1,p}_{x,t}(R^n\times(0,T])\cap C^{\frac{\alpha}{2},\alpha}(R^n)$ to*

(2.1)–(2.3) *with* $\partial\{\varphi^\epsilon < 0\} = \partial\{\varphi^\epsilon > 0\}$*, for any* $p \in (1,\infty)$ *and* $\varphi^\epsilon \in L^\infty(R^n \times (0,T]) \cap BV(R^n \times (0,T])$*. Furthermore,* $\partial\{\varphi^\epsilon < 0\}$ *is a Lipschitz graph in time-space independent of* ϵ*, i.e.,* $|\gamma(y_1,\varphi^\epsilon) - \gamma(y_2,\varphi^\epsilon)| \le \frac{|y_1-y_2|}{C_1}$.

Then we establish a surprising theorem in two spatial dimensions that the one-dimensional Hausdorff measure of the moving front is finite if the normal velocity $f(x,t)$ is Hölder continuous with respect to x and is bounded from below. In other words, there is a discontinuous L^∞ solution φ of the following level-set equation, whose level set $\partial\{\varphi < 0\} = \partial\{\varphi > 0\}$ is of finite codimension-one Hausdorff measure for every time $t > 0$:

$$\varphi_t + f(x,t)|D\varphi| = 0, \qquad x \in \mathbb{R}^2,\ t > 0, \tag{2.6}$$

$$\varphi(x,0) = \phi(x), \tag{2.7}$$

where $0 < C_1 \le f(x,t) \le C_2$.

Theorem 2.3. *Let the moving front* $\Gamma(t)$ *be the boundary of the reachable set* $\Omega(t)$ *driven by the normal velocity* $f(x,t)$*, which is Hölder continuous with* $|f(x_1,t) - f(x_2,t)| \le \kappa|x_1 - x_2|^\alpha$ *and* $0 < C_1 \le f(x,t) \le C_2$*. Assume that the initial front* $\Gamma(0)$ *is* $\bigcup_{i=1}^m \Gamma_i^0$*, where* Γ_i^0 *is the curve satisfying the local flattening condition: for any points* A *and* B *on* Γ_i^0 *with* $L(\overline{AB}) < 1$, $\sup_{x\in\widetilde{AB}} \operatorname{dist}(x,\overline{AB}) \le \xi L(\overline{AB})^{1+\beta}$*, where* $\widetilde{AB}$ *is the connecting part of* Γ_i^0 *between the points* A *and* B*. Then for any* $t > 0$*,* $\Gamma(t)$ *satisfies*

$$\mathcal{H}^1(\Gamma(t)\cap B_r(y_0)) \le C(m,\xi,\beta,\alpha,C_1,C_2)r \quad \text{if } r \in (0,\kappa^{\frac{-2+\alpha^2}{\alpha}}], \tag{2.8}$$

$$\mathcal{H}^1(\Gamma(t)\cap B_r(y_0)) \le C(m,\xi,\beta,\alpha,C_1,C_2)r^2\kappa^{\frac{2-\alpha^2}{\alpha}} \quad \text{if } r \ge \kappa^{\frac{-2+\alpha^2}{\alpha}}, \tag{2.9}$$

for any $y_0 \in \mathbb{R}^2$.

The argument is quite lengthy due to geometric complexity. Roughly speaking, in the first step, we show that the "convex" part of the moving front is bounded by the lower boundary of the convex hull of the curve; we show that the complement of the "convex" part, which consists of countably many "vaguely concave" parts, is locally nicely behaved; that is, each "vaguely concave" part is close to a straight line in the sense that the "concavity" is small. Then we decompose each "vaguely concave" part into no more than three subparts; the union of the lower boundaries of the convex hulls of the subparts serves as the first approximation of the moving front. We iterate the argument to treat each "vaguely concave" subpart to produce the nth approximation of the moving front. As the iteration number of the above procedure goes to infinity, the moving front becomes locally flat, and thus completely restored as a 1-dimensional curve, and its length is finite.

For the purpose of an a priori regularity estimate of temperature, we must weaken the assumption on the Lipschitz constant. From now on, we assume that the velocity function $f(x,t)$ satisfies

$$|f(x_1,t) - f(x_2,t)| \le \kappa(t)|x_1 - x_2|^\alpha, \text{ where } \kappa \in L^p(0,T), \tag{2.10}$$

Denote by $\mathcal{K} = \|\kappa\|_{L^p(0,T)}$ for $p > 1$.

Theorem 2.4. *Let the moving front* $\Gamma(t)$ *be the boundary of the reachable set* $\Omega(t)$ *driven by the normal velocity* $f(x,t)$*, which is Hölder continuous in space with* $|f(x_1,t) - f(x_2,t)| \le \kappa(t)|x_1 - x_2|^\alpha$*,* $0 < C_1 \le f(x,t) \le C_2$*, and* $\kappa \in L^p(0,T)$ *for sufficiently large* $p > 1$*. Denote* $\|\kappa\|_{L^p(0,T)}$ *by* $\mathcal{K}$*. Assume that the initial front* $\Gamma(0)$

is $\bigcup_{i=1}^m \Gamma_i^0$*, where* Γ_i^0 *is the curve satisfying the local flattening condition: for any points* A *and* B *on* Γ_i^0 *with* $L(\overline{AB}) < 1$, $\sup_{x\in\widetilde{AB}} \operatorname{dist}(x, \overline{AB}) \le \xi L(\overline{AB})^{1+\beta}$*, where* $\widetilde{AB}$ *is the connecting part of* Γ_i^0 *between the point* A *and* B*. Then for any* $t > 0$ *and any* $y_0 \in \mathbb{R}^2$, $\Gamma(t)$ *satisfies*

$$\mathcal{H}^1(\Gamma(t) \cap B_r(y_0)) \le C(m, \xi, \beta, \alpha, C_1, C_2) r \quad \text{if } r \in (0, \mathcal{K}^{\frac{-2+\alpha^2}{\alpha(1+\frac{1}{p})-\frac{1}{p}}}]; \tag{2.11}$$

$$\mathcal{H}^1(\Gamma(t) \cap B_r(y_0)) \le C(m, \xi, \beta, \alpha, C_1, C_2) r^2 \mathcal{K}^{\frac{2-\alpha^2}{\alpha(1+\frac{1}{p})-\frac{1}{p}}} \quad \text{if } r \ge \mathcal{K}^{\frac{-2+\alpha^2}{\alpha(1+\frac{1}{p})-\frac{1}{p}}}. \tag{2.12}$$

Based upon Theorem 2.4 and Lemma 2.2, using a solution formula, we employ a decomposition argument to derive the desired Hölder estimate for $u^\epsilon(x,t)$ and $\nabla u^\epsilon(x,t)$.

Theorem 2.5. *Assume that the initial front* $\Gamma(0)$ *is* $\bigcup_{i=1}^m \Gamma_i^0$*, where* Γ_i^0 *is the curve satisfying the local flattening condition: for any points* A *and* B *on* Γ_i^0 *with* $L(\overline{AB}) < 1$, $\sup_{x\in\widetilde{AB}} \operatorname{dist}(x, \overline{AB}) \le \xi L(\overline{AB})^{1+\beta}$*, where* $\widetilde{AB}$ *is the connecting part of* Γ_i^0 *between the point* A *and* B*. Let* $\|u_0^\epsilon\|_{C^{0,1}(\mathbb{R}^2)} \le C$*. Then the temperature* u^ϵ *of* (2.1)–(2.3) *enjoys*

$$\|u^\epsilon\|_{L^\infty(\mathbb{R}^2\times[0,T])} \le C, \tag{2.13}$$

$$\|u^\epsilon(\cdot, t_1) - u^\epsilon(\cdot, t_2)\|_{L^p(\mathbb{R}^2\times[0,T])} \le C|t_1 - t_2|^\beta \text{ for any } \beta \in (0, \frac{1}{2}), \tag{2.14}$$

$$|u^\epsilon(x_1, t) - u^\epsilon(x_2, t)| \le C(\alpha)|x_1 - x_2|^\alpha \text{ for any } \alpha \in (0, 1), \tag{2.15}$$

for any $x_1, x_2 \in \mathbb{R}^2$ *and* $t_1, t_2 > 0$.

Because of finite propagation speed of the moving front, the decomposition is conducted in such a way that the contribution from the parts away from the point (x,t) in the large scale is bounded by a coarse estimate of the moving front based upon Lemma 2.2, and that of the parts close to (x,t) in the small scale is estimated with the aid of Theorem 2.4. In our argument, the finite speed of propagation of the moving front plays an essential role.

It follows from Theorem 2.5, Lemma 2.2, the BV_{loc} estimate for the level-set function φ^ϵ (see [12, 32]), and the definition of L^∞ solution φ^ϵ of the Hamilton-Jacobi equation that the a priori $C^{0,\alpha}$ estimate in space of the temperature u^ϵ, the $C^{0,\frac{1}{3}}$ estimate in time of u^ϵ, and the uniform BV_{loc} estimate of φ^ϵ ensure that the approximate solution $(u^\epsilon, \varphi^\epsilon)$ of (2.1)–(2.3) converges to the solution (u, φ) of (1.1)–(1.3) as $\epsilon \to 0$. Thus we conclude our main Theorem 1.1.

3. Discussions

In this paper we have developed a new concept of weak solutions for a non-isothermal phase-change model of polymeric materials, which allows us, in particular, to treat the growth and impingement of multiple crystals in a sound mathematical way. As a major ingredient of an existence proof we have derived results on the geometric properties of the front and the Hölder continuity of the temperature.

The important physical question is the stability of the moving front. In the manufacturing process, the interface between crystal (solid phase) and melt (liquid phase) should be well controlled so that high-quality polymers can be produced. The lately discovered scaling law of crystallization of polymers is different from that of metal solidification (see [35]). The study of the moving front for metal solidification was extensive, though a rigorous mathematical justification is completely open. We will address the stability of the non-isothermal moving front for polymerical materials in the forthcoming work [33].

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