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ORBITS IN THE FLAG VARIETY AND IMAGES OF THE MOMENT MAP FOR CLASSICAL GROUPS I

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Abstract. We propose algorithms to get representatives and the images of the moment map of conormal bundles of $GL(p,\mathbf{C})\times GL(q,\mathbf{C})$-orbits in the flag variety of $GL(p+q,\mathbf{C})$, and $GL(p+q,\mathbf{C})$-orbits and $Sp(p,\mathbf{C})\times Sp(q,\mathbf{C})$-orbits in the flag variety of $Sp(p+q,\mathbf{C})$ and their signed Young diagrams.

Contents

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1. Introduction

For an admissible representation of a real semisimple Lie group, the associated variety is one of the most important invariants of representations. Borho-Brylinski [1] clarify the relation between representations in a $\mathcal{D}$-module picture and their associated varieties. The associated variety of a representation (Harish-Chandra module) is exactly the image of the moment map of the characteristic variety of the $\mathcal{D}$-module corresponding to the representation. In particular, the associated variety of a discrete series representation is nothing but the image of the moment map of the conormal bundle of the corresponding closed orbit in the flag variety. Motivated by these facts, it seems to be useful to determine the image of the moment map of the conormal bundle of various orbits in the flag variety. In this paper, for indefinite unitary groups, real symplectic groups, and indefinite symplectic groups, we will determine the image as *explicitly* as possible. Garfinkle gave another algorithm in [3]. By her algorithm we can also get signed Young diagrams from parameters of K-orbits in the flag variety. It remains to examine how the two algorithms agree.

In order to describe the results in this paper, we introduce the following notation. Let $G_{\mathbf{R}}$ be a real classical group. Let K be the complexification of a maximal compact subgroup of $G_{\mathbf{R}}$. The image of the moment map on a conormal bundle of a K-orbit in the flag variety is known to contain a (unique) dense nilpotent K-orbit in the orthogonal subspace $\mathfrak{p}$ of the Lie algebra $\mathfrak{k}$ of K in the Lie algebra $\mathfrak{g}$ of G (see [1], [2]). What we want to write down is the map from a K-orbit in the flag variety to a nilpotent K-orbit in $\mathfrak{p}$. For a K-orbit Q in the flag variety X, the image of the moment map μ of the conormal bundle $T^*_Q X$ is a K-saturation of the image of a fiber $(T^*_Q X)_x$ at a point $x \in Q$. Then the nilpotent K-orbit $\mathcal{O}$ in $\mathfrak{p}$ is a K-saturation of the subspace $\mu((T^*_Q X)_x)$ in $\mathfrak{p}$. The K-orbits in $\mathfrak{p}$ are parametrized by symbols called *signed Young diagrams* from which we can read off corresponding orbits (see [2]). Similarly, Matsuki and Rossmann [5], [8], [9] classified K-orbits in the flag variety, and moreover, for classical groups, Matsuki and Oshima introduce symbols (called *clans*) parametrizing these orbits [6]. Hence the map of K-orbits in the flag variety into nilpotent K-orbits in $\mathfrak{p}$ defined above can be expressed in terms of the map of clans into signed Young diagrams.

In this paper, we give the subspaces $\mu((T^*_Q X)_x) \subset \mathfrak{p}$ directly from the clan which parametrizes the corresponding K-orbit in the flag variety. We can determine the corresponding signed Young diagram from a generic point of this space.

In this paper we treat the cases $G_{\mathbf{R}} = U(p,q)$, $Sp(n,\mathbf{R})$, and $Sp(p,q)$. The same argument can be applied to groups which have a compact Cartan subgroup. The precise results will appear elsewhere.

We give a brief organization of this paper.

In Section 2, we treat the case of indefinite unitary groups $U(p,q)$. After we recall a standard notation of $U(p,q)$, we treat K-orbit ($GL(p,\mathbf{C}) \times GL(q,\mathbf{C})$-orbit) decomposition in the flag variety X in Section 2.1.

In Section 2.2, we recall a symbolic parametrization (called *clan*) of K-orbits in the flag variety in terms of clans along [6]. Since no proofs are given in [6], we give proofs for completeness. We give representatives of K-orbits for clans also.

In Section 2.3, we give a dimension formula of K-orbits for clans, and in Section 2.4 a graph (called *M-O graph*) which has clans for vertices. The graph shows closure relations between K-orbits.

In Section 2.5, after we recall the moment map, we give images of the moment map of the conormal bundles of K-orbits for clans.

In Section 2.6, we recall the parametrization of nilpotent K-orbits in $\mathfrak{p}$ by signed Young diagrams. Although we partly follow the treatment of [2], we do not use the Kostant-Sekiguchi correspondence. Our method stands on the following observation: for an element X of a K-orbit in $\mathfrak{p}$, if we can compute the dimension of the kernel of X^i for all $1 \leq i \leq n$, then we can obtain the signed Young diagram of the orbit using such information. We write down tables of signed Young diagrams for clans for $U(1,1)$, $U(2,1)$, $U(2,2)$, and $U(p,2)$ by way of examples.

In Section 3 (resp. in Section 4), we treat the case of real symplectic group $Sp(n,\mathbf{R})$ (resp. indefinite symplectic group $Sp(p,q)$). For applying the case of $U(2p,2q)$ to the case of $Sp(p,q)$, we use another realization of $U(2p,2q)$ in Section 4).

In Section 3.2-1 (resp. in Section 4.3-1), we define a symbolic parametrization (called *generalized clan*) of K-orbits ($GL(n,\mathbf{C})$-orbits (resp. $Sp(p,\mathbf{C}) \times Sp(q,\mathbf{C})$-orbits)) in the flag variety in the same way as in the case of $U(n,n)$ (resp. $U(2p,2q)$). Generalized clans for $Sp(n,\mathbf{R})$ (resp. $Sp(p,q)$) are given as elements of a subset of clans for $U(n,n)$ (resp. $U(2p,2q)$) satisfying some conditions. The notion of generalized clans is an improvement of the notion of clans in [6].

In Section 3.2-2 (resp. in Section 4.3-2), we give representatives of K-orbits for generalized clans.

In Section 3.3 (resp. in Section 4.4), we give a graph (called *M-O graph*) in terms of [6]. Instead of clans, the graph has generalized clans for vertices.

In Section 3.4 (resp. in Section 4.5), we give a dimension formula of K-orbits for generalized clans.

In Section 3.5 (resp. in Section 4.6), we give images of the moment map of the conormal bundle of K-orbits for generalized clans.

In Section 3.6 (resp. in Section 4.7), we recall the parametrization of nilpotent K-orbits in $\mathfrak{p}$ by signed Young diagrams. Lastly, we write down tables of signed Young diagrams for clans for $Sp(1,\mathbf{R})$, $Sp(2,\mathbf{R})$, and $Sp(3,\mathbf{R})$ (resp. $Sp(1,1)$, $Sp(2,1)$, and $Sp(2,2)$) by way of examples.

Notation 1.0.1. Let $\mathbf{N}$ denote the set of positive integers; $\mathbf{N} = \{1, 2, \dots\}$. For $n \in \mathbf{N}$, let an $n \times n$ matrix E_{ij} $(1 \leq i, j \leq n)$ denote the matrix unit which has 1 for the (i,j)-entry and 0 for other entries. Let an n-column vector e_i be the vector which has 1 for the i-th entry and 0 for other entries. For a matrix A, let A_{st} be the (s,t)-entry of A. Let $\mathrm{Mat}(m,n)$ be the set of $m \times n$-matrices over $\mathbf{C}$. Let $I_n \in \mathrm{Mat}(n,n)$ be the identity matrix, $J_n \in \mathrm{Mat}(n,n)$ satisfy $(J_n)_{st} = \delta_{s+t,n+1}$:

$$J_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ 0 & . \cdot \cdot & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

Let $\mathrm{diag}(a_1, \dots, a_n) \in \mathrm{Mat}(n,n)$ be a diagonal matrix

$$\mathrm{diag}(a_1, \dots, a_n) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}.$$

For a matrix $A \in \mathrm{Mat}(n,n)$ and a subset $\{i(1), i(2), \dots, i(m)\}$ of $\{1, \dots, n\}$, we denote by $A_{(i(1),i(2),\dots,i(m))}$ an $m \times m$-matrix whose (s,t)-entry is $A_{i(s)i(t)}$. Let $\#S$ denote cardinality of a finite set S. For m vectors $\{g_1, \dots, g_m \mid g_i \in \mathbf{C}^n\}$ $(m < n)$ let $\langle g_1, \dots, g_m \rangle$ be vector space spanned by $\{g_1, \dots, g_m\}$. Let $\mathfrak{S}_n$ be the set of permutations of $\{1, \dots, n\}$.

2. The case of $U(p,q)$

In this section we treat $G_{\mathbf{R}} = U(p,q)$, i.e., $GL(p,\mathbf{C}) \times GL(q,\mathbf{C})$-orbits in the flag variety of $GL(n,\mathbf{C})$ for $n = p+q$. Although we restrict ourselves to the case $G_{\mathbf{R}} = U(p,q)$, $Sp(n,\mathbf{R})$, and $Sp(p,q)$ in the main body of the paper, a preliminary discussion holds for an arbitrary linear connected reductive Lie group $G_{\mathbf{R}}$.

Let $G_{\mathbf{R}}$ be a real classical Lie group with Lie algebra $\mathfrak{g}_{\mathbf{R}}$, G the complexification of $G_{\mathbf{R}}$, θ a Cartan involution of $\mathfrak{g}_{\mathbf{R}}$. Let $\mathfrak{g}_{\mathbf{R}} = \mathfrak{k}_{\mathbf{R}} + \mathfrak{p}_{\mathbf{R}}$ be the Cartan decomposition corresponding to θ, $\mathfrak{k}$ the complexification of $\mathfrak{k}_{\mathbf{R}}$, K the analytic subgroup of G for $\mathfrak{k}$, and B a Borel subgroup of G.

We realize the indefinite unitary group $G_{\mathbf{R}} = U(p,q)$ as a group of matrices g in $GL(n,\mathbf{C})$ which leave invariant a Hermitian form of the signature (p,q) in $G = GL(n,\mathbf{C})$

$$x_1\overline{x_1} + \cdots + x_p\overline{x_p} - x_{p+1}\overline{x_{p+1}} - \cdots - x_n\overline{x_n},$$

i.e.,

$$U(p,q) = \left\{ g \in GL(n,\mathbf{C}) \;\middle|\; {}^t g \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \bar{g} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \right\}.$$

We fix a Cartan involution θ of $G_{\mathbf{R}}$ as follows:

$$\theta : g \mapsto \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} g \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

Then we have

$$\mathfrak{k} = \left\{ \begin{pmatrix} K_{11} & 0 \\ 0 & K_{22} \end{pmatrix} \;\middle|\; \begin{array}{l} K_{11} \in \mathrm{Mat}(p,p) \\ K_{22} \in \mathrm{Mat}(q,q) \end{array} \right\}, \tag{1}$$

$$\begin{aligned} K &= \left\{ \begin{pmatrix} K_{11} & 0 \\ 0 & K_{22} \end{pmatrix} \;\middle|\; \begin{array}{l} K_{11} \in GL(p,\mathbf{C}) \\ K_{22} \in GL(q,\mathbf{C}) \end{array} \right\}, \\ &\simeq GL(p,\mathbf{C}) \times GL(q,\mathbf{C}), \end{aligned} \tag{2}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & P_{12} \\ P_{21} & 0 \end{pmatrix} \;\middle|\; \begin{array}{l} P_{12} \in \mathrm{Mat}(p,q) \\ P_{21} \in \mathrm{Mat}(q,p) \end{array} \right\}. \tag{3}$$

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ consisting of all diagonal matrices of $\mathfrak{g}$, $\mathfrak{b}$ a Borel subalgebra consisting of all upper triangular matrices. This choice corresponds to the choice of simple root system Ψ:

$$\Psi := \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\} \subset \mathfrak{a}^*$$

where $\alpha_i \in \mathfrak{a}^*$ satisfies $\alpha_i(E_{jj}) = \delta_{ij} - \delta_{i+1,j}$.

2.1. $GL(p,\mathbf{C})\times GL(q,\mathbf{C})$**-orbits in the flag variety of** $GL(n,\mathbf{C})$**.** In this section we treat orbit decomposition in a flag variety.

Definition 2.1.1. A flag x of $GL(n,\mathbf{C})$ is a sequence of $n+1$ vector spaces

$$x = (V_0, V_1, V_2, \dots, V_n),$$

satisfying the following two conditions.

1. $\dim V_i = i$ for all $0 \le i \le n$.
2. $\{0\} = V_0 \subset V_1 \subset V_2 \subset \dots \subset V_n = \mathbf{C}^n$.

We denote the set of flags by X. We call X a *flag variety* of $GL(n,\mathbf{C})$.

Remark 2.1.2. We fix a G-equivariant natural isomorphism between X and G/B. A left coset $gB \in G/B$ of an element $g = (g_1 \dots g_n) \in G$ (here g_i $(1 \le i \le n)$ are elements of $\mathbf{C}^n$) can be identified with a flag $x = (V_0, V_1, \dots, V_n)$ in the following manner:

$V_0 = \{0\}$. V_i is spanned by i-vectors $g_1, g_2, \dots, g_{i-1}$, and g_i, i.e.,

$$V_i = \langle g_1, \dots, g_i\rangle$$

for all $1 \le i \le n$.

Notation 2.1.3. Let $V := \mathbf{C}^n$ and let θ be an involution of V such that

$$\theta : v \mapsto \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} v.$$

Let V_+ and V_- be the eigenspaces in V under θ for eigenvalues $+1$ and -1, respectively:

$$V_+ := \langle e_1, \dots, e_p\rangle \quad \text{and} \quad V_- := \langle e_{p+1}, \dots, e_n\rangle.$$

Remark 2.1.4. This gives a decomposition into a direct sum

$$V = V_+ \oplus V_-. \tag{4}$$

Notation 2.1.5. Let π_+ be the first projection and π_- the second projection with respect to the decomposition (4). Let $V_{i,+}$, $V_{i,-}$ be subspaces of V_i such that

$$V_{i,+} \ := \ V_i \cap V_+ \quad \text{and} \quad V_{i,-} \ := \ V_i \cap V_-.$$

We have

$$\pi_-(V_i) + V_j = \pi_+(V_i) + V_j = (V_i + V_-) \cap V_+ + V_j$$

for $i < j$. Because $K = \{g \in GL(n,\mathbf{C}) \mid gV_+ = V_+,\ gV_- = V_-\}$, we have

$$\dim(kV_i \cap V_+) = \dim(V_i \cap k^{-1}V_+) = \dim(V_i \cap V_+)$$

for all $k \in K$, and so on. Therefore we have the following proposition.

Proposition 2.1.6. *For a flag* $x = (V_0, V_1, \dots, V_n) \in X$*,* $\dim V_{i,+}$*,* $\dim V_{i,-}$*, and* $\dim(\pi_+(V_i) + V_j)$ *for* $0 \le i \le j \le n$ *are invariant under the actions of* K*.*

Definition 2.1.7. For a flag $x = (V_0, V_1, \dots, V_n) \in X$, $0 \le i \le n$, and $i < j$, we put

$$\begin{aligned}
(i;+) &:= \dim V_{i,+},\\
(i;-) &:= \dim V_{i,-},\\
(i;\mathbf{N}) &:= i - (i;+) - (i;-), \quad \text{and}\\
(i;\,j\,) &:= \dim(\pi_+(V_i) + V_j).
\end{aligned}$$

Remark 2.1.8. We have $\dim \pi_+(V_i) = (i;+) + (i;\mathbf{N})$ and $\dim \pi_-(V_i) = (i;-) + (i;\mathbf{N})$.

2.2. **A symbolic parametrization of $GL(p,\mathbf{C}) \times GL(q,\mathbf{C})$-orbits.** In this section we recall a symbolic parametrization of K-orbits in X and give representatives of K-orbits for the parameters.

2.2-1. *Clans for $U(p,q)$.* We give a set consisting of sequences of n symbols in the following definition. This set will parametrize K-orbits in the flag variety X.

Definition 2.2.1 (Clan)**.** (See [6].) An *indication* for $U(p,q)$ is an ordered set $(c_1 \ \cdots \ c_n)$ of n symbols satisfying the following four conditions.

1. A symbol c_i is $+$, $-$, or an element of $\mathbf{N}$ for $1 \le i \le n$.
2. If $c_i \in \mathbf{N}$, then there exists a unique $j \ne i$ with $c_j = c_i$, i.e.,
$$\#\{ \ i \mid c_i = a \ \} = 0 \text{ or } 2 \quad \text{for any} \ \ a \in \mathbf{N}.$$
3. The difference between numbers of $+$ and $-$ in indications $(c_1 \ \cdots \ c_n)$ coincides with the difference of signatures of the Hermitian form defining the group $G_{\mathbf{R}}$:
$$\#\{ \ i \mid c_i = +\} - \#\{ \ i \mid c_i = -\} = p - q.$$
4. If $c_i \in \mathbf{N}$, $a \in \mathbf{N}$, and $a < c_i$, then there exist some j such that $c_j = a$.

We define an equivalence relation between two indications as follows. Two indications $(c_1 \ \cdots \ c_n)$ and $(c'_1 \ \cdots \ c'_n)$ are regarded as equivalent if and only if there exists a permutation $\sigma \in \mathfrak{S}_m$ with $m := \max\{c'_i \in \mathbf{N}\}$ such that

$$c_i = \begin{cases} \sigma(c'_i) & \text{if } \ c'_i \in \mathbf{N}, \\ + & \text{if } \ c'_i = +, \\ - & \text{if } \ c'_i = -, \end{cases}$$

for $1 \le i \le n$. A *clan* is an equivalence class of the indications with respect to the equivalence relation. For example, $(2\ 2\ 1\ +\ 1\ -) = (1\ 1\ 2\ +\ 2\ -)$ as a clan. We denote the set of clans for $U(p,q)$ by $\mathcal{C}(U(p,q))$. By abuse of notation sometimes we represent a clan γ by an indication belonging to the clan γ.

Definition 2.2.2 (Standard indication)**.** If an indication $(c_1 \ \cdots \ c_n)$ of a clan satisfies the following condition, we call it *standard*.

If $c_i = c_j = a \in \mathbf{N}$ for $i < j$, $c_s = c_t = b \in \mathbf{N}$ for $s < t$, and $i < s$, then $a < b$.

Obviously, every clan has a unique standard indication.

Example 2.2.3. The set $\mathcal{C}(U(2,1))$ consists of six clans:

$$\mathcal{C}(U(2,1)) = \left\{ \begin{matrix} + & + & -, & + & - & +, & - & + & + \\ + & 1 & 1, & 1 & 1 & +, & 1 & + & 1 \end{matrix} \right\}.$$

Here $(+\ +\ -)$, for example, is denoted by $+\ +\ -$, for simplicity.

Example 2.2.4. The set $\mathcal{C}(U(2,2))$ consists of 21 clans:

$$\mathcal{C}(U(2,2)) = \left\{ \begin{array}{llll} + + - -, & + - + -, & + - - +, & - + + - \\ - + - +, & - - + +, & 1\ 1 + -, & 1\ 1 - + \\ + 1\ 1 -, & - 1\ 1 +, & + - 1\ 1, & - + 1\ 1 \\ 1 + 1 -, & 1 - 1 +, & + 1 - 1, & - 1 + 1 \\ 1 + - 1, & 1 - + 1, & 1\ 1\ 2\ 2, & 1\ 2\ 1\ 2 \\ 1\ 2\ 2\ 1 & & & \end{array} \right\}.$$

Proposition 2.2.5. *For a clan* $(c_1 \cdots c_n)$*, suppose* $c_i = c_j \in \mathbf{N}$ *and* $i < j$*. If* $m = j$*, then*

$$\begin{aligned} &\#\{ a \in \mathbf{N} \mid c_s = c_t = a \ \ for \ \ s \le i < m < t \} \\ &\qquad = \#\{ a \in \mathbf{N} \mid c_s = c_t = a \ \ for \ \ s \le i-1 < m < t \}. \end{aligned}$$

If $m < j$*, then*

$$\begin{aligned} &\#\{ a \in \mathbf{N} \mid c_s = c_t = a \ \ for \ \ s \le i < m < t \} \\ &\qquad = \#\{ a \in \mathbf{N} \mid c_s = c_t = a \ \ for \ \ s \le i-1 < m < t \} + 1. \end{aligned}$$

Proposition 2.2.6. *For an* $x = (V_0, V_1, \ldots, V_n) \in X$*, there exists a clan* $\gamma = (c_1 \cdots c_n) \in \mathcal{C}(U(p,q))$ *satisfying the following conditions.*

(5) $\quad (i;\mathbf{N}) = \#\{ a \in \mathbf{N} \mid c_s = c_t = a \textit{ for } s \le i < t \},$

(6) $\quad (i;+) = \#\{ l \mid c_l = + \textit{ for } l \le i \} + \#\{ a \in \mathbf{N} \mid c_s = c_t = a \textit{ for } s < t \le i \},$

(7) $\quad (i;-) = \#\{ l \mid c_l = - \textit{ for } l \le i \} + \#\{ a \in \mathbf{N} \mid c_s = c_t = a \textit{ for } s < t \le i \},$

(8) $\quad (i;j) = j + \#\{ a \in \mathbf{N} \mid c_s = c_t = a \textit{ for } s \le i < j < t \}$

for all $1 \le i \le n$ *and* $i < j$*. Such a clan* γ *is determined uniquely by the flag* x*. We call* γ *the clan of* x*.*

Proof. For a flag $x = (V_0, V_1, \ldots, V_n) \in X$, such a clan will be given inductively by the following procedure.

1. First of all, we remark that one of $(1;+)$, $(1;-)$, and $(1;\mathbf{N})$ is 1, and others are 0.
   (a) If $(1;+) = 1$, then we set $c_1 = +$.
   (b) If $(1;-) = 1$, then we set $c_1 = -$.
   (c) If $(1;\mathbf{N}) = 1$, then we set $c_1 = 1$.
2. Suppose we have already obtained $(c_1 \ldots c_{i-1})$. We remark that

$$(i;\mathbf{N}) - (i-1;\mathbf{N}) = 1,\ 0, \ \text{ or } \ -1.$$

   (a) If $(i;\mathbf{N}) - (i-1;\mathbf{N}) = 1$, then we set $c_i = a$. Here,

$$a = 1 + \max\{ b \in \mathbf{N} \mid c_j = b,\ j < i \}.$$

   (b) If $(i;\mathbf{N}) - (i-1;\mathbf{N}) = 0$, then we have either $(i;+) - (i-1;+) = 1$ or $(i;-) - (i-1;-) = 1$.
      (b-i) If $(i;+) - (i-1;+) = 1$, then we set $c_i = +$.
      (b-ii) If $(i;-) - (i-1;-) = 1$, then we set $c_i = -$.
   (c) If $(i;\mathbf{N}) - (i-1;\mathbf{N}) = -1$, then we have

(9) $$(i;+) - (i-1;+) = 1 \quad \text{and}$$

(10) $$(i;-) - (i-1;-) = 1.$$

We will choose a $j < i$ and put $c_i = c_j \in \mathbf{N}$ as follows. It follows from (9) and (10), we have a nonzero $v_+ \in \pi_+(V_{i-1})$ and a nonzero $v_- \in \pi_-(V_{i-1})$ such that $v_+ \notin V_{i-1}$, $v_- \notin V_{i-1}$ and

$$V_i = V_{i-1} + \langle v_+, v_- \rangle.$$

We have a unique $j \leq i-1$ such that

$$v_+ \in \pi_+(V_j) + V_{i-1} \quad \text{and} \quad v_+ \notin \pi_+(V_{j-1}) + V_{i-1}.$$

We set $c_i = c_j$. We will show that $(c_1 c_2 \dots c_n)$ is a clan and satisfies the conditions of Proposition 2.2.6.

Let $v_+ = v'_+ + v''_+$, such that $v'_+ \in \pi_+(V_j)$ and $v''_+ \in V_{i-1}$. Then $v'_+ \notin \pi_+(V_{j-1})$ and $v'_+ \in V_i$. Because of $v'_+ \notin V_{i'}$ for all $j < i' < i$,

$$\begin{aligned}(j;\, i') &= \dim(\pi_+(V_j) + V_{i'}) \\ &= \dim(\pi_+(V_{j-1}) + V_{i'} + \langle\, v'_+ \,\rangle\,) \\ &= \dim(\pi_+(V_{j-1}) + V_{i'}) + 1 \\ &= (j-1;\, i') + 1.\end{aligned}$$

Thus, if $i' < i$, then $c_j = c_t \in \mathbf{N}$ for some $i' < t$. So, $c_j = c_t$ for some $i \leq t$.

On the other hand, we have $V_j = V_{j-1} + \langle v'_+ + v'_- \rangle$ for some nonzero $v'_- \in V_-$. Then

$$\begin{aligned}(j;\, i\,) &= \dim(\pi_+(V_j) + V_i) \\ &= \dim(\pi_+(V_{j-1} + \langle v'_+ + v'_- \rangle) + V_i) \\ &= \dim(\pi_+(V_{j-1}) + V_i + \langle v'_+ \rangle) \\ &= \dim(\pi_+(V_{j-1}) + V_i) \\ &= (j-1;\, i\,).\end{aligned}$$

Thus, $c_j \neq c_t$ for $i < t$. This means that $c_j = c_i$.

By induction, we have constructed a clan $\gamma = (c_1 \dots c_n)$ of x. □

Proposition 2.2.7. *Conversely, for any clan $\gamma \in \mathcal{C}(U(p,q))$, there exists a flag $x \in X$ such that γ is the clan of x.*

Proof. For a clan $\gamma = (c_1 \ \cdots \ c_n) \in \mathcal{C}(U(p,q))$, such a flag will be given inductively as follows. For $1 \leq i \leq n$ we will choose a nonzero $v_i \in V$ such that $V_i = \langle v_1, \dots, v_i \rangle$.

1. Let $V_0 = \{0\}$.
2. Suppose we have already obtained $(V_0, V_1, \dots, V_{i-1})$.
   (a) If $c_i = +$, then we set $v_i = e_s \in V_+$ for

$$s = \min\{\, l \mid 1 \leq l \leq p \ \text{ and } \ e_l \notin \pi_+(V_{i-1})\,\}. \tag{11}$$

   (b) If $c_i = -$, then we set $v_i = e_t \in V_-$ for

$$t = \min\{\, l \mid p < l \leq n \ \text{ and } \ e_l \notin \pi_-(V_{i-1})\,\}. \tag{12}$$

   (c) If $c_i = c_j \in \mathbf{N}$ for some $i < j$, then we set $v_i = e_s + e_t$ for an s satisfying (11) and a t satisfying (12).
   (d) If $c_i = c_j \in \mathbf{N}$ for $j < i$, then we set $v_i = \theta(v_j)$.
3. Let $V_i = V_{i-1} \oplus \langle v_i \rangle$.

We have constructed a flag $x = (V_0, V_1, \dots, V_n) \in X$ such that γ is a clan of x. □

For $\{v_1, \dots, v_n\}$ given in previous proof, let $g = (v_1 \ \cdots \ v_n) \in G$. Since all flags in X corresponding to the clan $\gamma \in \mathcal{C}(U(p,q))$ belong to the K-orbit $KgB = \{\, kgb \mid k \in K,\ b \in B \,\}$, we have the following theorem.

Theorem 2.2.8. *Clans in $\mathcal{C}(U(p,q))$ parametrize K-orbits in the flag variety X via the correspondence in Proposition 2.2.6.*

2.2-2. *Representatives of $GL(p, \mathbf{C}) \times GL(q, \mathbf{C})$-orbits in $GL(n, \mathbf{C})/B$.* For a clan γ, we will give an element $g \in G$ such that γ is a clan of the flag $x = gB$.

Definition 2.2.9 (Signed clan)**.** A *signed clan* of a clan $\gamma = (c_1 \ \cdots \ c_n)$ is an ordered set $(d_1 \dots d_n)$ of n symbols $+,\ -,\ a_+$ and a_- for some $a \in \mathbf{N}$ satisfying the following two conditions.

1. If $c_i = +$, then $d_i = +$. If $c_i = -$, then $d_i = -$.
2. If $c_i = c_j = a$ for some $a \in \mathbf{N}$, then $(d_i, d_j) = (a_+, a_-)$ or (a_-, a_+).

Example 2.2.10. There are four signed clans of $(\,1\,1\,2\,2\,)$:

$$(1_+\,1_-\,2_+\,2_-\,),\ (\,1_+\,1_-\,2_-\,2_+\,),\ (\,1_-\,1_+\,2_+\,2_-\,),\quad \text{and} \quad (\,1_-\,1_+\,2_-\,2_+\,).$$

Remark 2.2.11. As is seen above, there are several choices of signatures for a clan. We can always adapt the special choice such that $(d_i, d_j) = (a_+, a_-)$ for $i < j$, which causes no problem for $U(p,q)$. However, other choices of signature are appropriate for other classical groups, hence we admit an ambiguity of signatures.

Definition 2.2.12. For a signed clan $\delta = (d_1 \dots d_n)$, we say a signature of d_i is *plus* if $d_i = +$ or a_+, *minus* if $d_i = -$ or a_-.

Notation 2.2.13. For each clan γ, let Q_γ be a K-orbit in the flag variety X corresponding to γ via the parametrization of Theorem 2.2.8, i.e.,

$$Q_\gamma = KgB.$$

Although we already have representatives $g \in G$ of K-orbits for clans in the proof of Proposition 2.2.7, we give representatives in the following theorem.

Theorem 2.2.14. *For a clan $\gamma \in \mathcal{C}(U(p,q))$ and a signed clan $\delta = (d_1 \dots d_n)$ of γ, fix a permutation $\sigma \in \mathfrak{S}_n$ satisfying the following condition.*

$$\begin{array}{rcccl} 1 & \le & \sigma(i) & \le & p \qquad \textit{if the signature of } d_i \textit{ is plus}, \\ p+1 & \le & \sigma(i) & \le & n \qquad \textit{if the signature of } d_i \textit{ is minus}. \end{array} \tag{13}$$

Then the representative $g(\delta) := g(\delta, \sigma) = (g_1 \ \dots \ g_n)$ is a representative of Q_γ, i.e., $Q_\gamma = Kg(\delta)B$. Here $g_i \in V$ are the column vectors defined as follows:

- *If $d_i = \pm$, then $g_i = e_{\sigma(i)}$.*
- *If $d_i = a_+$, $d_j = a_-$, then*

$$g_i = \frac{1}{\sqrt{2}}(\,e_{\sigma(i)} + e_{\sigma(j)}\,) \quad \textit{and} \quad g_j = \frac{1}{\sqrt{2}}(\,-e_{\sigma(i)} + e_{\sigma(j)}\,).$$

Remark 2.2.15. The representative $g(\delta) \in GL(n, \mathbf{C})$ given in Theorem 2.2.14 is real and orthogonal:

$$g(\delta)^{-1} = {}^t g(\delta).$$

Example 2.2.16. For a signed clan $\delta = \begin{pmatrix} + & 1_+ & - & 2_+ & 1_- & 2_- \end{pmatrix}$, if we choose a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 2 & 6 & 4 \end{pmatrix},$$

we get a representative

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

Here $\sigma = \begin{pmatrix} 1 & 2 & \dots \\ i_1 & i_2 & \dots \end{pmatrix}$ means $\sigma(1) = i_1$, $\sigma(2) = i_2$, and so on.

2.3. **Dimensions of $GL(p, \mathbf{C}) \times GL(q, \mathbf{C})$-orbits in $GL(n, \mathbf{C})/B$.** In this section we will give a dimension formula of K-orbits for clans. For calculation of a dimension, we need some preparations.

Notation 2.3.1. Suppose $x = (V_1, \dots, V_n)$ be a flag. We take a complementary subspace $W_{i,+}$ of $\pi_+(V_i)$ in V_+ (resp. $W_{i,-}$ of $\pi_-(V_i)$ in V_-):

$$V_+ = \pi_+(V_i) \oplus W_{i,+} \quad (\text{resp. } V_- = \pi_-(V_i) \oplus W_{i,-}).$$

We also take a complementary subspace W_i' of V_i in $\pi_+(V_i) \oplus \pi_-(V_i)$:

$$\pi_+(V_i) \oplus \pi_-(V_i) = V_i \oplus W_i'. \tag{14}$$

We have

$$V = V_i \oplus W_{i,+} \oplus W_{i,-} \oplus W_i' \quad \text{and} \quad \dim W_i' = (i; \mathbf{N}).$$

Remark 2.3.2. If $v \in V_i$, then $\pi_+(v) \in \pi_+(V_i)$. If $v \in W_{i,+}$ and $v \neq 0$, then $\pi_+(v) = v \notin \pi_+(V_i)$. If $v \in W_i'$, then $\pi_+(v) \in \pi_+(V_i)$.

Proposition 2.3.3. *Let $(c_1 \ \cdots \ c_n)$ be the clan of a flag $x = (V_0, \cdots, V_n)$. If*

$$v = v' + w_+ + w_- + w',$$

$v' \in V_i$, $w_+ \in W_{i,+}$, $w_- \in W_{i,-}$, $w' \in W_i'$, and

$$V_{i+1} = V_i + \langle v \rangle,$$

then c_{i+1} is as follows:

1. *If $w_+ \neq 0$ and $w_- = 0$, then we have $c_{i+1} = +$.*
2. *If $w_+ = 0$ and $w_- \neq 0$, then we have $c_{i+1} = -$.*
3. *If $w_+ \neq 0$ and $w_- \neq 0$, then we have*
$$c_{i+1} = 1 + \max\{\, c_s \mid c_s \in \mathbf{N} \ \text{ and } \ 1 \leq s \leq i \,\}.$$
4. *If $w_+ = 0$ and $w_- = 0$, then we have a unique j such that*
$$v \in \pi_+(V_j) + V_i \quad \text{and} \quad v \notin \pi_+(V_{j-1}) + V_i.$$
Then we have $c_{i+1} = c_j \in \mathbf{N}$.

Proof. 1. If $w_+ \neq 0$ and $w_- = 0$, then

$$\begin{aligned}\pi_+(V_{i+1}) &= \pi_+(V_i) + \langle \pi_+(v') + w_+ + \pi_+(w') \rangle \\ &= \pi_+(V_i) \oplus \langle w_+ \rangle\end{aligned}$$

and $\pi_-(V_{i+1}) = \pi_-(V_i)$ by Remark 2.3.2. Then,

$$\dim \pi_+(V_{i+1}) = \dim \pi_+(V_i) + 1 \quad \text{and} \quad \dim \pi_-(V_{i+1}) = \dim \pi_-(V_i).$$

Thus,

$$\begin{aligned}\dim W'_{i+1} &= \dim \pi_+(V_{i+1}) + \dim \pi_-(V_{i+1}) - \dim V_{i+1} \\ &= \dim \pi_+(V_i) + 1 + \dim \pi_-(V_i) - (i+1) \\ &= \dim \pi_+(V_i) + \dim \pi_-(V_i) - \dim V_i \\ &= \dim W'_i\end{aligned}$$

by equation (14). So $(i+1;\mathbf{N}) = (i;\mathbf{N})$. On the other hand,

$$\begin{aligned}(i+1;+) &= \dim \pi_+(V_{i+1}) - (i+1;\mathbf{N}) \\ &= \dim \pi_+(V_i) + 1 - (i;\mathbf{N}) \;=\; (i;+) + 1\end{aligned}$$

by Remark 2.1.8. Therefore $c_{i+1} = +$.

2. If $w_+ = 0$ and $w_- \neq 0$, then $c_{i+1} = -$ by the same argument of 1.
3. If $w_+ \neq 0$ and $w_- \neq 0$, then

$$\dim \pi_+(V_{i+1}) = \dim \pi_+(V_i) + 1 \quad \text{and} \quad \dim \pi_-(V_{i+1}) = \dim \pi_-(V_i) + 1.$$

Thus, $(i+1;\mathbf{N}) = (i;\mathbf{N}) + 1$ by the same argument of 1. Therefore,

$$c_{i+1} = 1 + \max\{\, c_s \mid c_s \in \mathbf{N} \ \text{ and } \ 1 \leq s \leq i \,\}.$$

4. By the end of the proof of Proposition 2.2.6, it is clear for $w_+ = 0$ and $w_- = 0$.

□

Corollary 2.3.4. *For a flag* $x = (V_0, \dots, V_n)$ *and a clan* $(c_1 \ \cdots \ c_n)$ *of* x, *if* $c_i = c_j \in \mathbf{N}$ *for* $i < j$, *then there exist* $v_+ \in V_+$ *and* $v_- \in V_-$ *such that*

$$V_i = V_{i-1} + \langle v_+ + v_- \rangle \quad \textit{and} \quad V_j = V_{j-1} + \langle v_+,\ v_- \rangle.$$

Proof. By the proof of Proposition 2.3.3, there exist $v_i \in V$ such that

$$\pi_+(v_i) \in V_+ - \pi_+(V_{i-1}), \qquad \pi_-(v_i) \in V_- - \pi_-(V_{i-1}),$$

$$V_i = V_{i-1} + \langle v_i \rangle,$$

and $v_j \in V_{j-1} \oplus W'_{j-1}$ such that

$$v_j \in \pi_+(V_i) + V_{j-1} \quad \text{and} \quad v_j \notin \pi_+(V_{i-1}) + V_{j-1}.$$

Since

$$v_j \in \pi_+(V_i) + \pi_-(V_i) + V_{j-1} \quad \text{and} \quad v_j \notin \pi_+(V_{i-1}) + \pi_-(V_{i-1}) + V_{j-1},$$

there exist $v'_+ \in \pi_+(V_i)$, $v'_- \in \pi_-(V_i)$ and $v_{j-1} \in V_{j-1}$ such that

$$v_j = v'_+ + v'_- + v_{j-1}.$$

So, we have

$$V_j = V_{j-1} + \langle\, v'_+ + v'_- \,\rangle.$$

Since there exist $v_s \in V_{i-1}$ $(1 \leq s \leq i-1)$ with v_i span V_i;

$$V_i = \langle\, v_s \mid 1 \leq s \leq i \,\rangle,$$

We have

$$\pi_+(V_i) = \langle\, \pi_+(v_s) \mid 1 \le s \le i \,\rangle, \quad \text{and} \quad \pi_-(V_i) = \langle\, \pi_-(v_s) \mid 1 \le s \le i \,\rangle.$$

Thus, there exist a_s, $b_s \in \mathbf{C}$ for $1 \le s \le i$ such that

$$v_i' = \sum_{s=1}^{i} a_s \pi_+(v_s) \quad \text{and} \quad v_-' = \sum_{s=1}^{i} b_s \pi_-(v_s).$$

Here, $a_i \ne b_i$. We have

$$V_j = V_{j-1} + \langle \sum_{s=1}^{i} (a_s \pi_+(v_s) + b_s \pi_-(v_s)) \,\rangle.$$

By simple calculations, we have

$$\begin{aligned}
V_i &= V_{i-1} + \langle \sum_{s=1}^{i} (a_s - b_s) v_s \,\rangle \\
V_j &= V_{j-1} + \langle \sum_{s=1}^{i} (-a_s - b_s) v_s + 2 \sum_{s=1}^{i} (a_s \pi_+(v_s) + b_s \pi_-(v_s)) \,\rangle \\
&= V_{j-1} + \langle \sum_{s=1}^{i} (a_s - b_s)(\pi_+(v_s) - \pi_-(v_s)) \,\rangle \\
&= V_{j-1} + \langle \sum_{s=1}^{i} (a_s - b_s) \pi_+(v_s),\ \sum_{s=1}^{i} (a_s - b_s) \pi_-(v_s) \,\rangle
\end{aligned}$$

The two vectors

$$v_+ = \sum_{s=1}^{i} (a_s - b_s) \pi_+(v_s) \quad \text{and} \quad v_- = \sum_{s=1}^{i} (a_s - b_s) \pi_-(v_s)$$

satisfy the conditions of the corollary. □

Proposition 2.3.5. *For a clan $\gamma \in \mathcal{C}(U(p,q))$, we have the following equation.*

$$\begin{aligned}
&\sum_{c_i=+} (p - \#\{\, l \,|\, c_l = + \ \ \mathit{for} \ \ l \le i \,\} \\
&\qquad\qquad\qquad - \#\{\, a \in \mathbf{N} \,|\, c_s = c_t = a \ \ \mathit{for} \ \ s < t \le i \,\}) \\
&\quad + \sum_{c_i=-} (q - \#\{\, l \,|\, c_l = - \ \ \mathit{for} \ \ l \le i \,\} \\
&\qquad\qquad\qquad - \#\{\, a \in \mathbf{N} \,|\, c_s = c_t = a \ \ \mathit{for} \ \ s < t \le i \,\}) \\
&\quad + \sum_{\substack{c_i = c_j \in \mathbf{N} \\ j < i}} (n - \#\{l \,|\, c_l = + \ \mathit{or} \ - \ \mathit{for} \ l \le i\} \\
&\qquad\qquad\qquad - 2\#\{a \in \mathbf{N} \,|\, c_s = c_t = a \ \mathit{for} \ s < t \le i\}) \\
&= \frac{1}{2}(p(p-1) + q(q-1)).
\end{aligned} \tag{15}$$

Proof. The left hand side of the equation is

$$\begin{aligned}
&\sum_{c_i=+}(p-\#\{l\,|\,c_l=+\text{ for } l\leq i\}-\#\{a\in\mathbf{N}\,|\,c_s=c_t=a \ \text{ for } \ s<t\leq i\,\}) \\
&\ + \sum_{\substack{c_i=c_j\in\mathbf{N}\\ j<i}}(p-\#\{\,l\,|\,c_l=+ \ \text{ for } \ l\leq i\,\}-\#\{\,a\in\mathbf{N}\,|\,c_s=c_t=a \ \text{ for } \ s<t\leq i\,\}) \\
&\ + \sum_{c_i=-}(q-\#\{\,l\,|\,c_l=- \ \text{ for } \ l\leq i\,\}-\#\{\,a\in\mathbf{N}\,|\,c_s=c_t=a \ \text{ for } \ s<t\leq i\,\}) \\
&\ + \sum_{\substack{c_i=c_j\in\mathbf{N}\\ j<i}}(q-\#\{\,l\,|\,c_l=- \ \text{ for } \ l\leq i\,\}-\#\{\,a\in\mathbf{N}\,|\,c_s=c_t=a \ \text{ for } \ s<t\leq i\,\}) \\
&= \sum_{i=1}^{p}(p-i)+\sum_{i=1}^{q}(q-i)=\frac{1}{2}(p(p-1)+q(q-1)). \quad \square
\end{aligned}$$

Proposition 2.3.6. *For a clan $\gamma\in\mathcal{C}(U(p,q))$, we have the following equation.*

$$\begin{aligned}
&\sum_{\substack{c_i=c_j\in\mathbf{N}\\ i<j}}(j-i-\#\{\,\tilde{c}_t\in\mathbf{N}\,|\,i<t<j \ \ \textit{and} \ \ \tilde{c}_t<\tilde{c}_i\,\}) \\
&\qquad = \sum_{\substack{c_i=c_j\in\mathbf{N}\\ i<j}}(j-i-\#\{\,a\in\mathbf{N}\,|\,c_s=c_t=a \ \ \textit{for} \ \ s<i<t<j\,\}),
\end{aligned} \tag{16}$$

where, $(\tilde{c}_1\ \cdots\ \tilde{c}_n)$ is the standard indication.

Proof. A clan $(c_1\ \cdots\ c_n)$ and its standard indication $(\tilde{c}_1\ \cdots\ \tilde{c}_n)$ satisfy

$$\begin{aligned}
&\sum_{c_i=c_j\in\mathbf{N},\,i<j}\#\{\,\tilde{c}_t\in\mathbf{N}\,|\,i<t<j \ \ \text{and} \ \ \tilde{c}_t<\tilde{c}_i\,\} \\
&\qquad = \sum_{c_i=c_j\in\mathbf{N},\,i<j}\#\{\,a\in\mathbf{N}\,|\,c_s=c_t=a \ \ \text{for} \ \ s<i<t<j\,\}. \quad \square
\end{aligned}$$

Definition 2.3.7. We define a *length* $\ell(\gamma)$ of a clan $\gamma=(c_1\ \cdots\ c_n)\in\mathcal{C}(U(p,q))$ by the value in the previous proposition:

$$\begin{aligned}
\ell(\gamma) &:= \sum_{\substack{c_i=c_j\in\mathbf{N}\\ i<j}}(j-i-\#\{\,\tilde{c}_t\in\mathbf{N}\,|\,i<t<j \ \ \text{and} \ \ \tilde{c}_i>\tilde{c}_t\,\}) \\
&= \sum_{\substack{c_i=c_j\in\mathbf{N}\\ i<j}}(j-i-\#\{\,a\in\mathbf{N}\,|\,c_s=c_t=a \ \ \text{for} \ \ s<i<t<j\,\}),
\end{aligned}$$

where, $(\tilde{c}_1\ \cdots\ \tilde{c}_n)$ is the standard indication of γ.

Proposition 2.3.8. *For $\gamma\in\mathcal{C}(U(p,q))$, we have the dimension and the codimension of $Q_\gamma=Kg(\delta)B$:*

$$\begin{aligned}
\dim Q_\gamma &= \ell(\gamma)+\tfrac{1}{2}(p(p-1)+q(q-1)), \\
\operatorname{codim} Q_\gamma &= pq-\ell(\gamma).
\end{aligned}$$

Proof. By Proposition 2.3.3, flags $(V_1, \dots, V_n)$ in Q_γ satisfy the following conditions.

1. If $c_i = +$, then
$$V_i/V_{i-1} \subset (V_{i-1} \oplus W_{i-1,+} \oplus W'_{i-1})/V_{i-1}.$$
2. If $c_i = -$, then
$$V_i/V_{i-1} \subset (V_{i-1} \oplus W_{i-1,-} \oplus W'_{i-1})/V_{i-1}.$$
3. If $c_i = c_j \in \mathbf{N}$ for $i < j$, then
$$V_i/V_{i-1} \subset V/V_{i-1},$$
$$V_j/V_{j-1} \subset (V_{j-1} + \pi_+(V_{i-1}))/V_{j-1}.$$

The dimension of K-orbit Q_γ is the sum of the following four values.

$$\sum_{c_i=+} (\dim(V_{i-1} \oplus W_{i-1,+} \oplus W'_{i-1}) - \dim V_{i-1} - 1), \tag{17}$$

$$\sum_{c_i=-} (\dim(V_{i-1} \oplus W_{i-1,-} \oplus W'_{i-1}) - \dim V_{i-1} - 1), \tag{18}$$

$$\sum_{\substack{c_i=c_j\in\mathbf{N}\\ i<j}} (\dim V - \dim V_{i-1} - 1), \tag{19}$$

$$\sum_{\substack{c_i=c_j\in\mathbf{N}\\ i<j}} (\dim(V_{j-1} + \pi_+(V_{i-1})) - \dim V_{j-1} - 1): \tag{20}$$

$$\dim Q_\gamma = (17) + (18) + (19) + (20).$$

Four equations

$$(17) = \sum_{c_i=+} (p - \#\{l \mid c_l = + \text{ for } l \le i\} - \#\{a \in \mathbf{N} \mid c_s = c_t = a \text{ for } s < t \le i\}),$$

$$(18) = \sum_{c_i=-} (q - \#\{l \mid c_l = - \text{ for } l \le i\} - \#\{a \in \mathbf{N} \mid c_s = c_t = a \text{ for } s < t \le i\}),$$

$$(19) = \sum_{\substack{c_i=c_j\in\mathbf{N}\\ i<j}} (n - i),$$

and

$$(20) = \sum_{\substack{c_i=c_j\in\mathbf{N}\\ i<j}} \#\{\, a \in \mathbf{N} \,|\, c_s = c_t = a \ \text{ for } \ s < i < j < t \,\},$$

and Proposition 2.3.5, lead

$$\begin{aligned}(17) + (18) + \sum_{\substack{c_i=c_j\in\mathbf{N}\\ i<j}} (n - \#\{l \mid c_l = + \text{ or } - \text{ for } l \le j\}\\ - 2\#\{\, a \in \mathbf{N} \,|\, c_s = c_t = a \ \text{ for } s < t \le j \,\})\\ = \frac{1}{2}(p(p-1) + q(q-1))\end{aligned}$$

and

$$
\begin{aligned}
(19) + (20) - \sum_{\substack{c_i = c_j \in \mathbf{N} \\ i<j}} &(n - \#\{l \mid c_l = + \text{ or } - \text{ for } l \le j\} \\
&\qquad - 2\#\{\, a \in \mathbf{N} \,|\, c_s = c_t = a \ \text{ for } s < t \le j \,\}) \\
= \sum_{\substack{c_i = c_j \in \mathbf{N} \\ i<j}} &(-i + \#\{l \mid c_l = + \text{ or } - \text{ for } l \le j\} + \#\{l \mid c_l \in \mathbf{N} \text{ for } l \le j\} \\
&\qquad - \#\{s \mid c_s = c_t \in \mathbf{N} \text{ for } i < s < j < t\}) \\
= \sum_{\substack{c_i = c_j \in \mathbf{N} \\ i<j}} &(-i + \#\{l \,|\, l \le j\} - \#\{c_s \in \mathbf{N} \,|\, c_s = c_t \in \mathbf{N} \text{ for } i < s < j < t\}) \\
= \sum_{\substack{c_i = c_j \in \mathbf{N} \\ i<j}} &(-i + j - \#\{\, a \in \mathbf{N} \,|\, c_s = c_t = a \ \text{ for } \ i < s < j < t \,\}) \\
= \sum_{\substack{c_i = c_j \in \mathbf{N} \\ i<j}} &(-i + j - \#\{\, a \in \mathbf{N} \,|\, c_s = c_t = a \ \text{ for } \ s < i < t < j \,\}) \\
= \ell(\gamma).&
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\dim Q_\gamma &= (17) + (18) + (19) + (20) \\
&= \frac{1}{2}(p(p-1) + q(q-1)) + \ell(\gamma).
\end{aligned}
$$

Since $\dim(G/B) = \frac{1}{2}n(n-1)$, the codimension of Q_γ is

$$\operatorname{codim}(Q_\gamma) = pq - \ell(\gamma). \quad \square$$

2.4. **M-O graph of $U(p,q)$.** We define an oriented graph from which we can read off closure relations between K-orbits. The graph has clans as vertices in which an edge is labeled by i for some $1 \le i \le n-1$.

Definition 2.4.1. Let $\Gamma'(U(p,q))$ be the set of all triples (γ, γ', i), with $\gamma, \gamma' \in \mathcal{C}(U(p,q))$, $1 \le i \le n-1$, satisfying the following conditions. Let $\gamma = (c_1 \ \cdots \ c_n)$ and $\gamma' = (c'_1 \ \cdots \ c'_n)$.

1. $c_i \ne c_{i+1}$.
2. If $(c_i, c_{i+1}) = (+,-)$ or $(-,+)$, then $c'_i = c'_{i+1} \in \mathbf{N}$ and $c'_j = c_j$ otherwise:

$$\gamma' = \begin{pmatrix} c_1 & \cdots & c_{i-1} & a & a & c_{i+2} & \cdots & c_n \end{pmatrix}$$

   for $a = 1 + \max\{\, b \in \mathbf{N} \mid \ c_j = b \ \text{ for } \ 1 \le j \le n \,\}$.
3. If $(c_i, c_{i+1}) \ne (+,-)$ or $(-,+)$, then $(c'_i, c'_{i+1}) = (c_{i+1}, c_i)$ and $c'_j = c_j$ otherwise:

$$\gamma' = \begin{pmatrix} c_1 & \cdots & c_{i-1} & c_{i+1} & c_i & c_{i+2} & \cdots & c_n \end{pmatrix}.$$

Example 2.4.2.

$$\Gamma'(U(2,1)) = \left\{ \begin{array}{llllll} ((+ \; - \; +), & (1 \; 1 \; +), & 1), & ((1 \; 1 \; +), & (1 \; + \; 1), & 2) \\ ((+ \; - \; +), & (+ \; 1 \; 1), & 2), & ((1 \; + \; 1), & (1 \; 1 \; +), & 2) \\ ((+ \; + \; -), & (+ \; 1 \; 1), & 2), & ((+ \; 1 \; 1), & (1 \; + \; 1), & 1) \\ ((- \; + \; +), & (1 \; 1 \; +), & 1), & ((1 \; + \; 1), & (+ \; 1 \; 1), & 1) \end{array} \right\}.$$

Remark 2.4.3. If $(\gamma, \gamma', i) \in \Gamma'(U(p,q))$, $\gamma = (c_1 \; \cdots \; c_n)$ and $(c_i, c_{i+1}) \neq (+,-)$ or $(-,+)$, then (γ', γ, i) is also an element of $\Gamma'(U(p,q))$.

Proposition 2.4.4. *If* $(\gamma, \gamma', i) \in \Gamma'(U(p,q))$, *then*

$$\dim Q_{\gamma'} - \dim Q_\gamma = \pm 1.$$

Proof. We will prove $\ell(\gamma') - \ell(\gamma) = \pm 1$. Suppose $(\gamma, \gamma', i) \in \Gamma'(U(p,q))$ and $\gamma = (c_1 \dots c_n)$. Then the clans γ and γ' are

$$\gamma' = (c_1 \cdots c_{i-1} \; c'_i \; c'_{i+1} \; c_{i+2} \cdots c_n)$$

and

$$(c_i, c_{i+1}; c'_i, c'_{i+1}) = (\pm, \mp\,; a, a), (a, \pm\,; \pm, a), (\pm, a\,; a, \pm), \text{or } (a, b\,; b, a)$$

for some $a, b \in \mathbf{N}$. We can take $(c_1 \; \cdots \; c_n)$ to be the standard indication.

1. If $(c_i, c_{i+1}) = (\pm, \mp)$, then $c'_i = c'_{i+1} = a$ for some $a \in \mathbf{N}$. Then we have

$$\sum_{c'_i = c'_j \in \mathbf{N}, i<j} (j - i - \#\{\, a' \in \mathbf{N} \,|\, c'_s = c'_t = a' \;\text{ for }\; s < i < t < j \,\})$$
$$- \sum_{c_i = c_j \in \mathbf{N}, i<j} (j - i - \#\{\, a' \in \mathbf{N} \,|\, c_s = c_t = a' \;\text{ for }\; s < i < t < j \,\})$$
$$= (i+1) - i - \#\{\, a' \in \mathbf{N} \,|\, c'_s = c'_t = a' \;\text{ for }\; s < i < t < i+1 \,\} = 1.$$

   Thus, $\ell(\gamma') - \ell(\gamma) = 1$.

2. If $(c_i, c_{i+1}) = (a, \pm)$ or $(\pm, a)$ for some $a \in \mathbf{N}$, the second case follows the first case by Remark 2.4.3. We prove that for the case of $(c_i, c_{i+1}) = (a, \pm)$.

   If $(c_i, c_{i+1}) = (a, \pm)$, there exist $j \neq i, i+1$ such that $c_j = a$.

   (a) If $i + 1 < j$, then

$$\begin{aligned} &(j - (i+1) - \#\{\, a' \in \mathbf{N} \,|\, c'_s = c'_t = a' \;\text{ for }\; s < i+1 < t < j \,\}) \\ &-(j - i - \#\{\, a' \in \mathbf{N} \,|\, c_s = c_t = a' \;\text{ for }\; s < i < t < j \,\}) \\ &\qquad = (j - (i+1) - \#\{\, a' \in \mathbf{N} \,|\, c_s = c_t = a' \;\text{ for }\; s < i < t < j \,\}) \\ &\qquad\quad -(j - i - \#\{\, a' \in \mathbf{N} \,|\, c_s = c_t = a' \;\text{ for }\; s < i < t < j \,\}) \\ &\qquad = (j - (i+1)) - (j - i) = -1. \end{aligned}$$

   There is no difference in other terms. So, $\ell(\gamma') - \ell(\gamma) = -1$.

   (b) If $j < i$, then

$$\begin{aligned} &((i+1) - j - \#\{\, a' \in \mathbf{N} \,|\, c'_s = c'_t = a' \;\text{ for }\; s < j < t < i+1 \,\}) \\ &-(i - j - \#\{\, a' \in \mathbf{N} \,|\, c_s = c_t = a' \;\text{ for }\; s < j < t < i \,\}) \\ &\qquad = ((i+1) - j - \#\{\, a' \in \mathbf{N} \,|\, c_s = c_t = a' \;\text{ for }\; s < j < t < i \,\}) \\ &\qquad\quad -(i - j - \#\{\, a' \in \mathbf{N} \,|\, c_s = c_t = a' \;\text{ for }\; s < j < t < i \,\}) \\ &\qquad = ((i+1) - j) - (i - j) = 1. \end{aligned}$$

   There is no difference in other terms. Therefore, $\ell(\gamma') - \ell(\gamma) = 1$.

3. If $(c_i, c_{i+1}) = (a, b)$ for some $a, b \in \mathbf{N}$ there exist $j, j' \notin \{i, i+1\}$ such that $c_j = a$ and $c_{j'} = b$.
   (a) For $i+1 < j < j'$ or $i+1 < j' < j$, the second case follows the first case by Remark 2.4.3.
   If $i+1 < j < j'$, then we have
$$\begin{aligned}
&(j-(i+1)-\#\{\, a' \in \mathbf{N} \mid c'_s = c'_t = a', s < i+1 < t < j \,\} \\
&\quad +j'-i-\#\{\, b' \in \mathbf{N} \mid c'_s = c'_t = b', s < i < t < j' \,\}) \\
&\qquad -(j-i-\#\{\, a' \in \mathbf{N} \mid c_s = c_t = a', s < i < t < j \,\} \\
&\qquad\quad +j'-(i+1)-\#\{\, b' \in \mathbf{N} \mid c_s = c_t = b', s < i+1 < t < j' \,\}) \\
&= (j-(i+1)-\#\{\, a' \in \mathbf{N} \mid c_s = c_t = a', s < i < t < j \,\} \\
&\quad +j'-i-\#\{\, b' \in \mathbf{N} \mid c_s = c_t = b', s < i+1 < t < j' \,\} - \#\{\, a \,\}) \\
&\qquad -(j-i-\#\{\, a' \in \mathbf{N} \mid c_s = c_t = a', s < i < t < j \,\} \\
&\qquad\quad +j'-(i+1)-\#\{\, b' \in \mathbf{N} \mid c_s = c_t = b', s < i+1 < t < j' \,\}) \\
&= (j-(i+1)+j'-i-(-1)) - (j-i+j'-(i+1)) = 1.
\end{aligned}$$
   There is no difference in other terms. Hence, $\ell(\gamma') - \ell(\gamma) = 1$.
   (b) For $j < i < i+1 < j'$ or $j' < i < i+1 < j$, the second case follows the first case by Remark 2.4.3.
   If $j < i < i+1 < j'$, then
$$\begin{aligned}
&((i+1)-j-\#\{\, a' \in \mathbf{N} \mid c'_s = c'_t = a', s < j < t < i+1 \,\} \\
&\quad +j'-i-\#\{\, b' \in \mathbf{N} \mid c'_s = c'_t = b', s < i < t < j' \,\}) \\
&\qquad -(i-j-\#\{\, a' \in \mathbf{N} \mid c_s = c_t = a', s < j < t < i \,\} \\
&\qquad\quad +j'-(i+1)-\#\{\, b' \in \mathbf{N} \mid c_s = c_t = b', s < i+1 < t < j' \,\}) \\
&= ((i+1)-j-\#\{\, a' \in \mathbf{N} \mid c_s = c_t = a', s < j < t < i \,\} \\
&\quad +j'-i-(\#\{\, b' \in \mathbf{N} \mid c_s = c_t = b', s < i+1 < t < j' \,\} + \#\{\, a \,\})) \\
&\qquad -(i-j-\#\{\, a' \in \mathbf{N} \mid c_s = c_t = a', s < j < t < i \,\} \\
&\qquad\quad +j'-(i+1)-\#\{\, b' \in \mathbf{N} \mid c_s = c_t = b', s < i+1 < t < j' \,\}) \\
&= ((i+1)-j+j'-i-1) - (i-j+j'-(i+1)) = 1.
\end{aligned}$$
   There is no difference in other terms. Thus, $\ell(\gamma') - \ell(\gamma) = 1$.
   (c) For $j < j' < i$ or $j' < j < i$, the second case follows the first case by Remark 2.4.3.
   If $j < j' < i$, then
$$\begin{aligned}
&((i+1)-j-\#\{\, a' \in \mathbf{N} \mid c'_s = c'_t = a', s < j < t < i+1 \,\} \\
&\quad +i-j'-\#\{\, b' \in \mathbf{N} \mid c'_s = c'_t = b', s < j' < t < i \,\}) \\
&\qquad -(i-j-\#\{\, a' \in \mathbf{N} \mid c_s = c_t = a', s < j < t < i \,\} \\
&\qquad\quad +(i+1)-j'-\#\{\, b' \in \mathbf{N} \mid c_s = c_t = b', s < j' < t < i+1 \,\}) \\
&= ((i+1)-j-\#\{\, a' \in \mathbf{N} \mid c_s = c_t = a', s < j < t < i \,\} \\
&\quad +i-j'-(\#\{\, b' \in \mathbf{N} \mid c_s = c_t = b', s < j' < t < i+1 \,\} - \#\{\, a \,\})) \\
&\qquad -(i-j-\#\{\, a' \in \mathbf{N} \mid c_s = c_t = a', s < j < t < i \,\} \\
&\qquad\quad +(i+1)-j'-\#\{\, b' \in \mathbf{N} \mid c_s = c_t = b', s < j' < t < i+1 \,\}) \\
&= ((i+1)-j+i-j'-(-1)) - (i-j+(i+1)-j') = 1.
\end{aligned}$$

Therefore we have $\ell(\gamma') - \ell(\gamma) = \pm 1$ for all $(\gamma, \gamma', i) \in \Gamma'(U(p,q))$. □

We define subsets of $\mathcal{C}(U(p,q))$ of which, elements have the same length.

Definition 2.4.5. Let $\mathcal{C}_m(U(p,q))$ be the set of clans of the length m:

$$\mathcal{C}_m(U(p,q)) := \{\, \gamma \in \mathcal{C}(U(p,q)) \,|\, \ell(\gamma) = m \,\}.$$

Then we have

$$\mathcal{C}(U(p,q)) = \bigsqcup_{m \geq 0} \mathcal{C}_m(U(p,q)). \tag{21}$$

Remark 2.4.6. If $\gamma \in \mathcal{C}_0(U(p,q))$, i.e., $\ell(\gamma) = 0$, then Q_γ is a closed orbit.

Proposition 2.4.7. *If* $(c_1 \dots c_n) \in \mathcal{C}_0(U(p,q))$*, then* $c_i = +$ *or* $-$ *for all* i*, i.e.,*

$$\mathcal{C}_0(U(p,q)) = \{\, (c_1 \dots c_n) \in \mathcal{C}(U(p,q)) \,|\, c_i \notin \mathbf{N} \text{ for all } 1 \leq i \leq n \,\}.$$

Proof. For $\gamma = (c_1 \dots c_n) \in \mathcal{C}(U(p,q))$, if $c_i = c_j \in \mathbf{N}$ for some $i < j$,

$$j - i - \#\{\, a \in \mathbf{N} \,|\, c_s = c_t = a,\ s < i < t < j \,\} > 0.$$

Therefore if $c_i = c_j \in \mathbf{N}$, then $\ell(\gamma) > 0$, and if $c_i \notin \mathbf{N}$ for all $1 \leq i \leq n$, then $\ell(\gamma) = 0$. □

Notation 2.4.8. We denote by B_i the parabolic subgroup of G for the root $-\alpha_i$ and all positive roots:

$$B_i = \{X \in GL(n, \mathbf{C}) \mid X_{st} = 0 \text{ if } s > t \text{ and } (s,t) \neq (i+1, i)\}.$$

Let

$$\pi_i : G/B \to G/B_i$$

be the canonical projection.

Remark 2.4.9. The projection π_i sends $(V_0, \dots, V_n) \in G/B$ to

$$(V_0, \dots, V_{i-1}, V_{i+1}, \dots, V_n).$$

For $(\gamma, \gamma', i) \in \Gamma'(U(p,q))$, there are two flags

$$(V_0, \dots, V_{i-1}, V_i, V_{i+1}, \dots, V_n)$$

having the clan γ and

$$(V_0, \dots, V_{i-1}, V_i', V_{i+1}, \dots, V_n)$$

having the clan γ', by Theorem 2.2.14. Therefore we have the following proposition.

Proposition 2.4.10. *If* $(\gamma, \gamma', i) \in \Gamma'(U(p,q))$*, then*

$$\pi_i(Q_\gamma) = \pi_i(Q_\gamma').$$

By following [6], we introduce an oriented graph.

Definition 2.4.11 (M-O graph of $U(p,q)$)**.** We give subsets $\Gamma_m(U(p,q))$ of $\Gamma'(U(p,q))$ for $m \geq 1$ as follows:

$$\Gamma_m(U(p,q)) := \{\, (\gamma, \gamma', i) \in \Gamma'(U(p,q)) \mid \gamma \in \mathcal{C}_{m-1}(U(p,q)), \gamma' \in \mathcal{C}_m(U(p,q)) \,\}.$$

The *M-O graph* of $U(p,q)$ is a finite oriented graph whose vertices are $\mathcal{C}(U(p,q))$ and whose oriented edges are $\Gamma(U(p,q)) := \bigcup_{m \in \mathbf{N}} \Gamma_m(U(p,q))$.

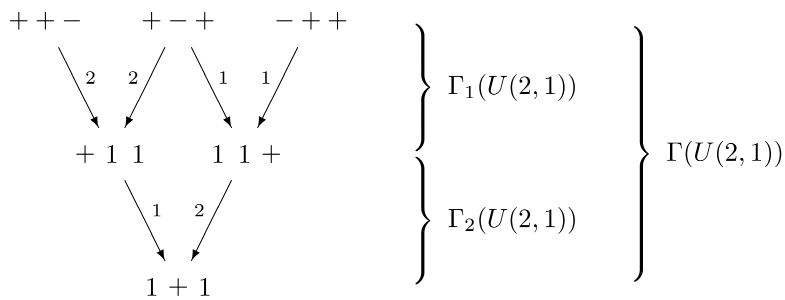

FIGURE 1

Example 2.4.12.

$$\Gamma_1(U(2,1)) = \left\{ \begin{array}{ll} ((-++),(11+),\quad 1), & ((+-+),(11+),\quad 1), \\ ((++-),(+11),\quad 2), & ((+-+),(+11),\quad 2) \end{array} \right\}$$
$$\Gamma_2(U(2,1)) = \left\{ \; ((11+),(1+1),\quad 2),\quad ((+11),(1+1),\quad 1) \; \right\}$$

$$\Gamma(U(2,1)) = \Gamma_1(U(2,1)) \cup \Gamma_2(U(2,1)).$$

If we denote (γ, γ', i) by $\gamma \xrightarrow{i} \gamma'$, the M-O graph of $U(2,1)$ is as in Figure 1.

Remark 2.4.13. If $(\gamma, \gamma', i) \in \Gamma(U(p,q))$, then

$$\dim Q_\gamma + 1 = \dim Q_{\gamma'} \quad \text{and} \quad \pi_i(Q_\gamma) = \pi_i(Q_{\gamma'}). \tag{22}$$

2.5. **Images of the moment map.** In this section we give the image of the moment map of a fiber of the conormal bundle of the K-orbit for each clan.

For each point $x \in X$ the morphism $G \to X$ sending $g \in G$ to gx gives rise to a linear map $\mathfrak{g} \to T_x(X)$, the tangent-map at x, and to a dual map $T_x^*(X) \to \mathfrak{g}^*$ from the cotangent bundle into the dual of the Lie algebra, the cotangent-map for $x \in X$ at x. The collection of tangent-maps $\mathfrak{g} \to T_x(X)$ at the various points $x \in X$ gives rise to an algebraic map $\mathfrak{g} \times X \to T(X)$, compatible with the projections to the base space X. Composing the cotangent-map with the map forgetting the base point in X, we obtain a canonical map $T^*(X) \to \mathfrak{g}^*$ from the cotangent-bundle into the dual of the Lie algebra, denoted by μ. We call it the *moment map* of the G-space X. (for example [1].)

Let φ be the morphism $G \to X$ sending $g \in G$ to gx for $x \in X$. Then we obtain a linear map $d\varphi : \mathfrak{g} = T_e G \to T_x(X)$ and a dual map $(d\varphi)^* : T_x^*(X) \to T_e^* G = \mathfrak{g}^*$.

Let G_x be the stabilizer of x in G and $\mathfrak{g}_x$ its Lie algebra. Since the kernel of $d\varphi$ is $\mathfrak{g}_x$, the image of $(d\varphi)^*$ is $(\mathfrak{g}_x)^\perp$. Here

$$(\mathfrak{g}_x)^\perp := \{ \, \xi \in \mathfrak{g}^* \mid \langle \xi, a \rangle = 0 \;\; \text{for all} \;\; a \in \mathfrak{g}_x \, \} \tag{23}$$

and $\langle \cdot , \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbf{C}$.

Let ψ be the morphism $K \to X$ sending $k \in K$ to kx. Then we have $\psi = \varphi|_K$ and $d\psi = d\varphi|_{\mathfrak{k}}$. The image of ψ is the K-orbit Q of x in X: $Q = \{kx \mid k \in K\}$.

The fiber $(T^*_Q X)_x$ of conormal bundle $T^*_Q X$ at x is

$$
\begin{aligned}
(T^*_Q X)_x &= \{\, \eta \in (T^*X)_x \mid \langle \eta, u\rangle = 0 \ \text{ for all } \ u \in T_x Q \,\} \\
&= \{\, \eta \in (T^*X)_x \mid \langle \eta, d\psi(a)\rangle = 0 \ \text{ for all } \ a \in \mathfrak{k} \,\} \\
&= \{\, \eta \in (T^*X)_x \mid \langle \eta, d\varphi(a)\rangle = 0 \ \text{ for all } \ a \in \mathfrak{k} \,\} \\
&= \{\, \eta \in (T^*X)_x \mid \langle (d\varphi)^*(\eta), a\rangle = 0 \ \text{ for all } \ a \in \mathfrak{k} \,\} \\
&= \{\, \eta \in T^*_Q X \mid (d\varphi)^*(\eta) \in \mathfrak{k}^\perp \,\}.
\end{aligned} \tag{24}
$$

By (23) and (24) we have

$$(d\varphi)^*((T^*_Q X)_x) = (\mathfrak{g}_x)^\perp \cap \mathfrak{k}^\perp.$$

Hence, the image of the moment map $\mu((T^*_Q X)_x)$ is $(\mathfrak{g}_x)^\perp \cap \mathfrak{k}^\perp$, where

$$
\begin{aligned}
(\mathfrak{g}_x)^\perp &= \{\, \xi \in \mathfrak{g}^* \mid \langle \xi, a\rangle = 0 \ \text{ for all } \ a \in \mathfrak{g}_x \,\}, \\
\mathfrak{k}^\perp &= \{\, \xi \in \mathfrak{g}^* \mid \langle \xi, a\rangle = 0 \ \text{ for all } \ a \in \mathfrak{k} \,\}.
\end{aligned}
$$

For a representative $g' \in G$ with $x = g'B$, we have $G_x = g' \cdot B \cdot g'^{-1}$ and $\mathfrak{g}_x = \mathrm{Ad}(g')\mathfrak{b}$. Thus, we have

$$(\mathfrak{g}_x)^\perp \cap \mathfrak{k}^\perp = (\mathrm{Ad}^*(g')\mathfrak{b})^\perp \cap \mathfrak{k}^\perp.$$

By identifying $\mathfrak{g}^*$ with the dual of $\mathfrak{g}$ by means of a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$, the image of the moment map is identified with the vector subspace $(\mathrm{Ad}({}^t g'^{-1})\mathfrak{b}^\perp) \cap \mathfrak{p}$ of $\mathfrak{g}$. Here $\mathfrak{b}^\perp$ is a subalgebra of $\mathfrak{g}$ that is orthogonal to $\mathfrak{b}$:

$$
\begin{aligned}
\mathfrak{b}^\perp &= \{\, b \in \mathfrak{g} \mid \beta(b, b') = 0 \ \text{ for all } \ b' \in \mathfrak{b} \,\}, \\
&= \{\, b \in \mathfrak{g} \mid b_{ij} = 0 \ \text{ if } \ i \le j \,\}.
\end{aligned}
$$

where $\beta(x, y)$ is the trace of ${}^t xy$ on $\mathfrak{g}$:

$$\beta(x, y) = \mathrm{tr}\, {}^t xy \ \text{ for } \ x, y \in \mathfrak{g}.$$

For a clan $\gamma = (c_1 \ldots c_n) \in \mathcal{C}(U(p,q))$, fix a signed clan δ of γ. Let $g(\delta)$ be a representative given as in Theorem 2.2.14. Then ${}^t g(\delta)^{-1} = g(\sigma)$. We will describe the image of the moment map $\mu((T^*_{Q_\gamma} X)_x) = (\mathrm{Ad}(g(\delta))\mathfrak{b}^\perp) \cap \mathfrak{p}$ for $x = g(\delta)B$.

We regard $\mathfrak{S}_p \times \mathfrak{S}_q$ as a subgroup of $\mathfrak{S}_n$ as follows:

$$
\begin{aligned}
&\mathfrak{S}_p \times \mathfrak{S}_q \subset \mathfrak{S}_n \\
&(\sigma_1, \sigma_2) = \sigma' \ ; \ \sigma'(i) = \begin{cases} \sigma_1(i) & \text{if } 1 \le i \le p, \\ \sigma_2(i-p) + p & \text{if } p+1 \le i \le n. \end{cases}
\end{aligned}
$$

We regard $\sigma \in \mathfrak{S}_n$ as an element of $GL(n, \mathbf{C})$ such that $\sigma(e_i) = e_{\sigma(i)}$ for all $1 \le i \le n$, i.e.,

$$\sigma = \begin{pmatrix} e_{\sigma(1)} & \cdots & e_{\sigma(n)} \end{pmatrix} \in GL(n, \mathbf{C}).$$

Then $\mathfrak{S}_p \times \mathfrak{S}_q$ is a subgroup of K. The theorem following the next definition gives the image of the moment map.

Definition 2.5.1 (Driving space for $U(p,q)$)**.** For a signed clan $\delta = (d_1 \ldots d_n)$ of a clan γ, we give the following vector subspace $\mathrm{Dri}(\delta)$ of $\mathfrak{g}$ and call it *driving space* of δ (for $U(p,q)$).

$$\mathrm{Dri}(\delta) := \{\, Y \in \mathfrak{g} \mid Y \text{ satisfies the following conditions } \}.$$

1. If d_i and d_j have the same signature, then $Y_{ij} = Y_{ji} = 0$.

2. If $c_i = c_j \in \mathbf{N}$, then $Y_{ij} = Y_{ji} = 0$.
3. Let $d_i = a_+$, $d_j = a_-$, $d_s = b_+$, $d_t = b_-$, $d_l = +$, $d_m = -$.
   (a) $Y_{\min(l,m),\max(l,m)} = 0$.
   (b) If $\min(i,j) < l < \max(i,j)$, then $Y_{jl} = Y_{lj} = 0$.
   (c) If $\min(i,j) < m < \max(i,j)$, then $Y_{im} = Y_{mi} = 0$.
   (d) If $l < \min(i,j)$, then $Y_{lj} = 0$.
   (e) If $m < \min(i,j)$, then $Y_{mi} = 0$.
   (f) If $\max(i,j) < l$, then $Y_{jl} = 0$.
   (g) If $\max(i,j) < m$, then $Y_{im} = 0$.
   (h) If $\max(i,j) < \min(s,t)$, then $Y_{it} = Y_{js} = 0$.
   (i) If $\min(i,j) < \min(s,t) < \max(i,j) < \max(s,t)$, then
      - $Y_{it} = Y_{js} = 0$, $Y_{ti} = Y_{sj}$ if the signature of $d_{\min(i,j)}$ is equal to the signature of $d_{\min(s,t)}$.
      - $Y_{it} = Y_{js} = 0$, $Y_{ti} + Y_{sj} = 0$ if the signature of $d_{\min(i,j)}$ is not equal to the signature of $d_{\min(s,t)}$.
   (j) If $\min(i,j) < \min(s,t) < \max(s,t) < \max(i,j)$, then $Y_{it} = Y_{ti} = Y_{js} = Y_{sj} = 0$.

Proposition 2.5.2. *For a clan $\gamma = (c_1 \ \cdots \ c_n) \in \mathcal{C}(U(p,q))$, fix a signed clan $\delta = (d_1 \ \cdots \ d_n)$ of γ. Let the representative $g := g(\delta,\sigma)$ be given as in Theorem 2.2.14 and $x = gB \in Q_\gamma$. We can read off $\mu(T^*_{Q_\gamma}X)_x$ from $\mathrm{Dri}(\delta)$ as follows:*

$$\mu\left((T^*_{Q_\gamma}X)_x\right) \quad = \quad \{\, Y_{(\sigma^{-1}(1),\dots,\sigma^{-1}(n))} \mid Y \in \mathrm{Dri}(\delta)\}.$$

Proposition 2.5.2 means

$$\begin{aligned}\mu\left((T^*_{Q_\gamma}X)_x\right) = (\mathrm{Ad}(g)\mathfrak{b}^\perp) \cap \mathfrak{p} \quad &= \quad \{\, Y_{(\sigma^{-1}(1),\dots,\sigma^{-1}(n))} \mid Y \in \mathrm{Dri}(\delta)\} \\ &= \quad \{\sigma Y \sigma^{-1} \mid Y \in \mathrm{Dri}(\delta)\} \\ &= \quad \mathrm{Ad}(\sigma)\,\mathrm{Dri}(\delta).\end{aligned}$$

Corollary 2.5.3. *For a representative $g = g(\delta,\sigma)$ of Q_γ given in Theorem 2.2.14 and $x = gB$, the driving space is the following space.*

$$\begin{aligned}\mathrm{Dri}(\delta) \quad &= \quad \mathrm{Ad}(\sigma^{-1})\mu\left((T^*_{Q_\gamma}X)_x\right) \\ &= \quad \{\, A_{(\sigma(1),\dots,\sigma(n))} \mid A \in \mu\left((T^*_{Q_\gamma}X)_x\right)\}.\end{aligned}$$

Proof of Proposition 2.5.2. For two representatives $g(\delta,\sigma)$ and $g(\delta,\sigma')$ there exists $\sigma'' \in \mathfrak{S}_p \times \mathfrak{S}_q \subset \mathfrak{S}_n$ such that $\sigma' = \sigma''\sigma$. Then

$$g(\delta,\sigma') = \sigma'' g(\delta,\sigma).$$

Suppose Proposition 2.5.2 is true for $g(\delta,\sigma)$, i.e.,

$$(\mathrm{Ad}(g(\delta,\sigma))\mathfrak{b}^\perp) \cap \mathfrak{p} = \mathrm{Ad}(\sigma)\,\mathrm{Dri}(\delta).$$

Then

$$\begin{aligned}(\mathrm{Ad}(g(\delta,\sigma')) \cdot \mathfrak{b}^\perp) \cap \mathfrak{p} &= (\mathrm{Ad}(\sigma'' \cdot g(\delta,\sigma)) \cdot \mathfrak{b}^\perp) \cap \mathfrak{p} \\ &= \mathrm{Ad}(\sigma'')((\mathrm{Ad}(g(\delta,\sigma)) \cdot \mathfrak{b}^\perp) \cap \mathfrak{p}) \\ &= \mathrm{Ad}(\sigma'')\,\mathrm{Ad}(\sigma)\,\mathrm{Dri}(\delta) \\ &= \mathrm{Ad}(\sigma''\sigma)\,\mathrm{Dri}(\delta) \\ &= \mathrm{Ad}(\sigma')\,\mathrm{Dri}(\delta).\end{aligned}$$

Thus, Proposition 2.5.2 is also true for $g(\delta, \sigma')$. So, we prove Proposition 2.5.2 for one case.

We calculate each element of $g\mathfrak{b}^{\perp}g^{-1} \cap \mathfrak{p}$.

Let $b \in \mathfrak{b}^{\perp}$. We have

$$(gbg^{-1})_{st} = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{si}\, b_{ij}\, g_{tj} = \sum_{i,j \in \{1,\dots,n\}, i>j} g_{si}\, b_{ij}\, g_{tj}. \tag{25}$$

1. If $(d_i, d_j) = (+,+)$ or $(-,-)$ for $i < j$, then we may assume $\sigma(i) < \sigma(j)$. Since
$$\{\, A'_{(\sigma(i),\sigma(j))} \mid A' \in \mathfrak{p} \,\} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\},$$
we have
$$\begin{aligned}\{\, A_{(\sigma(i),\sigma(j))} \mid A \in (\mathrm{Ad}(g)\mathfrak{b}^{\perp}) \cap \mathfrak{p} \,\} &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \\ &= \{ Y_{(i,j)} \mid Y \in \mathrm{Dri}(\delta) \}.\end{aligned}$$

2. If $(d_i, d_j) = (a_+, a_-)$, then $\sigma(i) \le p < \sigma(j)$. Let $i' = \min(i,j)$ and $j' = \max(i,j)$. We have
$$\begin{aligned}\{\, (\mathrm{Ad}(g)b)_{(\sigma(i),\sigma(j))} \mid b \in \mathfrak{b}^{\perp} \,\} &= \left\{ g' \begin{pmatrix} 0 & 0 \\ b_{j'i'} & 0 \end{pmatrix} {}^{t}g' \right\} \\ &= \left\{ \begin{pmatrix} y & \pm y \\ \mp y & -y \end{pmatrix} \right\}\end{aligned}$$
and
$$\{\, A'_{(\sigma(i),\sigma(j))} \mid A' \in \mathfrak{p} \,\} = \left\{ \begin{pmatrix} 0 & z_{\sigma(i)\sigma(j)} \\ z_{\sigma(j)\sigma(i)} & 0 \end{pmatrix} \right\},$$
where $g' = \tilde{g}$ or $k\tilde{g}$ for
$$\tilde{g} := \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad k := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in GL(1,\mathbf{C}) \times GL(1,\mathbf{C}) \hookrightarrow K.$$
So,
$$\begin{aligned}\{\, A_{(\sigma(i),\sigma(j))} \mid A \in (\mathrm{Ad}(g)\mathfrak{b}^{\perp}) \cap \mathfrak{p} \,\} &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \\ &= \{ Y_{(i,j)} \mid Y \in \mathrm{Dri}(\delta) \}.\end{aligned}$$

3. If $(d_l, d_m) = (+,-)$ for $l < m$, then $\sigma(l) \le p < \sigma(m)$. We have
$$\{\, (\mathrm{Ad}(g)b)_{(\sigma(l),\sigma(m))} \mid b \in \mathfrak{b}^{\perp} \,\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b_{ml} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$
and
$$\{\, A'_{(\sigma(l),\sigma(m))} \mid A' \in \mathfrak{p} \,\} = \left\{ \begin{pmatrix} 0 & z_{\sigma(l)\sigma(m)} \\ z_{\sigma(m)\sigma(l)} & 0 \end{pmatrix} \right\}.$$
Therefore,
$$\begin{aligned}\{\, A_{(\sigma(l),\sigma(m))} \mid A \in (\mathrm{Ad}(g)\mathfrak{b}^{\perp}) \cap \mathfrak{p} \,\} &= \left\{ \begin{pmatrix} 0 & 0 \\ y_{ml} & 0 \end{pmatrix} \right\} \\ &= \{\, Y_{(l,m)} \mid Y \in \mathrm{Dri}(\delta) \,\}.\end{aligned}$$

4. If $(d_m, d_l) = (-,+)$ for $m < l$, then $\sigma(l) \le p < \sigma(m)$. By the same argument of 3, we have

$$\begin{aligned}\{ A_{(\sigma(l),\sigma(m))} \mid A \in (\mathrm{Ad}(g)\mathfrak{b}^{\perp}) \cap \mathfrak{p} \} \\ &= \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b_{lm} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \cap \left\{ \begin{pmatrix} 0 & z_{\sigma(l)\sigma(m)} \\ z_{\sigma(m)\sigma(l)} & 0 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} 0 & y_{lm} \\ 0 & 0 \end{pmatrix} \right\} = \{ Y_{(l,m)} \mid Y \in \mathrm{Dri}(\delta) \}.\end{aligned}$$

5. If $(d_i, d_j) = (a_+, a_-)$ and $d_l = +$ for $\min(i,j) < l < \max(i,j)$, then we may assume $\sigma(i) < \sigma(l) \le p < \sigma(j)$. Let $i', j', l' \in \mathbf{N}$ satisfy

$$\{i', j', l'\} = \{i, j, l\} \quad \text{and} \quad i' < j' < l'. \tag{26}$$

We have

$$\{ (gbg^{-1})_{(\sigma(i),\sigma(l),\sigma(j))} \mid b \in \mathfrak{b}^{\perp} \} = \left\{ g' \begin{pmatrix} 0 & 0 & 0 \\ b_{j'i'} & 0 & 0 \\ b_{l'i'} & b_{l'j'} & 0 \end{pmatrix} {}^{t}g' \right\} \tag{27}$$

and

$$\{ A'_{(\sigma(i),\sigma(l),\sigma(j))} \mid A' \in \mathfrak{p} \} = \left\{ \begin{pmatrix} 0 & 0 & z_{\sigma(i)\sigma(j)} \\ 0 & 0 & z_{\sigma(l)\sigma(j)} \\ z_{\sigma(j)\sigma(i)} & z_{\sigma(j)\sigma(l)} & 0 \end{pmatrix} \right\} \tag{28}$$

for $g' = \tilde{g}$ or $k\tilde{g}$. Here,

$$\tilde{g} := \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and

$$k := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in GL(2, \mathbf{C}) \times GL(1, \mathbf{C}) \hookrightarrow K. \tag{29}$$

Therefore,

$$\begin{aligned}\{A_{(\sigma(i),\sigma(l),\sigma(j))} \mid A \in (\mathrm{Ad}(g)\mathfrak{b}^{\perp}) \cap \mathfrak{p} \} &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \\ &= \{Y_{(i,l,j)} \mid Y \in \mathrm{Dri}(\delta)\}.\end{aligned}$$

6. If $(d_i, d_j) = (a_+, a_-)$ and $d_m = -$ for $\min(i,j) < m < \max(i,j)$, then we may assume $\sigma(i) \le p < \sigma(m) < \sigma(j)$. Let $i', j', m' \in \mathbf{N}$ satisfy

$$\{ i', j', m' \} = \{ i, j, m \} \quad \text{and} \quad i' < j' < m'. \tag{30}$$

We have

$$\{ \left(gbg^{-1}\right)_{(\sigma(i),\sigma(m),\sigma(j))} \mid b \in \mathfrak{b}^{\perp} \} = \left\{ g' \begin{pmatrix} 0 & 0 & 0 \\ b_{j'i'} & 0 & 0 \\ b_{m'i'} & b_{m'j'} & 0 \end{pmatrix} {}^{t}g' \right\} \tag{31}$$

and

$$\{ A'_{(\sigma(i),\sigma(m),\sigma(j))} \mid A' \in \mathfrak{p} \} = \left\{ \begin{pmatrix} 0 & z_{\sigma(i)\sigma(m)} & z_{\sigma(i)\sigma(j)} \\ z_{\sigma(m)\sigma(i)} & 0 & 0 \\ z_{\sigma(j)\sigma(i)} & 0 & 0 \end{pmatrix} \right\} \tag{32}$$

for $g' = \tilde{g}$ or $k\tilde{g}$. Here,

$$\tilde{g} := \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and

$$k := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in GL(1, \mathbf{C}) \times GL(2, \mathbf{C}) \hookrightarrow K. \tag{33}$$

Hence,

$$\{A_{(\sigma(i),\sigma(m),\sigma(j))} \mid A \in (\mathrm{Ad}(g)\mathfrak{b}^{\perp}) \cap \mathfrak{p}\} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$
$$= \{Y_{(i,m,j)} \mid Y \in \mathrm{Dri}(\delta)\}.$$

7. If $(d_i, d_j) = (a_+, a_-)$ and $d_l = +$ for $l < \min(i, j)$, then we may assume $\sigma(i) < \sigma(l) \leq p < \sigma(j)$. Let $i', j', l' \in \mathbf{N}$ satisfy (26). We have (27) and (28), where $g' = \tilde{g}$ or $k\tilde{g}$ for

$$\tilde{g} := \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and k is as in (29). Hence,

$$\{ A_{(\sigma(i),\sigma(l),\sigma(j))} \mid (\mathrm{Ad}(g)\mathfrak{b}^{\perp}) \cap \mathfrak{p} \} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & y_{jl} & 0 \end{pmatrix} \right\}$$
$$= \{ Y_{(i,l,j)} \mid Y \in \mathrm{Dri}(\delta) \}.$$

8. If $(d_i, d_j) = (a_+, a_-)$ and $d_m = -$ for $m < \min(i, j)$, then we may assume $\sigma(i) \leq p < \sigma(m) < \sigma(j)$. Let $i', j', m' \in \mathbf{N}$ satisfy (30). We have (31) and (32), where $g' = \tilde{g}$ or $k\tilde{g}$ for

$$\tilde{g} := \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and k is as in (33). Thus,

$$\{ A_{(\sigma(i),\sigma(m),\sigma(j))} \mid A \in (\mathrm{Ad}(g)\mathfrak{b}^{\perp}) \cap \mathfrak{p} \} = \left\{ \begin{pmatrix} 0 & y_{im} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$
$$= \{ Y_{(i,m,j)} \mid Y \in \mathrm{Dri}(\delta) \}.$$

9. If $(d_i, d_j) = (a_+, a_-)$ and $d_l = +$ for $\max(i, j) < l$, then we may assume $\sigma(i) < \sigma(l) \leq p < \sigma(j)$. Let $i', j', l' \in \mathbf{N}$ satisfy (26). We have (27) and (28), where $g' = \tilde{g}$ or $k\tilde{g}$ for

$$\tilde{g} := \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

and k is as (29). So,

$$\{ A_{(\sigma(i),\sigma(l),\sigma(j))} \mid A \in (\mathrm{Ad}(g)\mathfrak{b}^{\perp}) \cap \mathfrak{p} \} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y_{lj} \\ 0 & 0 & 0 \end{pmatrix} \right\} = \{ Y_{(i,l,j)} \mid Y \in \mathrm{Dri}(\delta) \}.$$

10. If $(d_i, d_j) = (a_+, a_-)$ and $d_m = -$ for $\max(i,j) < m$, then we may assume $\sigma(i) \le p < \sigma(m) < \sigma(j)$. Let $i', j', m' \in \mathbf{N}$ satisfy (30). We have (31) and (32), where $g' = \tilde{g}$ or $k\tilde{g}$ for

$$\tilde{g} := \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

and k is as in (33). Therefore,

$$\{A_{(\sigma(i),\sigma(m),\sigma(j))} \mid A \in (\mathrm{Ad}(g)\mathfrak{b}^{\perp}) \cap \mathfrak{p}\} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ y_{mi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} = \{Y_{(i,m,j)} \mid Y \in \mathrm{Dri}(\delta)\}.$$

11. If $(d_i, d_j) = (a_+, a_-)$ and $(d_s, d_t) = (b_+, b_-)$ for $\max(i,j) < \min(t,s)$, then we may assume $\sigma(i) < \sigma(s) \le p < \sigma(j) < \sigma(t)$. Let $i', j', s', t' \in \mathbf{N}$ satisfy

$$\{i', j', s', t'\} = \{i, j, s, t\} \quad \text{and} \quad i' < j' < s' < t'. \tag{34}$$

We have

$$\{ \left(gbg^{-1}\right)_{(\sigma(i),\sigma(s),\sigma(j),\sigma(t))} \mid b \in \mathfrak{b}^{\perp} \} = \left\{ g \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_{j'i'} & 0 & 0 & 0 \\ b_{s'i'} & b_{s'j'} & 0 & 0 \\ b_{t'i'} & b_{t'j'} & b_{t's'} & 0 \end{pmatrix} {}^t g \right\} \tag{35}$$

and

$$\{ A_{(\sigma(i),\sigma(s),\sigma(j),\sigma(t))} \mid A \in \mathfrak{p} \} = \left\{ \begin{pmatrix} 0 & 0 & z_{\sigma(i)\sigma(j)} & z_{\sigma(i)\sigma(t)} \\ 0 & 0 & z_{\sigma(s)\sigma(j)} & z_{\sigma(s)\sigma(t)} \\ z_{\sigma(j)\sigma(i)} & z_{\sigma(j)\sigma(s)} & 0 & 0 \\ z_{\sigma(t)\sigma(i)} & z_{\sigma(t)\sigma(s)} & 0 & 0 \end{pmatrix} \right\} \tag{36}$$

for $g' = \tilde{g}$, $k_1\tilde{g}$, $k_2\tilde{g}$ or $k_1k_2\tilde{g}$. Here,

$$\tilde{g} := \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and k_1, $k_2 \in K$ are

$$k_1 := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(2, \mathbf{C}) \times GL(2, \mathbf{C}) \hookrightarrow K \tag{37}$$

and

$$k_2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(2,\mathbf{C}) \times GL(2,\mathbf{C}) \hookrightarrow K. \tag{38}$$

Hence,

$$\{ A_{(\sigma(i),\sigma(s),\sigma(j),\sigma(t))} \mid A \in (\mathrm{Ad}(g)\mathfrak{b}^{\perp}) \cap \mathfrak{p} \} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & y_{sj} & 0 \\ 0 & 0 & 0 & 0 \\ y_{ti} & 0 & 0 & 0 \end{pmatrix} \right\}$$
$$= \{ Y_{(i,s,j,t)} \mid Y \in \mathrm{Dri}(\delta) \}.$$

12. If $(d_i, d_j) = (a_+, a_-)$ and $(d_s, d_t) = (b_+, b_-)$ for $\min(i,j) < \min(t,s) < \max(i,j) < \max(t,s)$, then we may assume $\sigma(i) < \sigma(s) \leq p < \sigma(j) < \sigma(t)$. Let $i', j', s', t' \in \mathbf{N}$ satisfy (34). We have (36) and (35) for $g' = \tilde{g}$, $k_1\tilde{g}$, $k_2\tilde{g}$ or $k_1k_2\tilde{g}$. Here,

$$\tilde{g} := \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and k_1, k_2 are as (37) and (38). Thus,

$$\{ A_{(\sigma(i),\sigma(s),\sigma(j),\sigma(t))} \mid A \in (\mathrm{Ad}(g)\mathfrak{b}^{\perp}) \cap \mathfrak{p} \}$$
$$= \begin{cases} \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & y_{ti} & 0 \\ 0 & 0 & 0 & 0 \\ y_{ti} & 0 & 0 & 0 \end{pmatrix} \right\} & \text{if } g' = \tilde{g} \text{ or } k_1k_2\tilde{g}, \\ \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -y_{ti} & 0 \\ 0 & 0 & 0 & 0 \\ y_{ti} & 0 & 0 & 0 \end{pmatrix} \right\} & \text{if } g' = k_1\tilde{g} \text{ or } k_2\tilde{g}. \end{cases}$$

So,

$$\{ A_{(\sigma(i),\sigma(s),\sigma(j),\sigma(t))} \mid A \in (\mathrm{Ad}(g)\mathfrak{b}^{\perp}) \cap \mathfrak{p} \} = \{ Y_{(i,s,j,t)} \mid Y \in \mathrm{Dri}(\delta) \}.$$

13. If $(d_i, d_j) = (a_+, a_-)$, $(d_s, d_t) = (b_+, b_-)$ for $\min(i,j) < \min(s,t) < \max(s,t) < \max(i,j)$, then we may assume $\sigma(i) < \sigma(s) \leq p < \sigma(j) < \sigma(t)$. Let $i', j', s', t' \in \mathbf{N}$ satisfy (34). We have (36) and (35) for $g' = \tilde{g}$, $k_1\tilde{g}$, $k_2\tilde{g}$, or $k_1k_2\tilde{g}$. Here,

$$\tilde{g} := \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

and k_1 and k_2 are as (37) and (38). Thus,

$$\{ A_{(\sigma(i),\sigma(s),\sigma(j),\sigma(t))} \mid A \in (\mathrm{Ad}(g)\mathfrak{b}^{\perp}) \cap \mathfrak{p} \} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$
$$= \{ Y_{(i,s,j,t)} \mid Y \in \mathrm{Dri}(\delta) \}.$$

We have finished proving Proposition 2.5.2 for all the elements. □

Since $g(\delta) = (e_{\sigma(1)} \dots e_{\sigma(n)}) = \sigma$, if a clan γ corresponds to a closed orbit, we have the following lemma.

Lemma 2.5.4. *If a clan γ is an element of $\mathcal{C}_0(U(p,q))$, then*

$$\mathrm{Dri}(\delta) = \mathfrak{b}^{\perp} \cap ({}^t g(\delta)\, \mathfrak{p}\, g(\delta)).$$

Definition 2.5.5 (Driving matrix for $U(p,q)$)**.** Let $\delta = (d_1 \dots d_n)$ be a signed clan of $\gamma = (c_1 \dots c_n)$ satisfying the following condition.

(39) If $c_i = c_j = a \in \mathbf{N}$ for $i < j$, then $(d_i, d_j) = (a_+, a_-)$.

Let F be a field $\mathbf{C}(y_{st} \mid 1 \le s,t \le n)$ generated by algebraically independents y_{st} for $1 \le s,t \le n$. Let $Y(\gamma)$ and $Y(\gamma,m)$, $1 \le m \le 6$ be elements of $F \otimes_{\mathbf{C}} \mathrm{Mat}(n,n)$ satisfying the following conditions.

1. $$Y(\gamma,1)_{ij} = \begin{cases} 1 & \text{if } i > j, \\ 0 & \text{if } i \le j. \end{cases}$$
2. $$Y(\gamma,2)_{ij} = \begin{cases} 0 & \text{if } (d_i,d_j) = (+,+) \text{ or } (-,-), \\ Y(\gamma,1)_{ij} & \text{otherwise.} \end{cases}$$
3. If $(d_s, d_t) = (a_+, a_-)$ for some $a \in \mathbf{N}$, then
   - $Y(\gamma,3)_{s,k_1} = 0$ if $1 \le k_1 \le s$,
   - $Y(\gamma,3)_{k_2,s} = Y(\gamma,3)_{t,k_2} = 0$ if $s < k_2 \le t$,
   - $Y(\gamma,3)_{k_3,t} = 0$ if $t < k_3 \le n$,

   and $Y(\gamma,3)_{ij} = Y(\gamma,2)_{ij}$ otherwise.

Remark 2.5.6. If $Y(\gamma,3)_{ij} = 1$, then the following conditions are satisfied.

(a) $j < i$.
(b) $(d_i,d_j) = (a_-,b_+)$, $(a_-,+)$, $(a_-,-)$, $(+,a_+)$, $(-,a_+)$, $(+,-)$, or $(-,+)$ for some a, $b \in \mathbf{N}$, $a \ne b$.
(c) If $d_i = a_-$ for some $a \in \mathbf{N}$, then we have $d_k = a_+$ for some $j < k < i$.
(d) If $d_j = a_+$ for some $a \in \mathbf{N}$, then we have $d_k = a_-$ for some $j < k < i$.

4. If $Y(\gamma,3)_{ij} = 1$, $(d_i,d_j) = (a_-,b_+)$ and $(d_k,d_l) = (a_+,b_-)$ for some $a,b \in \mathbf{N}$, then
   $$Y(\gamma,4)_{ij} = y_{ij},$$
   $$Y(\gamma,4)_{kl} = \begin{cases} y_{ij} & \text{if } j < k < l < i, \\ y_{kl} & \text{if } j < l < k < i, \end{cases}$$
   and $Y(\gamma,4)_{i'j'} = Y(\gamma,3)_{i'j'}$ otherwise.
5. If $Y(\gamma,4)_{ij} = 1$, $(d_i,d_j) = (a_-,-)$, and $d_k = a_+$ for some $a \in \mathbf{N}$, then
   $$Y(\gamma,5)_{ij} = 0, \qquad Y(\gamma,5)_{kj} = y_{kj},$$
   and $Y(\gamma,5)_{i'j'} = Y(\gamma,4)_{i'j'}$ otherwise.

TABLE 1

	1_+	$+$	2_+	1_-	3_+	$+$	3_-	2_-
1_+	0	0	0	0	0	0	0	0
$+$	1	0	0	0	0	0	0	0
2_+	1	1	0	0	0	0	0	0
1_-	1	1	1	0	0	0	0	0
3_+	1	1	1	1	0	0	0	0
$+$	1	1	1	1	1	0	0	0
3_-	1	1	1	1	1	1	0	0
2_-	1	1	1	1	1	1	1	0

Figure for $Y(\gamma, 1)$

	1_+	$+$	2_+	1_-	3_+	$+$	3_-	2_-
1_+	0	0	0	0	0	0	0	0
$+$	1	0	0	0	0	0	0	0
2_+	1	1	0	0	0	0	0	0
1_-	1	1	1	0	0	0	0	0
3_+	1	1	1	1	0	0	0	0
$+$	1	0	1	1	1	0	0	0
3_-	1	1	1	1	1	1	0	0
2_-	1	1	1	1	1	1	1	0

Figure for $Y(\gamma, 2)$

	1_+	$+$	2_+	1_-	3_+	$+$	3_-	2_-
1_+	0	0	0	0	0	0	0	0
$+$	0	0	0	0	0	0	0	0
2_+	0	0	0	0	0	0	0	0
1_-	0	0	0	0	0	0	0	0
3_+	0	0	0	0	0	0	0	0
$+$	1	0	0	0	0	0	0	0
3_-	1	1	0	0	0	0	0	0
2_-	1	1	0	0	0	0	0	0

Figure for $Y(\gamma, 3)$

	1_+	$+$	2_+	1_-	3_+	$+$	3_-	2_-
1_+	0	0	0	0	0	0	0	0
$+$	0	0	0	0	0	0	0	0
2_+	0	0	0	y_{81}	0	0	0	0
1_-	0	0	0	0	0	0	0	0
3_+	0	0	0	y_{54}	0	0	0	0
$+$	1	0	0	0	0	0	0	0
3_-	y_{71}	1	0	0	0	0	0	0
2_-	y_{81}	1	0	0	0	0	0	0

Figure for $Y(\gamma, 4)$

	1_+	$+$	2_+	1_-	3_+	$+$	3_-	2_-
1_+	0	0	0	0	0	0	0	0
$+$	0	0	0	0	0	0	0	0
2_+	0	0	0	y_{81}	0	0	0	0
1_-	0	0	0	0	0	0	0	0
3_+	0	0	0	y_{54}	0	0	0	0
$+$	0	0	0	y_{64}	0	0	0	0
3_-	y_{71}	1	0	0	0	0	0	0
2_-	y_{81}	1	0	0	0	0	0	0

Figure for $Y(\gamma, 6)$

	1_+	$+$	2_+	1_-	3_+	$+$	3_-	2_-
1_+	0	0	0	0	0	0	0	0
$+$	0	0	0	0	0	0	0	0
2_+	0	0	0	y_{81}	0	0	0	0
1_-	0	0	0	0	0	0	0	0
3_+	0	0	0	y_{54}	0	0	0	0
$+$	0	0	0	y_{64}	0	0	0	0
3_-	y_{71}	y_{72}	0	0	0	0	0	0
2_-	y_{81}	y_{82}	0	0	0	0	0	0

Figure for $Y(\gamma)$

6. If $Y(\gamma, 5)_{ij} = 1$, $(d_i, d_j) = (+, a_+)$, and $d_k = a_-$ for some $a \in \mathbf{N}$, then
$$Y(\gamma, 6)_{ij} = 0, \qquad Y(\gamma, 6)_{ik} = y_{ik},$$
and $Y(\gamma, 6)_{i'j'} = Y(\gamma, 5)_{i'j'}$ otherwise.
7. $Y(\gamma)_{ij} = \begin{cases} y_{ij} & \text{if } Y(\gamma, 6)_{ij} = 1, \\ Y(\gamma, 6)_{ij} & \text{otherwise.} \end{cases}$

We call $Y(\gamma)$ the *driving matrix* of γ (for $U(p, q)$).

Example 2.5.7. For a clan $\gamma = (1 + 2\,1\,3 + 3\,2)$, the signed clan of γ is $\delta = (1_+ + 2_+ 1_- 3_+ + 3_- 2_-)$. We give figures for $Y(\gamma, m)$, $1 \le m \le 6$ and $Y(\gamma)$ in Table 1. In these figures, the entry which is in the same row of c_i and the same column of c_j is the (i, j)-entry of the matrix. Therefore the driving matrix $Y(\gamma)$ is as follows:

$$Y(\gamma) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_{81} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_{54} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_{64} & 0 & 0 & 0 & 0 \\ y_{71} & y_{72} & 0 & 0 & 0 & 0 & 0 & 0 \\ y_{81} & y_{82} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can read off $\mathrm{Dri}(\delta)$ from $Y(\gamma)$.

Proposition 2.5.8. *Let δ be the signed clan satisfying condition* (39) *of a clan γ, then*

$$\mathrm{Dri}(\delta) = \{\, Y(\gamma) \mid y_{ij} \in \mathbf{C} \ \ \textit{for} \ \ 1 \le i, j \le n \,\}.$$

Proof. Let δ be the signed clan of γ satisfying condition (39). We compare $Y(\gamma)$ and $Y \in \mathrm{Dri}(\delta)$.

1. If $(d_i, d_j) = (+,+)$ or $(-,-)$ for $i < j$, then
$$Y(\gamma)_{(i,j)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(i,j)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$
2. If $(d_i, d_j) = (a_+, a_-)$, then $i < j$ and
$$Y(\gamma)_{(i,j)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(i,j)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$
3. If $(d_i, d_j) = (+,-)$ or $(-,+)$, and $i < j$, then
$$Y(\gamma)_{(i,j)} = \begin{pmatrix} 0 & 0 \\ y_{ji} & 0 \end{pmatrix} \quad \text{and} \quad Y_{(i,j)} = \begin{pmatrix} 0 & 0 \\ x_{ji} & 0 \end{pmatrix}$$
for some $x_{ji} \in \mathbf{C}$.
4. If $(d_i, d_l, d_j) = (a_+, +, a_-)$ and $i < l < j$, then
$$Y(\gamma)_{(i,l,j)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(i,l,j)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
5. If $(d_i, d_m, d_j) = (a_+, -, a_-)$ and $i < m < j$, then
$$Y(\gamma)_{(i,m,j)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(i,m,j)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
6. If $(d_l, d_i, d_j) = (+, a_+, a_-)$ and $l < i < j$, then
$$Y(\gamma)_{(l,i,j)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_{jl} & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(l,i,j)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_{jl} & 0 & 0 \end{pmatrix}$$
for some $x_{jl} \in \mathbf{C}$.
7. If $(d_m, d_i, d_j) = (-, a_+, a_-)$ and $m < i < j$, then
$$Y(\gamma)_{(m,i,j)} = \begin{pmatrix} 0 & 0 & 0 \\ y_{im} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(m,i,j)} = \begin{pmatrix} 0 & 0 & 0 \\ x_{im} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
for some $x_{im} \in \mathbf{C}$.
8. If $(d_i, d_j, d_l) = (a_+, a_-, +)$ and $i < j < l$, then
$$Y(\gamma)_{(i,j,l)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & y_{lj} & 0 \end{pmatrix} \quad \text{and} \quad Y_{(i,j,l)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & x_{lj} & 0 \end{pmatrix}$$
for some $x_{lj} \in \mathbf{C}$.

9. If $(d_i, d_j, d_m) = (a_+, a_-, -)$ and $i < j < m$, then

$$Y(\gamma)_{(i,j,m)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_{mi} & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(i,j,m)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_{mi} & 0 & 0 \end{pmatrix}$$

for some $x_{mi} \in \mathbf{C}$.

10. If $(d_i, d_j, d_s, d_t) = (a_+, a_-, b_+, b_-)$ and $i < j < s < t$, then

$$Y(\gamma)_{(i,j,s,t)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & y_{sj} & 0 & 0 \\ y_{ti} & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(i,j,s,t)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x_{sj} & 0 & 0 \\ x_{ti} & 0 & 0 & 0 \end{pmatrix}$$

for some x_{sj}, $x_{ti} \in \mathbf{C}$.

11. If $(d_i, d_s, d_j, d_t) = (a_+, b_+, a_-, b_-)$ and $i < s < j < t$, then

$$Y(\gamma)_{(i,s,j,t)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & y_{ti} & 0 \\ 0 & 0 & 0 & 0 \\ y_{ti} & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(i,s,j,t)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x_{ti} & 0 \\ 0 & 0 & 0 & 0 \\ x_{ti} & 0 & 0 & 0 \end{pmatrix}$$

for some $x_{ti} \in \mathbf{C}$.

12. If $(d_i, d_s, d_t, d_j) = (a_+, b_+, b_-, a_-)$ and $i < s < t < j$, then

$$Y(\gamma)_{(i,s,t,j)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_{(i,s,t,j)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We got $\mathrm{Dri}(\delta) = \{\, Y(\gamma) \mid y_{ij} \in \mathbf{C} \ \text{ for } \ 1 \le i, j \le n \,\}$. □

Example 2.5.9. For $\gamma = (- + +)$, we have

$$\mathrm{Dri}(\delta) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ y_{21} & 0 & 0 \\ y_{31} & 0 & 0 \end{pmatrix} \middle| \; y_{ij} \in \mathbf{C} \right\}.$$

By Proposition 2.5.2 and Proposition 2.5.8, we get the following theorem.

Theorem 2.5.10. *For a clan $\gamma \in \mathcal{C}(U(p,q))$, fix a signed clan δ of γ. Let the representative $g := g(\delta,\sigma)$ be given as in Theorem 2.2.14 and $x = gB \in Q_\gamma$. We can read off $\mu\left((T^*_{Q_\gamma}X)_x\right)$ from the driving matrix $Y(\gamma)$ for $U(p,q)$ as follows:*

$$\left(\mu(T^*_{Q_\gamma}X)_x\right) = \{\, Y(\gamma)_{(\sigma^{-1}(1),\dots,\sigma^{-1}(n))} \mid y_{ij} \in \mathbf{C} \ \ \textit{for} \ \ 1 \le i,j \le n \,\}.$$

Definition 2.5.11 (Generic element)**.** We call A in $\mu(T^*_{Q_\gamma}X)$ a *generic element* if A satisfies

$$\dim(\ker A^i) = \min\{\, \dim(\ker(A')^i) \mid A' \in \mu\left((T^*_{Q_\gamma}X)_x\right)\}$$

for all $1 \le i \le n$. Similarly, for a signed clan δ, we call Y in $Dri(\delta)$ a *generic element* if Y satisfies

$$\dim(\ker Y^i) = \min\{\, \dim(\ker(Y')^i) \mid Y' \in \mathrm{Dri}(\delta)\,\}$$

for all $1 \le i \le n$.

2.6. **Signed Young diagrams.** In this section we give tables of signed Young diagrams for clans of $U(1,1)$, $U(2,1)$, $U(2,2)$ and $U(p,2)$ by way of examples.

A *signed Young diagram* is a Young diagram in which every box is labeled with a $+$ or $-$ sign in such a way that signs alternate across rows and they need not alternate down columns. Two signed Young diagrams are regarded as equal if and only if one can be obtained from the other by interchanging rows of equal length. The signature of a signed Young diagram is the ordered pair (i,j) where i is the number of boxes labeled $+$ and j is the number of boxes labeled $-$. For $G_{\mathbf{R}} = U(p,q)$, nilpotent K-orbits in $\mathfrak{p}$ are parametrized by signed Young diagrams of signature (p,q) (see [2]).

An element A of $\mathfrak{p}$ satisfies

$$A \cdot V_+ \subset V_- \quad \text{and} \quad A \cdot V_- \subset V_+.$$

Definition 2.6.1. The signed Young diagram of a nilpotent orbit is defined as follows. We remark that a signed Young diagram is determined by the number of boxes labeled $+$ and the number of boxes labeled $-$ in each column. Let $D_{i,+}$ be the number of boxes labeled $+$ in i-column and $D_{i,-}$ be the number of boxes labeled $-$ in i-column.

For an element A of an orbit, the signed Young diagram of the orbit satisfies

$$\sum_{j=1}^{i} D_{j,+} = \dim(\ker(A^i|_{V_+})) \quad \text{and} \quad \sum_{j=1}^{i} D_{j,-} = \dim(\ker(A^i|_{V_-})).$$

So, we have

$$D_{i,+} = \dim(\ker(A^i|_{V_+})) - \dim(\ker(A^{i-1}|_{V_+}))$$

and

$$D_{i,-} = \dim(\ker(A^i|_{V_-})) - \dim(\ker(A^{i-1}|_{V_-})).$$

Proposition 2.6.2. *Under the conditions of Theorem 2.2.14, we put*

$$\begin{aligned} V'_+ &:= \sigma^{-1}V_+ = (e_{\sigma^{-1}(1)} \dots e_{\sigma^{-1}(n)})V_+ = \langle e_{\sigma^{-1}(1)}, \dots, e_{\sigma^{-1}(p)} \rangle, \\ V'_- &:= \sigma^{-1}V_- = (e_{\sigma^{-1}(1)} \dots e_{\sigma^{-1}(n)})V_- = \langle e_{\sigma^{-1}(p+1)}, \dots, e_{\sigma^{-1}(n)} \rangle. \end{aligned}$$

Then, we have

$$\dim(\ker(Y^i|_{V'_+})) = \dim(\ker(A^i|_{V_+})),$$
$$\dim(\ker(Y^i|_{V'_-})) = \dim(\ker(A^i|_{V_-}))$$

for a generic element Y *in* $\mathrm{Dri}(\delta)$ *and* A *in* $\mu(T^*_{Q_\gamma}X)$*.*

We consider linear equations

$$Y(\gamma)^i\vec{a} = \vec{0} \qquad \text{for} \qquad \vec{a} \in F \otimes_{\mathbf{C}} V'_+. \tag{40}$$

The set of solutions of (40) is an F-subspace of $F \otimes_{\mathbf{C}} V'_+$. Because

$$\dim_F\{\,\vec{a} \in F \otimes_{\mathbf{C}} V'_+ \mid Y(\gamma)^i\vec{a} = \vec{0}\,\} = \dim_{\mathbf{C}}(\ker(Y^i|_{V'_+}))$$

for a generic element $Y \in \mathrm{Dri}(\gamma)$, we have the following proposition

Proposition 2.6.3. *Let* $Y(\gamma)^0 = I_n$*. We have the following equations.*

$$D_{i,+} = \dim_F\{\vec{a} \in F \otimes_{\mathbf{C}} V'_+ \mid Y(\gamma)^i\vec{a} = \vec{0}\} - \dim_F\{\vec{a} \in F \otimes_{\mathbf{C}} V'_+ \mid Y(\gamma)^{i-1}\vec{a} = \vec{0}\},$$
$$D_{i,-} = \dim_F\{\vec{b} \in F \otimes_{\mathbf{C}} V'_- \mid Y(\gamma)^i\vec{b} = \vec{0}\} - \dim_F\{\vec{b} \in F \otimes_{\mathbf{C}} V'_- \mid Y(\gamma)^{i-1}\vec{b} = \vec{0}\}.$$

Example 2.6.4. For a clan $\gamma = (\,+\,1\,1\,-\,+\,)$, and a signed clan $\delta = (\,+\,1_+\,1_-\,-\,+\,)$. We get a permutation, a driving matrix $Y(\gamma)$, and two vectors $\vec{a}$ and $\vec{b}$ are as follows:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$Y(\gamma) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 & 0 \\ y_{41} & y_{42} & 0 & 0 & 0 \\ 0 & 0 & y_{53} & y_{54} & 0 \end{pmatrix}, \ \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ a_5 \end{pmatrix}, \quad \text{and} \quad \vec{b} = \begin{pmatrix} 0 \\ 0 \\ b_3 \\ b_4 \\ 0 \end{pmatrix}.$$

We can calculate $Y(\gamma)\vec{a} + Y(\gamma)\vec{b}$ as follows:

$$Y(\gamma)\vec{a} + Y(\gamma)\vec{b} = Y(\gamma)\begin{pmatrix} a_1 \\ a_2 \\ b_3 \\ b_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ y_{31}a_1 \\ y_{41}a_1 + y_{42}a_2 \\ y_{53}b_3 + y_{54}b_4 \end{pmatrix}.$$

The equations $Y(\gamma)\vec{a} = \vec{0}$ and $Y(\gamma)\vec{b} = \vec{0}$ lead $\begin{cases} a_1 = 0,\ a_2 = 0, \\ x_{53}b_3 + x_{54}b_4 = 0. \end{cases}$ So, we have

$$(\,D_{1,+}\,,\,D_{1,-}\,) \,=\, (\,1-0\,,\,1-0\,) \,=\, (\,1\,,\,1\,).$$

+
−

FIGURE A

+	−
−	+

FIGURE B

+	−	+
−	+	

FIGURE C

Thus, the first column of the signed Young diagram of γ is as Figure A.

We can calculate $Y(\gamma)^2\vec{a} + Y(\gamma)^2\vec{b}$ as follows:

$$\begin{aligned} Y(\gamma)^2\vec{a} + Y(\gamma)^2\vec{b} &= Y(\gamma)(Y(\gamma)\vec{a} + Y(\gamma)\vec{b}) \\ &= Y(\gamma)\begin{pmatrix} 0 \\ 0 \\ y_{31}a_1 \\ y_{41}a_1 + y_{42}a_2 \\ y_{53}b_3 + y_{54}b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (y_{53}y_{31} + y_{54}y_{41})a_1 + y_{54}y_{42}a_2 \end{pmatrix}. \end{aligned}$$

The equations $Y(\gamma)^2\vec{a} = \vec{0}$ and $Y(\gamma)^2\vec{b} = \vec{0}$ lead $(y_{53}y_{31} + y_{54}y_{41})a_1 + y_{54}y_{42}a_2 = 0$. Therefore, we have

$$(\, D_{2,+}\,,\, D_{2,-}\,) \,=\, (\, 2-1\,,\, 2-1\,) \,=\, (\,1\,,\,1\,).$$

Thus, the first and second columns of the signed Young diagram of γ is as Figure B.

We can calculate $Y(\gamma)^3\vec{a} + Y(\gamma)^3\vec{b}$ as follows:

$$\begin{aligned} Y(\gamma)^3\vec{a} + Y(\gamma)^3\vec{b} &= Y(\gamma)(Y(\gamma)^2\vec{a} + Y(\gamma)^2\vec{b}) \\ &= Y(\gamma)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (y_{53}y_{31} + y_{54}y_{41})a_1 + y_{54}y_{42}a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

By the equations $Y(\gamma)^3\vec{a} = \vec{0}$ and $Y(\gamma)^3\vec{b} = \vec{0}$, we have

$$(\, D_{3,+}\,,\, D_{3,-}\,) \,=\, (\, 3-2\,,\, 2-2\,) \,=\, (\,1\,,\,0\,).$$

Therefore, the signed Young diagram for the clan γ is as Figure C. We give tables of signed Young diagrams for clans by way of examples.

Example 2.6.5. This is the table of the case of $G_{\mathbf{R}} = U(1,1)$.

clan γ	a representative $g(\gamma)$ of K-orbit	driving matrix $Y(\gamma)$	signed Young diagram
$+\,-$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ y_{21} & 0 \end{pmatrix}$	[−][+]
$-\,+$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ y_{21} & 0 \end{pmatrix}$	[+][−]
$1\ 1$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	[+] [−]

Example 2.6.6. This is the table of the case of $G_{\mathbf{R}} = U(2,1)$.

clan γ	a representative	driving matrix $Y(\gamma)$	signed Young diagram
$+\,-\,+$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ y_{21} & 0 & 0 \\ 0 & y_{32} & 0 \end{pmatrix}$	[+][−][+]
$+\,+\,-$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_{31} & y_{32} & 0 \end{pmatrix}$	[−][+] [+]
$+\ 1\ 1$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_{31} & 0 & 0 \end{pmatrix}$	
$-\,+\,+$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ y_{21} & 0 & 0 \\ y_{31} & 0 & 0 \end{pmatrix}$	[+][−] [+]
$1\ 1\ +$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & y_{32} & 0 \end{pmatrix}$	
$1\ +\ 1$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	[+] [+] [−]

Example 2.6.7. This is the table of the case of $G_{\mathbf{R}} = U(2,2)$.

clan γ	a representative	driving matrix $Y(\gamma)$	signed Young diagram
$+-+-$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 \\ 0 & y_{32} & 0 & 0 \\ y_{41} & 0 & y_{43} & 0 \end{pmatrix}$	$-+-+$
$-+-+$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 \\ 0 & y_{32} & 0 & 0 \\ y_{41} & 0 & y_{43} & 0 \end{pmatrix}$	$+-+-$
$+--+$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 \\ 0 & y_{42} & y_{43} & 0 \end{pmatrix}$	$+-+$ $-$
$+-11$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 \\ 0 & y_{32} & 0 & 0 \\ y_{41} & 0 & 0 & 0 \end{pmatrix}$	
$11-+$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 \\ 0 & y_{42} & y_{43} & 0 \end{pmatrix}$	
$-++-$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 \\ 0 & y_{42} & y_{43} & 0 \end{pmatrix}$	$-+-$ $+$
$-+11$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 \\ 0 & y_{42} & 0 & 0 \end{pmatrix}$	
$11+-$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & y_{32} & 0 & 0 \\ y_{41} & 0 & y_{43} & 0 \end{pmatrix}$	
$++--$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_{31} & y_{32} & 0 & 0 \\ y_{41} & y_{42} & 0 & 0 \end{pmatrix}$	$-+$ $-+$
$+11-$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 \\ y_{41} & y_{42} & 0 & 0 \end{pmatrix}$	

clan γ	a representative	driving matrix $Y(\gamma)$	signed Young diagram
$--++$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_{31} & y_{32} & 0 & 0 \\ y_{41} & y_{42} & 0 & 0 \end{pmatrix}$	$\begin{array}{\|c\|c\|}\hline + & - \\ \hline + & - \\ \hline \end{array}$
$-\,1\;1\,+$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_{41} & 0 & y_{43} & 0 \end{pmatrix}$	
$1+1-$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_{41} & y_{42} & 0 & 0 \end{pmatrix}$	$\begin{array}{\|c\|c\|}\hline - & + \\ \hline + \\ \cline{1-1} - \\ \cline{1-1} \end{array}$
$+1-1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 \\ y_{41} & 0 & 0 & 0 \end{pmatrix}$	
$1+-1$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & y_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	
$1-1+$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & y_{42} & y_{43} & 0 \end{pmatrix}$	$\begin{array}{\|c\|c\|}\hline + & - \\ \hline + \\ \cline{1-1} - \\ \cline{1-1} \end{array}$
$-1+1$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	
$1-+1$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & y_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	
$1\;1\;2\;2$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & y_{32} & 0 & 0 \\ y_{41} & 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{\|c\|c\|}\hline + & - \\ \hline - & + \\ \hline \end{array}$
$1\;2\;1\;2$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & y_{41} & 0 \\ 0 & 0 & 0 & 0 \\ y_{41} & 0 & 0 & 0 \end{pmatrix}$	

clan γ	a representative	driving matrix $Y(\gamma)$	signed Young diagram
1 2 2 1	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{\|c\|} \hline + \\ \hline + \\ \hline - \\ \hline - \\ \hline \end{array}$

Example 2.6.8. This is the table for the case of $G_{\mathbf{R}} = U(p,2)$.

signed Young diagram	clan $(i,j,k,l \geq 1,\ a,b,c \geq 0)$
$+-+-+$ $+$ $\vdots$ $\}\,p-3$ $+$	$\underset{i}{+}-\underset{j}{+}-\underset{k}{+}$
	$\underset{i}{+}\,1\,\underset{a}{+}\,1\,\underset{j}{+}-\underset{k}{+}$
	$\underset{i}{+}-\underset{j}{+}\,1\,\underset{a}{+}\,1\,\underset{k}{+}$
	$\underset{i}{+}\,1\,\underset{a}{+}\,1\,\underset{j}{+}\,2\,\underset{b}{+}\,2\,\underset{k}{+}$
$+-+-$ $+$ $\vdots$ $\}\,p-2$ $+$	$-\underset{i}{+}-\underset{j}{+}$
	$1\,\underset{a}{+}\,1\,\underset{i}{+}-\underset{j}{+}$
	$-\underset{i}{+}\,1\,\underset{a}{+}\,1\,\underset{j}{+}$
	$1\,\underset{a}{+}\,1\,\underset{i}{+}\,2\,\underset{b}{+}\,2\,\underset{j}{+}$
$-+-+$ $+$ $\vdots$ $\}\,p-2$ $+$	$\underset{i}{+}-\underset{j}{+}-$
	$\underset{i}{+}-\underset{j}{+}\,1\,\underset{a}{+}\,1$
	$\underset{i}{+}\,1\,\underset{a}{+}\,1\,\underset{j}{+}-$
	$\underset{i}{+}\,1\,\underset{a}{+}\,1\,\underset{j}{+}\,2\,\underset{b}{+}\,2$
$+-+$ $+-+$ $+$ $\vdots$ $\}\,p-4$ $+$	$\underset{i}{+}\underset{j}{+}--\underset{k}{+}\underset{l}{+}$
	$\underset{i}{+}\,1\,\underset{a}{+}\,1\,-\underset{j}{+}\underset{k}{+}$
	$\underset{i}{+}\underset{j}{+}-\,1\,\underset{a}{+}\,1\,\underset{k}{+}$
	$\underset{i}{+}\,1\,\underset{a}{+}-\underset{b}{+}\,1\,\underset{j}{+}$ $(i+a \geq 2,\ j+b \geq 2)$
	$\underset{i}{+}\,1\,\underset{a}{+}\,1\;2\,\underset{b}{+}\,2\,\underset{j}{+}$
	$\underset{i}{+}\,1\,\underset{a}{+}\,2\,\underset{b}{+}\,1\,\underset{c}{+}\,2\,\underset{j}{+}$
	$\underset{i}{+}\,1\,\underset{a}{+}\,2\,\underset{b}{+}\,2\,\underset{c}{+}\,1\,\underset{j}{+}$ $(i+a \geq 2,\ j+c \geq 2)$

signed Young diagram	clan $(i,j,k \geq 1,\ a,b,c \geq 0)$
$+-+$ $+-$ $+$ $\vdots$ $\}\,p-3$ $+$	$+--\underset{i}{+}\underset{j}{+}$
	$+-\,1\,\underset{a}{+}\,1\,\underset{i}{+}$
	$1\,\underset{a}{+}\,1\,-\underset{i}{+}\underset{j}{+}$
	$+\,1\,-\underset{a}{+}\,1\,\underset{i}{+}$ $(i+a \geq 2)$
	$1\,\underset{i}{+}-\underset{a}{+}\,1\,\underset{j}{+}$ $(j+a \geq 2)$
	$1\,\underset{a}{+}\,1\;2\,\underset{b}{+}\,2\,\underset{i}{+}$
	$1\,\underset{a}{+}\,2\,\underset{b}{+}\,1\,\underset{c}{+}\,2\,\underset{i}{+}$
	$+\,1\;2\,\underset{a}{+}\,2\,\underset{b}{+}\,1\,\underset{i}{+}$ $(i+b \geq 2)$
	$1\,\underset{i}{+}\,2\,\underset{a}{+}\,2\,\underset{b}{+}\,1\,\underset{j}{+}$ $(j+b \geq 2)$
$+-+$ $-+$ $+$ $\vdots$ $\}\,p-3$ $+$	$\underset{i}{+}\underset{j}{+}--+$
	$\underset{i}{+}\,1\,\underset{a}{+}\,1\,-+$
	$\underset{i}{+}\underset{j}{+}-\,1\,\underset{a}{+}\,1$
	$\underset{i}{+}\,1\,\underset{a}{+}-\,1\,+$ $(i+a \geq 2)$
	$\underset{i}{+}\,1\,\underset{a}{+}-\underset{j}{+}\,1$ $(i+a \geq 2)$
	$\underset{i}{+}\,1\,\underset{a}{+}\,1\;2\,\underset{b}{+}\,2$
	$\underset{i}{+}\,1\,\underset{a}{+}\,2\,\underset{b}{+}\,1\,\underset{c}{+}\,2$
	$\underset{i}{+}\,1\,\underset{a}{+}\,2\,\underset{b}{+}\,2\;1\,+$ $(i+a \geq 2)$
	$\underset{i}{+}\,1\,\underset{a}{+}\,2\,\underset{b}{+}\,2\,\underset{j}{+}\,1$ $(i+a \geq 2)$

signed Young diagram	clan $(i,j \geq 1,\ a,b \geq 0)$
$-+-$ $+$ $\vdots$ $+$ $\}\,p-1$	$- \underset{i}{+} \underset{j}{+} -$
	$1 \underset{a}{+} 1 \underset{i}{+} -$
	$- \underset{i}{+} 1 \underset{a}{+} 1$
	$1 \underset{a}{+} 1 \underset{i}{+} 2 \underset{b}{+} 2$
$+-+$ $-$ $+$ $\vdots$ $+$ $\}\,p-2$	$1 \underset{i}{+} 1 - +$
	$+ - 1 \underset{i}{+} 1$
	$+ 1 - 1 +$ $(p=3)$
	$1 \underset{i}{+} - 1 +$
	$+ 1 - \underset{i}{+} 1$
	$1 \underset{i}{+} - \underset{j}{+} 1$
	$+ 1\ 2 \underset{a}{+} 2\ 1 +$
	$1 \underset{i}{+} 2 \underset{a}{+} 2\ 1 +$
	$+ 1\ 2 \underset{a}{+} 2 \underset{i}{+} 1$
	$1 \underset{i}{+} 2 \underset{a}{+} 2 \underset{j}{+} 1$
$+-$ $+-$ $+$ $\vdots$ $+$ $\}\,p-2$	$- - \underset{i}{+} \underset{j}{+}$
	$- 1 \underset{a}{+} 1 \underset{i}{+}$
	$1 - \underset{a}{+} 1 \underset{i}{+}$ $(i+a \geq 2)$
	$1\ 2 \underset{a}{+} 2 \underset{b}{+} 1 \underset{i}{+}$ $(i+b \geq 2)$

signed Young diagram	clan $(i,j \geq 1,\ a,b,c \geq 0)$
$-+$ $-+$ $+$ $\vdots$ $+$ $\}\,p-2$	$\underset{i}{+} \underset{j}{+} - -$
	$\underset{i}{+} 1 \underset{a}{+} 1 -$
	$\underset{i}{+} 1 \underset{a}{+} - 1$ $(i+a \geq 2)$
	$\underset{i}{+} 1 \underset{a}{+} 2 \underset{b}{+} 2\ 1$ $(i+a \geq 2)$
$+-$ $-+$ $+$ $\vdots$ $+$ $\}\,p-2$	$1 \underset{a}{+} 1\ 2 \underset{b}{+} 2$
	$1 \underset{a}{+} 2 \underset{b}{+} 1 \underset{c}{+} 2$
$+-$ $-$ $+$ $\vdots$ $+$ $\}\,p-1$	$- 1 \underset{i}{+} 1$
	$1 - \underset{i}{+} 1$
	$1\ 2 \underset{a}{+} 2 \underset{i}{+} 1$
	$1\ 2 \underset{a}{+} 2\ 1 +$
$-+$ $-$ $+$ $\vdots$ $+$ $\}\,p-1$	$1 \underset{i}{+} 1 -$
	$1 \underset{i}{+} - 1$
	$1 \underset{i}{+} 2 \underset{a}{+} 2\ 1$
	$+ 1\ 2 \underset{a}{+} 2\ 1$
$-$ $-$ $+$ $\vdots$ $+$ $\}\,p$	$1\ 2 \underset{a}{+} 2\ 1$

3. The case of $Sp(n, \mathbf{R})$

In this section, we treat the case of $G_{\mathbf{R}} = Sp(n, \mathbf{R})$, i.e., we treat $GL(n, \mathbf{C})$-orbits in the flag variety of $Sp(n, \mathbf{C})$. We will apply the case of $U(n, n)$ to the case of $Sp(n, \mathbf{R})$.

Definition 3.0.9. For a clan $\gamma = (c_1\ \cdots\ c_n) \in \mathcal{C}(U(p,q))$, transpose ${}^t\gamma$, of γ is a clan in $\mathcal{C}(U(p,q))$ such that

$$ {}^t\gamma = (c_n\ \cdots\ c_1) $$

and minus $-\gamma = (c'_1 \ \cdots c'_n)$ is a clan in $\mathcal{C}(U(q,p))$ such that

$$c'_i = \begin{cases} - & \text{if } c_i = +, \\ + & \text{if } c_i = -, \\ c_i & \text{if } c_i \in \mathbf{N}. \end{cases}$$

We call a clan γ *symmetric* if the transpose of γ is equal to γ as a clan, i.e.,

$${}^t\gamma \ = \ \gamma.$$

We call a clan γ *skew symmetric* if the transpose of γ is equal to minus of γ as a clan, i.e.,

$${}^t\gamma \ = \ -\gamma.$$

Example 3.0.10. A clan $\gamma_1 = \left(+ \quad 1 \quad - \quad 2 \quad 1 \quad - \quad 2 \quad + \right)$ is symmetric because

$$\begin{aligned} {}^t\gamma_1 &= \left(+ \quad 2 \quad - \quad 1 \quad 2 \quad - \quad 1 \quad + \right) \\ &= \left(+ \quad 1 \quad - \quad 2 \quad 1 \quad - \quad 2 \quad + \right) \\ &= \gamma_1. \end{aligned}$$

A clan $\gamma_2 = \left(+ \quad 1 \quad - \quad 2 \quad 1 \quad + \quad 2 \quad - \right)$ is skew symmetric because

$$\begin{aligned} {}^t\gamma_2 &= \left(- \quad 2 \quad + \quad 1 \quad 2 \quad - \quad 1 \quad + \right) \\ &= -\left(+ \quad 1 \quad - \quad 2 \quad 1 \quad - \quad 2 \quad + \right) \\ &= -\gamma_2. \end{aligned}$$

From now on, we denote G, B, K, $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{k}$, and $\mathfrak{p}$ for $G_{\mathbf{R}} = U(p',q')$ by G_{AIII}, B_{AIII}, K_{AIII}, $\mathfrak{g}_{\text{AIII}}$, $\mathfrak{b}_{\text{AIII}}$, $\mathfrak{k}_{\text{AIII}}$, and $\mathfrak{p}_{\text{AIII}}$. Let $U(p',q') = U(n,n)$. We denote $U(n,n)$ by $G_{\mathbf{R}\text{AIII}}$.

3.1. **Flags of $Sp(n,\mathbf{C})$.** In this section, after realizing a real form $Sp(n,\mathbf{R})$ of $Sp(n,\mathbf{C})$, we recall flags of $Sp(n,\mathbf{C})$. We realize a complex symplectic group $Sp(n,\mathbf{C})$ as a group of matrices g in $GL(2n,\mathbf{C})$ which leave invariant an exterior form

$$x_1 \wedge x_{2n} + x_2 \wedge x_{2n-1} + \cdots + x_n \wedge x_{n+1},$$

i.e.,

$$Sp(n,\mathbf{C}) = \left\{ g \in GL(2n,\mathbf{C}) \;\middle|\; {}^tg \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} \right\}. \tag{41}$$

By this realization, a Borel subgroup can be upper triangular.

We realize a real symplectic group $G_{\mathbf{R}} = Sp(n,\mathbf{R})$ as a group of matrices g in $Sp(n,\mathbf{C})$ which leave invariant a Hermitian form of the signature (n,n)

$$x_1\overline{x_1} + \cdots + x_n\overline{x_n} - x_{n+1}\overline{x_{n+1}} - \cdots - x_{2n}\overline{x_{2n}},$$

i.e.,

$$Sp(n,\mathbf{R}) = \left\{ g \in Sp(n,\mathbf{C}) \;\middle|\; {}^tg \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \overline{g} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \right\}. \tag{42}$$

By this realization of the real form, a compact Cartan subgroup can be diagonal. This realization is not only the author's taste but also corresponds to $U(n,n)$. (See (43).)

Remark 3.1.1. The group (42) and a group

$$Sp(n, \mathbf{C}) \cap GL(2n, \mathbf{R})$$

are conjugate by a matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} I_n & J_n \\ -iJ_n & iI_n \end{pmatrix}.$$

We fix a Cartan involution θ of $G_{\mathbf{R}}$:

$$\theta : g \ \mapsto \ \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} g \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

Then we have

$$\mathfrak{k} = \mathfrak{g} \cap \mathfrak{k}_{\mathrm{AIII}}, \qquad \mathfrak{p} = \mathfrak{g} \cap \mathfrak{p}_{\mathrm{AIII}}, \qquad \mathfrak{b} = \mathfrak{g} \cap \mathfrak{b}_{\mathrm{AIII}}. \tag{43}$$

and

$$K = \left\{ \begin{pmatrix} K' & 0 \\ 0 & J_n\, {}^t{K'}^{-1} J_n \end{pmatrix} \middle| \ K' \in GL(n, \mathbf{C}) \right\} \simeq GL(n, \mathbf{C}).$$

Notation 3.1.2. Let $V = \mathbf{C}^{2n}$ and θ be an involution of V such that

$$\theta : v \mapsto \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} v.$$

Let V_+ and V_- be the eigenspaces in V under θ for eigenvalues $+1$ and -1, respectively:

$$V_+ := \langle e_1, \dots, e_n \rangle, \quad \text{and} \quad V_- := \langle e_{n+1}, \dots, e_{2n} \rangle.$$

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ consisting of all diagonal matrices of $\mathfrak{g} = \mathfrak{sp}(n, \mathbf{C})$, $\mathfrak{b}$ a Borel subalgebra consisting of all upper triangular matrices of $\mathfrak{g}$. This choice corresponds to the choice of simple root system Ψ:

$$\Psi := \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \mathfrak{a}^*;$$

where $\alpha_i \in \mathfrak{a}^*$ satisfies

$$\alpha_i(E_{jj} - E_{2n+1-j, 2n+1-j}) = \begin{cases} \delta_{ij} - \delta_{i+1,j} & \text{if } 1 \leq i \leq n-1, \\ 2\delta_{nj} & \text{if } i = n. \end{cases}$$

Let ω be a nondegenerate skew-symmetric bilinear form on $V = \mathbf{C}^{2n}$ such that

$$\omega(v, w) = {}^t v \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} w.$$

If W is a subspace of V, we write

$$W^{\perp} = \{\, v \in V \mid \omega(v, w) = 0 \ \text{ for all } \ w \in W \,\}.$$

This is the *orthogonal subspace* of W, relative to ω. We say that W is *isotropic* if $W \subset W^{\perp}$.

Remark 3.1.3. The orthogonal subspace of V_+ and V_- are V_+ and V_- respectively:

$$(V_+)^{\perp} = V_+ \qquad \text{and} \qquad (V_-)^{\perp} = V_-.$$

Remark 3.1.4. A flag x of $Sp(n, \mathbf{C})$ is a sequence of $2n + 1$ vector spaces

$$x = (V_0, V_1, V_2, \dots, V_{2n}),$$

satisfying the following three conditions.

1. $\dim V_i = i$ for all $0 \le i \le 2n$.
2. $\{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{2n} = \mathbf{C}^{2n}$.
3. For $0 \le i \le 2n$, V_{2n-i} is the orthogonal subspace of V_i:

$$V_{2n-i} = V_i^{\perp},$$

i.e., for $1 \le i \le n$, V_i are isotropic subspaces.

We denote the set of flags by X.

Remark 3.1.5. We fix a G-equivariant natural isomorphism between X and G/B via the manner in Remark 2.1.2.

3.2. **A symbolic parametrization of $GL(n,\mathbf{C})$-orbits in $Sp(n,\mathbf{C})/B$.** In this section we give a symbolic parametrization of K-orbits in the flag variety X and give representatives of K-orbits for the parameters. Our parameters are an improvement of the Matsuki-Oshima's parameters [6]. The set of parameters of the K-orbits in the flag variety X is a subset of $\mathcal{C}(U(n,n))$. The parameter we give for a K-orbit Q in the flag variety for $G_{\mathbf{R}} = Sp(n,\mathbf{R})$ coincides with a clan in $\mathcal{C}(U(n,n))$ which includes Q.

3.2-1. *Generalized clans for $Sp(n,\mathbf{R})$.* We give a subset of $\mathcal{C}(U(n,n))$. The subset parametrizes K-orbits in the flag variety X.

Proposition 3.2.1. *For a flag $x = (V_0, V_1, \ldots, V_{2n}) \in X$, there exists a clan $\gamma = (c_1 \ \cdots \ c_{2n}) \in \mathcal{C}(U(n,n))$ via the conditions in Proposition 2.2.6.*

Proposition 3.2.2. *Let $x \in X$ and $\gamma = (c_1 \ \cdots \ c_{2n}) \in \mathcal{C}(U(n,n))$ be the clan of x. Then the clan is skew symmetric, i.e., ${}^t\gamma = -\gamma$.*

Proof. We remark that

$$\begin{aligned} V_{2n-i,\pm} &= V_{2n-i} \cap V_{\pm} = (V_i)^{\perp} \cap (V_{\pm})^{\perp} \\ &= (V_i + V_{\pm})^{\perp} = (\pi_{\mp}(V_i) \oplus V_{\pm})^{\perp}. \end{aligned}$$

1. If $c_i = +$, then there exists a vector v_+ in V_+ such that $v_+ \notin \pi_+(V_{i-1})$ and $V_i = V_{i-1} \oplus \langle v_+ \rangle$. Since

$$\begin{aligned} V_{2n-i+1,+} &= (\pi_-(V_{i-1}) \oplus V_+)^{\perp}, \quad \text{and} \\ V_{2n-i,+} &= (\pi_-(V_i) \oplus V_+)^{\perp} = (\pi_-(V_{i-1}) \oplus V_+)^{\perp}, \end{aligned}$$

we have

$$(2n-i+1;+) - (2n-i;+) = 0.$$

Since

$$\begin{aligned} V_{2n-i+1,-} &= (\pi_+(V_{i-1}) \oplus V_-)^{\perp} \quad \text{and} \\ V_{2n-i,-} &= (\pi_+(V_i) \oplus V_-)^{\perp} = (\pi_+(V_{i-1}) \oplus \langle v_+ \rangle \oplus V_-)^{\perp}, \end{aligned}$$

we have

$$(2n-i+1;-) - (2n-i;-) = 1.$$

Thus, we have $c_{2n+1-i} = -$.

2. If $c_i = -$, then we have $c_{2n+1-i} = +$ by the same argument of 1.

3. We will prove that $c_{2n+1-i} = c_{i'} \in \mathbf{N}$ for some $i' < 2n+1-i$ if $c_i \in \mathbf{N}$ and $c_i \neq c_j$ for all $j < i$. If $c_i \in \mathbf{N}$ and $c_i \neq c_j$ for all $j < i$, then there exists a $v_i \in \mathbf{C}^{2n}$ such that $\pi_+(v_i) \notin \pi_+(V_{i-1})$, $\pi_-(v_i) \notin \pi_-(V_{i-1})$, and
$$V_i = V_{i-1} \oplus \langle\, v_i \,\rangle.$$
Since
$$V_{2n-i+1,\pm} = (\pi_\mp(V_{i-1}) \oplus V_\pm)^\perp \quad \text{and}$$
$$V_{2n-i,\pm} = (\pi_\mp(V_i) \oplus V_\pm)^\perp = (\pi_\mp(V_{i-1}) \oplus \langle\, \pi_\mp(v_i) \,\rangle \oplus V_\pm)^\perp,$$
we have
$$(2n-i+1,\pm) - (2n-i,\pm) = 1.$$
Therefore, $c_{2n+1-i} = c_{i'} \in \mathbf{N}$ for some $i' < 2n+1-i$.
4. We will prove that $c_{2n+1-j} \in \mathbf{N}$ and $c_{j'} \neq c_{2n+1-j}$ for all $j' < 2n+1-j$ if $c_i = c_j \in \mathbf{N}$ for $i < j$. Then there exists a $v_i \in \mathbf{C}^{2n}$ such that $\pi_+(v_i) \notin \pi_+(V_{i-1})$, $\pi_-(v_i) \notin \pi_-(V_{i-1})$, and
$$V_i = V_{i-1} \oplus \langle\, v_i \,\rangle \subset V_j = V_{j-1} \oplus \langle\, \pi_+(v_i) \,\rangle.$$
Since $\pi_\pm(v_i) \in \pi_\pm(V_i) \subset \pi_\pm(V_{j-1})$, we have
$$\begin{aligned} V_{2n-j,\pm} &= (\pi_\mp(V_j) \oplus V_\pm)^\perp \\ &= ((\pi_\mp(V_{j-1}) + \langle\, \pi_\mp(v_i) \,\rangle) \oplus V_\pm)^\perp \\ &= (\pi_\mp(V_{j-1}) \oplus V_\pm)^\perp \;=\; V_{2n-j+1,\pm}. \end{aligned}$$
Thus,
$$(2n-j+1,\pm) - (2n-j,\pm) = 0.$$
Therefore $c_{2n+1-j} \in \mathbf{N}$ and $c_{2n+1-j} \neq c_{j'}$ for all $j' < 2n+1-j$.
5. It is enough to prove that $c_i = c_j \in \mathbf{N}$ if $c_{2n+1-i} = c_{2n+1-j} \in \mathbf{N}$ for $2n+1-i \leq j < i$. Let $2n+1-i \leq j < i$.

   We prove that by induction on i. Suppose that if $c_k = c_l \in \mathbf{N}$ for $2n+1-i < k < l < i$, then $c_{2n+1-k} = c_{2n+1-l} \in \mathbf{N}$.

   (a) We assume now $c_i = c_s \in \mathbf{N}$ for $s < j$ hoping to show a contradiction. Then there exist v_i, $v_s \in \mathbf{C}^{2n}$, such that
   $$V_{2n+1-i} = V_{2n-i} \oplus \langle\, v_i \,\rangle \;\subset\; V_{2n+1-j} = V_{2n-j} \oplus \langle\, \pi_+(v_i) \,\rangle$$
   and
   $$V_s = V_{s-1} \oplus \langle\, v_s \,\rangle \;\subset\; V_{j-1} \;\subset\; V_i = V_{i-1} \oplus \langle\, \pi_-(v_s) \,\rangle.$$
   Thus,
   $$V_{2n-i} = V_i^\perp \subset \langle\, \pi_-(v_s) \,\rangle^\perp. \tag{44}$$
   Since
   $$v_s \;\in\; V_s \;\subset\; V_{j-1} = V_{2n-j+1}^\perp = V_{2n-j}^\perp \cap \langle\, \pi_+(v_i) \,\rangle^\perp \;\subset\; \langle\, \pi_+(v_i) \,\rangle^\perp,$$
   we have
   $$\omega(\pi_-(v_s), v_i) = \omega(\pi_-(v_s), \pi_+(v_i)) = \omega(v_s, \pi_+(v_i)) = 0.$$
   So
   $$\langle\, v_i \,\rangle \subset \langle\, \pi_-(v_s) \,\rangle^\perp. \tag{45}$$

On the other hand,

$$\begin{aligned} V_{2n-i} &= V_i^{\perp} = V_{i-1}^{\perp} \cap \langle\, \pi_-(v_s)\,\rangle^{\perp} \\ &= V_{2n-i+1} \cap \langle\, \pi_-(v_s)\,\rangle^{\perp} = (V_{2n-i} \oplus \langle\, v_i\,\rangle) \cap \langle\, \pi_-(v_s)\,\rangle^{\perp} \\ &= V_{2n-i} \oplus \langle\, v_i\,\rangle = V_{2n-i+1} \end{aligned}$$

by (44) and (45). That contradicts $V_{2n-i} \neq V_{2n-i+1}$. We have proved $c_i \neq c_s$ for all $s < j$.

(b) We will assume that $c_i = c_s$ for $j < s < i$ and see what happens. Then $c_{2n+1-s} \in \mathbf{N}$ and there exists $1 \leq t < 2n+1-s$ such that $c_t = c_{2n+1-s}$. By assumptions, t satisfies $1 \leq t < 2n+1-i$. Then there exist v_t, $v_s \in \mathbf{C}^n$ satisfying

$$V_{t-1} \oplus \langle\, v_t\,\rangle \subset V_{2n-i} \subset V_{2n-s+1} = V_{2n-s} \oplus \langle\, \pi_+(v_t)\,\rangle$$

and

$$V_s = V_{s-1} \oplus \langle\, v_s\,\rangle \subset V_i = V_{i-1} \oplus \langle\, \pi_-(v_s)\,\rangle.$$

Thus,

$$V_{2n-s} = V_s^{\perp} \subset \langle\, v_s\,\rangle^{\perp}. \tag{46}$$

Since

$$v_t \in V_{2n-i} = V_i^{\perp} = V_{i-1}^{\perp} \cap \langle\, \pi_-(v_s)\,\rangle^{\perp} \subset \langle\, \pi_-(v_s)\,\rangle^{\perp},$$

we have

$$\begin{aligned} \omega(\pi_+(v_t), v_s) &= \omega(\pi_+(v_t), \pi_-(v_s)) \\ &= \omega(v_t, \pi_-(v_s)) = 0. \end{aligned}$$

So

$$\langle\, \pi_+(v_t)\,\rangle \subset \langle\, v_s\,\rangle^{\perp}. \tag{47}$$

On the other hand

$$\begin{aligned} V_{2n-s} &= V_s^{\perp} = V_{s-1}^{\perp} \cap \langle\, v_s\,\rangle^{\perp} \\ &= V_{2n-s+1} \cap \langle\, v_s\,\rangle^{\perp} \\ &= (V_{2n-s} \oplus \langle\, \pi_+(v_t)\,\rangle) \cap \langle\, v_s\,\rangle^{\perp} \\ &= V_{2n-s} \oplus \langle\, \pi_+(v_t)\,\rangle = V_{2n-s+1} \end{aligned}$$

by (46) and (47). That contradicts $V_{2n-s} \neq V_{2n-s+1}$.

Therefore $c_{2n+1-j} = c_{2n+1-i} \in \mathbf{N}$ if $c_i = c_j \in \mathbf{N}$ by induction. □

Definition 3.2.3 (Generalized clan)**.** We call a clan satisfying the condition in Proposition 3.2.2 a *generalized clan* for $Sp(n, \mathbf{R})$. We denote the set of generalized clans for $Sp(n, \mathbf{R})$ by $\mathcal{C}(Sp(n, \mathbf{R}))$:

$$\mathcal{C}(Sp(n, \mathbf{R})) = \left\{ \gamma \in \mathcal{C}(U(n,n)) \;\middle|\; {}^t\gamma = -\gamma \right\}.$$

Example 3.2.4. The set $\mathcal{C}(Sp(2, \mathbf{R}))$ consists of 11 generalized clans:

$$\left\{ \begin{array}{llll} + + - -, & + - + -, & - + - +, & - - + + \\ + 1 1 -, & - 1 1 +, & 1 + - 1, & 1 - + 1 \\ 1 1 2 2, & 1 2 1 2, & 1 2 2 1 & \end{array} \right\}.$$

We recall the notion of clans in [6] in order to compare our notion of generalized clans.

Definition 3.2.5. [6] A *clan* for $Sp(n,\mathbf{R})$ is an equivalence class of ordered sets $(c_1 \cdots c_n)$ of n symbols satisfying the following five conditions.

1. For $1 \le i \le n$, A symbol c_i is $+$, $-$, 0, a, or $\overline{a}$ for $a \in \mathbf{N}$.
2. If $c_i = a \in \mathbf{N}$, then there exists a unique $j \ne i$ with $c_j = a$ and $c_t \ne \overline{a}$ for all t.
3. If $c_i = \overline{a}$ for $a \in \mathbf{N}$, then there exists a unique $j \ne i$ with $c_j = \overline{a}$ and $c_t \ne a$ for all t.
4. If $c_i = a$ or $\overline{a}$ for $a \in \mathbf{N}$, $b \in \mathbf{N}$ and $b < a$, then there exists some j such that $c_j = b$ or $\overline{b}$.
5. Two ordered sets $(c_1 \ \cdots \ c_n)$ and $(c'_1 \ \cdots \ c'_n)$ are regarded as equivalent if and only if there exists a permutation $\sigma \in \mathfrak{S}_m$ for
$$m := \max\{\, b \in \mathbf{N} \mid b \text{ or } \overline{b} \text{ for some } i\,\}$$
such that
$$c_i = \begin{cases} \sigma(a) & \text{if } c'_i = a \in \mathbf{N}, \\ + & \text{if } c'_i = +, \\ - & \text{if } c'_i = -, \\ 0 & \text{if } c'_i = 0, \\ \overline{\sigma(a)} & \text{if } c'_i = \overline{a} \ (a \in \mathbf{N}) \end{cases}$$
for $1 \le i \le n$.

For example, $(\overline{2}\ 1\ 1\ +\ \overline{2}\ -) = (\overline{1}\ 2\ 2\ +\ \overline{1}\ -)$ as a clan.

The following one-to-one correspondence exists between clans for $Sp(n,\mathbf{R})$ and generalized clans for $Sp(n,\mathbf{R})$.

A clan $(c'_1 \ \cdots \ c'_n)$ corresponds to a generalized clan $(c_1 \ \cdots \ c_{2n})$ satisfying the following five conditions for $1 \le i, j \le n$:

1. If $c'_i = +$, then $(c_i, c_{2n+1-i}) = (+,-)$.
2. If $c'_i = -$, then $(c_i, c_{2n+1-i}) = (-,+)$.
3. If $c'_i = c'_j \in \mathbf{N}$, then $c_i = c_j \in \mathbf{N}$ and $c_{2n+1-i} = c_{2n+1-j} \in \mathbf{N}$.
4. If $c'_i = c'_j = \overline{a}$, then $c_i = c_{2n+1-j} \in \mathbf{N}$ and $c_{2n+1-i} = c_j \in \mathbf{N}$.
5. If $c_i = 0$, then $c_i = c_{2n+1-i} \in \mathbf{N}$.

For example, a clan $(\overline{1}\ 2\ 2\ +\ 0\ \overline{1})$ corresponds to a generalized clan
$$(1\ 2\ 2\ +\ 3\ 4\ 1\ 3\ -\ 5\ 5\ 4).$$

Therefore $\mathcal{C}(Sp(n,\mathbf{R}))$ is a parametrization of K-orbits in X by [6, Theorem 4.1].

Corollary 3.2.6. *Generalized clans in $\mathcal{C}(Sp(n,\mathbf{R}))$ parametrize K-orbits in the flag variety X via the correspondence in Proposition 2.2.6.*

Notation 3.2.7. For each generalized clan γ, let Q_γ be the K-orbit in the flag variety X corresponding to γ via the parametrization of Corollary 3.2.6.

3.2-2. *Representatives of $GL(n,\mathbf{C})$-orbits in $Sp(n,\mathbf{C})/B$.* We will give an element $g \in G$ such that the flag $x = gB$ in $X = Sp(n,\mathbf{C})/B$ corresponds to a generalized clan $\gamma \in \mathcal{C}(Sp(n,\mathbf{R}))$ and the flag $x = gB_{\mathrm{AIII}}$ in $GL(2n,\mathbf{C})/B_{\mathrm{AIII}}$ corresponds to the clan $\gamma \in \mathcal{C}(U(n,n))$.

Definition 3.2.8 (Skew symmetric signed clan)**.** A *skew symmetric signed clan* $\gamma_- = (d_1 \cdots d_m)$ of a generalized clan $\gamma = (c_1 \cdots c_m)$ is a signed clan (in Definition 2.2.9) of the clan γ satisfying the following condition.

$$\text{If } c_i = c_j = a \in \mathbf{N} \text{ for } i < j, \text{ then } (d_i, d_j) = (a_+, a_-). \tag{48}$$

Example 3.2.9. The skew symmetric signed clan of $\gamma = (1\,2\,1\,-\,3\,2\,+\,4\,3\,4)$ is

$$\gamma_- = (\ 1_+\ \ 2_+\ \ 1_-\ \ -\ \ 3_+\ \ 2_-\ \ +\ \ 4_+\ \ 3_-\ \ 4_-\ ).$$

Corollary 3.2.10. *In a skew symmetric signed clan $(d_1 \cdots d_m)$, the signature of d_i is not equal to the signature of d_{m+1-i} for $1 \le i \le m$.*

In this section, we call a skew symmetric signed clan just a *signed clan*.

We will give a representative $g \in G$ of $Q_\gamma = K \cdot g \cdot B$.

Theorem 3.2.11. *For $\gamma \in \mathcal{C}(Sp(n, \mathbf{R}))$ and the signed clan $\gamma_- = (d_1 \cdots d_{2n})$ of γ, fix a permutation $\sigma'' \in \mathfrak{S}_n$ and a permutation $\sigma' \in \mathfrak{S}_{2n}$ satisfying the following condition.*

For $i \le n$,

$$\sigma'(i) = \sigma''(i) \ \text{ and } \ \sigma'(2n+1-i) = 2n+1-\sigma''(i) \quad \text{if the signature of } d_i \text{ is plus,}$$

$$\sigma'(i) = 2n+1-\sigma''(i) \ \text{ and } \ \sigma'(2n+1-i) = \sigma''(i) \quad \text{if the signature of } d_i \text{ is minus.}$$

The matrix $g(\gamma) = (g_1\ g_2\ \cdots\ g_{2n}) \in \mathrm{Mat}(2n, 2n)$ is a representative of Q_γ, i.e., $Q_\gamma = Kg(\gamma)B$. Here $g_i \in V$ are column vectors defined as follows:

- *If $d_i = -$, then $g_i = e_{\sigma'(i)}$.*
- *If $d_i = +$, then*

$$g_i = \begin{cases} e_{\sigma'(i)} & \text{if } i \le n, \\ -e_{\sigma'(i)} & \text{if } n < i. \end{cases}$$

- *If $(d_i, d_j) = (a_+, a_-)$ for some $a \in \mathbf{N}$, then*

$$g_i = \begin{cases} \frac{1}{\sqrt{2}}(e_{\sigma'(i)} + e_{\sigma'(j)}) & \text{if } i \le n, \\ -\frac{1}{\sqrt{2}}(e_{\sigma'(i)} + e_{\sigma'(j)}) & \text{if } n < i, \end{cases}$$

  and

$$g_j = \frac{1}{\sqrt{2}}(-e_{\sigma'(i)} + e_{\sigma'(j)}).$$

Proof. The permutation σ' satisfies the conditions (13) of σ in Theorem 2.2.14 for $U(n,n)$. Thus, $g(\gamma)$ is a representative of K_{AIII}-orbit $Q_{\gamma\mathrm{AIII}}$, i.e., $Q_{\gamma\mathrm{AIII}} = K_{\mathrm{AIII}} \cdot g(\gamma) \cdot B_{\mathrm{AIII}}$, and $g(\gamma) \in G$. Therefore, we have

$$Q_\gamma = Kg(\gamma)B. \quad \square$$

Example 3.2.12. For a signed clan $\gamma_- = (\,1_+\,2_+\,-\,1_-\,+\,3_+\,2_-\,+\,4_+\,-\,3_-\,4_-\,)$, for example, if we choose an identity permutation as σ'';

$$\sigma'' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix},$$

then σ' is as follows:

$$\sigma' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 2 & 10 & 9 & 5 & 6 & 7 & 8 & 4 & 3 & 11 & 12 \end{pmatrix}.$$

Remark 3.2.13. The representative $g(\gamma)$ is real and orthogonal:

$$g(\gamma)^{-1} = {}^t g(\gamma).$$

3.3. **M-O graph for $Sp(n, \mathbf{R})$.** We define an oriented graph which has generalized clans as vertices and in which an edge is labeled by i for some $1 \leq i \leq n$. We can read off closure relations in the graph.

Definition 3.3.1. Let $\Gamma(Sp(n, \mathbf{R}))$ be the set of all triples (γ, γ', i) with $\gamma, \gamma' \in \mathcal{C}(Sp(n, \mathbf{R}))$ and $1 \leq i \leq n$, satisfying one of the following two conditions.

1. $(\gamma, \gamma', i) \in \Gamma(U(n, n))$. (This case happens only if $i = n$ or $i \neq n$ and $(c_i, c_{i+1}) = (c_{2n-i}, c_{2n+1-i}) = (a, b)$ for $a, b \in \mathbf{N}$.)
2. There exists a clan $\gamma'' \in \mathcal{C}(U(n, n))$ such that

$$\{(\gamma, \gamma'', i),\ (\gamma'', \gamma', 2n - i)\} \subset \Gamma(U(n, n)).$$

Example 3.3.2. Since $(+ - + -)$, $(+\,1\,1\,-) \in \mathcal{C}(Sp(2, \mathbf{R}))$, and

$$((+ - + -), (+\,1\,1\,-), 2) \in \Gamma(U(2, 2)),$$

$$((+ - + -), (+\,1\,1\,-), 2) \in \Gamma(Sp(2, \mathbf{R})).$$

Since $(1\,2\,1\,2)$, $(1\,2\,2\,1) \in \mathcal{C}(Sp(2, \mathbf{R}))$, and $((1\,2\,1\,2), (1\,2\,2\,1), 1) \in \Gamma(U(2, 2))$,

$$((1\,2\,1\,2), (1\,2\,2\,1), 1) \in \Gamma(Sp(2, \mathbf{R})).$$

Since $(+ - + -)$, $(1\,1\,2\,2) \in \mathcal{C}(Sp(2, \mathbf{R}))$, $(1\,1\,+\,-) \in \mathcal{C}(U(2, 2))$, and

$$\{((+ - + -), (1\,1\,+\,-), 1),\ ((1\,1\,+\,-), (1\,1\,2\,2), 3)\} \in \Gamma(U(2, 2)),$$

$$((+ - + -), (1\,1\,2\,2), 1) \in \Gamma(Sp(2, \mathbf{R})).$$

Notation 3.3.3. We denote by $B_{i\text{AIII}}$ the parabolic subgroup of G_{AIII} in Notation 2.4.8. We denote by B_i the parabolic subgroup of G for the root $-\alpha_i$ and all positive roots:

$$B_i = (B_{i\text{AIII}} B_{2n-i\text{AIII}}) \cap G.$$

Let

$$\pi_i : G/B \to G/B_i$$

be the canonical projection.

Remark 3.3.4. The projection π_i sends $(V_0, \dots, V_{2n}) \in G/B$ to

$$\begin{cases} (V_0, \dots, V_{i-1}, V_{i+1}, \dots, V_{2n-i-1}, V_{2n-i+1}, \dots, V_{2n}) & \text{if } 1 \leq i < n, \\ (V_0, \dots, V_{n-1}, V_{n+1}, \dots, V_{2n}) & \text{if } i = n. \end{cases}$$

By Theorem 2.2.14 and Remark 3.3.4, we have the following proposition.

Proposition 3.3.5. *If* $(\gamma, \gamma', i) \in \Gamma(Sp(n, \mathbf{R}))$, *then*

$$\pi_i(Q_\gamma) = \pi_i(Q_{\gamma'}).$$

Proposition 3.3.6. *If* $(\gamma, \gamma', i) \in \Gamma(Sp(n, \mathbf{R}))$*, then*

$$\dim Q_\gamma = \dim Q_{\gamma'} - 1$$

and

$$Kg(\gamma)B \subset \overline{Kg(\gamma')B}.$$

Proof. By [6], if $(\gamma, \gamma', i) \in \Gamma(Sp(n, \mathbf{R}))$, then we have

$$\dim Q_\gamma = \dim Q_{\gamma'} \pm 1$$

and

$$Kg(\gamma)B \subset \overline{Kg(\gamma')B}, \quad \text{or} \quad Kg(\gamma')B \subset \overline{Kg(\gamma)B}.$$

If $i = n$, then $(\gamma, \gamma', n) \in \Gamma(U(n, n))$. Therefore

$$K_{\mathrm{AIII}}g(\gamma)B_{\mathrm{AIII}} \subset \overline{K_{\mathrm{AIII}}g(\gamma')B_{\mathrm{AIII}}}.$$

Thus, we have $Kg(\gamma)B \subset \overline{Kg(\gamma')B}$.

If $i \neq n$ and both of (γ, γ'', i) and $(\gamma'', \gamma', 2n - i)$ are elements of $\Gamma(U(n, n))$ for some $\gamma'' \in \mathcal{C}(U(n, n))$, then we have $Kg(\gamma)B \subset \overline{Kg(\gamma')B}$ by the same argument of the case of $i = n$. Similarly, if $(\gamma, \gamma', i) \in \Gamma(U(n, n))$, then we have $Kg(\gamma)B \subset \overline{Kg(\gamma')B}$. Therefore $\dim Q_\gamma = \dim Q_{\gamma'} - 1$ for all $(\gamma, \gamma', i) \in \Gamma(Sp(n, \mathbf{R}))$. □

Although the definition of the following graph follows [6], we use generalized clans instead of clans as vertices.

Definition 3.3.7 (M-O graph of $Sp(n, \mathbf{R})$)**.** Let $\mathcal{C}_0(Sp(n, \mathbf{R})) = \mathcal{C}(Sp(n, \mathbf{R})) \cap \mathcal{C}_0(U(n, n))$:

$$\mathcal{C}_0(Sp(n, \mathbf{R})) = \{\, (c_1 \cdots c_n) \in \mathcal{C}(Sp(n, \mathbf{R})) \,|\, c_i \notin \mathbf{N} \text{ for all } 1 \leq i \leq n \,\}.$$

We give subsets $\Gamma_m(Sp(n, \mathbf{R}))$ of $\Gamma(Sp(n, \mathbf{R}))$ and subsets $\mathcal{C}_m(Sp(n, \mathbf{R}))$ of $\mathcal{C}(Sp(n, \mathbf{R}))$ for $m \geq 1$ by induction as follows:

$$\Gamma_m(Sp(n, \mathbf{R})) := \{(\gamma', \gamma, i) \in \Gamma'(Sp(n, \mathbf{R}))|\ \gamma' \in \mathcal{C}_{m-1}(Sp(n, \mathbf{R}))\},$$
$$\mathcal{C}_m(Sp(n, \mathbf{R})) := \{\gamma \in \mathcal{C}(Sp(n, \mathbf{R}))|\ (\gamma', \gamma, i) \in \Gamma_m(Sp(n, \mathbf{R}))\}.$$

The *M-O graph* with generalized clans of $Sp(n, \mathbf{R})$ is a finite oriented graph whose vertices are generalized clans in $\mathcal{C}(Sp(n, \mathbf{R}))$ and whose oriented edges are $\Gamma(Sp(n, \mathbf{R})) = \bigcup_{m \in \mathbf{N}} \Gamma_m(Sp(n, \mathbf{R}))$.

Example 3.3.8. The $\Gamma(Sp(2, \mathbf{R}))$ is as Figure 2.

Remark 3.3.9. If $(\gamma, \gamma', i) \in \Gamma(Sp(n, \mathbf{R}))$, then

$$\pi_i(Q_\gamma) = \pi_i(Q_{\gamma'}) \quad \text{and} \quad \dim Q_\gamma + 1 = \dim Q_{\gamma'}.$$

3.4. **Dimensions of $GL(n, \mathbf{C})$-orbits in $Sp(n, \mathbf{C})/B$.** In this section we will give a dimension formula of K-orbits for generalized clans.

Definition 3.4.1. We define the *length* $\ell_{\mathrm{C}}(\gamma)$ of a generalized clan $\gamma = (c_1 \cdots c_{2n}) \in \mathcal{C}(Sp(n, \mathbf{R}))$ such that

$$(49) \quad \ell_{\mathrm{C}}(\gamma) = \frac{1}{2}\left(\ell(\gamma) + \#\{t \in \mathbf{N} \,|\, c_s = c_t \in \mathbf{N} \quad \text{and} \quad s \leq n < t \leq 2n + 1 - s\}\right).$$

Where $\ell(\gamma)$ is the length of the clan γ which is given in Definition 2.3.7.

Proposition 3.4.2. *If* $\gamma \in \mathcal{C}_m(Sp(n, \mathbf{R}))$*, then* $\ell_{\mathrm{C}}(\gamma) = m$*.*

Proof. We prove the proposition by induction on m. Let $\gamma = (c_1 \cdots c_{2n})$.

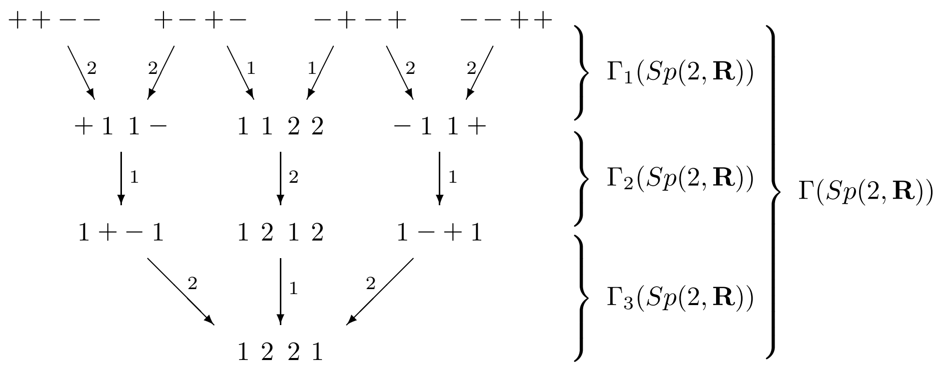

FIGURE 2

1. If $\gamma \in \mathcal{C}_0(Sp(n,\mathbf{R}))$, then $\ell_{\mathrm{C}}(\gamma)=0$.
2. Suppose $\ell_{\mathrm{C}}(\gamma')=j$, if $\gamma' \in \mathcal{C}_j(Sp(n,\mathbf{R}))$. For $\gamma \in \mathcal{C}_{j+1}(Sp(n,\mathbf{R}))$, there exists $\gamma'=(c'_1 \ \cdots \ c'_{2n}) \in \mathcal{C}_j(Sp(n,\mathbf{R}))$ such that $(\gamma',\gamma,i) \in \Gamma(Sp(n,\mathbf{R}))$.
   (a) If $i=n$, then $\ell(\gamma)=\ell(\gamma')+1$ and

$$\begin{aligned}&\#\{t \in \mathbf{N} \mid c_s = c_t \in \mathbf{N} \ \text{ and } \ s \leq n < t \leq 2n+1-s\} - \#\{\, n+1 \,\}\\ &\qquad = \#\{t \in \mathbf{N} \mid c'_s = c'_t \in \mathbf{N} \ \text{ and } \ s \leq n < t \leq 2n+1-s\}.\end{aligned}$$

   So, $\ell_{\mathrm{C}}(\gamma')=j+1$.
   (b) If $i \neq n$ and $(c_i, c_{i+1}) = (c_{2n-i}, c_{2n-i+1}) = (a,b)$ for some $a \in \mathbf{N}$ and $b \in \mathbf{N}$, then

$$\ell(\gamma)=\ell(\gamma')+1$$

   and

$$\begin{aligned}&\#\{t \in \mathbf{N} \mid c_s = c_t \in \mathbf{N} \ \text{ and } \ s \leq n < t \leq 2n+1-s\} - \#\{\, 2n+1-i \,\}\\ &\qquad = \#\{t \in \mathbf{N} \mid c'_s = c'_t \in \mathbf{N} \ \text{ and } \ s \leq n < t \leq 2n+1-s\}\end{aligned}$$

   (c) If $i \neq n$ and not $(c_i, c_{i+1}) = (c_{2n-i}, c_{2n-i+1}) = (a,b)$, then there exists a clan $\gamma'' \in \mathcal{C}(U(n,n))$ such that (γ',γ'',i), $(\gamma'',\gamma,2n-i) \in \Gamma(U(n,n))$. So, we have

$$\ell(\gamma)=\ell(\gamma'')+1=\ell(\gamma')+2$$

   and

$$\begin{aligned}&\#\{t \in \mathbf{N} \mid c_s = c_t \in \mathbf{N} \ \text{ and } \ s \leq n < t \leq 2n+1-s\}\\ &\qquad = \#\{t \in \mathbf{N} \mid c'_s = c'_t \in \mathbf{N} \ \text{ and } \ s \leq n < t \leq 2n+1-s\}.\end{aligned}$$

   Therefore $\ell_{\mathrm{C}}(\gamma)=j+1$.

   So, $\ell_{\mathrm{C}}(\gamma)=m$ if $\gamma \in \mathcal{C}_m(Sp(n,\mathbf{R}))$ for all $m \in \mathbf{N}$. □

A generalized clan

$$\gamma = (\ 1\ 2\ \cdots\ n\ n\ \cdots\ 2\ 1\ )$$

corresponds to the open orbit and

$$\ell_{\mathrm{C}}(\gamma) = \frac{1}{2}n(n+1).$$

Since the dimension of a closed orbit is $\frac{1}{2}n(n-1)$, we have the following dimension formulas.

Proposition 3.4.3. *For $\gamma \in \mathcal{C}(Sp(n, \mathbf{R}))$, we have the dimension and the codimension of $Q_\gamma = Kg(\delta)B \subset X$:*

$$\begin{aligned} \dim Q_\gamma &= \ell_{\mathrm{C}}(\gamma) + \tfrac{1}{2}n(n-1), \\ \operatorname{codim} Q_\gamma &= \tfrac{1}{2}n(n+1) - \ell_{\mathrm{C}}(\gamma). \end{aligned}$$

3.5. **Images of the moment map.** We denote $(\mathfrak{b}_{\mathrm{AIII}})^\perp$ by $\mathfrak{b}_{\mathrm{AIII}}^\perp$. We have

$$\begin{aligned} (\mathrm{Ad}(g)\mathfrak{b}^\perp) \cap \mathfrak{p} &= (\mathrm{Ad}(g) \cdot \mathfrak{b}_{\mathrm{AIII}}^\perp \cap \mathfrak{g}) \cap (\mathfrak{p}_{\mathrm{AIII}} \cap \mathfrak{g}) \\ &= ((\mathrm{Ad}(g) \cdot \mathfrak{b}_{\mathrm{AIII}}^\perp) \cap \mathfrak{p}_{\mathrm{AIII}}) \cap \mathfrak{g}, \end{aligned}$$

and elements X in $\mathfrak{p}$ satisfy

$$X_{ij} = X_{2n+1-j, 2n+1-i}.$$

Since we gave the image $(\mathrm{Ad}(g) \cdot \mathfrak{b}_{\mathrm{AIII}}^\perp) \cap \mathfrak{p}_{\mathrm{AIII}}$ of the moment map μ in Proposition 2.5.2 and $\sigma := \sigma'$ in Theorem 3.2.11, we get the following proposition.

Proposition 3.5.1. *For a generalized clan $\gamma = (c_1 \ \cdots \ c_{2n}) \in \mathcal{C}(Sp(n, \mathbf{R}))$, fix a permutation $\sigma' \in \mathfrak{S}_{2n}$ as in Theorem 3.2.11. Let the representative $g := g(\gamma)$ be given as in Theorem 3.2.11, and $x = gB \in Q_\gamma$. We can read off $\mu(T^*_{Q_\gamma}X)_x$ from a vector subspace $\mathrm{Dri}(\gamma)$ of $\mathfrak{g}$.*

$$\mu(T^*_{Q_\gamma}X)_x = \{\, Y_{(\sigma'^{-1}(1), \ldots, \sigma'^{-1}(2n))} \mid Y \in \mathrm{Dri}(\gamma) \}.$$

Here, $\mathrm{Dri}(\gamma)$ is the subspace of $\mathrm{Dri}(\gamma_-)$, the driving space of the signed clan γ_- for $U(n,n)$ in Definition 2.5.1, is defined as follows:

$$\begin{aligned} \mathrm{Dri}(\gamma) &:= \mathrm{Dri}(\gamma_-) \cap \mathrm{Ad}({\sigma'}^{-1}) \cdot \mathfrak{g} \\ &= \{\, Y \in \mathrm{Dri}(\gamma_-) \mid Y_{ij} = Y_{2n+1-j, 2n+1-i} \,\}. \end{aligned}$$

We call the space $\mathrm{Dri}(\gamma)$ the driving space *of the generalized clan γ (for $Sp(n, \mathbf{R})$).*

Proof. For the signed clan $\gamma_- = (d_1 \cdots d_{2n})$ and the permutation σ' in Theorem 3.2.11 and a representative $g_{\mathrm{AIII}}(\gamma_-, \sigma')$ in Theorem 2.2.14, we have

$$g = g(\gamma_-, \sigma') \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_{2n}).$$

Here,

$$\varepsilon_i = \begin{cases} -1 & \text{if } n < i \ \text{ and the signature of } \ d_i \ \text{ is plus,} \\ \ \ 1 & \text{otherwise.} \end{cases}$$

So,

$$\begin{aligned} &(\mathrm{Ad}(g)\mathfrak{b}_{\mathrm{AIII}}^\perp) \cap \mathfrak{p}_{\mathrm{AIII}} \\ &\quad = (\mathrm{Ad}(g(\gamma_-, \sigma'))(\mathrm{Ad}(\operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_{2n})) \cdot \mathfrak{b}_{\mathrm{AIII}}^\perp) \cap \mathfrak{p}_{\mathrm{AIII}} \\ &\quad = (\mathrm{Ad}(g(\gamma_-, \sigma')) \cdot \mathfrak{b}_{\mathrm{AIII}}^\perp) \cap \mathfrak{p}_{\mathrm{AIII}} = \mathrm{Ad}(\sigma') \cdot \mathrm{Dri}(\gamma_-) \end{aligned}$$

Therefore,

$$\begin{aligned} \mu(T^*_{Q_\gamma}X)_x &= (\mathrm{Ad}(\sigma')\cdot \mathrm{Dri}(\gamma_-))\cap \mathfrak{g} \\ &= \mathrm{Ad}(\sigma')\cdot(\mathrm{Dri}(\gamma_-)\cap \mathrm{Ad}({\sigma'}^{-1})\mathfrak{g}) \\ &= \mathrm{Ad}(\sigma')\cdot \mathrm{Dri}(\gamma). \end{aligned}$$

is as stated in the proposition. □

Corollary 3.5.2. *For the representative $g := g(\gamma)$ of Q_γ given as in Theorem 3.2.11, we have the following equation.*

$$\begin{aligned} \mathrm{Dri}(\gamma) &= \mathrm{Ad}({\sigma'}^{-1})((\mathrm{Ad}(g)\mathfrak{b}^\perp)\cap\mathfrak{p}) \\ &= \{\, A_{(\sigma'(1),\dots,\sigma'(2n))} \mid A\in(\mathrm{Ad}(g)\mathfrak{b}^\perp)\cap\mathfrak{p}\,\}. \end{aligned}$$

Since $g(\gamma) = (e_{\sigma'(1)}\ \cdots\ e_{\sigma'(2n)}) = \sigma'$ if a generalized clan γ corresponds to a closed orbit, we have the following lemma.

Lemma 3.5.3. *If a generalized clan γ is an element of $\mathcal{C}_0(Sp(n,\mathbf{R}))$, then*

$$\mathrm{Dri}(\gamma) = \mathfrak{b}^\perp \cap (\mathrm{Ad}(g(\gamma)^{-1})\cdot\mathfrak{p}).$$

Definition 3.5.4 (Driving matrix)**.** For a generalized clan $\gamma\in\mathcal{C}(Sp(n,\mathbf{R}))$, let $Y(\gamma)$ and $Y(\gamma,m)$, $1\le m\le 8$ be elements of $F\otimes_{\mathbf{C}}\mathrm{Mat}(2n,2n)$ satisfying the following conditions. Let $\gamma_- = (d_1\ \cdots\ d_{2n})$.

1. $Y(\gamma,1)_{ij} = \begin{cases} 1 & \text{if } j<i\le 2n+1-j,\\ 0 & \text{otherwise.}\end{cases}$
2. $Y(\gamma,2)_{ij} = \begin{cases} 0 & \text{if } (d_i,d_j)=(+,+) \text{ or } (-,-),\\ Y(\gamma,1)_{ij} & \text{otherwise.}\end{cases}$
3. If $(d_i,d_j)=(a_+,a_-)$, then
   - $Y(\gamma,3)_{i,k_1}=0$ if $1\le k_1\le i$,
   - $Y(\gamma,3)_{k_2,i}=Y(\gamma,3)_{j,k_2}=0$ if $i<k_2\le j$,
   - $Y(\gamma,3)_{k_3,j}=0$ if $j<k_3\le 2n$,

   for all $a\in\mathbf{N}$ and $Y(\gamma,3)_{i'j'}=Y(\gamma,2)_{i'j'}$ otherwise.

Remark 3.5.5. If $Y(\gamma,3)_{ij}=1$, then the following conditions are satisfied.

(a) $j<i\le 2n+1-j$ (so $j\le n$).
(b) $(d_i,d_j)=(a_-,b_+)$, $(a_-,+)$, $(a_-,-)$, $(+,a_+)$, $(-,a_+)$, $(+,-)$, or $(-,+)$ for some a, $b\in\mathbf{N}$, $a\ne b$.
(c) If $d_i=a_-$ for some $a\in\mathbf{N}$, then we have $d_k=a_+$ for some $j<k<i$.
(d) If $d_j=a_+$ for some $a\in\mathbf{N}$, then we have $d_k=a_-$ for some $j<k<i$.

4. If $Y(\gamma,3)_{ij}=1$, $(d_i,d_j)=(a_-,b_+)$ and $(d_k,d_l)=(a_+,b_-)$ for some a, $b\in\mathbf{N}$, then

$$Y(\gamma,4)_{ij}=Y(\gamma,4)_{2n+1-j,2n+1-i}=y_{ij},$$

$$Y(\gamma,4)_{kl}=Y(\gamma,4)_{2n+1-l,2n+1-k}=\begin{cases} y_{ij} & \text{if } j<k<l<i\\ y_{kl} & \text{if } j<l<k<i\end{cases}$$

for all a, $b\in\mathbf{N}$, and $Y(\gamma,4)_{i'j'}=Y(\gamma,3)_{i'j'}$ otherwise.

5. If $Y(\gamma,4)_{ij}=1$, $(d_i,d_j)=(a_-,-)$, and $d_k=a_+$ for $a\in\mathbf{N}$, then
$$\begin{aligned}&Y(\gamma,5)_{ij}=0 \quad \text{and}\\ &Y(\gamma,5)_{kj}=Y(\gamma,5)_{2n+1-j,2n+1-k} \;=\; y_{kj},\end{aligned}$$
and $Y(\gamma,5)_{i'j'}=Y(\gamma,4)_{i'j'}$ otherwise.
6. If $Y(\gamma,5)_{ij}=1$, $(d_i,d_j)=(+,a_+)$, and $d_k=a_-$ for $a\in\mathbf{N}$, then
$$\begin{aligned}&Y(\gamma,6)_{ij}=0 \quad \text{and}\\ &Y(\gamma,6)_{ik}=Y(\gamma,6)_{2n+1-k,2n+1-i}=y_{ik},\end{aligned}$$
and $Y(\gamma,6)_{i'j'}=Y(\gamma,5)_{i'j'}$ otherwise.
7. $Y(\gamma)_{ij}=Y(\gamma)_{2n+1-j,2n+1-i}=\begin{cases}y_{ij} & \text{if } Y(\gamma,6)_{ij}=1,\\ Y(\gamma,6)_{ij} & \text{otherwise.}\end{cases}$

We call $Y(\gamma)$ the *driving matrix* of γ (for $Sp(n,\mathbf{R})$).

By the same arguments of Proposition 2.5.8, we get the following proposition.

Proposition 3.5.6. *For a generalized clan $\gamma\in\mathcal{C}(Sp(n,\mathbf{R}))$, the Driving space* $\mathrm{Dri}(\gamma)$ *for $Sp(n,\mathbf{R})$ and the Driving matrix $Y(\gamma)$ of γ for $Sp(n,\mathbf{R})$ satisfy the following condition.*

$$\mathrm{Dri}(\gamma)=\{\,Y(\gamma)\mid y_{ij}\in\mathbf{C},\quad \textit{for}\;\; 1\le i,j\le 2n\,\}.$$

By Proposition 3.5.1 and Proposition 3.5.6, we got the following theorem which means we can read off $\mu(T^*_{Q_\gamma}X)_x$ from the driving matrix $Y(\gamma)$.

Theorem 3.5.7. *For a generalized clan $\gamma\in\mathcal{C}(Sp(n,\mathbf{R}))$, fix a permutation $\sigma'\in\mathfrak{S}_{2n}$ in Theorem 3.2.11. Let the representative $g:=g(\gamma)$ be given as in Theorem 3.2.11, and $x=gB\in Q_\gamma$. We can read off $\mu(T^*_{Q_\gamma}X)_x$ from the driving matrix $Y(\gamma)$ for $Sp(n,\mathbf{R})$ as follows:*

$$\mu(T^*_{Q_\gamma}X)_x=\{\,Y(\gamma)_{(\sigma'^{-1}(1),\dots,\sigma'^{-1}(2n))}\mid y_{ij}\in\mathbf{C}\;\;\textit{for}\;\;1\le i,j\le 2n\,\}.$$

3.6. **Signed Young diagrams.** In this section, after we recall a parametrization of nilpotent K-orbits in $\mathfrak{p}$, we give tables of signed Young diagrams for the clans of $Sp(1,\mathbf{R})$, $Sp(2,\mathbf{R})$, and $Sp(3,\mathbf{R})$ by way of examples.

Nilpotent K-orbits in $\mathfrak{p}$ are parametrized by signed Young diagrams of signature (n,n) satisfying the following conditions.

> If $m\in\mathbf{N}$ is odd, then the number of rows of which the length are m and which are labeled $+$ in the first column is equal to the number of rows of which the length are m and which are labeled $-$ in the first column.

For an element A of an orbit, the signed Young diagram of the orbit satisfies

$$\sum_{j=1}^{i}D_{j,+}=\dim(\ker(A^i|_{V_+}))\quad\text{and}\quad\sum_{j=1}^{i}D_{j,-}=\dim(\ker(A^i|_{V_-})).$$

Proposition 3.6.1. *Under the conditions of Theorem 3.2.11, we put*

$$\begin{aligned}V'_+ &:= {\sigma'}^{-1}V_+=(e_{\sigma'^{-1}(1)}\cdots e_{\sigma'^{-1}(2n)})V_+\\ &= \langle e_{\sigma'^{-1}(1)},\cdots,e_{\sigma'^{-1}(n)}\rangle,\\ V'_- &:= {\sigma'}^{-1}V_-=(e_{\sigma'^{-1}(1)}\cdots e_{\sigma'^{-1}(2n)})V_-\\ &= \langle e_{\sigma'^{-1}(n+1)},\dots,e_{\sigma'^{-1}(2n)}\rangle.\end{aligned}$$

By the same argument of the case of $U(n,n)$, we have the following proposition.

Proposition 3.6.2. *Let $Y(\gamma)^0 = I_n$. We have the following equations:*

$$D_{i,+} = \dim_F\{\,\vec{a} \in F \otimes_{\mathbf{C}} V'_+ \mid Y(\gamma)^i \vec{a} = \vec{0}\,\} - \dim_F\{\,\vec{a} \in F \otimes_{\mathbf{C}} V'_+ \mid Y(\gamma)^{i-1} \vec{a} = \vec{0}\,\},$$
$$D_{i,-} = \dim_F\{\,\vec{b} \in F \otimes_{\mathbf{C}} V'_- \mid Y(\gamma)^i \vec{b} = \vec{0}\,\} - \dim_F\{\,\vec{b} \in F \otimes_{\mathbf{C}} V'_- \mid Y(\gamma)^{i-1} \vec{b} = \vec{0}\,\}.$$

At last we give tables of signed Young diagrams for the clans of $Sp(1,\mathbf{R})$ in Example 3.6.3, $Sp(2,\mathbf{R})$ in Example 3.6.4, and $Sp(2,\mathbf{R})$ in Example 3.6.5.

Example 3.6.3. This is the table of the case of $G_{\mathbf{R}} = Sp(1,\mathbf{R})$.

clan γ	a representative $g(\gamma)$ of K-orbit	driving matrix $Y(\gamma)$	signed Young diagram
$+\,-$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ y_{21} & 0 \end{pmatrix}$	$\boxed{-}\boxed{+}$
$-\,+$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ y_{21} & 0 \end{pmatrix}$	$\boxed{+}\boxed{-}$
$1\ 1$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{array}{c}\boxed{+}\\ \boxed{-}\end{array}$

Example 3.6.4. This is the table of the case of $G_{\mathbf{R}} = Sp(2, \mathbf{R})$.

clan γ	a representative $g(\gamma)$ of K-orbit	driving matrix $Y(\gamma)$	signed Young diagram
$+-+-$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 \\ 0 & y_{32} & 0 & 0 \\ y_{41} & 0 & y_{21} & 0 \end{pmatrix}$	$\begin{array}{\|c\|c\|c\|c\|}\hline - & + & - & + \\ \hline\end{array}$
$-+-+$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 \\ 0 & y_{32} & 0 & 0 \\ y_{41} & 0 & y_{21} & 0 \end{pmatrix}$	$\begin{array}{\|c\|c\|c\|c\|}\hline + & - & + & - \\ \hline\end{array}$
$++--$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_{31} & y_{32} & 0 & 0 \\ y_{41} & y_{31} & 0 & 0 \end{pmatrix}$	$\begin{array}{\|c\|c\|}\hline - & + \\ \hline - & + \\ \hline\end{array}$
$+\ 1\ 1\ -$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 \\ y_{41} & y_{31} & 0 & 0 \end{pmatrix}$	
$--++$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_{31} & y_{32} & 0 & 0 \\ y_{41} & y_{31} & 0 & 0 \end{pmatrix}$	$\begin{array}{\|c\|c\|}\hline + & - \\ \hline + & - \\ \hline\end{array}$
$-\ 1\ 1\ +$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 \\ y_{41} & y_{31} & 0 & 0 \end{pmatrix}$	
$1\ 1\ 2\ 2$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & y_{32} & 0 & 0 \\ y_{41} & 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{\|c\|c\|}\hline + & - \\ \hline - & + \\ \hline\end{array}$
$1\ 2\ 1\ 2$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & y_{41} & 0 \\ 0 & 0 & 0 & 0 \\ y_{41} & 0 & 0 & 0 \end{pmatrix}$	

clan γ	a representative $g(\gamma)$ of K-orbit	driving matrix $Y(\gamma)$	signed Young diagram
$1 + - 1$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & y_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{\|c\|c\|} - & + \\ + \\ - \end{array}$
$1 - + 1$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & y_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{\|c\|c\|} + & - \\ + \\ - \end{array}$
$1\ 2\ 2\ 1$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{\|c\|} + \\ + \\ - \\ - \end{array}$

Example 3.6.5. This is the table of the case of $G_{\mathbf{R}} = Sp(3, \mathbf{R})$.

clan γ	signed Young diagram
$+ - + - + -$	$- + - + - +$
$- + - + - +$	$+ - + - + -$
$+ - - + + -$	$- + - +$ $+ -$
$+ - 1\ 1 + -$	
$1\ 1 - + 2\ 2$	
$- + + - - +$	$+ - + -$ $- +$
$- + 1\ 1 - +$	
$1\ 1 + - 2\ 2$	
$+ - - + + -$	$- + - +$ $- +$
$+ 1\ 1\ 2\ 2 -$	
$+ 1\ 2\ 1\ 2 -$	
$- - + - + +$	$+ - + -$ $+ -$
$- 1\ 1\ 2\ 2 +$	
$- 1\ 2\ 1\ 2 +$	
$1\ 1\ 2\ 2\ 3\ 3$	$+ - +$ $- + -$
$1\ 2\ 1\ 3\ 2\ 3$	
$1\ 2\ 3\ 1\ 2\ 3$	
$+ 1 - + 1 -$	$- + - +$ $+$ $-$
$1 + - + - 1$	
$- 1 + - 1 +$	$+ - + -$ $+$ $-$
$1 - + - + 1$	
$+ + + - - -$	$- +$ $- +$ $- +$
$+ + 1\ 1 - -$	
$+ 1 + - 1 -$	
$- - - + + +$	$+ -$ $+ -$ $+ -$
$- - 1\ 1 + +$	
$- 1 - + 1 +$	

clan γ	signed Young diagram
$1 + 1\ 2 - 2$	$+ -$ $- +$ $- +$
$1 + 2\ 1 - 2$	
$1\ 2 + - 1\ 2$	
$1 - 1\ 2 + 2$	$+ -$ $+ -$ $- +$
$1 - 2\ 1 + 2$	
$1\ 2 - + 1\ 2$	
$1 + + - - 1$	$- +$ $- +$ $+$ $-$
$+ 1\ 2\ 2\ 1 -$	
$1 + 2\ 2 - 1$	
$1 - - + + 1$	$+ -$ $+ -$ $+$ $-$
$- 1\ 2\ 2\ 1 +$	
$1 - 2\ 2 + 1$	
$1\ 2\ 2\ 3\ 3\ 1$	$+ -$ $- +$ $+$ $-$
$1\ 2\ 3\ 3\ 1\ 2$	
$1\ 2\ 3\ 2\ 3\ 1$	
$1\ 2 + - 2\ 1$	$- +$ $+$ $+$ $-$ $-$
$1\ 2 - + 2\ 1$	$+ -$ $+$ $+$ $-$ $-$
$1\ 2\ 3\ 3\ 2\ 1$	$+$ $+$ $+$ $-$ $-$ $-$

4. The case of $Sp(p,q)$

In this section, we treat the case of $G_{\mathbf{R}} = Sp(p,q)$, i.e., $Sp(p,\mathbf{C})\times Sp(q,\mathbf{C})$-orbits in the flag variety of $Sp(n,\mathbf{C})$. Here, $n = p+q$. We will apply the case of $U(2p,2q)$ to the case of $Sp(p,q)$. For this, we change the realization of $U(2p,2q)$. We denote $U(2p,2q)$ by $G_{\mathbf{R}\mathrm{AIII}}$.

4.1. **Changing the realization of $U(2p,2q)$.** From now on, we use the following realization of $U(2p,2q)$.

We realize $G_{\mathbf{R}\mathrm{AIII}} = U(2p,2q)$ as the group of matrices g in $GL(2n,\mathbf{C})$ which leave invariant the Hermitian form of the signature $(2p,2q)$

$$x_1\overline{x_1} + \cdots + x_p\overline{x_p} - x_{p+1}\overline{x_{p+1}} - \cdots - x_{p+2q}\overline{x_{p+2q}} + x_{p+2q+1}\overline{x_{p+2q+1}} + \cdots + x_{2n}\overline{x_{2n}}, \tag{50}$$

i.e.,

$$U(2p,2q) = \left\{ g \in GL(2n,\mathbf{C}) \,\middle|\, {}^t g \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix} \bar{g} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix} \right\}.$$

We fix a Cartan involution θ of $G_{\mathbf{R}\mathrm{AIII}}$ as follows:

$$\theta : g \mapsto \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix} \cdot g \cdot \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix}$$

for $g \in G_{\mathbf{R}\mathrm{AIII}}$. Then we have

$$\mathfrak{k}_{\mathrm{AIII}} = \left\{ \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & 0 \\ K_{31} & 0 & K_{33} \end{pmatrix} \,\middle|\, \begin{array}{l} K_{11},\ K_{13},\ K_{31},\ K_{33} \in \mathrm{Mat}(p,p) \\ K_{22} \in \mathrm{Mat}(2q,2q) \end{array} \right\},$$

$$K_{\mathrm{AIII}} = \left\{ \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & 0 \\ K_{31} & 0 & K_{33} \end{pmatrix} \,\middle|\, \begin{array}{l} K_{11},\ K_{13},\ K_{31},\ K_{33} \in \mathrm{Mat}(p,p) \\ \begin{pmatrix} K_{11} & K_{13} \\ K_{31} & K_{33} \end{pmatrix} \in GL(2p,\mathbf{C}) \\ K_{22} \in GL(2q,\mathbf{C}) \end{array} \right\}, \tag{51}$$

and

$$\mathfrak{p}_{\mathrm{AIII}} = \left\{ \begin{pmatrix} 0 & P_{12} & 0 \\ P_{21} & 0 & P_{23} \\ 0 & P_{32} & 0 \end{pmatrix} \,\middle|\, \begin{array}{l} P_{12}, P_{32} \in \mathrm{Mat}(p,2q) \\ P_{21}, P_{23} \in \mathrm{Mat}(2q,p) \end{array} \right\}.$$

Notation 4.1.1. Let $V = \mathbf{C}^{2n}$ and θ be an involution of V such that

$$\theta : v \mapsto \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix} v.$$

Let V_+ and V_- be the eigenspaces in V under θ for eigenvalues $+1$ and -1 respectively:

$$\begin{aligned} V_+ &:= \langle e_1, \ldots, e_p, e_{p+2q+1}, \ldots, e_{2n} \rangle, \\ V_- &:= \langle e_{p+1}, \ldots, e_{p+2q} \rangle. \end{aligned}$$

Under this realization, Theorem 2.2.14 changes into the following proposition.

Proposition 4.1.2. *For* $\gamma \in \mathcal{C}(U(2p, 2q))$ *and a signed clan* $\delta = (d_1 \ldots d_{2n})$ *of* γ, *fix a permutation* $\sigma \in \mathfrak{S}_{2n}$ *satisfying the following condition.*

$$(52)\quad \begin{array}{ll} 1 \le \sigma(i) \le p \ \text{ or } \ p+2q+1 \le \sigma(i) \le 2n & \text{if the signature of } d_i \text{ is plus,} \\ p+1 \le \sigma(i) \le p+2q & \text{if the signature of } d_i \text{ is minus.} \end{array}$$

Then the matrix $g_{\mathrm{AIII}}(\delta) := g_{\mathrm{AIII}}(\delta, \sigma) = (g_1\ g_2\ \ldots\ g_n)$ *is a representative of* Q_γ. *Here* $g_i \in V$ *are the column vectors defined as follows*:

- *If* $c_i = \pm$, *then* $g_i = e_{\sigma(i)}$.
- *If* $c_i = a_+$, $c_j = a_-$, *then*

$$g_i = \frac{1}{\sqrt{2}}(\, e_{\sigma(i)} + e_{\sigma(j)}\,) \quad \text{and} \quad g_j = \frac{1}{\sqrt{2}}(\, -e_{\sigma(i)} + e_{\sigma(j)}\,).$$

4.2. **The case of** $Sp(p, q)$**.** We use the realization of $Sp(n, \mathbf{C})$ as (41) in Section 3. We realize an indefinite symplectic group $G_{\mathbf{R}} = Sp(p, q)$ as a group of matrices g in $Sp(n, \mathbf{C})$ which leave invariant the Hermitian form (50) of the signature $(2p, 2q)$, i.e.,

$$Sp(p,q) = \left\{ g \in Sp(n,\mathbf{C}) \;\middle|\; {}^t g \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix} \bar{g} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix} \right\}.$$

We fix a Cartan involution θ of $G_{\mathbf{R}}$ as follows:

$$\theta : g \ \mapsto \ \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix} g \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2q} & 0 \\ 0 & 0 & I_p \end{pmatrix}.$$

Then we have

$$\mathfrak{k} = \mathfrak{g} \cap \mathfrak{k}_{\mathrm{AIII}}, \qquad \mathfrak{p} = \mathfrak{g} \cap \mathfrak{p}_{\mathrm{AIII}}, \qquad \mathfrak{b} = \mathfrak{g} \cap \mathfrak{b}_{\mathrm{AIII}},$$

and

$$\begin{aligned} K &= \left\{ \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & 0 \\ K_{31} & 0 & K_{33} \end{pmatrix} \;\middle|\; \begin{array}{l} K_{11},\ K_{13},\ K_{31},\ K_{33} \in \mathrm{Mat}(p,p) \\ \begin{pmatrix} K_{11} & K_{13} \\ K_{31} & K_{33} \end{pmatrix} \in Sp(p,\mathbf{C}) \\ K_{22} \in Sp(q,\mathbf{C}) \end{array} \right\} \\ &\simeq \ Sp(p,\mathbf{C}) \times Sp(q,\mathbf{C}). \end{aligned}$$

Remark 4.2.1. We have $(V_+)^\perp = V_-$ and $(V_-)^\perp = V_+$.

Remark 4.2.2. We fix a G-equivariant natural isomorphism between X and G/B via the manner in Remark 2.1.2.

4.3. **A symbolic parametrization of** $Sp(p, \mathbf{C}) \times Sp(q, \mathbf{C})$**-orbits.** In this section we give a symbolic parametrization of K-orbits in X and give representatives of K-orbits for the parameters. Our parameters are an improvement of the Matsuki-Oshima's parameters [6]. The set of parameters for $G_{\mathbf{R}} = Sp(p, q)$ is a subset of $\mathcal{C}(U(2p, 2q))$. The parameter we give for a K-orbit Q in the flag variety for $G_{\mathbf{R}} = Sp(p, q)$ coincides with a clan in $\mathcal{C}(U(2p, 2q))$ which includes Q.

4.3-1. *Generalized clans for $Sp(p,q)$.* We give a subset of $\mathcal{C}(U(2p,2q))$. The subset parametrizes K-orbits in the flag variety X.

Proposition 4.3.1. *For a flag $x = (V_0, V_1, \dots, V_{2n}) \in X$, there exists a clan $\gamma = (c_1 \ \cdots \ c_{2n}) \in \mathcal{C}(U(2p,2q))$ via the conditions in Proposition 2.2.6.*

Proposition 4.3.2. *Let $x \in X$ and $\gamma = (c_1 \ \cdots \ c_{2n}) \in \mathcal{C}(U(2p,2q))$ be the clan of x. Then the clan γ is symmetric, i.e., ${}^t\gamma = \gamma$ and $i + j \neq 2n+1$ if $c_i = c_j \in \mathbf{N}$.*

Proof. We remark that

$$\begin{aligned} V_{2n-i,\pm} &= V_{2n-i} \cap V_{\pm} = (V_i)^{\perp} \cap (V_{\mp})^{\perp} \\ &= (V_i + V_{\mp})^{\perp} = (\pi_{\pm}(V_i) \oplus V_{\mp})^{\perp}. \end{aligned}$$

1. If $c_i = +$, then there exists a vector v_+ in V_+ such that $v_+ \notin \pi_+(V_{i-1})$ and $V_i = V_{i-1} \oplus \langle v_+ \rangle$. Since

$$\begin{aligned} V_{2n-i+1,+} &= (\pi_+(V_{i-1}) \oplus V_-)^{\perp}, \quad \text{and} \\ V_{2n-i,+} &= (\pi_+(V_i) \oplus V_-)^{\perp}, \\ &= (\pi_+(V_{i-1}) \oplus \langle v_+ \rangle \oplus V_-)^{\perp}, \end{aligned}$$

we have

$$(2n-i+1;+) - (2n-i;+) = 1.$$

Since

$$\begin{aligned} V_{2n-i+1,-} &= (\pi_-(V_{i-1}) \oplus V_+)^{\perp} \quad \text{and} \\ V_{2n-i,-} &= (\pi_-(V_i) \oplus V_+)^{\perp} \\ &= (\pi_-(V_{i-1}) \oplus V_+)^{\perp}, \end{aligned}$$

we have

$$(2n-i+1;-) - (2n-i;-) = 0.$$

Thus, we have $c_{2n+1-i} = +$.

2. If $c_i = -$, then we have $c_{2n+1-i} = -$ by the same argument of 1.
3. We prove $(c_i, c_{2n+1-i}) \neq (a,a)$ for any $a \in \mathbf{N}$. Suppose $c_i = c_{2n+1-i} \in \mathbf{N}$ and $1 \leq i \leq n$, then there exists a vector v_i in $\mathbf{C}^{2n}$ such that

$$V_i = V_{i-1} \oplus \langle v_i \rangle \subset V_{2n-i+1} = V_{2n-i} \oplus \langle \pi_+(v_i) \rangle.$$

Then we have

$$\begin{aligned} v_i \notin V_{i-1} &= V_{2n-i+1}^{\perp} \\ &= (V_{2n-i} \oplus \langle \pi_+(v_i) \rangle)^{\perp} \\ &= V_{2n-i}^{\perp} \cap \langle \pi_+(v_i) \rangle^{\perp}. \end{aligned}$$

That contradicts

$$v_i \in V_i = V_{2n-i}^{\perp} \quad \text{and} \quad v_i \in \langle \pi_+(v_i) \rangle^{\perp}.$$

So, we got $(c_i, c_j) \neq (a,a)$ for any $a \in \mathbf{N}$.

4. We will prove that $c_{2n+1-i} = c_{i'} \in \mathbf{N}$ for some $i' < 2n+1-i$ if $c_i \in \mathbf{N}$ and $c_i \neq c_j$ for all $j < i$. If $c_i \in \mathbf{N}$ and $c_i \neq c_j$ for all $j < i$, then there exists a $v_i \in \mathbf{C}^{2n}$ such that $\pi_+(v_i) \notin \pi_+(V_{i-1})$, $\pi_-(v_i) \notin \pi_-(V_{i-1})$, and
$$V_i = V_{i-1} \oplus \langle\, v_i \,\rangle.$$
Since
$$V_{2n-i+1,\pm} = (\pi_\pm(V_{i-1}) \oplus V_\mp)^\perp \quad \text{and}$$
$$V_{2n-i,\pm} = (\pi_\pm(V_i) \oplus V_\mp)^\perp = (\pi_\pm(V_{i-1}) \oplus \langle\, \pi_\pm(v_i) \,\rangle \oplus V_\mp)^\perp$$
we have
$$(2n-i+1, \pm) - (2n-i, \pm) = 1.$$
Therefore, $c_{2n+1-i} = c_{i'} \in \mathbf{N}$ for some $i' < 2n+1-i$.
5. We will prove that $c_{2n+1-j} \in \mathbf{N}$ and $c_{j'} \neq c_{2n+1-j}$ for all $j' < 2n+1-j$ if $c_i = c_j \in \mathbf{N}$ for $i < j$. Then there exists a $v_i \in \mathbf{C}^{2n}$ such that $\pi_+(v_i) \notin \pi_+(V_{i-1})$, $\pi_-(v_i) \notin \pi_-(V_{i-1})$, and
$$V_i = V_{i-1} \oplus \langle\, v_i \,\rangle \subset V_j = V_{j-1} \oplus \langle\, \pi_+(v_i) \,\rangle.$$
Since $\pi_\pm(v_i) \in \pi_\pm(V_i) \subset \pi_\pm(V_{j-1})$, we have
$$\begin{aligned} V_{2n-j,\pm} &= (\pi_\pm(V_j) \oplus V_\mp)^\perp \\ &= ((\pi_\pm(V_{j-1}) + \langle\, \pi_\pm(v_i) \,\rangle) \oplus V_\mp)^\perp \\ &= (\pi_\pm(V_{j-1}) \oplus V_\mp)^\perp \;=\; V_{2n-j+1,\pm}. \end{aligned}$$
Thus,
$$(2n-j+1, \pm) - (2n-j, \pm) = 0.$$
Therefore $c_{2n+1-j} \in \mathbf{N}$ and $c_{2n+1-j} \neq c_{j'}$ for all $j' < 2n+1-j$.
6. It is enough to prove that $c_i = c_j \in \mathbf{N}$ if $c_{2n+1-i} = c_{2n+1-j} \in \mathbf{N}$ for $2n+1-i \leq j < i$. Let $2n+1-i \leq j < i$.

   We prove that by induction on i. We suppose that if $c_k = c_l \in \mathbf{N}$ for $2n+1-i < k < l < i$, then $c_{2n+1-k} = c_{2n+1-l} \in \mathbf{N}$.
   (a) We assume now $c_i = c_s \in \mathbf{N}$ for $s < j$ hoping to show a contradiction. Then there exist v_i, $v_s \in \mathbf{C}^{2n}$, such that
$$V_{2n+1-i} = V_{2n-i} \oplus \langle\, v_i \,\rangle \subset V_{2n+1-j} = V_{2n-j} \oplus \langle\, \pi_+(v_i) \,\rangle \quad \text{and}$$
$$V_s = V_{s-1} \oplus \langle\, v_s \,\rangle \subset V_{j-1} \subset V_i = V_{i-1} \oplus \langle\, \pi_+(v_s) \,\rangle.$$
   Thus,
$$V_{2n-i} = V_i^\perp \subset \langle\, \pi_+(v_s) \,\rangle^\perp. \tag{53}$$
   Since
$$v_s \in V_s \subset V_{j-1} = V_{2n-j+1}^\perp = V_{2n-j}^\perp \cap \langle\, \pi_+(v_i) \,\rangle^\perp \subset \langle\, \pi_+(v_i) \,\rangle^\perp,$$
   we have
$$\omega(\pi_+(v_s), v_i) = \omega(\pi_+(v_s), \pi_+(v_i)) = \omega(v_s, \pi_+(v_i)) = 0.$$
   So
$$\langle\, v_i \,\rangle \subset \langle\, \pi_+(v_s) \,\rangle^\perp. \tag{54}$$

On the other hand,

$$\begin{aligned} V_{2n-i} &= V_i^{\perp} = V_{i-1}^{\perp} \cap \langle \pi_+(v_s) \rangle^{\perp} \\ &= V_{2n-i+1} \cap \langle \pi_+(v_s) \rangle^{\perp} = (V_{2n-i} \oplus \langle v_i \rangle) \cap \langle \pi_+(v_s) \rangle^{\perp} \\ &= V_{2n-i} \oplus \langle v_i \rangle = V_{2n-i+1} \end{aligned}$$

by (44) and (45). That contradicts $V_{2n-i} \neq V_{2n-i+1}$. We have proved $c_i \neq c_s$ for all $s < j$.

(b) We will assume that $c_i = c_s$ for $j < s < i$ and see what happens. Then $c_{2n+1-s} \in \mathbf{N}$ and there exists $1 \leq t < 2n+1-s$ such that $c_t = c_{2n+1-s}$. By assumptions, t satisfies $1 \leq t < 2n+1-i$. Then there exist v_t, $v_s \in \mathbf{C}^n$ satisfying

$$V_{t-1} \oplus \langle v_t \rangle \subset V_{2n-i} \subset V_{2n-s+1} = V_{2n-s} \oplus \langle \pi_+(v_t) \rangle$$

and

$$V_s = V_{s-1} \oplus \langle v_s \rangle \subset V_i = V_{i-1} \oplus \langle \pi_+(v_s) \rangle.$$

Thus,

$$V_{2n-s} = V_s^{\perp} \subset \langle v_s \rangle^{\perp}. \tag{55}$$

Since

$$v_t \in V_{2n-i} = V_i^{\perp} = V_{i-1}^{\perp} \cap \langle \pi_+(v_s) \rangle^{\perp} \subset \langle \pi_+(v_s) \rangle^{\perp},$$

we have

$$\begin{aligned} \omega(\pi_+(v_t), v_s) &= \omega(\pi_+(v_t), \pi_+(v_s)) \\ &= \omega(v_t, \pi_+(v_s)) = 0. \end{aligned}$$

So

$$\langle \pi_+(v_t) \rangle \subset \langle v_s \rangle^{\perp}. \tag{56}$$

On the other hand

$$\begin{aligned} V_{2n-s} &= V_s^{\perp} = V_{s-1}^{\perp} \cap \langle v_s \rangle^{\perp} \\ &= V_{2n-s+1} \cap \langle v_s \rangle^{\perp} \\ &= (V_{2n-s} \oplus \langle \pi_+(v_t) \rangle) \cap \langle v_s \rangle^{\perp} \\ &= V_{2n-s} \oplus \langle \pi_+(v_t) \rangle = V_{2n-s+1} \end{aligned}$$

by (46) and (47). That contradicts $V_{2n-s} \neq V_{2n-s+1}$.

Therefore $c_{2n+1-j} = c_{2n+1-i} \in \mathbf{N}$ if $c_i = c_j \in \mathbf{N}$ by induction. □

Definition 4.3.3 (Generalized clan)**.** We call a clan satisfying the condition in Proposition 4.3.2 a *generalized clan* for $Sp(p,q)$. We denote the set of generalized clans for $Sp(p,q)$ by $\mathcal{C}(Sp(p,q))$:

$$\mathcal{C}(Sp(p,q)) = \left\{ \gamma \in \mathcal{C}(U(2p,2q)) \;\middle|\; \begin{array}{l} {}^t\gamma = \gamma \text{ and } \; i+j \neq 2n+1 \\ \text{if } \; c_i = c_j \notin \mathbf{N} \end{array} \right\}.$$

Example 4.3.4. The set $\mathcal{C}(Sp(1,1))$ consists of four generalized clans:

$$\left\{ \begin{matrix} + & - & - & +, & - & + & + & - \\ 1 & 1 & 2 & 2, & 1 & 2 & 1 & 2 \end{matrix} \right\}.$$

Example 4.3.5. The set $\mathcal{C}(Sp(2,1))$ consists of nine generalized clans:

$$\left\{\begin{array}{cccccc} + & + & - & - & + & +, \\ + & 1 & 1 & 2 & 2 & +, \\ + & 1 & 2 & 1 & 2 & +, \end{array}\quad\begin{array}{cccccc} + & - & + & + & - & +, \\ 1 & + & 1 & 2 & + & 2, \\ 1 & + & 2 & 1 & + & 2, \end{array}\quad\begin{array}{cccccc} - & + & + & + & + & - \\ 1 & 1 & + & + & 2 & 2 \\ 1 & 2 & + & + & 1 & 2 \end{array}\right\}.$$

We recall the definition of clans in [6] in order to compare our notion of generalized clans.

Definition 4.3.6. [6] A *clan* for $Sp(p,q)$ is an equivalence class of ordered sets $(c_1 \ \cdots \ c_n)$ of n symbols satisfying the following six conditions.

1. For $1 \leq i \leq n$, a symbol c_i is $+$, $-$, a, or $\overline{a}$ for $a \in \mathbf{N}$.
2. If $c_i = a \in \mathbf{N}$, then there exists a unique $j \neq i$ with $c_j = a$ and $c_t \neq \overline{a}$ for all t.
3. If $c_i = \overline{a}$ for $a \in \mathbf{N}$, then there exists a unique $j \neq i$ with $c_j = \overline{a}$ and $c_t \neq a$ for all t.
4. The difference between numbers of $+$ and $-$ in clans $(c_1 \ \cdots \ c_n)$ coincides with half of the difference of signatures of the Hermitian form defining the group $G_{\mathbf{R}}$:
$$\#\{\, i \mid c_i = +\} - \#\{\, i \mid c_i = -\} = p - q.$$
5. If $c_i = a$ or $\overline{a}$ for $a > 1$, then there exist some j such that $c_j = a-1$ or $\overline{a-1}$.
6. Two ordered sets $(c_1 \ \cdots \ c_n)$ and $(c'_1 \ \cdots \ c'_n)$ are regarded as equivalent if and only if there exists a permutation $\sigma \in \mathfrak{S}_m$ with $m := \max\{\, c'_i \in \mathbf{N} \,\}$ such that
$$c_i = \begin{cases} \sigma(a) & \text{if } c'_i = a \in \mathbf{N} \\ + & \text{if } c'_i = + \\ - & \text{if } c'_i = - \\ \overline{\sigma(a)} & \text{if } c'_i = \overline{a} \ (a \in \mathbf{N}) \end{cases}$$
for $1 \leq i \leq n$.

For example, $(2\ 1\ 1\ +\ \overline{2}\ -) = (1\ 2\ 2\ +\ \overline{1}\ -)$ as a clan.

The following one-to-one correspondence exists between clans for $Sp(p,q)$ and generalized clans for $Sp(p,q)$. A clan $(c'_1 \ \cdots \ c'_n)$ corresponds to a generalized clan $(c_1 \ \cdots \ c_{2n})$ satisfying the following four conditions for $1 \leq i, j \leq n$:

1. If $c'_i = +$, then $(c_i, c_{2n+1-i}) = (+,+)$.
2. If $c'_i = -$, then $(c_i, c_{2n+1-i}) = (-,-)$.
3. If $c'_i = c'_j \in \mathbf{N}$, then $c_i = c_j \in \mathbf{N}$ and $c_{2n+1-i} = c_{2n+1-j} \in \mathbf{N}$.
4. If $c'_i = c'_j = \overline{a}$, then $c_i = c_{2n+1-j} \in \mathbf{N}$ and $c_j = c_{2n+1-i} \in \mathbf{N}$.

For example, a clan $(1\ 2\ 2\ +\ \overline{1})$ corresponds to a generalized clan

$$(1\ 2\ 2\ +\ 3\ 1\ +\ 4\ 4\ 3).$$

Therefore $\mathcal{C}(Sp(p,q))$ is a parametrization of K-orbits in X by [6, Theorem 4.1].

Corollary 4.3.7. *Generalized clans in $\mathcal{C}(Sp(p,q))$ parametrize K-orbits in the flag variety X via the correspondence in Proposition 2.2.6.*

Notation 4.3.8. For each generalized clan γ, let Q_γ be the K-orbit in the flag variety X corresponding to γ via the parametrization of Corollary 4.3.7.

4.3-2. *Representatives of $Sp(p,\mathbf{C}) \times Sp(q,\mathbf{C})$-orbits in $Sp(n,\mathbf{C})/B$.* We will give an element $g \in G$ such that the flag $x = gB$ corresponds to a generalized clan $\gamma \in \mathcal{C}(Sp(p,q))$ and the flag $x = gB_{\mathrm{AIII}}$ in X corresponds to the clan $\gamma \in \mathcal{C}(U(2p,2q))$.

Definition 4.3.9 (Symmetric signed clan)**.** A *symmetric signed clan* $\gamma_+ = (d_1 \ \cdots \ d_{2n})$ of a generalized clan $\gamma = (c_1 \cdots c_{2n})$ is a signed clan (in Definition 2.2.9) of the clan γ satisfying the following condition. If $c_i = c_j = a \in \mathbf{N}$, then we have $c_{2n+1-j} = c_{2n+1-i} = b \in \mathbf{N}$. We can assume $\min(i, 2n+1-i) < \min(j, 2n+1-j)$. Then

$$(d_i, d_j) = (a_+, a_-) \quad \text{and} \quad (d_{2n+1-j}, d_{2n+1-i}) = (b_-, b_+). \tag{57}$$

Example 4.3.10. The symmetric signed clan of $\gamma = (1\ 2\ 1\ +\ 3\ 2\ +\ 4\ 3\ 4)$ is

$$\gamma_+ = (1_+\ 2_+\ 1_-\ +\ 3_-\ 2_-\ +\ 4_-\ 3_+\ 4_+).$$

Corollary 4.3.11. *In a symmetric signed clan $\gamma_+ = (d_1 \ \cdots \ d_{2n})$, the signature of d_i is equal to the signature of d_{2n+1-i} for $1 \le i \le 2n$.*

In this section, we call a skew symmetric signed clan just a *clan*.

We give a representative $g \in G$ of Q_γ.

Theorem 4.3.12. *For $\gamma \in \mathcal{C}(Sp(p,q))$ and the signed clan $\gamma_+ = (d_1 \ \cdots \ d_{2n})$ of γ, fix two permutations $\sigma'' \in \mathfrak{S}_n$ such that*

$$\begin{aligned} 1 \le \sigma''(i) \le p & \quad \text{if the signature of } d_i \text{ is plus,} \\ p+1 \le \sigma''(i) \le n & \quad \text{if the signature of } d_i \text{ is minus,} \end{aligned}$$

and $\sigma' \in \mathfrak{S}_{2n}$ such that

if the signature of d_i is plus or $d_i = -$, then

$$\sigma'(i) = \begin{cases} \sigma''(i) & \text{if } i \le n, \\ 2n+1-\sigma''(i) & \text{if } n < i, \end{cases}$$

and if $d_i = a_-$ and $d_j = a_+$ for some $a \in \mathbf{N}$, then

$$\sigma'(i) = \begin{cases} \sigma''(i) & \text{if } j \le n, \\ 2n+1-\sigma''(i) & \text{if } n < j. \end{cases}$$

The matrix $g(\gamma) = (g_1\ g_2\ \cdots\ g_{2n}) \in \mathrm{Mat}(2n,2n)$ is a representative of Q_γ, i.e., $Q_\gamma = Kg(\gamma)B$. Here $g_i \in V$ are column vectors defined as follows:

- *If $c_i = \pm$, then $g_i = e_{\sigma'(i)}$.*
- *If $(c_i, c_j) = (a_+, a_-)$ for some $a \in \mathbf{N}$, then*

$$g_i = \frac{1}{\sqrt{2}}(e_{\sigma'(i)} + e_{\sigma'(j)}) \quad \text{and} \quad g_j = \begin{cases} -\frac{1}{\sqrt{2}}(-e_{\sigma'(i)} + e_{\sigma'(j)}) & \text{if } j \le n < i, \\ \frac{1}{\sqrt{2}}(-e_{\sigma'(i)} + e_{\sigma'(j)}) & \text{otherwise.} \end{cases}$$

Proof. The permutation σ' satisfies the conditions (52) of σ in Proposition 4.1.2. Thus, $g(\gamma)$ is a representative of K_{AIII}-orbit $Q_{\gamma\mathrm{AIII}}$, i.e., $Q_{\gamma\mathrm{AIII}} = K_{\mathrm{AIII}}g(\gamma)B_{\mathrm{AIII}}$, and $g(\gamma) \in G$. Therefore, we have

$$Q_\gamma = Kg(\gamma)B. \quad \square$$

Example 4.3.13. For a signed clan

$$\gamma_+ = (1_+ \; 2_+ \; - \; 1_- \; + \; 3_- \; 2_- \; + \; 4_- \; - \; 3_+ \; 4_+),$$

for example, if we choose

$$\sigma'' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 6 & 4 & 3 & 5 \end{pmatrix},$$

then σ' is

$$\sigma' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 2 & 6 & 4 & 3 & 8 & 5 & 10 & 9 & 7 & 11 & 12 \end{pmatrix}.$$

Remark 4.3.14. The representation $g(\gamma)$ is real and orthogonal:

$$g(\gamma)^{-1} = {}^t g(\gamma).$$

4.4. **M-O graph of $Sp(p,q)$.** We define an oriented graph which has generalized clans as vertices and in which an edge is labeled by i for some $1 \leq i \leq n$. We can read off closure relations from the graph.

Definition 4.4.1. Let $\Gamma(Sp(p,q))$ be the set of all triples (γ, γ', i) with $\gamma, \gamma' \in \mathcal{C}(Sp(p,q))$ and $1 \leq i \leq n$, satisfying one of the following two conditions.

1. $(\gamma, \gamma', i) \in \Gamma(U(2p, 2q))$. (This case happens only if $i = n$.)
2. There exists a clan $\gamma'' \in \mathcal{C}(U(2p, 2q))$ such that

$$\{(\gamma, \gamma'', i),\ (\gamma'', \gamma', 2n - i)\} \subset \Gamma(U(2p, 2q)).$$

By Theorem 2.2.14 and Remark 3.3.4, we have the following proposition.

Proposition 4.4.2. *If* $(\gamma, \gamma', i) \in \Gamma(Sp(p,q))$*, then*

$$\pi_i(Q_\gamma) = \pi_i(Q_{\gamma'}).$$

Proposition 4.4.3. *If* $(\gamma, \gamma', i) \in \Gamma(Sp(p,q))$*, then*

$$\dim Q_\gamma = \dim Q_{\gamma'} - 1$$

and

$$Kg(\gamma)B \subset \overline{Kg(\gamma')B}.$$

Proof. By [6], if $(\gamma, \gamma', i) \in \Gamma(Sp(p,q))$, then we have

$$\dim Q_\gamma = \dim Q_{\gamma'} \pm 1$$

and

$$Kg(\gamma)B \subset \overline{Kg(\gamma')B}, \quad \text{or} \quad Kg(\gamma')B \subset \overline{Kg(\gamma)B}.$$

If $i = n$, then $(\gamma, \gamma', n) \in \Gamma(U(2p, 2q))$. Therefore

$$K_{\mathrm{AIII}}g(\gamma)B_{\mathrm{AIII}} \subset \overline{K_{\mathrm{AIII}}g(\gamma')B_{\mathrm{AIII}}} \quad \textit{and} \quad K_{\mathrm{AIII}}g(\gamma')B_{\mathrm{AIII}} \not\subset \overline{K_{\mathrm{AIII}}g(\gamma)B_{\mathrm{AIII}}}.$$

Thus, we have $Kg(\gamma)B \subset \overline{Kg(\gamma')B}$.

If $i \neq n$, then both of (γ, γ'', i) and $(\gamma'', \gamma', 2n - i)$ are elements of $\Gamma(U(2p, 2q))$ for some $\gamma'' \in \mathcal{C}(U(2p, 2q))$. By the same argument of the case of $i = n$, $Kg(\gamma)B \subset \overline{Kg(\gamma')B}$. Therefore $\dim Q_\gamma = \dim Q_{\gamma'} - 1$ for all $(\gamma, \gamma', i) \in \Gamma(Sp(p,q))$. □

Although the definition of the following graph follows [6], we use generalized clans instead of clans as vertices.

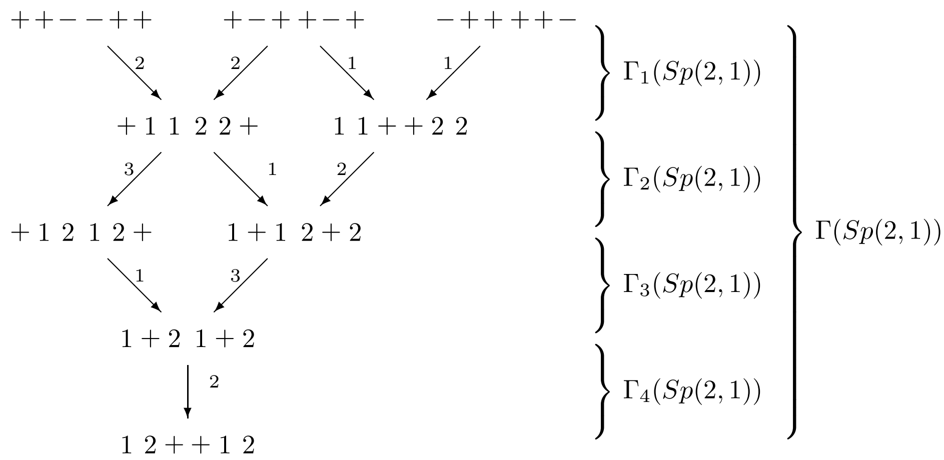

FIGURE 3

Definition 4.4.4 (M-O graph of $Sp(p,q)$)**.** Let

$$\mathcal{C}_0(Sp(p,q)) = \mathcal{C}(Sp(p,q)) \cap \mathcal{C}_0(U(2p,2q)) :$$

$$\mathcal{C}_0(Sp(p,q)) = \{\, (c_1 \dots c_n) \in \mathcal{C}(Sp(p,q)) \,|\, c_i \notin \mathbf{N} \text{ for all } 1 \le i \le n \,\}.$$

We give subsets $\Gamma_m(Sp(p,q))$ of $\Gamma(Sp(p,q))$ and subsets $\mathcal{C}_m(Sp(p,q))$ of $\mathcal{C}(Sp(p,q))$ for $m \ge 1$ by induction as follows:

$$\Gamma_m(Sp(p,q)) := \{\, (\gamma',\gamma,i) \in \Gamma'(Sp(p,q)) \mid \gamma \in \mathcal{C}_{m-1}(Sp(p,q)) \,\},$$
$$\mathcal{C}_m(Sp(p,q)) := \{\, \gamma \in \mathcal{C}(Sp(p,q)) \mid (\gamma',\gamma,i) \in \Gamma_m(Sp(p,q)) \,\}.$$

The *M-O graph* with generalized clans of $Sp(p,q)$ is a finite oriented graph whose vertices are generalized clans in $\mathcal{C}(Sp(p,q))$ and whose oriented edges are

$$\Gamma(Sp(p,q)) \;=\; \bigcup_{m\in\mathbf{N}} \Gamma_m(Sp(p,q)) \,.$$

Example 4.4.5. The $\Gamma(Sp(2,1))$ is as Figure 3.

Remark 4.4.6. If $(\gamma,\gamma',i) \in \Gamma(Sp(p,q))$, then

$$\pi_i(Q_\gamma) \;=\; \pi_i(Q_{\gamma'}) \quad \text{and} \quad \dim Q_\gamma + 1 \;=\; \dim Q_{\gamma'}.$$

4.5. **Dimensions of $Sp(p,\mathbf{C}) \times Sp(q,\mathbf{C})$-orbits in $Sp(n,\mathbf{C})$.** In this section we will give a dimension formula of K-orbits for generalized clans. We defined the *length* $\ell_{\mathrm{C}}(\gamma)$ of a generalized clan in Definition 3.4.1

$$\ell_{\mathrm{C}}(\gamma) = \frac{1}{2}\,(\,\ell(\gamma) + \#\{t \in \mathbf{N} \,|\, c_s = c_t \in \mathbf{N} \;\text{ and }\; s \le n < t \le 2n+1-s\})\,.$$

Proposition 4.5.1. *If* $\gamma \in \mathcal{C}_m(Sp(p,q))$*, then* $\ell_{\mathrm{C}}(\gamma) = m$.

Proof. We prove the proposition by induction on m. Let $\gamma = (c_1 \dots c_{2n})$.

1. If $\gamma \in \mathcal{C}_0(Sp(p,q))$, then $\ell_{\mathrm{C}}(\gamma) = 0$.

2. Suppose $\ell_{\mathrm{C}}(\gamma') = j$, if $\gamma' \in \mathcal{C}_j(Sp(p,q))$. For $\gamma \in \mathcal{C}_{j+1}(Sp(p,q))$, there exists $\gamma' = (c'_1 \dots c'_{2n}) \in \mathcal{C}_j(Sp(p,q))$ such that $(\gamma', \gamma, i) \in \Gamma(Sp(p,q))$ for some i.
   (a) If $i = n$, then $\ell(\gamma) = \ell(\gamma') + 1$, and
   $$\begin{aligned}\#\{t \in \mathbf{N} \mid c_s = c_t \in \mathbf{N} \quad \text{and} \quad s \le n < t \le 2n+1-s\} - \#\{\, n+1 \,\} \\ = \#\{t \in \mathbf{N} \mid c'_s = c'_t \in \mathbf{N} \quad \text{and} \quad s \le n < t \le 2n+1-s\}.\end{aligned}$$
   So, $\ell_{\mathrm{C}}(\gamma') = j+1$.
   (b) If $i \neq n$, then there exists a clan $\gamma'' \in \mathcal{C}(U(2p,2q))$ such that
   $$(\gamma', \gamma'', i), \quad (\gamma'', \gamma, 2n-i) \in \Gamma(U(2p,2q)).$$
   So, we have
   $$\ell(\gamma) = \ell(\gamma'') + 1 = \ell(\gamma') + 2$$
   and
   $$\begin{aligned}\#\{t \in \mathbf{N} \mid c_s = c_t \in \mathbf{N} \quad \text{and} \quad s \le n < t \le 2n+1-s\} \\ = \#\{t \in \mathbf{N} \mid c'_s = c'_t \in \mathbf{N} \quad \text{and} \quad s \le n < t \le 2n+1-s\}.\end{aligned}$$
   Therefore $\ell_{\mathrm{C}}(\gamma) = j+1$.

   So, $\ell_{\mathrm{C}}(\gamma) = m$ if $\gamma \in \mathcal{C}_m(Sp(p,q))$ for all $m \in \mathbf{N}$. □

A generalized clan
$$\gamma = (\underbrace{1 \ \ 2 \ \ \cdots \ \ (2p)}_{2p} \ \ \underbrace{- \ \ \cdots \ \ -}_{2n-4p} \ \ \underbrace{(2p-1) \ \ (2p) \ \ \cdots \ \ 5 \ \ 6 \ \ 3 \ \ 4 \ \ 1 \ \ 2}_{2p})$$
corresponds to the open orbit and
$$\ell_{\mathrm{C}}(\gamma) = 2pq.$$
Since the dimension of a closed orbit is $p^2 + q^2$, we have the following dimension formulas.

Proposition 4.5.2. *For $\gamma \in \mathcal{C}(Sp(p,q))$, we have the dimension and the codimension of $Q_\gamma = Kg(\delta)B \subset X$:*
$$\begin{aligned}\dim Q_\gamma &= \ell_{\mathrm{C}}(\gamma) + p^2 + q^2, \\ \operatorname{codim} Q_\gamma &= 2pq - \ell_{\mathrm{C}}(\gamma).\end{aligned}$$

4.6. **Images of the moment map.** We have
$$\begin{aligned}(\mathrm{Ad}(g(\gamma))\mathfrak{b}^\perp) \cap \mathfrak{p} &= (\mathrm{Ad}(g(\gamma)) \cdot (\mathfrak{b}^\perp_{\mathrm{AIII}} \cap \mathfrak{g})) \cap (\mathfrak{p}_{\mathrm{AIII}} \cap \mathfrak{g}) \\ &= (\mathrm{Ad}(g(\gamma)) \cdot \mathfrak{b}^\perp_{\mathrm{AIII}}) \cap \mathfrak{p}_{\mathrm{AIII}}) \cap \mathfrak{g},\end{aligned}$$

and elements X in $\mathfrak{g}$ satisfy
$$X_{ij} = \begin{cases} -X_{2n+1-j,2n+1-i} & \text{if } 1 \le i,j \le n \text{ or } n+1 \le i,j \le 2n, \\ \ \ X_{2n+1-j,2n+1-i} & \text{if } i \le n < j \text{ or } j \le n < i, \end{cases}.$$

Since we gave the image $(\mathrm{Ad}(g) \cdot \mathfrak{b}^\perp_{\mathrm{AIII}}) \cap \mathfrak{p}_{\mathrm{AIII}}$ of the moment map μ in Proposition 2.5.2 with $g := g_{\mathrm{AIII}}(\gamma)$ in Proposition 4.1.2 and $\sigma := \sigma'$ in Theorem 4.3.12, we get the following proposition.

Proposition 4.6.1. *For a generalized clan $\gamma = (c_1 \ \cdots \ c_{2n}) \in \mathcal{C}(Sp(p,q))$, fix a $\sigma' \in \mathfrak{S}_{2n}$ given as in Theorem 4.3.12. Let the representative $g := g(\gamma)$ be given as in Theorem 4.3.12, and $x = gB \in Q_\gamma$. We can read off $\mu(T^*_{Q_\gamma}X)_x$ from a vector subspace* $\mathrm{Dri}(\delta)$ *of* $\mathfrak{g}$.

$$\begin{aligned}\mu(T^*_{Q_\gamma}X)_x &= \mathrm{Ad}(\sigma')\cdot \mathrm{Dri}(\gamma)\\ &= \{\sigma' Y\, {}^t\sigma' \mid Y \in \mathrm{Dri}(\gamma)\}\\ &= \{\, Y_{(\sigma'^{-1}(1),\dots,\sigma'^{-1}(n))} \mid Y \in \mathrm{Dri}(\gamma)\}.\end{aligned}$$

Here, $\mathrm{Dri}(\gamma)$ *is the subspace of* $\mathrm{Dri}(\gamma_+)$*, the driving space of the signed clan γ_+ for $U(2p,2q)$ in Definition 2.5.1, defined as follows:*

$$\begin{aligned}\mathrm{Dri}(\gamma) \ &:= \ \mathrm{Dri}(\gamma_+) \cap \mathrm{Ad}(\sigma'^{-1})\cdot \mathfrak{g}\\ &= \ \{\, Y \in \mathrm{Dri}(\gamma_+) \mid Y \textit{ satisfies the following two conditions }\}.\end{aligned}$$

1. *If* $1 \le \sigma'(i), \sigma'(j), \le n$ *or* $n+1 \le \sigma'(i), \sigma'(j) \le 2n$*, then*
$$Y_{i,j} = -Y_{2n+1-j,2n+1-i}.$$
2. *If* $\sigma'(i) \le n < \sigma'(j)$ *or* $\sigma'(j) \le n < \sigma'(i)$*, then*
$$Y_{i,j} = Y_{2n+1-j,2n+1-i}.$$

We call the space $\mathrm{Dri}(\gamma)$ *the* driving space *of the generalized clan $\gamma \in \mathcal{C}(Sp(p,q))$.*

Proof. For the signed clan $\gamma_+ = (d_1 \ \cdots \ d_{2n})$ and the permutation σ' in Theorem 4.3.12 and a representative $g_{\mathrm{AIII}}(\gamma_+,\sigma')$ in Proposition 4.1.2, we have

$$g = g(\gamma_+,\sigma')\cdot \mathrm{diag}(\varepsilon_1 \ \cdots \ \varepsilon_{2n}).$$

Here,

$$\varepsilon_i = \begin{cases} -1 & \text{if } \ (d_i,d_j) = (a_-,a_+) \ \text{ for } \ i \le n < j,\\ 1 & \text{otherwise.}\end{cases}$$

So,

$$\begin{aligned}&(\mathrm{Ad}(g)\mathfrak{b}^\perp_{\mathrm{AIII}}) \cap \mathfrak{p}_{\mathrm{AIII}}\\ &\quad= (\mathrm{Ad}(g(\gamma_+,\sigma'))(\mathrm{Ad}(\mathrm{diag}(\varepsilon_1 \dots \varepsilon_{2n}))\cdot \mathfrak{b}^\perp_{\mathrm{AIII}}) \cap \mathfrak{p}_{\mathrm{AIII}}\\ &\quad= (\mathrm{Ad}(g(\gamma_+,\sigma'))\cdot \mathfrak{b}^\perp_{\mathrm{AIII}}) \cap \mathfrak{p}_{\mathrm{AIII}}.\end{aligned}$$

Therefore, $\mu(T^*_{Q_\gamma}X)_x = \mathrm{Ad}(\sigma')\cdot \mathrm{Dri}(\gamma)$ is as above. □

Corollary 4.6.2. *For the representative $g(\gamma)$ of Q_γ given as in Theorem 4.3.12, we have the following equation.*

$$\begin{aligned}\mathrm{Dri}(\gamma) \ &= \ \mathrm{Ad}(\sigma'^{-1})(\mathrm{Ad}(g(\gamma))\cdot \mathfrak{b}^\perp) \cap \mathfrak{p}\\ &= \ \{\, A_{(\sigma'(1),\dots,\sigma'(2n))} \mid A \in (\mathrm{Ad}(g(\gamma))\cdot \mathfrak{b}^\perp) \cap \mathfrak{p}\,\}.\end{aligned}$$

Since $g(\gamma) = (e_{\sigma'(1)} \dots e_{\sigma'(2n)}) = \sigma'$ if a generalized clan γ corresponds to a closed orbit, we have the following lemma.

Lemma 4.6.3. *If a generalized clan γ is an element of $\mathcal{C}_0(Sp(p,q))$, then*

$$\mathrm{Dri}(\gamma) = \mathfrak{b}^\perp \cap (\mathrm{Ad}(g(\gamma)^{-1})\cdot \mathfrak{p}).$$

Definition 4.6.4 (Driving matrix)**.** For a generalized clan $\gamma \in \mathcal{C}(Sp(p,q))$, let $Y(\gamma)$ and $Y(\gamma,m)$, $1 \le m \le 8$ be elements of $F \otimes_{\mathbf{C}} \mathrm{Mat}(2n,2n)$ satisfying the following conditions. Let $\gamma_+ = (d_1 \ \cdots \ d_{2n})$.

1. $Y(\gamma,1)_{ij} = \begin{cases} 1 & \text{if } \ j < i \leq 2n+1-j \\ 0 & \text{otherwise.} \end{cases}$
2. $Y(\gamma,2)_{ij} = \begin{cases} 0 & \text{if } \ (d_i, d_j) = (+,+) \ \text{ or } \ (-,-), \\ Y(\gamma,1)_{ij} & \text{otherwise.} \end{cases}$
3. If $(d_s, d_t) = (a_+, a_-)$ or (a_-, a_+) for $s < t$, then
   - $Y(\gamma,3)_{s,k_1} = 0$ if $1 \leq k_1 \leq s$,
   - $Y(\gamma,3)_{k_2,s} = Y(\gamma,3)_{t,k_2} = 0$ if $s \leq k_2 \leq t$,
   - $Y(\gamma,3)_{k_3,t} = 0$ if $t \leq k_3 \leq 2n$,

   for all $a \in \mathbf{N}$ and $Y(\gamma,3)_{ij} = Y(\gamma,2)_{ij}$ otherwise.

Remark 4.6.5. If $Y(\gamma,3)_{ij} = 1$, then the following conditions are satisfied.

(a) $j < i \leq 2n+1-j$ (so $j \leq n$).
(b) $(d_i, d_j) = (a_+, b_+)$, (a_-, b_+), $(a_+, +)$, $(a_+, -)$, $(a_-, +)$, $(a_-, -)$, $(+, a_+)$, $(-, a_+)$, $(+, -)$, or $(-, +)$ for some a, $b \in \mathbf{N}$, $a \neq b$.
(c) If $i + j = 2n+1$, then $(d_i, d_j) = (a_+, b_+)$ for some a, $b \in \mathbf{N}$, $a \neq b$.
(d) If $d_i = +$ or $-$, then $\sigma'(i) \leq n$ if $i \leq n$, $\sigma'(i) > n$ if $i > n$.
(e) If $d_i = a_+$, then $d_k = a_-$ for some $j < k < i$, Then $n < \sigma'(k) < \sigma'(i)$.
(f) If $d_i = a_-$, then $d_k = a_+$ for some $j < k < i$. Then $\sigma'(k) < \sigma'(i) \leq n$.
(g) If $d_j = +$ or $-$, then $\sigma'(j) < n$.
(h) If $d_j = a_+$, then $d_k = a_-$ for $j < k < i$. Then $\sigma'(j)$, $\sigma'(k) \leq n$.

4. If $Y(\gamma,3)_{i,2n+1-i} = 1$, $(d_i, d_{2n+1-i}) = (a_+, b_+)$, and $(d_j, d_{2n+1-j}) = (a_-, b_-)$, for some a, $b \in \mathbf{N}$, then $Y(\gamma,4)_{i,2n+1-i} = 0$,
$$\begin{aligned} &(\ Y(\gamma,4)_{i,2n+1-j}, \ Y(\gamma,4)_{j,2n+1-i} \ ) \\ &\quad = \begin{cases} (y_{i,2n+1-j}, y_{i,2n+1-j}) & \text{if } \ 2n+1-j < j, \\ (0,0) & \text{if } \ j < 2n+1-j, \end{cases} \end{aligned}$$
   and $Y(\gamma,4)_{i'j'} = Y(\gamma,3)_{i'j'}$ otherwise (i', j').
5. If $Y(\gamma,4)_{ij} = 1$, $i+j \neq 2n+1$, $(d_i, d_j) = (a_+, b_+)$, and $(d_k, d_l) = (a_-, b_-)$ for some a, $b \in \mathbf{N}$, then $Y(\gamma,5)_{ij} = 0$,
$$(\ Y(\gamma,5)_{il}, \ Y(\gamma,5)_{2n+1-l,2n+1-i} \ ) = \ (y_{il}, y_{il}),$$
   and
$$(\ Y(\gamma,5)_{kj}, \ Y(\gamma,5)_{2n+1-j,2n+1-j} \ ) = \begin{cases} (-y_{il}, -y_{il}) & \text{if } k < l, \\ (y_{kj}, y_{kj}) & \text{if } l < k, \end{cases}$$
   and $Y(\gamma,5)_{i'j'} = Y(\gamma,4)_{i'j'}$ otherwise.
6. If $Y(\gamma,5)_{ij} = 1$, $(d_i, d_j) = (a_-, b_+)$ and $(d_k, d_l) = (a_+, b_-)$ for some a, $b \in \mathbf{N}$, then
$$(\ Y(\gamma,6)_{ij}, \ Y(\gamma,6)_{2n+1-j,2n+1-i} \ ) = \ (y_{ij}, -y_{ij}),$$
$$(\ Y(\gamma,6)_{kl}, \ Y(\gamma,6)_{2n+1-l,2n+1-k} \ ) = \begin{cases} (-y_{ij}, y_{ij}) & \text{if } k < l, \\ (y_{kl}, -y_{kl}) & \text{if } l < k, \end{cases}$$
   and $Y(\gamma,6)_{i'j'} = Y(\gamma,5)_{i'j'}$ otherwise.

7. If $Y(\gamma,6)_{ij} = 1$, $(d_i, d_j) = (a_+, +)$ or $(d_i, d_j) = (a_-, -)$, $d_k = a_-$ or $d_k = a_+$, for some $a \in \mathbf{N}$, then $Y(\gamma,7)_{ij} = 0$,
$$(\; Y(\gamma,7)_{kj}, \; Y(\gamma,7)_{2n+1-j,2n+1-k} \;) = \begin{cases} (y_{kj}, -y_{kj}) & \text{if } \sigma'(k) \leq n, \\ (y_{kj}, y_{kj}) & \text{if } \sigma'(k) > n, \end{cases}$$
and $Y(\gamma,7)_{i'j'} = Y(\gamma,6)_{i'j'}$ otherwise.
8. If $Y(\gamma,7)_{ij} = 1$, $(d_i, d_j) = (+, a_+)$, and $d_k = a_-$ for some $a \in \mathbf{N}$, then $Y(\gamma,8)_{ij} = 0$,
$$(\; Y(\gamma,8)_{ik}, \; Y(\gamma,8)_{2n+1-k,2n+1-i} \;) = \begin{cases} (y_{ik}, -y_{ik}) & \text{if } \sigma'(i) \leq n, \\ (y_{ik}, y_{ik}) & \text{if } \sigma'(i) > n, \end{cases}$$
and $Y(\gamma,8)_{i'j'} = Y(\gamma,7)_{i'j'}$ otherwise.
9. If $Y(\gamma,8)_{ij} = 1$, then
$$(\; Y(\gamma)_{ij}, \; Y(\gamma)_{2n+1-j,2n+1-i} \;) = \begin{cases} (y_{ij}, -y_{ij}) & \text{if } \sigma'(i) \leq n, \\ (y_{ij}, y_{ij}) & \text{if } \sigma'(i) > n, \end{cases}$$
and $Y(\gamma)_{i'j'} = Y(\gamma,8)_{i'j'}$ for other (i', j').

We call $Y(\gamma)$ the *driving matrix* of γ (for $Sp(p,q)$).

By the same arguments of Proposition 2.5.8, we get the following proposition.

Proposition 4.6.6. *For a generalized clan $\gamma \in \mathcal{C}(Sp(p,q))$, the driving space* $\mathrm{Dri}(\gamma)$ *for $Sp(p,q)$ and the driving matrix $Y(\gamma)$ of γ for $Sp(p,q)$ satisfy the following condition.*
$$\mathrm{Dri}(\gamma) = \{ \, Y(\gamma) \mid y_{ij} \in \mathbf{C} \quad \textit{for} \quad 1 \leq i,j \leq 2n \, \}.$$

By Proposition 4.6.1 and Proposition 4.6.6, we got the following theorem which means we can read off $\mu(T^*_{Q_\gamma} X)_x$ from the driving matrix $Y(\gamma)$.

Theorem 4.6.7. *For a generalized clan $\gamma \in \mathcal{C}(Sp(p,q))$, fix a $\sigma' \in \mathfrak{S}_{2n}$ given as in Theorem 4.3.12. Let the representative $g := g(\gamma)$ be given as in Theorem 4.3.12, and $x = gB \in Q_\gamma$. We can read off $\mu(T^*_{Q_\gamma} X)_x$ from the driving matrix $Y(\gamma)$ for $Sp(p,q)$ as follows:*
$$\mu(T^*_{Q_\gamma} X)_x = \{ \, Y(\gamma)_{(\sigma'^{-1}(1),\dots,\sigma'^{-1}(2n))} \mid y_{ij} \in \mathbf{C} \quad \textit{for} \quad 1 \leq i,j \leq 2n \, \}.$$

4.7. **Signed Young diagrams.** In this section, after we recall a parametrization of nilpotent K-orbits in $\mathfrak{p}$, we give tables of signed Young diagrams for the clans of $Sp(1,1)$, $Sp(2,1)$, and $Sp(2,2)$ by way of examples.

Nilpotent K-orbits in $\mathfrak{p}$ are parametrized by signed Young diagrams of signature $(2p, 2q)$ satisfying the following conditions.

1. If $m \in \mathbf{N}$ is even, then the number of rows of which the length are m and which are labeled $+$ in the first column is equal to the number of rows of which the length are m and which are labeled $-$ in the first column.

2. If $m \in \mathbf{N}$ is odd, the number of rows of which the length are m and which are labeled $+$ in the first column is even.
3. If $m \in \mathbf{N}$ is odd, the number of rows of which the length are m and which are labeled $-$ in the first column is even.

For an element A of an orbit, the signed Young diagram of the orbit satisfies

$$\sum_{j=1}^{i} D_{j,+} = \dim(\ker(A^i|_{V_+})) \quad \text{and} \quad \sum_{j=1}^{i} D_{j,-} = \dim(\ker(A^i|_{V_-})).$$

Proposition 4.7.1. *Under the conditions of Theorem 4.3.12, we put*

$$\begin{aligned} V'_+ &:= {\sigma'}^{-1}V_+ = (e_{\sigma'^{-1}(1)} \dots e_{\sigma'^{-1}(2n)})V_+ \\ &= \langle e_{\sigma'^{-1}(1)}, \dots, e_{\sigma'^{-1}(p)} \rangle \oplus \langle e_{\sigma'^{-1}(p+2q+1)}, \dots, e_{\sigma'^{-1}(2n)} \rangle, \\ V'_- &:= {\sigma'}^{-1}V_- = (e_{\sigma'^{-1}(1)} \dots e_{\sigma'^{-1}(2n)})V_- \\ &= \langle e_{\sigma'^{-1}(p+1)}, \dots, e_{\sigma'^{-1}(p+2q)} \rangle. \end{aligned}$$

By the same argument of the case of $U(2p, 2q)$, we have the following proposition.

Proposition 4.7.2. *Let $Y(\gamma)^0 = I_n$. We have the following equations:*

$$D_{i,+} = \dim_F\{ \vec{a} \in F \otimes_{\mathbf{C}} V'_+ \mid Y(\gamma)^i \vec{a} = \vec{0} \} - \dim_F\{ \vec{a} \in F \otimes_{\mathbf{C}} V'_+ \mid Y(\gamma)^{i-1} \vec{a} = \vec{0} \},$$
$$D_{i,-} = \dim_F\{ \vec{b} \in F \otimes_{\mathbf{C}} V'_- \mid Y(\gamma)^i \vec{b} = \vec{0} \} - \dim_F\{ \vec{b} \in F \otimes_{\mathbf{C}} V'_- \mid Y(\gamma)^{i-1} \vec{b} = \vec{0} \}.$$

At last we give tables of signed Young diagrams for the clans of $Sp(1,1)$ in Example 4.7.3, $Sp(2,1)$ in Example 4.7.4, $Sp(2,2)$ in Example 4.7.5, and a subgraph of $\Gamma(Sp(2,2))$ in Figure 4 (which has edges (γ, γ', i) such that the first signature of the clan $\gamma = (c_1 \dots c_8)$ is plus, i.e., $c_m = +$ for

$$m = \min\{ j \mid c_j = + \text{ or } - \}$$

and the corresponding signed Young diagrams). In Figure 4, clans and signed Young diagrams are separated by dots. A clan and its signed Young diagram are in the same component. In Figure 4,

$$(\ + - - + + - - +\ ) \overset{1}{\dashrightarrow} (\ 1\ 1 + - - + 2\ 2\ )$$

instead of

$$(\ + - - + + - - +\ ) \xrightarrow{1} (\ 1\ 1 - + + - 2\ 2\ ) \quad \in \Gamma(Sp(2,2)).$$

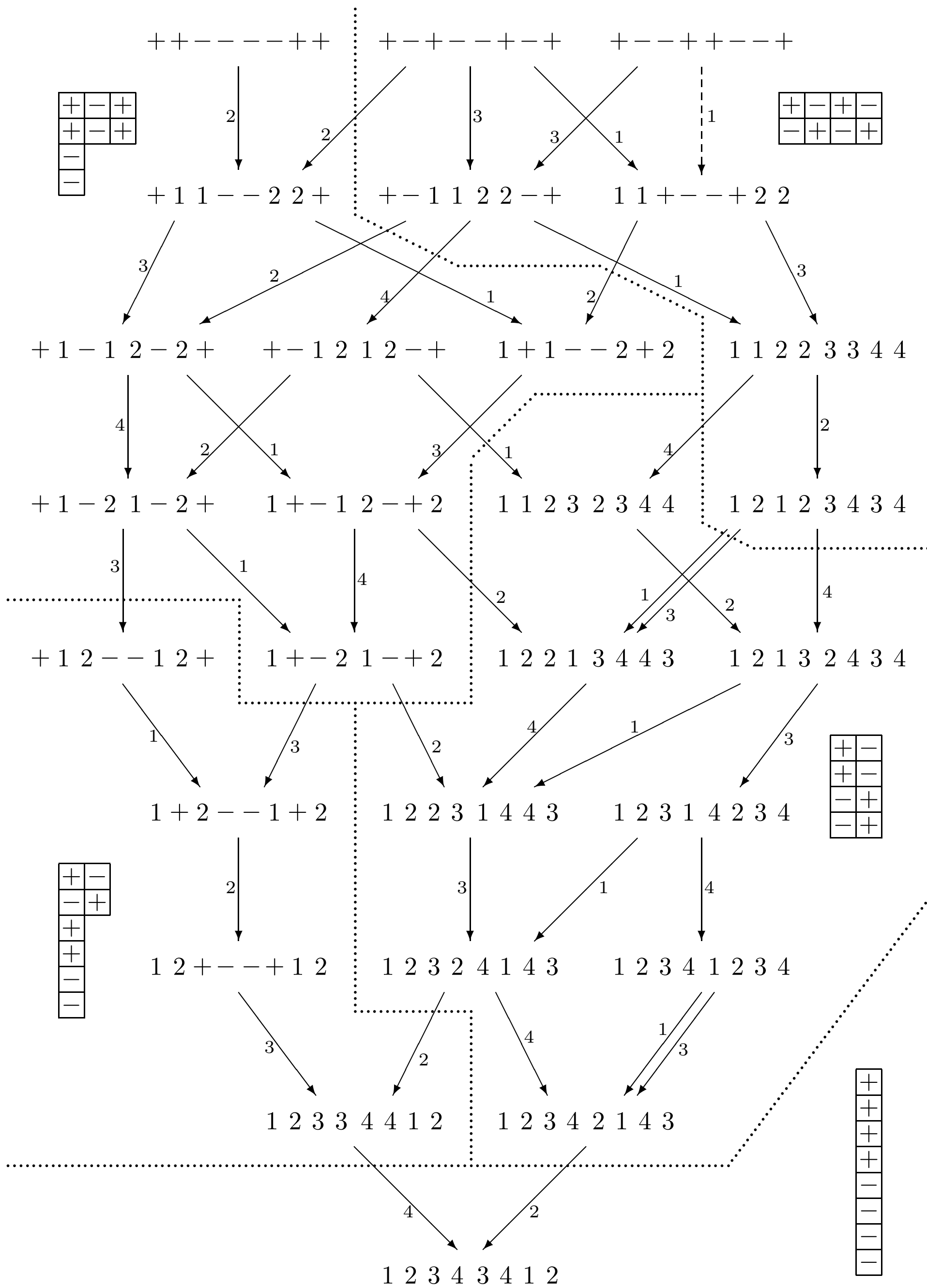
++−−−−++
+−+−−+−+
+−−++−−+
+11−−22+
+−1122−+
11+−−+22
+1−12−2+
+−1212−+
1+1−−2+2
11223344
+1−21−2+
1+−12−+2
11232344
12123434
+12−−12+
1+−21−+2
12213443
12132434
1+2−−1+2
12231443
12314234
12+−−+12
12324143
12341234
12334412
12342143
12343412

Figure 4

Example 4.7.3. This is the table of the case of $G_{\mathbf{R}} = Sp(1,1)$.

clan γ	a representative $g(\gamma)$ of K-orbit	driving matrix $Y(\gamma)$	signed Young diagram
$+--+$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 \\ 0 & y_{31} & -y_{21} & 0 \end{pmatrix}$	$\begin{array}{\|c\|c\|}\hline + & - \\ \hline - & + \\ \hline\end{array}$
$-++-$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 \\ 0 & y_{31} & -y_{21} & 0 \end{pmatrix}$	
$1\ 1\ 2\ 2$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_{42} & 0 & 0 & 0 \\ 0 & y_{42} & 0 & 0 \end{pmatrix}$	
$1\ 2\ 1\ 2$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{\|c\|}\hline + \\ \hline + \\ \hline - \\ \hline - \\ \hline\end{array}$

Example 4.7.4. This is the table of the case of $G_{\mathbf{R}} = Sp(2,1)$.

clan γ	a representative $g(\gamma)$ of K-orbit	driving matrix $Y(\gamma)$	signed Young diagram
$+ + - - + +$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ y_{31} & y_{32} & 0 & 0 & 0 & 0 \\ y_{41} & y_{42} & 0 & 0 & 0 & 0 \\ 0 & 0 & y_{42}-y_{32} & 0 & 0 \\ 0 & 0 & y_{41}-y_{31} & 0 & 0 \end{pmatrix}$	
$+ - + + - +$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & y_{32} & 0 & 0 & 0 & 0 \\ 0 & y_{42} & 0 & 0 & 0 & 0 \\ y_{51} & 0 & y_{42}-y_{32} & 0 & 0 \\ 0 & y_{51} & 0 & 0 & -y_{21} & 0 \end{pmatrix}$	$+ - +$ $+ - +$
$+\ 1\ 1\ 2\ 2\ +$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 & 0 & 0 \\ y_{41} & y_{53} & 0 & 0 & 0 & 0 \\ 0 & 0 & y_{53} & 0 & 0 & 0 \\ 0 & 0 & y_{41}-y_{31} & 0 & 0 \end{pmatrix}$	
$- + + + + -$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 & 0 & 0 \\ y_{41} & 0 & 0 & 0 & 0 & 0 \\ y_{51} & 0 & 0 & 0 & 0 & 0 \\ 0 & y_{51} & y_{41}-y_{31}-y_{21} & 0 \end{pmatrix}$	$+ -$ $- +$ $+$ $+$
$1\ 1 + + 2\ 2$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_{32} & 0 & 0 & 0 & 0 \\ 0 & y_{42} & 0 & 0 & 0 & 0 \\ y_{62} & 0 & y_{42}-y_{32} & 0 & 0 \\ 0 & y_{62} & 0 & 0 & 0 & 0 \end{pmatrix}$	

clan γ	a representative $g(\gamma)$ of K-orbit	driving matrix $Y(\gamma)$	signed Young diagram
$+\ 1\ 2\ 1\ 2\ +$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 & 0 & 0 \\ y_{41} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -y_{41} & y_{31} & 0 & 0 \end{pmatrix}$	$\begin{array}{\|c\|c\|} \hline + & - \\ \hline - & + \\ \hline + \\ \cline{1-1} + \\ \cline{1-1} \end{array}$
$1\ +\ 1\ 2\ +\ 2$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ y_{63} & y_{53} & 0 & 0 & 0 & 0 \\ 0 & 0 & y_{53} & 0 & 0 & 0 \\ 0 & 0 & y_{63} & 0 & 0 & 0 \end{pmatrix}$	
$1\ +\ 2\ 1\ +\ 2$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_{54} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_{54} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	
$1\ 2\ +\ +\ 1\ 2$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{\|c\|} \hline + \\ \hline + \\ \hline + \\ \hline + \\ \hline - \\ \hline - \\ \hline \end{array}$

Example 4.7.5. This is the table of $G_{\mathbf{R}} = Sp(2,2)$.

clan γ	signed Young diagram
$+ - + - - + - +$	$+ - + -$ $- + - +$
$- + - + + - + -$	
$+ - - + + - - +$	
$- + + - - + + -$	
$1\ 1 + - - + 2\ 2$	
$1\ 1 - + + - 2\ 2$	
$+ - 1\ 1\ 2\ 2 - +$	
$- + 1\ 1\ 2\ 2 + -$	
$1\ 1\ 2\ 2\ 3\ 3\ 4\ 4$	
$1\ 2\ 1\ 2\ 3\ 4\ 3\ 4$	
$+ + - - - - + +$	$+ - +$ $+ - +$ $-$ $-$
$+ 1\ 1 - - 2\ 2 +$	
$1 + 1 - - 2 + 2$	
$+ 1 - 1\ 2 - 2 +$	
$+ - 1\ 2\ 1\ 2 - +$	
$1 + - 1\ 2 - + 2$	
$+ 1 - 2\ 1 - 2 +$	
$1 + - 2\ 1 - + 2$	
$- - + + + + - -$	$- + -$ $- + -$ $+$ $+$
$- 1\ 1 + + 2\ 2 -$	
$1 - 1 + + 2 - 2$	
$- 1 + 1\ 2 + 2 -$	
$- + 1\ 2\ 1\ 2 + -$	
$1 - + 1\ 2 + - 2$	
$- 1 + 2\ 1 + 2 -$	
$1 - + 2\ 1 + - 2$	

clan γ	signed Young diagram
$1\ 1\ 2\ 3\ 2\ 3\ 4\ 4$	$+ -$ $+ -$ $- +$ $- +$
$1\ 2\ 2\ 1\ 3\ 4\ 4\ 3$	
$1\ 2\ 1\ 3\ 2\ 4\ 3\ 4$	
$1\ 2\ 2\ 3\ 1\ 4\ 4\ 3$	
$1\ 2\ 3\ 1\ 4\ 2\ 3\ 4$	
$1\ 2\ 3\ 2\ 4\ 1\ 4\ 3$	
$1\ 2\ 3\ 4\ 1\ 2\ 3\ 4$	
$1\ 2\ 3\ 4\ 2\ 1\ 4\ 3$	
$+ 1\ 2 - - 1\ 2 +$	$+ -$ $- +$ $+$ $+$ $-$ $-$
$- 1\ 2 + + 1\ 2 -$	
$1 - 2 + + 1 - 2$	
$1 + 2 - - 1 + 2$	
$1\ 2 + - - + 1\ 2$	
$1\ 2 - + + - 1\ 2$	
$1\ 2\ 3\ 3\ 4\ 4\ 1\ 2$	
$1\ 2\ 3\ 4\ 3\ 4\ 1\ 2$	$+$ $+$ $+$ $+$ $-$ $-$ $-$ $-$

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