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THE ENVELOPING ALGEBRA OF THE LIE SUPERALGEBRA $osp(1, 2r)$

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ABSTRACT. Let $\mathfrak{g}$ be the Lie superalgebra $osp(1, 2r)$ and $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$.

In this paper we obtain a description of the set of primitive ideals $\operatorname{Prim} U(\mathfrak{g})$ as an ordered set. We also obtain the multiplicities of composition factors of Verma modules over $U(\mathfrak{g})$, and of simple highest weight modules for $U(\mathfrak{g})$ when regarded as a $U(\mathfrak{g}_0)$-module by restriction.

0.1. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite dimensional complex classical simple Lie superalgebra. In [M1] we showed that any primitive ideal in $U(\mathfrak{g})$ is the annihilator of a simple highest weight module $\widetilde{L}(\lambda)$, (see 0.2–0.5 for notation). To complete the description of the set of primitive ideals $\operatorname{Prim} U(\mathfrak{g})$, it is necessary to say when two modules $\widetilde{L}(\lambda), \widetilde{L}(\mu)$ have the same annihilator. For Lie superalgebras of Type I, this was done in [L2] using a bijection between $\operatorname{Prim} U(\mathfrak{g}_0)$ and $\operatorname{Prim} U(\mathfrak{g})$. However this bijection does not preserve inclusions.

In this paper we study the case where $\mathfrak{g} = osp(1, 2r)$ and obtain a description of $\operatorname{Prim} U(\mathfrak{g})$ as an ordered set. We also obtain the multiplicities of composition factor of Verma modules over $U(\mathfrak{g})$, and of $\widetilde{L}(\lambda)$ when regarded as a $U(\mathfrak{g}_0)$-module by restriction.

The orthosymplectic Lie superalgebra $osp(V, \beta)$ may be defined as the Lie superalgebra of all linear operators on a $\mathbb{Z}_2$-graded vector space V preserving a nondegenerate even bilinear supersymmetric form β. We refer to [K1, 2.1.2] or [Sch, II. 4.3.A, page 129] for more details. In [M3] we give an alternative construction for $\mathfrak{g} = osp(1, 2r)$ using the rth Weyl algebra. This leads to a construction of an analog of the Joseph ideal in $U(\mathfrak{g})$.

There are several related reasons why we might expect $U(\mathfrak{g})$ to be structurally similar to $U(\mathfrak{g}_0)$ when $\mathfrak{g} = osp(1, 2r)$. For example the Harish-Chandra map yields an isomorphism $Z(\mathfrak{g}) \simeq S(\mathfrak{h})^W$, all weights in $\mathfrak{h}^*$ are typical, and all finite dimensional modules are completely reducible. The results of this paper tend to confirm this expectation.

Some of the proofs in this paper work for Lie superalgebras other than $osp(1, 2r)$. For example most results hold for typical representations of $s\ell(r, 1)$. In order to state our results in greater detail, we introduce some notation.

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0.2. **Weights and Roots.** Basic classical simple Lie superalgebras are defined in [K2, Section 1]. In [K1, 2.5] these algebras are called contragredient Lie superalgebras. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, be a basic classical simple Lie superalgebra, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We denote the set of roots of $\mathfrak{g}_i$ with respect to $\mathfrak{h}$ by Δ_i for $i = 0, 1$. Let $\alpha_1, \dots, \alpha_n$ be a basis of simple roots for $\mathfrak{g}$ and set $Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$, $Q^+ = \sum_{i=1}^{n} \mathbb{N}\alpha_i$ and $\Delta_i^+ = \Delta_i \cap Q^+$ for $i = 0, 1$. Also set $\overline{\Delta}_0^+ = \{\alpha \in \Delta_0^+ | \alpha/2 \notin \Delta_1^+\}$, and $\overline{\Delta}_1^+ = \{\alpha \in \Delta_1^+ | 2\alpha \notin \Delta_0^+\}$. If $\alpha, \beta \in \mathfrak{h}^*$, we write $\alpha \leq \beta$ if $\beta - \alpha \in Q^+$.

We fix an even nondegenerate $\mathfrak{g}$-invariant bilinear form $(\ ,\ )$ on $\mathfrak{g}$. As in [Sch, II.3.2], the restriction of $(\ ,\ )$ to $\mathfrak{h}$ is nondegenerate. Thus for $\lambda \in \mathfrak{h}^*$, there exists a unique element $h_\lambda \in \mathfrak{h}$ such that $\lambda(h) = (h_\lambda, h)$ for all $h \in \mathfrak{h}$. If $\lambda, \mu \in \mathfrak{h}^*$, we set $(\lambda, \mu) = (h_\lambda, h_\mu)$.

Let $\mathfrak{h}'$ be a Cartan subalgebra of the semisimple Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ and $\beta_1, \dots, \beta_r$ the unique basis of simple roots of $[\mathfrak{g}_0, \mathfrak{g}_0]$ contained in Δ_0^+. We set

$$Q_0 = \sum_{i=1}^{r} \mathbb{Z}\beta_i, \qquad Q_0^+ = \sum_{i=1}^{r} \mathbb{N}\beta_i.$$

Let $\omega_1, \dots, \omega_r \in (\mathfrak{h}')^*$ be the fundamental dominant weights defined by $(\omega_i, \beta_j^v) = \delta_{ij}$. If $\mathfrak{g}_0$ is semisimple we set

$$P_0 = \sum_{i=1}^{r} \mathbb{Z}\omega_i, \qquad P_0^+ = \sum_{i=1}^{r} \mathbb{N}\omega_i.$$

Otherwise $\mathfrak{h} = (\mathfrak{h}') \oplus \mathbb{C}z$ where $\mathbb{C}z$ is the center of $\mathfrak{g}_0$ and we identify $(\mathfrak{h}')^*$ with $(\mathbb{C}z)^\perp$. There is a nonzero element $\alpha \in \mathfrak{h}^*$, unique up to scalar, such that $(\alpha, \beta_i) = 0$ for $i = 1, \dots, r$. In this case we set

$$P_0 = \mathbb{C}\alpha + \sum_{i=1}^{r} \mathbb{Z}\omega_i, \qquad P_0^+ = \mathbb{C}\alpha + \sum_{i=1}^{r} \mathbb{N}\omega_i.$$

In addition $P_0 = \{\lambda \in \mathfrak{h}^* | (\lambda, \beta_i^v) \in \mathbb{Z} \text{ for } i = 1, \dots, r\}$. If M is a $U(\mathfrak{g})$-module, and $\alpha \in \mathfrak{h}^*$, we set

$$M^\alpha = \{m \in M | hm = \alpha(h)m \quad \text{for all } h \in \mathfrak{h}\}.$$

0.3. **Verma Modules and Primitive Ideals.** The choice of a basis of simple roots determines a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, [M1, Lemma 1.4]. Set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. For all $\lambda \in \mathfrak{h}^*$ the Verma modules $\widetilde{M}(\lambda)$,$M(\lambda)$ are defined as in [M1, 1.1]. These modules have unique graded simple factor modules $\widetilde{L}(\lambda)$ and $L(\lambda)$. We set $J(\lambda) = \operatorname{ann}_{U(\mathfrak{g})} \widetilde{L}(\lambda)$ and $I(\lambda) = \operatorname{ann}_{U(\mathfrak{g}_0)} L(\lambda)$. Note that $L(\lambda)$ is finite dimensional if and only if $\lambda \in P_0^+$.

In the study of Verma modules for $U(\mathfrak{g}_0)$ it is convenient to consider the category $\mathcal{O}$ of modules defined in [J1, 1.10]. As in [M1, 1.1] we also consider the category $\widetilde{\mathcal{O}}$ of graded $U(\mathfrak{g})$-modules which belong to the category $\mathcal{O}$ when regarded as $U(\mathfrak{g}_0)$-modules by restriction. The Grothendieck groups of these categories are denoted by $\mathcal{C}(\mathcal{O})$ and $\mathcal{C}(\widetilde{\mathcal{O}})$. If M is an object of $\mathcal{O}$, we denote the character of M by chM, see [J1, 1.11].

0.4. **Centers.** The centers of $U(\mathfrak{g})$ and $U(\mathfrak{g}_0)$ are denoted by $Z(\mathfrak{g})$ and $Z(\mathfrak{g}_0)$ respectively. The action of $Z(\mathfrak{g})$ on $\widetilde{M}(\lambda)$ determines the central character χ_λ and we set $m_\lambda = \ker \chi_\lambda$. We define the central character χ^0_λ of $Z(\mathfrak{g}_0)$ in a similar way, and set $m^0_\lambda = \ker \chi^0_\lambda$.

For $\lambda \in \mathfrak{h}^*$ and M a $\mathfrak{g}_0$-module, we set

$$M_{\langle\lambda\rangle} = \{v \in M |\text{ for all } z \in Z(\mathfrak{g}_0), (z - \chi^0_\lambda(z))^n v = 0, \text{ for } n \gg 0\}.$$

Similarly if M is a $\mathfrak{g}$-module, we set

$$M_{(\lambda)} = \{v \in M |\text{ for all } z \in Z(\mathfrak{g}),\ (z - \chi_\lambda(z))^n v = 0, \text{ for } n \gg 0\},$$

and let $\widetilde{\mathcal{O}}_\lambda$ be the full subcategory of the category $\widetilde{\mathcal{O}}$ consisting of modules M such that $M = M_{(\lambda)}$.

0.5. **The Weyl Group.** Set $\rho_0 = \frac{1}{2}\sum_{\beta \in \Delta^+_0} \beta,\ \rho_1 = \frac{1}{2}\sum_{\beta\in\Delta^+_1}\beta$ and $\rho = \rho_0 - \rho_1$. We say that $\lambda \in \mathfrak{h}^*$ is typical if $(\lambda + \rho, \alpha) \neq 0$ for all $\alpha \in \overline{\Delta}^+_1$, $\mathfrak{g}$-regular if $(\lambda+\rho,\alpha) \neq 0$ for all $\alpha \in \Delta^+_0$ and $\mathfrak{g}_0$-regular if $(\lambda + \rho_0, \alpha) \neq 0$ for all $\alpha \in \Delta^+_0$. If $\alpha \in \Delta^+_0 \cup (\Delta^+_1 \backslash \overline{\Delta}^+_1)$, we set $\alpha^v = \alpha/(\alpha,\alpha)$, and write s_α for the reflection in the hyperplane orthogonal to α.

We define translated actions of the Weyl group W on $\mathfrak{h}^*$ by

$$w \,.\, \lambda = w(\lambda + \rho) - \rho$$

$$w \circ \lambda = w(\lambda + \rho_0) - \rho_0$$

for $w \in W$ and $\lambda \in \mathfrak{h}^*$.

By [M2, Lemma 2.3] W acts on the set Γ of sums of distinct odd positive roots by the rule

$$w * \gamma = \rho_1 + w\gamma - w\rho_1.$$

These actions are related by

$$w \circ (\lambda - \gamma) = w.\lambda - w * \gamma$$

and

$$w.(\lambda + \gamma) = w \circ \lambda + w * \gamma.$$

For $\lambda \in \mathfrak{h}^*$, let $\Delta(\lambda)$ be the subroot system of Δ_0 given by

$$\Delta(\lambda) = \{\alpha \in \Delta_0 | (\lambda, \alpha^v) \in \mathbb{Z}\}$$

and set $W_\lambda = \{w \in W | w\lambda - \lambda \in Q_0\}$ the Weyl group of $\Delta(\lambda)$.

There is a unique basis B_λ of $\Delta(\lambda)$ contained in Q^+_0. The sets $\Delta(\lambda), W_\lambda$ and B_λ depend only on the coset $\Lambda = \lambda + P_0$ and we denote them also by $\Delta(\Lambda), W_\Lambda, B_\Lambda$. For a coset $\Lambda \in \mathfrak{h}^*/P_0$ we set

$$\Lambda^+ = \{\lambda \in \Lambda | (\lambda + \rho, \alpha^v) \geq 0 \text{ for all } \alpha \in B_\Lambda\}$$

and

$$\Lambda^{++} = \{\lambda \in \Lambda | (\lambda + \rho, \alpha^v) > 0 \text{ for all } \alpha \in B_\Lambda\}.$$

In addition Λ^+_0 and Λ^{++}_0 are defined in the same way by replacing ρ with ρ_0. We define

$$\mathcal{X}_\lambda = \{J(w.\lambda) | w \in W_\lambda\}$$

and

$$\mathcal{X}^0_\lambda = \{I(w \circ \lambda) | w \in W_\lambda\}.$$

For $\lambda \in \mathfrak{h}^*$, we set $B^0_\lambda = \{\alpha \in B_\lambda | (\lambda + \rho_0, \alpha) = 0\}, \widetilde{B}^0_\lambda = \{\alpha \in B_\lambda | (\lambda + \rho, \alpha) = 0\}$ and $W^0_\lambda = \{w \in W_\lambda | w.\lambda = \lambda\}$. For $w \in W$ we define $\tau_\Lambda(w) = \{\alpha \in B_\Lambda | w\alpha < 0\}$.

0.6. The main results of this paper are as follows

Theorem A. *Let* $\mathfrak{g} = osp(1,2r)$*. If* $\lambda \in \Lambda^{++}, \mu \in \Lambda_0^{++}$ *and* $w_1, w_2, \in W_\lambda$*, then*

$$|\widetilde{M}(w_1.\lambda) : \widetilde{L}(w_2.\lambda)| = |M(w_1 \circ \mu) : L(w_2 \circ \mu)|.$$

Theorem B. *With the same hypotheses as in Theorem A*

$$J(w_1.\lambda) \subseteq J(w_2.\lambda)$$

if and only if

$$I(w_1 \circ \mu) \subseteq I(w_2 \circ \mu).$$

A consideration of central characters shows that if $J(\lambda) \subseteq J(\mu)$, then $\lambda \in W.\mu$. Therefore to describe $\operatorname{Prim} U(\mathfrak{g})$ as a poset we have to describe combinatorically the relation between elements w_1, w_2 of the Weyl group determined by the inclusion $J(w_1.\lambda) \subseteq J(w_2.\lambda)$ for all $\lambda \in \Lambda^+$. Theorem 4.3 allows us to assume that $w_1, w_2 \in W_\Lambda$. For regular λ, the resulting relation on Weyl group elements is given by Theorem B, since the structure of the poset $\operatorname{Prim} U(\mathfrak{g}_0)$ is known.

For $\mathfrak{g}$ arbitrary classical simple, there is a version of the translation principle which implies that in order to describe the poset $\mathcal{X}_\lambda$ and the multiplicities $|\widetilde{M}(w_1.\lambda) : \widetilde{L}(w_2.\lambda)|$ for λ regular and typical, we can assume that λ is sufficiently far from the walls of the Weyl chamber. To make this precise we say that a statement depending on $\lambda \in \mathfrak{h}^*$ holds for all λ sufficiently far from the walls if there is a positive constant c, such that the statement is true for all λ with $|(\lambda, \beta_i)| > c$ for $i = 1, \dots r$.

By some ring theoretic arguments given in 4.1, in order to relate $\operatorname{Prim} U(\mathfrak{g}_0)$ to $\operatorname{Prim} U(\mathfrak{g})$, it is enough to study the effect of restriction and induction on simple highest weight modules. When the highest weight is sufficiently far from the walls, the structure of these modules is particularly simple (see Lemmas 3.4 and 3.6).

In addition the description of $\mathcal{X}_\lambda$ in the singular case can be obtained from the description in the regular case using the translation principle. The proof of the translation principle is an easy adaptation of the corresponding result for semisimple Lie algebras. Brief details are given in Section 1.

We note that although Theorem B suggests a strong resemblance between $\operatorname{Prim} U(\mathfrak{g}_0)$ and $\operatorname{Prim} U(\mathfrak{g})$, there are significant differences in the singular cases. To illustrate these we describe $\operatorname{Prim} U(\mathfrak{g})$ in detail when $\mathfrak{g} = osp(1,4)$.

To show that Theorem A gives the multiplicities of all composition factors of Verma modules in the regular case, we must show that $|\widetilde{M}(\lambda) : \widetilde{L}(\mu)| \neq 0$ implies $\mu \in W_\lambda.\lambda$ (see Corollary 2.8). This is done using the Jantzen filtration on Verma modules. The multiplicities in the singular case can then be obtained from the translation principle. In addition we describe all homomorphisms between Verma modules.

0.7. When $\mathfrak{g} = osp(1,2r)$, the main differences with the Lie algebra case are consequences of the following result.

Lemma. *If* $\lambda \in \mathfrak{h}^*$ *and* $\alpha \in \widetilde{B}_\lambda^0$*, then* $\alpha/2$ *is not a root.*

Proof. The hypotheses mean that $(\lambda, \alpha^v) \in \mathbb{Z}$ and $(\lambda + \rho, \alpha^v) = 0$. However using [M2, 0.3], it is easy to see that if $\alpha/2$ is a root, then $(\rho, \alpha^v) \in (1/2) + \mathbb{Z}$. □

1. The Translation Principle

1.1. Necessary and sufficient conditions for $\widetilde{L}(\lambda)$ to be finite dimensional are given in [K1, Theorem 8]. Note that if $\widetilde{L}(\lambda)$ is finite dimensional, then so too is $L(\lambda)$ and hence $\lambda \in P_0^+$. However if $\lambda \in P_0^+$, then $\widetilde{L}(\lambda)$ is finite dimensional if and only if some additional conditions hold. An examination of these conditions, stated in [K1, Theorem 8], shows that they are automatically satisfied if λ is sufficiently far from the walls. Hence we deduce the following result.

Lemma. *There exists a nonempty W-invariant open subset U of $\mathfrak{h}^*$ such that λ is typical and $\widetilde{L}(\lambda)$ is finite dimensional if and only if $\lambda \in P_0^+ \cap U$.*

If $\lambda \in P_0 \cap U$, we denote by $V(\lambda)$ the unique finite dimensional simple module with highest weight contained in $W(P_0^+ \cap U)$.

1.2. Suppose $\lambda, \mu \in \mathfrak{h}^*$ are typical and $\lambda - \mu \in P \cap U$. We define $T_\lambda^\mu : \widetilde{\mathcal{O}}_\lambda :\longrightarrow \widetilde{\mathcal{O}}_\mu$ to be the exact functor given by $T_\lambda^\mu(M) = (M \otimes V(\mu - \lambda))_{(\mu)}$.

Since T_λ^μ is exact, it induces a homomorphism from $\mathcal{C}(\widetilde{\mathcal{O}}_\lambda)$ to $\mathcal{C}(\widetilde{\mathcal{O}}_\mu)$, also denoted by T_λ^μ.

We can use the following lemma to weaken the condition on $\lambda - \mu$ in the last definition.

Lemma. *If $\lambda \in \Lambda^{++}$ and $\mu \in \Lambda^+$, then there exists $\nu \in \Lambda^{++}$ such that $\lambda - \nu$ and $\nu - \mu$ belong to $P_0 \cap U$.*

Proof. Since U is a nonempty open set, and $\mu + P_0$ is Zariski dense, we have

$$(\mu + P_0) \cap (\mu + U) \cap (\lambda - U) \neq \varnothing.$$

Thus there exists $x \in P_0 \cap U$ such that $\nu = \mu + x$ satisfies the conditions of the lemma. $\square$

In the situation of the lemma we define $T_\lambda^\mu : \mathcal{C}(\widetilde{\mathcal{O}}_\lambda) \longrightarrow \mathcal{C}(\widetilde{\mathcal{O}}_\mu)$ and $T_\mu^\lambda : \mathcal{C}(\widetilde{\mathcal{O}}_\mu) \longrightarrow \mathcal{C}(\widetilde{\mathcal{O}}_\lambda)$ by $T_\lambda^\mu = T_\nu^\mu T_\lambda^\nu$ and $T_\mu^\lambda = T_\nu^\lambda T_\mu^\nu$. As in [J1, Theorem 2.10], we can show these maps are well defined.

1.3. The main results on the functors T_λ^μ now follow with the same proofs as [J1] and [J2].

Theorem. *If $\lambda \in \Lambda^{++}, \mu \in \Lambda^+$ and λ, μ are typical, then for all $w \in W_\lambda$, we have*

(1) $T_\lambda^\mu \widetilde{M}(w.\lambda) \cong \widetilde{M}(w.\mu)$,

(2) $T_\lambda^\mu \widetilde{L}(w.\lambda) \cong \begin{cases} \widetilde{L}(w.\mu) & \text{if } \widetilde{B}_\mu^0 \subseteq \tau_\Lambda(w) \\ 0 & \text{otherwise,} \end{cases}$

(3) $ch T_\mu^\lambda \widetilde{M}(w.\mu) = \sum_{w_1 \in W_\mu^0} ch \widetilde{M}(ww_1.\lambda)$.

Proof. This follows as in [J1, 2.10, 2.11 and 2.17], see also [J2, 4.12 (2), (3) and 4.13 (1)]. $\square$

1.4. For $\lambda \in \mathfrak{h}^*$ typical, we set $\widehat{\mathcal{X}}_\lambda = \{\operatorname{Ann} M | M \in \widetilde{\mathcal{O}}_\lambda\}$. If T_λ^μ is defined, there is a map $T_\lambda^\mu : \widehat{\mathcal{X}}_\lambda \longrightarrow \widehat{\mathcal{X}}_\mu$ given by $T_\lambda^\mu(\operatorname{Ann} M) = \operatorname{Ann} T_\lambda^\mu M$ (see [J2, Lemma 5.4]). Fix $\Lambda \in \mathfrak{h}^*/P$ and $\lambda \in \Lambda^{++}$. As in [J2, Satz 5.7] there is a well defined map τ_Λ from $\mathcal{X}_\lambda$ onto the power set of B_Λ such that $\tau_\Lambda(J(w.\lambda)) = \tau_\Lambda(w)$.

Theorem. *Let $\Lambda \in \mathfrak{h}^*/P$, and suppose $\lambda \in \Lambda^{++}$ and $\mu \in \Lambda^+$ are typical. Then there is an isomorphism of ordered sets*

$$\phi : \{I \in \mathcal{X}_\lambda | \widetilde{B}^0_\mu \subseteq \tau_\Lambda(I)\} \xrightarrow{\sim} \mathcal{X}_\mu.$$

If $w \in W_\lambda$ and if $\widetilde{B}^0_\mu \subseteq \tau_\Lambda(w)$, then $\phi(J(w.\lambda)) = J(w.\mu)$.

Proof. By Lemma 1.2, there exists $\nu \in \Lambda^{++}$ such that the functors T^ν_λ and T^μ_ν are defined. The proof of [J2, Theorem 5.8] then shows that the map $I \longrightarrow T^\mu_\nu T^\nu_\lambda I$ is an isomorphism between the ordered sets. The last statement follows from Theorem 1.3(2). $\square$

2. The Structure of Verma Modules

2.1. If $\mathfrak{g}$ is a semisimple Lie algebra, several important properties of the Verma modules $M(\lambda)$ depend on the fact that $U(\mathfrak{n}^-)$ is a domain. For classical Lie superalgebras the analog of this result is false in general. However it remains true for $\mathfrak{g} = osp(1, 2r)$. In fact it follows from [AL] that $U(\mathfrak{g})$ is a domain in this case. We provide another proof that $U(\mathfrak{n}^-)$ is a domain for $\mathfrak{g} = osp(1, 2r)$, which gives additional information.

Theorem. *If $\mathfrak{g} = osp(1, 2r)$, then $U(\mathfrak{n}^-)$ is an iterated Ore extension.*

Proof. For each $\beta \in \Delta^+$, let $e_{-\beta}$ span the one-dimensional vector space $\mathfrak{g}^{-\beta}$. Let $\alpha_1, \ldots, \alpha_r$ be the basis for the positive roots given in [M2, 0.3]. If $\alpha = \sum k_i\alpha_i \in \Delta^+$ set $ht(\alpha) = \sum k_i$. Next if $\alpha \in \Delta^+_0$, set $h(\alpha) = ht(\alpha)$ and if $\alpha \in \Delta^+_1, h(\alpha) = h(2\alpha)$. Note that if α, β are distinct odd roots, then $h(\alpha) \neq h(\beta)$. It follows that if α, β and $\alpha + \beta \in \Delta^+$ with $\alpha \neq \beta$, then

$$h(\alpha + \beta) > \min\{h(\alpha), h(\beta)\}.$$

Now order the roots in $\overline{\Delta}^+_0 \cup \Delta^+_1$ as $\beta_1, \ldots, \beta_s$ in such a way that $i \leq j$ implies that $h(\beta_i) \geq h(\beta_j)$. For $1 \leq i \leq s$ set

$$\begin{aligned}\mathfrak{n}^i = \operatorname{span}\{&e_{-\beta_j} | j \leq i, \beta_j \in \overline{\Delta}^+_0\} \\ &\cup \{e_{-\beta_j}, e_{-2\beta_j} | j \leq i, \beta_j \in \Delta^+_1\}.\end{aligned}$$

We claim that each $\mathfrak{n}^i$ is a subalgebra of $\mathfrak{n}$ and that $\mathfrak{n}^{i-1}$ is an ideal in $\mathfrak{n}^i$. To see this, suppose that α and β are roots such that $e_{-\alpha} \in \mathfrak{n}^{i-1}$ and $e_{-\beta} \in \mathfrak{n}^i$. Then $h(\alpha) = h(\beta_j), h(\beta) = h(\beta_k)$, where $j \leq i-1, k \leq i$. Hence

$$\min\{h(\alpha), h(\beta)\} \geq h(\beta_i).$$

It follows that if $\alpha + \beta$ is a root, then $h(\alpha + \beta) = h(\beta_l)$ with $l \leq i - 1$ and hence $[e_{-\alpha}, e_{-\beta}] \in \mathfrak{n}^{i-1}$. This shows that $[\mathfrak{n}^{i-1}, \mathfrak{n}^i] \subseteq \mathfrak{n}^{i-1}$ and a similar argument shows that $[\mathfrak{n}^i, \mathfrak{n}^i] \subseteq \mathfrak{n}^i$. There are now two cases depending on whether β_i is even or odd.

If β_i is odd, we have $e^2_{-\beta_i} = e_{-2\beta_i}$ up to a nonzero scalar multiple. Also if x is a homogeneous element of $U(\mathfrak{n}^{i-1})$, we have

$$e_{-\beta_i}x = \sigma(x)e_{-\beta_i} + \delta(x)$$

where $\sigma(x) = (-1)^{\deg x}x$ and $\delta = ade_{-\beta_i}$. By the PBW theorem, monomials in $e_{-\beta_i}$ are linearly independent over $U(\mathfrak{n}^{i-1})$ and hence $U(\mathfrak{n}^i)$ is an Ore extension of $U(\mathfrak{n}^{i-1})$. The case where $\beta_i \in \overline{\Delta}^+_0$ is easier and left to the reader. $\square$

2.2. For the remainder of this section we assume that $\mathfrak{g} = osp(1,2r)$.

Corollary. *For $\lambda, \mu \in \mathfrak{h}^*$*

a) *$\widetilde{M}(\lambda)$ has a unique minimal nonzero submodule.*
b) *$\mathrm{Hom}_{\mathfrak{g}}(\widetilde{M}(\mu), \widetilde{M}(\lambda))$ has dimension ≤ 1 over $\mathbb{C}$.*
c) *Every nonzero element of $\mathrm{Hom}_{\mathfrak{g}}(\widetilde{M}(\mu), \widetilde{M}(\lambda))$ is injective.*

Proof. Since $U(\mathfrak{n}^-)$ is a domain, the same proofs as in [D, 7.6.3 and 7.6.6] work. □

2.3. It follows from Corollary 2.2 that if $\mathrm{Hom}_{\mathfrak{g}}(\widetilde{M}(\mu), \widetilde{M}(\lambda)) \neq 0$, then $\widetilde{M}(\mu)$ is uniquely embedded in $\widetilde{M}(\lambda)$ up to scalar. As in [D, 7.6.7] we write $\widetilde{M}(\mu) \subseteq \widetilde{M}(\lambda)$ in this situation. When this occurs $\widetilde{M}(\mu)$ and $\widetilde{M}(\lambda)$ have the same central character, so we have $\mu = w.\lambda$ for some $w \in W$.

2.4. Let $\zeta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{h})$ be the projection relative to the decomposition $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+)$. Let $x \longrightarrow {}^t x$ be the antiautomorphism of $U(\mathfrak{g})$ defined in [M1, 3.2]. We consider the bilinear form F on $U(\mathfrak{n}^-)$ given by

$$F(x,y) = \zeta({}^t xy) \quad \text{for } x, y \in U(\mathfrak{g}).$$

Let F_η denote the restriction of F to a bilinear form on $U(\mathfrak{n}^-)^{-\eta}$. We view $U(\mathfrak{h})$ as the algebra of polynomial functions on $\mathfrak{h}^*$. Then for $\lambda \in \mathfrak{h}^*, F_\eta(\lambda)$ can be interpreted as the bilinear form on $\widetilde{M}(\lambda)^{\lambda-\eta}$ given by

$$F_\eta(\lambda)(xv_\lambda, yv_\lambda) = \zeta({}^t xy)(\lambda)$$

for $x, y \in U(\mathfrak{n}^-)^{-\eta}$.

Now let T be an indeterminate and consider the Verma module $\widetilde{M}(\widetilde{\lambda})$ over $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[T]$ with highest weight $\widetilde{\lambda} = \lambda + T\rho$. We obtain as above a $\mathbb{C}[T]$-valued bilinear form $F_\eta(\widetilde{\lambda})$ on the $\mathbb{C}[T]$-weight spaces $\widetilde{M}(\widetilde{\lambda})_{\lambda-\eta}$. Then we set

$$\widetilde{M}_n(\widetilde{\lambda}) = \{u \in \widetilde{M}(\widetilde{\lambda}) | F_\eta(\widetilde{\lambda})(u, \widetilde{M}(\widetilde{\lambda})) \subseteq (T^n)\}$$

and let $\widetilde{M}_n(\lambda)$ be the image of $\widetilde{M}_n(\widetilde{\lambda})$ obtained by reducing mod T. Note that as in [J1, Satz 5.2], $\widetilde{M}_1(\lambda)$ is the unique maximal proper submodule of $\widetilde{M}(\lambda)$.

As usual, if $\eta \in Q^+$, a partition of η is a map $\pi : \Delta^+ \to \mathbb{N}$ such that $\pi(\alpha) = 0$ or 1 for all $\alpha \in \Delta_1^+$ and

$$\sum_{\alpha \in \Delta^+} \pi(\alpha)\alpha = \eta.$$

We denote by $\mathbf{P}(\eta)$ the set of partitions of η, and for $\alpha \in \Delta_1^+$ we define

$$\mathbf{P}_\alpha(\eta) = \{\pi \in \mathbf{P}(\eta) | \pi(\alpha) = 0\}.$$

Set $p(\eta) = |\mathbf{P}(\eta)|$ and $p_\alpha(\eta) = |\mathbf{P}_\alpha(\eta)|$. Fix an ordering on the set $|\Delta^+|$, and for $\pi \in \mathbf{P}(\eta)$, let $e_{-\pi} = \prod_{-\alpha \in \Delta^+} e_\alpha$, the product being taken with respect to this order.

Theorem. *Up to a nonzero constant factor*

$$\det F_\eta = \prod_{\alpha \in \overline{\Delta}_0^+} \prod_{r=1}^{\infty} (h_\alpha + \rho(h_\alpha) - r\frac{(\alpha,\alpha)}{2})^{p(\eta - r\alpha)}$$
$$\prod_{\alpha \in \Delta_1^+} \prod_{\substack{r=1 \\ r \text{ odd}}}^{\infty} (h_\alpha + \rho(h_\alpha) - r\frac{(\alpha,\alpha)}{2})^{p(\eta - r\alpha)}.$$

2.5. Theorem 2.4 is stated incorrectly in [K2] and [Jak, Theorem 2.4]. For this reason we make some brief comments on the proof. The only place where the proof differs significantly from the proof of [KK, Theorem 1] is in the computation of the leading term. We remark that the correct formula for all basic classical simple Lie superalgebras is stated in [K3].

Lemma. *Up to a nonzero constant factor, the leading term of* $\det F_\eta$ *is*

$$\prod_{\alpha\in\overline{\Delta}_0^+}\prod_{r=1}^{\infty} h_\alpha^{p(\eta-r\alpha)} \prod_{\alpha\in\Delta_1^+}\prod_{\substack{r=1\\ r\,\mathrm{odd}}}^{\infty} h_\alpha^{p(\eta-r\eta)}.$$

Proof. $U(\mathfrak{n}^-)^{-\eta}$ has basis $\{e_{-\pi}|\pi\in\mathbf{P}(\eta)\}$, and as in [S, Lemma 4] the leading term is

$$\prod_{\pi\in\mathbf{P}(\eta)}\prod_{\alpha\in\Delta^+} h_\alpha^{\pi(\alpha)}.$$

If $\alpha\in\overline{\Delta}_0^+$, it follows, as in [S, Lemma 4] again, that the multiplicity of h_α is $\sum_{r=1}^{\infty} p(\eta-r\alpha)$. If $\alpha\in\Delta_1^+$, then $\beta=2\alpha\in\Delta_0^+$, and the multiplicity of h_α is

$$\sum_{\pi\in\mathbf{P}(\eta)}\pi(\alpha)+\pi(\beta)=\sum_{\pi\in\mathbf{P}_\alpha(\eta)}\pi(\beta)+\sum_{\pi\in\mathbf{P}_\alpha(\eta-\alpha)}(1+\pi(\beta)).$$

The first sum here is equal to

$$\sum_{r=1}^{\infty} r(p_\alpha(\eta-r\beta)-p_\alpha(\eta-(r+1)\beta))=\sum_{r=1}^{\infty} p_\alpha(\eta-r\beta).$$

The second term equals

$$p_\alpha(\eta-\alpha)+\sum_{r=1}^{\infty} p_\alpha(\eta-\alpha-r\beta).$$

Using the fact that

$$p(\eta)=p_\alpha(\eta)+p_\alpha(\eta-\alpha)$$

this shows that the multiplicity of h_α in the leading term of $\det F_\eta$ is

$$\sum_{r=1}^{\infty} p_\alpha(\eta-r\alpha)=\sum_{\substack{r=1\\ r\,\mathrm{odd}}}^{\infty} p(\eta-r\alpha). \quad \square$$

2.6. Theorem. *For all* $\lambda\in\mathfrak{h}^*$

$$\sum_{n>0} ch\widetilde{M}_n(\lambda)=\sum_{\alpha\in\overline{\Delta}_0^+(\lambda)} ch\widetilde{M}(s_\alpha.\lambda)+\sum_{\alpha\in\Delta_1^+(\lambda)} ch\widetilde{M}(s_\alpha.\lambda)$$

where

$$\overline{\Delta}_0^+(\lambda)=\{\alpha\in\overline{\Delta}_0^+|(\lambda+\rho,\alpha^v)\in\mathbb{N}\backslash\{0\}\},$$
$$\Delta_1^+(\lambda)=\{\alpha\in\Delta_1^+|(\lambda+\rho,\alpha^v)\in 1+2\mathbb{N}\}.$$

Proof. By [J1, Lemma 5.1]

$$\sum_{n>0}\dim\widetilde{M}_n(\lambda)^{\lambda-\eta}=v_T(\det F_\eta(\widetilde{\lambda}))$$

where v_T denotes the T-adic valuation. Note that

$$v_T(h_\alpha + \rho(h_\alpha) - r(\alpha,\alpha)/2)(\widetilde{\lambda}) = \begin{cases} 1 & \text{if } (\lambda+\rho,\alpha^v) = r, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} \sum_{n>0} ch\widetilde{M}_n(\lambda) &= \sum_{n>0}\sum_{\eta} \dim \widetilde{M}_n(\lambda) e^{\lambda-\eta} \\ &= \sum_{\alpha\in\overline{\Delta}_0^+(\lambda)} p(\eta + s_\alpha.\lambda - \lambda)e^{\lambda-\eta} + \sum_{\alpha\in\Delta_1^+(\lambda)} p(\eta + s_\alpha.\lambda - \lambda)e^{\lambda-\eta} \\ &= \sum_{\alpha\in\overline{\Delta}_0^+(\lambda)} p(\eta)e^{s_\alpha.\lambda-\eta} + \sum_{\alpha\in\Delta_1^+(\lambda)} p(\eta)e^{s_\alpha.\lambda-\eta} \\ &= \sum_{\alpha\in\overline{\Delta}_+^0(\lambda)} ch\widetilde{M}(s_\alpha.\lambda) + \sum_{\alpha\in\Delta_+^1(\lambda)} ch\widetilde{M}(s_\alpha.\lambda). \qquad \square \end{aligned}$$

2.7. Theorem 2.6 motivates the following definitions, c.f. [J1, 2.19]. Suppose that $\lambda,\mu\in\mathfrak{h}^*$. We write $\mu\uparrow'\lambda$ to mean that $\mu = s_\alpha.\lambda$ for some $\alpha\in\overline{\Delta}_0^+(\lambda)\cup\Delta_1^+(\lambda)$ and $\mu\uparrow\lambda$ if there exist $\mu_0,\mu_1,\dots,\mu_t$ such that $\mu_{i-1}\uparrow'\mu_i$ for $i=1,\dots,t; \mu_0=\mu$ and $\mu_t=\lambda$.

Theorem. *For $\lambda,\mu\in\mathfrak{h}^*$, the following are equivalent.*

(1) $\mathrm{Hom}_{\mathfrak{g}}(\widetilde{M}(\mu),\widetilde{M}(\lambda))\neq 0$,
(2) $|\widetilde{M}(\lambda):\widetilde{L}(\mu)|\neq 0$,
(3) $\mu\uparrow\lambda$.

Proof. The implication (1) $\Rightarrow$ (2) is clear and (2) $\Rightarrow$ (3) follows as in [J1, 5.3] using Theorem 2.6. Finally (3) $\Rightarrow$ (1) follows as in [D, Lemma 7.6.13]. $\square$

2.8. Corollary. *If $|\widetilde{M}(\lambda):\widetilde{L}(\mu)|\neq 0$, then $\mu\in W_\lambda.\lambda$.*

Proof. We can assume that $\mu = s_\alpha.\lambda$ where $\alpha\in\overline{\Delta}_0^+(\lambda)$ or $\alpha\in\Delta_1^+(\lambda)$. In the first case $(\rho,\alpha^v)\in\mathbb{Z}$ and hence $(\lambda,\alpha^v)\in\mathbb{Z}$, while in the second case $(\rho,\alpha^v)\in 1+2\mathbb{N}$ and again $(\lambda,\alpha^v)\in\mathbb{Z}$. Thus in both cases $s_\alpha\in W_\lambda$. $\square$

3. Restriction and Induction

3.1. Set $R = U(\mathfrak{g}_0)$ and $S = U(\mathfrak{g})$. In order to relate Prim$R$ and Prim$S$, we study the behaviour of highest weight modules under restriction and induction. It is shown in [M2, Theorem 3.2] that if $\mathfrak{g}$ is classical simple, then as an R-module $\widetilde{M}(\lambda)$ has a finite filtration whose factors are Verma modules $M(\lambda-\gamma)$, with $\gamma\in\Gamma$. For the induced module we have a slightly weaker result. For $\gamma\in\Gamma$, we define $K(\gamma)\in\mathbb{N}$ through

$$\prod_{\beta\in\Delta_1^+}(1+e^\beta) = \sum_{\gamma\in\Gamma} K(\gamma)e^\gamma.$$

Lemma. *There is an equality of characters*

$$ch(S\otimes_R M(\lambda)) = \sum_{\gamma\in\Gamma} K(\gamma) ch\widetilde{M}(\lambda+\gamma).$$

Proof. We have

$$S \otimes_R M(\lambda) \cong \Lambda\mathfrak{g}_1 \otimes_{\mathbb{C}} M(\lambda)$$

as $\mathfrak{g}_0$-modules, where $\mathfrak{g}_0$ acts on $\Lambda\mathfrak{g}_1$ by the adjoint representation. Since

$$chM(\lambda) = e^\lambda \prod_{\alpha\in\Delta_0^+}(1-e^{-\alpha})^{-1}$$

it follows that

$$\begin{aligned} ch(S\otimes_R M(\lambda)) &= \prod_{\beta\in\Delta_1^+}(1+e^{\beta})(1+e^{-\beta})chM(\lambda) \\ &= \sum_{\gamma\in\Gamma} K(\gamma)e^{\lambda+\gamma}\prod_{\beta\in\Delta_1^+}(1+e^{-\beta})\prod_{\alpha\in\Delta_0^+}(1-e^{-\alpha})^{-1} \\ &= \sum_{\gamma\in\Gamma} K(\gamma)ch\widetilde{M}(\lambda+\gamma). \end{aligned}$$ ☐

3.2. Let $\mathfrak{g} = osp(1, 2r)$. In this case $S \otimes_R M(\lambda)$ is very often a direct sum of Verma modules. To show this we need a preliminary result.

Lemma. *Let $\mathfrak{g} = osp(1, 2r)$. If I is any nonzero left ideal of $U(\mathfrak{n}^-)$, then $I \cap U(\mathfrak{n}_0^-) \neq 0$.*

Proof. Let $\mathcal{C}$ be the set of nonzero elements of $U(\mathfrak{n}_0^-)$. Since $U(\mathfrak{n}^-)$ is a domain, elements of $\mathcal{C}$ are nonzero-divisors in $U(\mathfrak{n}^-)$. By the proof of [B, Theorem 1, page 20] $\mathcal{C}$ is an Ore set in $U(\mathfrak{n}^-)$ and $D = U(\mathfrak{n}^-)_{\mathcal{C}}$ is artinian. It follows that D is a division ring. Since $I\mathcal{C}^{-1}$ is a left ideal of D, the result follows. ☐

It is shown in [M2, Theorem 3.6], that if $\lambda \in \mathfrak{h}^*$ and all the central characters $\chi^0_{\lambda-\gamma}, \gamma \in \Gamma$ are distinct, then as an R-module $\widetilde{M}(\lambda) = \oplus_{\gamma\in\Gamma} M(\lambda-\gamma)$. We next prove a companion result for induced Verma modules.

Proposition. *If the central characters $\chi_{\lambda+\gamma}, \gamma \in \Gamma$ are distinct, then*

$$S \otimes_R M(\lambda) \cong \bigoplus_{\gamma\in\Gamma} \widetilde{M}(\lambda+\gamma).$$

Proof. If $M_\gamma = (S \otimes_R M(\lambda))_{(\lambda+\gamma)}$, then by hypothesis and Lemma 3.1,

$$(S\otimes_R M(\lambda)) \cong \bigoplus M_\gamma \quad \text{and} \quad chM_\gamma = ch\widetilde{M}(\lambda+\gamma).$$

Thus M_γ contains a highest weight vector v of weight $\lambda+\gamma$ and there is a homomorphism of $U(\mathfrak{g})$-modules $\phi : \widetilde{M}(\lambda+\gamma) \longrightarrow U(\mathfrak{g})v \subseteq M_\gamma$. We can identify $\operatorname{Ker}\phi$ with a left ideal I of $U(\mathfrak{n}^-)$. If $I \neq 0$, then by the above lemma, $I \cap U(\mathfrak{n}_0^-) \neq 0$, and then v would be a torsion element of M_γ as a left $U(\mathfrak{n}_0^-)$-module. However by [J1, Satz 2.2], $S\otimes_R M(\lambda)$ has a finite series of $U(\mathfrak{g}_0)$-submodules whose factors are Verma modules, so $S\otimes_R M(\lambda)$ is torsion free as a $U(\mathfrak{n}_0^-)$-module. This shows that $\operatorname{Ker}\phi = 0$, and the equality of characters implies that ϕ has image M_γ. ☐

3.3. Let $\mathfrak{g} = osp(1,2r)$. It is shown in [M2, Theorem 3.6] that if $\lambda \in \mathfrak{h}^*$, then λ is $\mathfrak{g}$-regular if and only if the central characters $\chi^0_{\lambda+\gamma}$, with $\gamma \in \Gamma$ are all distinct. There is a corresponding result for $\mathfrak{g}_0$-regularity.

Lemma. *If $\lambda \in \mathfrak{h}^*$, then λ is $\mathfrak{g}_0$-regular if and only if the central characters $\chi_{\lambda+\gamma}$, with $\gamma \in \Gamma$ are all distinct.*

Proof. Recall that the central characters χ_λ are related to the Harish Chandra map $\psi : Z(\mathfrak{g}) \longrightarrow S(\mathfrak{h})^W$ by the equation

$$\chi_\lambda(z) = \psi(z)(\lambda + \rho)$$

for $z \in Z(\mathfrak{g})$ and $\lambda \in \mathfrak{h}^*$.

Suppose first that $(\lambda + \rho_0, \alpha) = 0$, for some $\alpha \in \Delta_0^+$. By [M2, Lemma 3.8], there exist distinct elements γ, γ' of Γ such that $s_\alpha * \gamma = \gamma'$. As in [M2, Proposition 3.9], it follows that

$$\begin{aligned} \chi_{\lambda+\gamma}(z) &= \psi(z)(\lambda + \gamma + \rho) \\ &= \psi(z)(s_\alpha(\lambda + \gamma + \rho)) \\ &= \psi(z)(\lambda + \gamma' + \rho) \\ &= \chi_{\lambda+\gamma'}(z). \end{aligned}$$

Thus $\chi_{\lambda+\gamma} = \chi_{\lambda+\gamma'}$.

Conversely, suppose that γ and γ' are distinct elements of Γ such that $\chi_{\lambda+\gamma} = \chi_{\lambda+\gamma'}$. Then $w(\lambda + \rho + \gamma) = \lambda + \rho + \gamma'$ for some $w \in W$. If $\mu = -(\lambda + \rho)$, then $w(\mu - \gamma) = \mu - \gamma'$, so by [M2, Lemma 3.10], $(\mu - \rho_1, \alpha) = 0$ for some $\alpha \in \Delta_0$. Hence $(\lambda + \rho_0, \alpha) = 0$, so λ is not $\mathfrak{g}_0$-regular. $\square$

Remark. If $\mathfrak{g} = s\ell(r,1)$ and λ is such that $\lambda + \gamma$ is typical for all $\gamma \in \Gamma$, then λ is $\mathfrak{g}_0$-regular if and only if all the central characters $\chi_{\lambda+\gamma}$ are distinct. The proof is the same as Lemma 3.3, except that we use [M2, Remark 3.11] in place of [M2, Lemma 3.10].

3.4. To make further progress, we need to make additional hypotheses. For the remainder of Section 3 we assume that $\mathfrak{g} = osp(1,2r)$ or $s\ell(r,1)$. Note that this implies that $K(\gamma) = 1$ for all $\gamma \in \Gamma$.

Lemma. *Suppose that λ is typical and sufficiently far from the walls, then as a $U(\mathfrak{g}_0)$-module*

$$\widetilde{L}(\lambda) = \bigoplus_{\gamma\in\Gamma} L(\lambda - \gamma).$$

Proof. We may assume that $(\lambda + \rho, \alpha) \neq 0$ for all $\alpha \in \Delta_0^+$. Then by [M2, Lemma 3.7 and Corollary 3.11]

$$\widetilde{M}(\lambda) = \bigoplus_{\gamma\in\Gamma} M(\lambda - \gamma)$$

as a $U(\mathfrak{g}_0)$-module.

Thus $\widetilde{L}(\lambda) = \bigoplus_{\gamma\in\Gamma} \widetilde{L}(\lambda)_{\langle\lambda-\gamma\rangle}$ and $\widetilde{L}(\lambda)_{\langle\lambda-\gamma\rangle}$ is a factor module of $M(\lambda - \gamma)$. On the other hand we may construct a nondegenerate contravariant form (,) on $\widetilde{L}(\lambda)$ as in [J1, Satz 1.6]. By [J1, Satz 1.13e)] $\widetilde{L}(\lambda)_{\langle\lambda-\gamma\rangle}$ is orthogonal to $\widetilde{L}(\lambda)_{\langle\lambda-\gamma'\rangle}$ if $\gamma \neq \gamma'$. It follows that the restriction of (,) to the $U(\mathfrak{g}_0)$-module $\widetilde{L}(\lambda)_{\langle\lambda-\gamma\rangle}$ is nondegenerate. Thus $\widetilde{L}(\lambda)_{\langle\lambda-\gamma\rangle}$ is either zero or isomorphic to $L(\lambda - \gamma)$ by [J1, Satz 1.6].

If $\widetilde{L}(\lambda)_{\langle\lambda-\gamma\rangle} = 0$, then the highest weight vector for the $U(\mathfrak{g}_0)$-submodule $M(\lambda-\gamma)$ of $\widetilde{M}(\lambda)$ must belong to $\widetilde{M}_1(\lambda)$. By Theorem 2.6, this implies that $\lambda-\gamma \le s_\alpha.\lambda$ for some α, and hence $\gamma \ge (\lambda+\rho,\alpha^v)\alpha$, and this is not possible for λ sufficiently far from the walls. □

3.5. We require some properties of the Grothendieck group $\mathcal{C}(\widetilde{\mathcal{O}}_\lambda)$ which are entirely analogous to those of the Lie algebra case. For λ $\mathfrak{g}$-regular, the group $\mathcal{C}(\widetilde{\mathcal{O}}_\lambda)$ is free abelian on the classes of the modules $\widetilde{M}(w.\lambda)$ for $w \in W$. Since $|\widetilde{M}(\lambda) : \widetilde{L}(\mu)| \neq 0$ implies that $\mu \le \lambda$, and $|\widetilde{M}(\lambda) : \widetilde{L}(\lambda)| = 1$ it follows that the classes of the modules $\widetilde{L}(w.\lambda)$ also form a basis for $\mathcal{C}(\widetilde{\mathcal{O}}_\lambda)$. If $M \in \widetilde{\mathcal{O}}_\lambda$, and if in $\mathcal{C}(\widetilde{\mathcal{O}}_\lambda)$ we have

$$M = \sum_{w\in W} b_w \widetilde{L}(w.\lambda)$$

with $b_w \in \mathbb{Z}$, then we set

$$(\widetilde{L}(w.\lambda) : M) = b_w.$$

The matrix $(\widetilde{L}(w.\lambda) : \widetilde{M}(y.\lambda))$ is clearly the inverse of the matrix

$$|\widetilde{M}(y.\lambda) : \widetilde{L}(w.\lambda)|.$$

Lemma. *Assume that λ is typical and sufficiently far from the walls. Then*

$$|\widetilde{M}(y.\lambda) : \widetilde{L}(w.\lambda)| = |M(y\circ\lambda) : L(w\circ\lambda)|$$

and

$$(\widetilde{L}(y.\lambda) : \widetilde{M}(w.\lambda)) = (L(y\circ\lambda) : M(w\circ\lambda))$$

for all $y, w \in W$.

Proof. In the Grothendieck group $\mathcal{C}(\widetilde{\mathcal{O}}_\lambda)$ we can write

$$\widetilde{M}(y.\lambda) = \sum b_{y,w}\widetilde{L}(w.\lambda).$$

Note that $w\circ\lambda = w.\lambda - w*0$. Thus

$$\begin{aligned}\widetilde{L}(w.\lambda)_{\langle\lambda\rangle} &= \widetilde{L}(w.\lambda)_{\langle w\circ\lambda\rangle}\\ &= L(w.\lambda - w*0)\\ &= L(w\circ\lambda)\end{aligned}$$

where the second equality follows from Lemma 3.4. Similarly $\widetilde{M}(y.\lambda)_{\langle\lambda\rangle} = M(y\circ\lambda)$. Therefore

$$M(y\circ\lambda) = \sum b_{y,w} L(w\circ\lambda).$$

This proves the first statement, and the second now follows easily. □

3.6. Lemma. *Assume that λ is typical and sufficiently far from the walls. Then*

$$(S\otimes_R L(\lambda)) \cong \bigoplus_{\gamma\in\Gamma} \widetilde{L}(\lambda+\gamma).$$

Proof. In the group $\mathcal{C}(\mathcal{O}_\lambda)$ we can write

$$L(\lambda) = \sum_w b_w M(w\circ\lambda).$$

Hence

$$(S\otimes_R L(\lambda)) = \bigoplus (S\otimes_R L(\lambda))_{(\lambda+\gamma)}$$

where by Lemma 3.1

$$\begin{aligned}(S \otimes_R L(\lambda))_{(\lambda+\gamma)} &= \sum_w b_w (S \otimes_R M(w \circ \lambda))_{(\lambda+\gamma)} \\ &= \sum_{w,\gamma'} b_w \widetilde{M}(w \circ \lambda + w * \gamma')_{(\lambda+\gamma)} \\ &= \sum_w b_w \widetilde{M}(w.(\lambda+\gamma)).\end{aligned}$$

However $b_w = (\widetilde{L}(\lambda) : \widetilde{M}(w.\lambda)) = (\widetilde{L}(\lambda+\gamma) : \widetilde{M}(w.(\lambda+\gamma))$ using Lemma 3.5 for the first equality, and Theorem 1.3 for the second.

Thus in $\mathcal{C}(\widetilde{\mathcal{O}}_{\lambda+\gamma})$, we have

$$(S \otimes_R L(\lambda))_{(\lambda+\gamma)} = \widetilde{L}(\lambda+\gamma).$$

Since $\widetilde{L}(\lambda+\gamma)$ is irreducible, it follows that $(S \otimes_R L(\lambda))_{(\lambda+\gamma)}$ and $\widetilde{L}(\lambda+\gamma)$ are isomorphic. $\square$

3.7. Theorem A follows easily by combining Lemma 3.5 with the translation principles for $U(\mathfrak{g})$ and $U(\mathfrak{g}_0)$. Suppose that $\lambda \in \Lambda^{++}, \mu \in \Lambda_0^{++}$, and that $\nu \in \Lambda^{++}$ is sufficiently far from the walls. Then

$$\begin{aligned}|\widetilde{M}(w_1.\lambda) : \widetilde{L}(w_2.\lambda)| &= |\widetilde{M}(w_1.\nu) : \widetilde{L}(w_2.\nu)| \\ &= |M(w_1 \circ \nu) : L(w_2 \circ \nu)| \\ &= |M(w_1 \circ \mu) : L(w_2 \circ \mu)|.\end{aligned}$$

3.8. Suppose that $\Lambda \in \mathfrak{h}^*/P$ and $\mu \in \Lambda_0^{++}$. Define matrices $(a_{w,y}), (b_{y,w})$ by

$$M(y \circ \mu) = \sum_{w \in W_\Lambda} b_{y,w} L(w \circ \mu),$$

$$L(w \circ \mu) = \sum_{w \in W_\Lambda} a_{w,y} M(y \circ \mu).$$

By [BB] or [BK], the multiplicities $b_{y,w}$ are given by the Kazhdan-Lusztig conjecture. From [J1, 2.10a), 2.11], we have for $\nu \in \Lambda_0^+$

$$M(y \circ \nu) = \sum_{B_\nu^0 \subseteq \tau_\Lambda(w)} b_{y,w} L(w \circ \nu).$$

In addition if $\lambda \in \Lambda^+$, and $\widetilde{B}_\lambda^0 \subseteq \tau_\Lambda(w)$, then it follows from Theorem 1.3 and Theorem A that

$$\widetilde{L}(w.\lambda) = \sum_{y \in W_\Lambda} a_{w,y} \widetilde{M}(y.\lambda).$$

We now combine these results to obtain a formula for the restriction of $\widetilde{L}(w.\lambda)$ to $U(\mathfrak{g}_0)$ for $w \in W_\lambda$. Note that by [J2, 2.7] the set $\{w \in W_\Lambda | \widetilde{B}_\lambda^0 \subseteq \tau_\Lambda(w)\}$ is a left transversal to W_λ^0 in W_Λ. Therefore there is no loss of generality in assuming that $\widetilde{B}_\lambda^0 \subseteq \tau_\Lambda(w)$. Every element $\nu \in \Lambda$ is conjugate under W_Λ to a unique element $\overline{\nu}$ of Λ_0^+. Choose $w_\nu \in W_\Lambda$ such that

$$w_\nu^{-1} \circ \nu = \overline{\nu}.$$

Theorem. *If $\lambda \in \Lambda^+$ and $\widetilde{B}^0_\lambda \subseteq \tau_\Lambda(w)$, then in the Grothendieck group $\mathcal{C}(\mathcal{O})$, we have*

$$\widetilde{L}(w.\lambda) = \sum_{\gamma\in\Gamma}\sum a_{w,y} b_{yw_{\lambda-\gamma},v} L(v \circ (\overline{\lambda-\gamma}))$$

where the inner sum is taken over all $y, v \in W_\Lambda$ such that $B^0_{\overline{\lambda-\gamma}} \subseteq \tau_\Lambda(v)$.

Proof. Working in $\mathcal{C}(\mathcal{O})$, we have by the foregoing remarks and [M2, Theorem 3.2]

$$\begin{aligned}\widetilde{L}(w.\lambda) &= \sum_{y\in W} a_{w,y}\widetilde{M}(y.\lambda)\\ &= \sum_{y\in W,\gamma\in\Gamma} a_{w,y} M(y\circ(\lambda-\gamma))\\ &= \sum_{y\in W,\gamma\in\Gamma} a_{w,y} M(yw_{\lambda-\gamma}\circ(\overline{\lambda-\gamma}))\\ &= \sum a_{w,y} b_{yw_{\lambda-\gamma},v} L(v\circ(\overline{\lambda-\gamma})). \quad \square\end{aligned}$$

In certain circumstances the above formula can be simplified considerably.

Corollary. a) *If $\gamma \in \Gamma$, and $\lambda-\gamma \in \Lambda_0^+$, then the term in Theorem 3.8 corresponding to γ is equal to*

$$\begin{array}{ll} L(w\circ(\lambda-\gamma)) & \textit{if } \ B^0_{\lambda-\gamma} \subseteq \tau_\Lambda(w),\\ 0 & \textit{otherwise.}\end{array}$$

b) *Assume that $\lambda-\gamma \in \Lambda_0^+$ for all $\gamma \in \Gamma$, then*

$$\widetilde{L}(w.\lambda) = \sum L(w\circ(\lambda-\gamma))$$

where the sum is over all $\gamma \in \Gamma$ such that $B^0_{\lambda-\gamma} \subseteq \tau_\Lambda(w)$.

Proof. a) In this case we can take $w_{\lambda-\gamma} = 1$.
b) This follows from a). $\square$

3.9. We show the condition in the previous corollary is satisfied for $\lambda \in \Lambda^{++}$.

Lemma. *Suppose that α is an even root and $\gamma \in \Gamma$.*

a) *If $\alpha/2$ is a root, then $(\rho_1-\gamma, \alpha^v) = \pm(\frac{1}{2})$.*
b) *If $\alpha/2$ is not a root, then $(\rho_1-\gamma,\alpha^v) = \pm 1$ or 0.*

Proof. This is an easy calculation which we leave to the reader. Use the notation of [M2, 0.3] or [K1, 2.5.4]. $\square$

Corollary. *If $\lambda \in \Lambda^{++}$, then $\lambda-\gamma \in \Lambda_0^+$ for all $\gamma \in \Gamma$.*

Proof. By assumption $a = (\lambda+\rho,\alpha^v) > 0$ for all $\alpha \in B_\Lambda$. We must show that $b = (\lambda-\gamma+\rho_0,\alpha^v) \geq 0$. Note that $b-a = (\rho_1-\gamma,\alpha^v)$. Also if $\alpha/2$ is not a root, then $(\rho,\alpha^v) \in \mathbb{Z}$ and so $a \in \mathbb{Z}$, while if $\alpha/2$ is a root, then $(\rho,\alpha^v) \in \frac{1}{2}+\mathbb{Z}$ and $a \in \frac{1}{2}+\mathbb{Z}$. Thus the result follows from the lemma. $\square$

3.10. Finally we investigate the case where $\lambda \in \Lambda^+$, and $\lambda - \gamma \notin \Lambda_0^+$.

Lemma. *Suppose that* $\lambda \in \Lambda^+$, $\gamma \in \Gamma$, $\alpha \in B_\Lambda$ *and* $(\lambda - \gamma + \rho_0, \alpha^v) < 0$. *Then* $(\lambda+\rho,\alpha) = 0, (\lambda-\gamma+\rho_0,\alpha^v) = -1$ *and* $s_\alpha \circ (\lambda-\gamma) = \lambda-\gamma'$ *where* $\gamma' = \gamma - \alpha \in \Gamma$.

Proof. By assumption $a = (\lambda + \rho, \alpha^v) \geq 0$ and $b = (\lambda - \gamma + \rho_0, \alpha^v) < 0$. As in Corollary 3.9, this gives $a = 0$ and $b = -1$. (Note that $\alpha/2$ is not a root by Lemma 0.7.) Now $s_\alpha \circ (\lambda - \gamma) = s_\alpha.\lambda - s_\alpha * \gamma = \lambda - s_\alpha * \gamma$, and we get $s_\alpha * \gamma = \gamma - \alpha$, since $b = -1$. □

Corollary. *In Theorem 3.8 we have* $\overline{\lambda - \gamma} = \lambda - \gamma'$ *for some* $\gamma' \in \Gamma$.

Proof. Suppose that $\lambda \in \Lambda^+, \gamma \in \Gamma$ and choose $w \in W$ such that $w \circ (\lambda-\gamma) = \lambda-\gamma'$ with $\gamma' \in \Gamma$, and (γ', ρ_0) minimal. We claim that $\lambda - \gamma' \in \Lambda_0^+$. This follows from the lemma, since for $\alpha \in B_\Lambda$, we have $(\gamma' - \alpha, \rho_0) < (\gamma', \rho_0)$. □

4. Primitive Ideals

4.1. To describe $\operatorname{Prim} U(\mathfrak{g})$, we need to relate it to $\operatorname{Prim} U(\mathfrak{g}_0)$. To do this we need some ring theoretic background. Suppose that $R \subseteq S$ are Noetherian rings such that S is free as a right R-module.

For $Q \in \operatorname{Spec} R$ we set $X_Q = \{P \in \operatorname{Spec} S | P \text{ is minimal over } \operatorname{ann}_S(S/SQ)\}$. Note that, $SQ \neq S$ since S is a free right R-module. Also if $Q = \operatorname{ann}_R L$, then $\operatorname{ann}_S(S/SQ) = \operatorname{ann}_S(S \otimes_R L)$, by [M1, Lemma 2.5]. Finally observe that if $Q, Q' \in \operatorname{Spec} R$ such that $Q \subseteq Q'$, and $P' \in X_{Q'}$, then $P \subseteq P'$ for some $P \in X_Q$.

Now suppose that $P \in \operatorname{Spec} S$, and set

$$Y_P = \{Q \in \operatorname{Spec} R | Q \text{ is minimal over } P \cap R\}.$$

Clearly if $P, P' \in \operatorname{Spec} S$ are such that $P \subseteq P'$ and $Q' \in Y_{P'}$, then $Q \subseteq Q'$ for some $Q \in Y_P$.

Remark. The sets X_Q were introduced in [L1], where it is shown that, under suitable hypotheses on R and S we have $\operatorname{Prim} S = \bigcup_{Q \in \operatorname{Prim} R} X_Q$. This result was crucial in the proof of the main result of [M1], but we shall not need it in this paper.

4.2. **Proof of Theorem B.** Let $S = U(\mathfrak{g})$ and $R = U(\mathfrak{g}_0)$. We may assume that $\lambda \in \Lambda^{++}$ and $\mu \in \Lambda_0^{++}$ are sufficiently far from the walls. In particular we assume that all the central characters $\chi_{\mu+\gamma}, \gamma \in \Gamma$ are distinct, as are all the $\chi^0_{\lambda-\gamma}$.

Suppose first that $P_1 = J(w_1.\lambda) \subseteq P_2 = J(w_2.\lambda)$. By Lemma 3.4 we have as R-modules

$$\widetilde{L}(w_i.\lambda) = \bigoplus_{\gamma \in \Gamma} L(w_i \circ (\lambda - \gamma)),$$

for i = 1, 2; and hence

$$Y_{P_i} = \{I(w_i \circ (\lambda - \gamma)) | \gamma \in \Gamma\}.$$

Therefore by the remarks in 4.1, $I(w_1 \circ (\lambda - \gamma)) \subseteq I(w_2 \circ \lambda)$, for some $\gamma \in \Gamma$, and a consideration of central characters forces γ to be zero. Hence by [J2, Satz 5.8], $I(w_1 \circ \mu) \subseteq I(w_2 \circ \mu)$.

Conversely suppose that $Q_1 = I(w_1 \circ \mu) \subseteq Q_2 = I(w_2 \circ \mu)$. By Lemma 3.6 we have as S-modules

$$S \otimes_R L(w_i \circ \mu) = \bigoplus_{\gamma \in \Gamma} \widetilde{L}(w_i.(\mu + \gamma)).$$

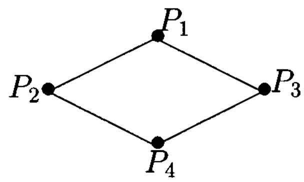

FIGURE 1

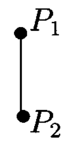

FIGURE 2

Thus

$$X_{Q_i} = \{J(w_i.(\mu + \gamma))|\gamma \in \Gamma\}.$$

Hence by the remarks in 4.1 and consideration of central characters we obtain $J(w_1.\mu) \subseteq J(w_2.\mu)$. Therefore by Theorem 1.4, $J(w_1.\lambda) \subseteq J(w_2.\lambda)$. $\square$

4.3. Fix $\Lambda \in \mathfrak{h}^*/P_0$ and $\lambda \in \Lambda^{++}$. Theorem B and Theorem 1.4 allow us to determine when $J(w_1.\lambda) \subseteq J(w_2.\lambda)$ for all $w_1, w_2 \in W_\Lambda$. The next result enables us to say when $J(w_1.\lambda) \subseteq J(w_2.\lambda)$ for all $w_1, w_2 \in W$ and to show that $\mathcal{X}_\lambda = \mathcal{X}_{w.\lambda}$ for all $w \in W$. As noted in [J2, 2.8], the set $W^\Lambda = \{w \in W|w(B_\Lambda) \subseteq R^+\}$ is a set of left coset representatives for W_Λ in W.

Theorem. *For all $\lambda \in \Lambda$ and all $w \in W^\Lambda$, we have $J(\lambda) = J(w.\lambda)$.*

Proof. This is shown in the same way as [J2, Satz 5.16]. $\square$

5. THE CASE WHERE $\mathfrak{g} = osp(1,4)$

5.1. If $\mathfrak{g} = osp(1,2)$, the description of $\operatorname{Prim} U(\mathfrak{g})$ as a partially ordered set is given in [P]. We illustrate our results in the smallest new case; that of $\mathfrak{g} = osp(1,4)$. An interesting feature is that there are fewer singular cases than for $\mathfrak{g}_0 = sp(4)$. As in [M2, 0.3] we may identify $\mathfrak{h}^*$ with $\mathbb{C}^2$ and (,) with the usual inner product. Then we can take $\alpha = (0,1)$ and $\beta = (1,-1)$ as simple roots. We have

$$\Delta_0^+ = \{2\alpha, \beta, 2\alpha + \beta, 2(\alpha + \beta)\}$$

and

$$\Delta_1^+ = \{\alpha, \alpha + \beta\}.$$

In addition $\rho_0 = (2,1), \rho = (\frac{1}{2})(3,1), P = \mathbb{Z}^2$ and $P^+ = \mathbb{N}(1,1) + \mathbb{N}(1,0)$. If $\lambda = (a,b)$ we have

$$(\lambda + \rho, (2\alpha)^v) = b + (\tfrac{1}{2}), \quad (\lambda + \rho, \beta^v) = a - b + 1$$

$$(\lambda + \rho, (2\alpha + \beta)^v) = a + b + 2, \quad (\lambda + \rho, (2\alpha + 2\beta)^v) = a + (\tfrac{3}{2}).$$

Fix $\Lambda \in \mathfrak{h}^*/P$. If $\Delta(\Lambda)$ has rank 2 and $\lambda \in \Lambda_0^{++}$, then by [J2, Satz 5.7 and Anhang 5A.2] the Hasse diagram for $\mathcal{X}_\lambda^0$ is as in Figure 1.

If $\operatorname{rank} \Delta(\Lambda) \leq 1$, then $\mathcal{X}_\lambda^0$ either is a singleton or has Hasse diagram as in Figure 2.

Now suppose $\lambda \in \Lambda^+$. If $|\mathcal{X}_\lambda| > 1$, the Hasse diagram of $\mathcal{X}_\lambda$ will be as in Figure 1 or Figure 2. In each case we describe the partition of W into the subsets

$$W_i = \{w \in W|P_i = J(w.\lambda)\}.$$

The longest element $s_\alpha s_\beta s_\alpha s_\beta$ of W is denoted by v.

Case 1. If $\Lambda = \mathbb{Z}^2$, then $B_\Lambda = \{2\alpha, \beta\}$. For $\lambda = (a,b) \in \mathbb{Z}^2$, we have $\lambda \in \Lambda^{++}$ if and only if $a - b + 1 > 0$ and $b + (\frac{1}{2}) > 0$. If $\lambda \in \Lambda^{++}$, then by Theorem B, $\mathcal{X}_\lambda$ has Hasse diagram as in Figure 1 with

$$W_1 = \{1\}, \quad W_2 = \{s_\alpha, s_\beta s_\alpha, s_{2\alpha+\beta}\}$$

$$W_3 = \{s_\beta, s_\alpha s_\beta, s_{\alpha+\beta}\}, \quad W_4 = \{v\}.$$

If $\lambda \in \Lambda^+ \backslash \Lambda^{++}$ then we must have $\widetilde{B}^0_\lambda = \{\beta\}$. This contrasts with the Lie algebra case where we have

$$\Lambda_0^+ \backslash \Lambda_0^{++} = \{\lambda \in \Lambda_0^+ | B^0_\lambda \text{ contains } \beta \text{ or } 2\alpha\}.$$

If $\lambda \in \Lambda^+ \backslash \Lambda^{++}$, then by Theorem 1.4, the Hasse diagram of $\mathcal{X}_\lambda$ is as in Figure 2 with $W_2 = \{v, vs_\beta = s_{2\alpha+\beta}\}$ and $W_1 = W \backslash W_2$.

Case 2. If $\Lambda = (1/2, 1/2) + \mathbb{Z}^2$, then $B_\Lambda = \{\beta, 2\alpha + \beta\}$. For $\lambda = (a,b) \in \Lambda$, we have $\lambda \in \Lambda^{++}$ if and only if $a + b + 2 > 0$ and $a - b + 1 > 0$. We have $W^\Lambda = \{1, s_\alpha\}$. Therefore if $\lambda \in \Lambda^{++}$, Theorem B and Theorem 4.3 imply that $\mathcal{X}_\lambda$ has Hasse diagram as in Figure 1 with

$$W_1 = W^\Lambda,\ W_2 = W^\Lambda s_\beta,\ W_3 = W^\Lambda s_{2\alpha+\beta},\ W_4 = W^\Lambda v.$$

Now $\lambda \in \Lambda^+ \backslash \Lambda^{++}$ if and only if $\lambda \in \Lambda^+$ and $\widetilde{B}^0_\lambda$ contains β or $2\alpha + \beta$. If $\widetilde{B}^0_\lambda = \{\gamma\}$ with $\gamma = \beta$ or $2\alpha + \beta$, then $\mathcal{X}_\lambda$ has Hasse diagram as in Figure 2 with

$$W_1 = W^\Lambda \{1, s_\gamma\}, \quad W_2 = W^\Lambda \{v, vs_\gamma\}.$$

Finally if $\widetilde{B}^0_\lambda = B_\Lambda$, then $\lambda = -\rho$ and hence $\mathcal{X}_\lambda = \{J(\lambda)\}$.

Observe that Cases 1 and 2 cover all cases where $\Delta(\Lambda)$ has rank two.

Case 3. If $\Delta(\Lambda)$ has rank 1, then $\Delta(\Lambda) = \{\pm\gamma\}$ where $\gamma \in \Delta_0^+$. For $\lambda \in \Lambda^{++}, \mathcal{X}_\lambda$ has Hasse diagram as in Figure 2 with $W_1 = W^\Lambda$ and $W_2 = W^\Lambda s_\gamma$. If $\widetilde{B}^0_\lambda = \{\gamma\}$, then $\mathcal{X}_\lambda = \{J(\lambda)\}$.

Case 4. Finally if $\Delta(\Lambda) = 0$, then $\mathcal{X}_\lambda = \{J(\lambda)\}$.

5.2. Let $\mathfrak{g} = osp(1,4)$ and retain the notation of 5.1. We describe the multiplicities of the composition factors of $\widetilde{L}(w.\lambda)$ for all $\lambda \in \Lambda^+, w \in W$. By Corollary 3.9 and [J2, 2.7] we may assume that $\lambda \in \Lambda^+ \backslash \Lambda^{++}$, and $\widetilde{B}^0_\lambda \subseteq \tau_\Lambda(w)$. Let $\leq$ denote the Bruhat order on W_Λ. From [J1, 3.6] we can deduce that

$$b_{y,w} = \begin{cases} 1 & \text{if } y \leq w, \\ 0 & \text{otherwise} \end{cases}$$

and

$$a_{w,y} = \begin{cases} (-1)^{\ell(w)+\ell(y)} & \text{if } y \leq w, \\ 0 & \text{otherwise.} \end{cases}$$

Case 1. If $\Lambda = \mathbb{Z}^2$ and $\lambda \in \Lambda^+ \backslash \Lambda^{++}$, then $\widetilde{B}^0_\lambda = \{\beta\}$. Set $W_1 = \{w \in W | w\beta < 0\}$ $= \{s_\beta, s_\alpha s_\beta, s_{\alpha+\beta}, v\}$. It suffices to describe $\widetilde{L}(w.\lambda)$ for all $w \in W_1$. Now if $\gamma \in \Gamma$, then $\lambda - \gamma \in \Lambda_0^+$ unless $\gamma = \alpha + \beta$. Also $s_\beta \circ (\lambda - (\alpha + \beta)) = \lambda - \alpha \in \Lambda_0^+$. Hence from 3.8–3.10 we obtain in the group $\mathcal{C}(\mathcal{O})$,

$$\widetilde{L}(w.\lambda) = \sum_{\substack{\gamma \in \Gamma, \gamma \neq \alpha+\beta \\ B^0_{\lambda-\gamma} \subseteq \tau_\Lambda(w)}} L(w \circ (\lambda - \gamma)) + \sum_{y,u \in W_\Lambda} a_{w,y} b_{ys_\beta,u} L(u \circ (\lambda - \alpha)).$$

We call the terms in the second sum, the extra terms. These may be found by computing $\sum_{y,u\in W_\Lambda} a_{w,y}b_{ys_\beta,u}u$ inside the group algebra of W_Λ. Set $\mu = \lambda - \alpha$. We find that if $w = s_{\alpha+\beta}$ or v, the extra terms are

$$L(w \circ \mu) + L(ws_\beta \circ \mu),$$

while if $w = s_\alpha s_\beta$ or s_β, the extra terms are

$$L(w \circ \mu) + L(ws_\beta \circ \mu) + L(ws_\alpha \circ \mu).$$

Case 2. If $\Lambda = (1/2, 1/2) + \mathbb{Z}^2$, we describe $\widetilde{L}(w.\lambda)$ for all $\lambda \in \Lambda^+$, and $w \in W_\Lambda$ such that $\widetilde{B}^0_\lambda \subseteq \tau_\Lambda(w)$. If $\gamma \in \Gamma$ is such that $\lambda - \gamma \in \Lambda^+_0$, then the term in Theorem 3.8 corresponding to γ can be simplified using a) of Corollary 3.8. Now assume that $\lambda \neq -\rho$. If $(\lambda - \gamma + \rho_0, \eta^v) = 0$ for $\eta \in B_\Lambda$, then $\eta \in \widetilde{B}^0_\lambda$ by Lemma 3.10, so as $\lambda \neq -\rho$ this can only happen for one such η. Also in this case $s_\eta \circ (\lambda - \gamma) = \lambda - s_\eta * \gamma$ and again, since $\lambda \neq -\rho$ we have $\lambda - s_\eta * \gamma \in \Lambda^+_0$. After some computations we find that the term in Theorem 3.8 corresponding to γ is equal to

$$L(w \circ (\lambda - s_\eta * \gamma)) + L(ws_\eta \circ (\lambda - s_\eta * \gamma)).$$

Finally assume that $\lambda = -\rho$. For $\gamma = 0$ or α we have $\lambda - \gamma \in \Lambda^+_0$, if while $\gamma = \alpha + \beta$, $s_\beta \circ (\lambda - \gamma) = \lambda - \alpha \in \Lambda^+_0$ and if $\gamma = 2\alpha + \beta$, $s_{2\alpha+\beta} \circ (\lambda - \gamma) = \lambda$. Using Theorem 3.8 with $w = v$ we obtain

$$\widetilde{L}(\lambda) = \widetilde{L}(v.\lambda) = L(\lambda) + L(\lambda - \alpha) + 2L(\lambda - \alpha - \beta) + 2L(\lambda - 2\alpha - \beta).$$

We leave it to the reader to find formulas for $\widetilde{L}(w.\lambda)$ when $\Delta(\Lambda)$ has rank less than two.

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