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# ROGAWSKI'S CONJECTURE ON THE JANTZEN FILTRATION FOR THE DEGENERATE AFFINE HECKE ALGEBRA OF TYPE $A$

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Abstract. The functors constructed by Arakawa and the author relate the representation theory of $\mathfrak{gl}_n$ and that of the degenerate affine Hecke algebra $H_\ell$ of $\mathrm{GL}_\ell$. They transform the Verma modules over $\mathfrak{gl}_n$ to the standard modules over $H_\ell$. In this paper we prove that they transform the simple modules to the simple modules (in more general situations than in the previous paper). We also prove that they transform the Jantzen filtration on the Verma modules to that on the standard modules. We obtain the following results for the representations of $H_\ell$ by translating the corresponding results for $\mathfrak{gl}_n$ through the functors: (i) the (generalized) Bernstein-Gelfand-Gelfand resolution for a certain class of simple modules, (ii) the multiplicity formula for the composition series of the standard modules, and (iii) its refinement concerning the Jantzen filtration on the standard modules, which was conjectured by Rogawski.

## Introduction

This paper is a continuation of the paper [AS], in which we gave functors from $\mathcal{O}(\mathfrak{gl}_n)$ to $\mathcal{R}(H_\ell)$. Here $\mathcal{O}(\mathfrak{gl}_n)$ denotes the Bernstein-Gelfand-Gelfand (in short, BGG) category of representations of the complex Lie algebra $\mathfrak{gl}_n$, and $\mathcal{R}(H_\ell)$ denotes the category of finite-dimensional representations of the degenerate affine Hecke algebra $H_\ell$ of $\mathrm{GL}_\ell$ introduced by Drinfeld [Dr].

Let us review the results in [AS]. Let $\mathfrak{t}_n^*$ and $W_n$ denote the space of weights and Weyl group of $\mathfrak{gl}_n$ respectively. For $\lambda \in \mathfrak{t}_n^*$, let $M(\lambda)$ denote the Verma module with highest weight $\lambda$ and $L(\lambda)$ its simple quotient. Let $V_n = \mathbb{C}^n$ denote the vector representation of $\mathfrak{gl}_n$. For each $\lambda \in \mathfrak{t}_n^*$ and $X \in \mathrm{obj}\mathcal{O}(\mathfrak{gl}_n)$, we define an action of $H_\ell$ on the finite-dimensional vector space $F_\lambda(X) = \mathrm{Hom}_{\mathfrak{gl}_n}(M(\lambda), X \otimes V_n^{\otimes \ell})$. Under the condition that $\lambda + \rho$ is dominant, we proved that the functor $F_\lambda$ is exact and $F_\lambda(M(\mu))$ is isomorphic to $\mathcal{M}(\lambda,\mu)$ unless it is zero. Here $\mathcal{M}(\lambda,\mu) \in \mathrm{obj}\mathcal{R}(H_\ell)$ denotes the standard module. With the restriction $\ell = n$, we proved that $F_\lambda(L(\mu))$ is isomorphic to the unique simple quotient $\mathcal{L}(\lambda,\mu)$ of $\mathcal{M}(\lambda,\mu)$ unless it is zero. Any simple $H_\ell$-module is thus obtained. To prove the irreducibility of $F_\lambda(L(\mu))$, we compared the multiplicities of the simple modules in the composition series of $M(\mu)$ and those in $\mathcal{M}(\lambda,\mu)$ by using the Kazhdan-Lusztig type multiplicity formulas known for $\mathcal{O}(\mathfrak{gl}_n)$ and $\mathcal{R}(H_\ell)$. (See (b), (c) below.)

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In the present paper, further properties of the functors are deduced from the key observation that the $\mathfrak{gl}_n$-contravariant bilinear form on a highest weight $\mathfrak{gl}_n$-module $X$ induces the $H_\ell$-contravariant bilinear form on $F_\lambda(X)$. The irreducibility of $F_\lambda(L(\mu))$ is deduced from the non-degeneracy of the bilinear form. As a consequence, we can determine the images of simple $\mathfrak{gl}_n$-modules (Theorem 3.2.2) without assuming $\ell = n$ or referring to the multiplicity formulas.

We also prove that $F_\lambda$ transforms the Jantzen filtration on $M(\mu)$ to that on $F_\lambda(M(\mu)) \cong \mathcal{M}(\lambda, \mu)$ (Theorem 4.3.5).

The following are the consequences of these results.

(i) We obtain a resolution for a certain class of simple $H_\ell$-modules by applying $F_\lambda$ to the BGG resolution [BGG] and its generalization by Gabber and Joseph [GJ1] for $\mathfrak{gl}_n$-modules. This generalizes the results of Cherednik [Ch1] and Zelevinsky [Ze4].

(ii) To simplify the descriptions, we assume $\lambda$ and $\mu$ are dominant integral weights. (More general cases are treated in §5.2.) Set $w \circ \mu = w(\mu + \rho) - \rho$ for $w \in W_n$ and let $w, y \in W_n$ be such that $\lambda - w \circ \mu$ and $\lambda - y \circ \mu$ are weights of $V_n^{\otimes \ell}$. We have a direct proof of the following formula:

$$\text{(a)} \qquad [M(w \circ \mu) : L(y \circ \mu)] = [\mathcal{M}(\lambda, w \circ \mu) : \mathcal{L}(\lambda, y \circ \mu)].$$

Let $P_{w,y}(q)$ denote the Kazhdan-Lusztig polynomial of $W_n$. The formula (a) implies the equivalence of the following two multiplicity formulas:

$$\text{(b)} \qquad [\mathcal{M}(\lambda, w \circ \mu) : \mathcal{L}(\lambda, y \circ \mu)] = P_{w,y}(1),$$

$$\text{(c)} \qquad [M(w \circ \mu) : L(y \circ \mu)] = P_{w,y}(1).$$

The formula (b) was proved by Ginzburg [Gi1] (see also [CG]) for affine Hecke algebras and by Lusztig for degenerate (or graded) affine Hecke algebras [Lu2], and (c) was proved by Beilinson and Bernstein [BB1] and Brylinski and Kashiwara [BK] by using the geometric method and the theory of perverse sheaves. We remark that our proof of (a) is purely algebraic.

(iii) We have a refinement of the formula (a): Let $\lambda, \mu$ and $w, y$ be as in (ii). (See §5.3 for more general cases.) Let

$$M(\mu) = M(\mu)_0 \supseteq M(\mu)_1 \supseteq M(\mu)_2 \supseteq \cdots,$$
$$\mathcal{M}(\lambda, \mu) = \mathcal{M}(\lambda, \mu)_0 \supseteq \mathcal{M}(\lambda, \mu)_1 \supseteq \mathcal{M}(\lambda, \mu)_2 \supseteq \cdots$$

be the Jantzen filtrations on $M(\mu)$ and $\mathcal{M}(\lambda, \mu)$, respectively. Since $F_\lambda$ preserves the Jantzen filtration, we have

$$\text{(a')} \qquad [M(w \circ \mu)_j : L(y \circ \mu)] = [\mathcal{M}(\lambda, w \circ \mu)_j : \mathcal{L}(\lambda, y \circ \mu)].$$

The Jantzen filtration on standard modules over affine Hecke algebras of GL was introduced by Rogawski [Ro]. He conjectured a refinement of the formula (b) concerning the Jantzen filtration. A proof of Rogawski's conjecture has been presented in [Gi2] without details[1]. A degenerate affine Hecke analogue of Rogawski's

[1] I. Grojnowski announced similar results in a series of his lectures at Kyoto in 1997. He also treated affine Hecke algebras at roots of unity by the geometric method. After the submission of this article, the author received a manuscript of Grojnowski concerning these results, which will form a part of his forthcoming paper "Affine and cyclotomic Hecke algebras at roots of unity".

conjecture is written as follows:

$$\text{(b')} \qquad \sum_{i\in\mathbb{Z}_{\geq 0}} [\mathrm{gr}_i\, \mathcal{M}(\lambda, w\circ\mu) : \mathcal{L}(\lambda, y\circ\mu)] q^{(l(y)-l(w)-i)/2} = P_{w,y}(q).$$

The formula (a') implies the equivalence between (b') and the improved Kazhdan-Lusztig multiplicity formula

$$\text{(c')} \qquad \sum_{i\in\mathbb{Z}_{\geq 0}} [\mathrm{gr}_i\, M(w\circ\mu) : L(y\circ\mu)] q^{(l(y)-l(w)-i)/2} = P_{w,y}(q),$$

which was proved in [BB2].

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## 1. Basic definitions

1.1. **Lie algebra $\mathfrak{gl}_n$.** Let $\mathfrak{gl}_n$ denote the Lie algebra consisting of all $n\times n$ matrices with entries in $\mathbb{C}$. Let $\mathfrak{t}_n$ be the Cartan subalgebra of $\mathfrak{gl}_n$ consisting of all diagonal matrices. An inner product is defined on $\mathfrak{gl}_n$ by

$$(x|y)_n = \mathrm{tr}(xy) \tag{1.1.1}$$

for $x, y \in \mathfrak{gl}_n$. Let $\mathfrak{t}_n^*$ denote the dual space of $\mathfrak{t}_n$. The natural pairing is denoted by $\langle\ ,\ \rangle_n : \mathfrak{t}_n^* \times \mathfrak{t}_n \to \mathbb{C}$. Let $E_{i,j}$ $(1 \leq i, j \leq n)$ denote the matrix with only nonzero entries 1 at the $(i, j)$-th component. Define a basis $\{\epsilon_i\}_{i=1,\dots,n}$ of $\mathfrak{t}_n^*$ by $\epsilon_i(E_{j,j}) = \delta_{i,j}$, and define the roots by $\alpha_{ij} = \epsilon_i - \epsilon_j$ and the simple roots by $\alpha_i = \epsilon_i - \epsilon_{i+1}$.

Put

$$R_n = \{\alpha_{ij} \mid 1 \leq i \neq j \leq n\}, \tag{1.1.2}$$

$$R_n^+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}, \quad R_n^- = R_n \setminus R_n^+, \tag{1.1.3}$$

$$\Pi_n = \{\alpha_i \mid i = 1, \dots n-1\}. \tag{1.1.4}$$

Then $R_n \subseteq \mathfrak{t}_n^*$ is a root system of type $A_{n-1}$. Since the restriction of $(\,|\,)_n$ to $\mathfrak{t}_n$ is non-degenerate, we have an isomorphism $\mathfrak{t}_n^* \xrightarrow{\sim} \mathfrak{t}_n$, whose image of $\xi \in \mathfrak{t}_n^*$ is denoted by $\xi^\vee$. In particular, we have $\epsilon_i^\vee = E_{i,i}$ and $\alpha_i^\vee = E_{i,i} - E_{i+1,i+1}$.

Putting $\mathfrak{n}_n^+ = \bigoplus_{i<j} \mathbb{C}E_{i,j}$, $\mathfrak{n}_n^- = \bigoplus_{i>j} \mathbb{C}E_{i,j}$, we have a triangular decomposition $\mathfrak{gl}_n = \mathfrak{n}_n^+ \oplus \mathfrak{t}_n \oplus \mathfrak{n}_n^-$ . We put $\mathfrak{b}_n^\pm = \mathfrak{n}_n^\pm \oplus \mathfrak{t}_n$.

Let $\sigma$ denote the involution on $\mathfrak{gl}_n$ given by the transposition: $\sigma(E_{i,j}) = E_{j,i}$. The inner product $(\,|\,)_n$ is invariant with respect to $\sigma$: $(\sigma(x)|\sigma(y))_n = (x|y)_n$ for all $x, y \in \mathfrak{gl}_n$.

Put $\rho = \frac{1}{2}\sum_{\alpha \in R_n^+} \alpha$ and define

$$Q_n = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i, \tag{1.1.5}$$

$$D_n = \{\lambda \in \mathfrak{t}_n^* \mid \langle \lambda + \rho, \alpha \rangle_n \notin \mathbb{Z}_{<0} \text{ for all } \alpha \in R_n^+\}, \tag{1.1.6}$$

$$D_n^\circ = \{\lambda \in \mathfrak{t}_n^* \mid \langle \lambda, \alpha \rangle_n \notin \mathbb{Z}_{<0} \text{ for all } \alpha \in R_n^+\}, \tag{1.1.7}$$

$$P_n = \bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i, \quad P_n^+ = P_n \cap D_n^\circ. \tag{1.1.8}$$

An element of $D_n^\circ$ (resp. $P_n$, $P_n^+$) is called a *dominant* (resp. *integral, dominant integral*) weight.

1.2. **Weyl group.** Let $W_n \subset \mathrm{GL}(\mathfrak{t}_n^*)$ be the Weyl group associated to the root system $(R_n, \Pi_n)$, which is by definition generated by the reflections $s_\alpha$ $(\alpha \in R_n)$ defined by

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle_n \alpha \quad (\lambda \in \mathfrak{t}_n^*). \tag{1.2.1}$$

We often write $s_i = s_{\alpha_i}$ for $\alpha_i \in \Pi_n$. Note that $W_n$ is isomorphic to the symmetric group $\mathfrak{S}_n$.

We often use another action of $W_n$ on $\mathfrak{t}_n^*$, which is given by

$$w \circ \lambda = w(\lambda + \rho) - \rho \quad (w \in W_n,\ \lambda \in \mathfrak{t}_n^*). \tag{1.2.2}$$

For $w, y \in W_n$, we write $w \geq y$ if and only if $y$ can be obtained as a subexpression of a reduced expression of $w$. The resulting relation in $W_n$ defines a partial order called the *Bruhat order*.

1.3. **Representations of $\mathfrak{gl}_n$.** For a $\mathfrak{t}_n$-module $X$ and $\lambda \in \mathfrak{t}_n^*$, put

$$X_\lambda = \{v \in X \mid hv = \langle \lambda, h \rangle_n v \text{ for all } h \in \mathfrak{t}_n\}, \tag{1.3.1}$$

$$P(X) = \{\lambda \in \mathfrak{t}_n^* \mid X_\lambda \neq 0\}. \tag{1.3.2}$$

The space $X_\lambda$ is called the *weight space* of weight $\lambda$ with respect to $\mathfrak{t}_n$, and an element of $P(X)$ is called a weight of $X$.

Let $U(\mathfrak{gl}_n)$ denote the universal enveloping algebra of $\mathfrak{gl}_n$. Let $\mathcal{O} = \mathcal{O}(\mathfrak{gl}_n)$ denote the category of $\mathfrak{gl}_n$-modules which are finitely generated over $U(\mathfrak{gl}_n)$, $\mathfrak{n}_n^+$-locally finite and $\mathfrak{t}_n$-semisimple (see [BGG]). The category $\mathcal{O}$ is closed under the operations such as forming subquotient modules, finite direct sums, and tensor products with finite-dimensional modules. For $\lambda \in \mathfrak{t}_n^*$, let $M(\lambda) = U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{b}_n^+)} \mathbb{C}v_\lambda$ denote the Verma module with highest weight $\lambda$, where $v_\lambda$ denotes the highest weight vector. The unique simple quotient of $M(\lambda)$ is denoted by $L(\lambda)$. The modules $M(\lambda)$ and $L(\lambda)$ are objects of $\mathcal{O}$.

Let $\chi_\lambda : Z(U(\mathfrak{gl}_n)) \to \mathbb{C}$ denote the infinitesimal character of $M(\lambda)$. We introduce an equivalence relation in $\mathfrak{t}_n^*$ by

$$\lambda \sim \mu \Leftrightarrow \lambda = w \circ \mu \ \text{ for some } w \in W_n. \tag{1.3.3}$$

Then it follows that $\chi_\lambda = \chi_\mu$ if and only if $\lambda \sim \mu$. Let $[\lambda]$ denote the equivalence class of $\lambda \in \mathfrak{t}_n^*$. Define the full subcategory $\mathcal{O}_{[\lambda]}$ of $\mathcal{O}$ by

$$\mathrm{obj}\mathcal{O}_{[\lambda]} = \{X \in \mathrm{obj}\mathcal{O} \mid (\mathrm{Ker}\,\chi_\lambda)^k X = 0 \text{ for some } k\}.$$

Then any $X \in \mathrm{obj}\mathcal{O}$ admits a decomposition

$$X = \bigoplus_{[\lambda]\in\mathfrak{t}_n^*/\sim} X^{[\lambda]} \tag{1.3.4}$$

such that $X^{[\lambda]} \in \mathrm{obj}\mathcal{O}_{[\lambda]}$. The correspondence $X \mapsto X^{[\lambda]}$ gives an exact functor on $\mathcal{O}$.

**Lemma 1.3.1.** *Let $\lambda \in D_n$. Then the natural map $(X^{[\lambda]})_\lambda \to (X/\mathfrak{n}_n^- X)_\lambda$ is bijective.*

*Remark* 1.3.2. (i) There also exists a canonical bijection $\mathrm{Hom}_{\mathfrak{gl}_n}(M(\lambda), X) \cong (X^{[\lambda]})_\lambda$ for $\lambda \in D_n$.

(ii) A proof of Lemma 1.3.1 for integral $\lambda$ is given in [AS]. The generalization to non-integral cases is similarly proved.

## 2. Degenerate affine Hecke algebras and their representations

2.1. **Degenerate affine Hecke algebras.** For a group $G$, let $\mathbb{C}[G]$ denote its group ring. Let $S(\mathfrak{t}_\ell)$ denote the symmetric algebra of $\mathfrak{t}_\ell$, which is isomorphic to the polynomial ring $\mathbb{C}[\epsilon_1^\vee, \dots, \epsilon_\ell^\vee]$.

**Definition 2.1.1.** The *degenerate* (*or graded*) *affine Hecke algebra* $H_\ell$ of $\mathrm{GL}_\ell$ is the unital associative algebra over $\mathbb{C}$ defined by the following properties:

(i) As a vector space, $H_\ell \cong \mathbb{C}[W_\ell] \otimes S(\mathfrak{t}_\ell)$.

(ii) The subspaces $\mathbb{C}[W_\ell] \otimes \mathbb{C}$ and $\mathbb{C} \otimes S(\mathfrak{t}_\ell)$ are subalgebras of $H_\ell$ in a natural fashion (their images will be identified with $\mathbb{C}[W_\ell]$ and $S(\mathfrak{t}_\ell)$ respectively).

(iii) The following relations hold in $H_\ell$:

$$s_\alpha \cdot \xi - s_\alpha(\xi) \cdot s_\alpha = -\langle \alpha, \xi \rangle_\ell \quad (\alpha \in \Pi_\ell,\ \xi \in \mathfrak{t}_\ell). \tag{2.1.1}$$

It is easy to verify the following lemma.

**Lemma 2.1.2.** *There exists a unique anti-involution $\iota$ on $H_\ell$ such that*

$$\iota(w) = w^{-1}\ (w \in W_\ell), \quad \iota(\xi) = \xi\ (\xi \in \mathfrak{t}_\ell).$$

2.2. **Induced modules.** For a pair $(a, b)$ of complex numbers such that $b - a + 1 \in \mathbb{Z}_{\geq 0}$, we put $[a, b] := \{a, a+1, \dots, b\} \subseteq \mathbb{C}$ ($[a, a-1] := \emptyset$) and call it a *segment*. For a segment $\Delta = [a, b]$ such that $b - a + 1 = \ell$, there exists a unique one-dimensional representation $\mathbb{C}_\Delta = \mathbb{C}\mathbf{1}_\Delta$ of $H_\ell$ (we put $H_0 = \mathbb{C}$) such that

$$w\mathbf{1}_\Delta = \mathbf{1}_\Delta \quad (w \in W_\ell), \tag{2.2.1}$$

$$\epsilon_i^\vee \mathbf{1}_\Delta = (a + i - 1)\mathbf{1}_\Delta \quad (i = 1, \dots, \ell). \tag{2.2.2}$$

Let $\Phi = (\Delta_1, \dots, \Delta_n)$ be an ordered sequence of segments with $\Delta_i = [a_i, b_i]$ such that $b_i - a_i + 1 = \ell_i$ and $\ell = \sum_{i=1}^n \ell_i$. Regard $H_{\ell_1} \otimes H_{\ell_2} \otimes \cdots \otimes H_{\ell_n}$ as a subalgebra of $H_\ell$. Define an $H_\ell$-module $\mathcal{M}(\Phi)$ by

$$\mathcal{M}(\Phi) = H_\ell \otimes_{H_{\ell_1} \otimes \cdots \otimes H_{\ell_n}} (\mathbb{C}_{\Delta_1} \otimes \cdots \otimes \mathbb{C}_{\Delta_n}). \tag{2.2.3}$$

Evidently $\mathcal{M}(\Phi)$ is a cyclic module with a cyclic weight vector

$$\mathbf{1}_\Phi := \mathbf{1}_{\Delta_1} \otimes \cdots \otimes \mathbf{1}_{\Delta_n}, \tag{2.2.4}$$

whose weight $\zeta_\Phi$ is given by

(2.2.5)
$$\langle \zeta_\Phi, \epsilon_j^\vee \rangle_\ell = a_i + j - \sum_{k=1}^{i-1} \ell_k - 1 \quad \text{for} \quad \sum_{k=1}^{i-1} \ell_k < j \leq \sum_{k=1}^{i} \ell_k.$$

It is also obvious that $\mathcal{M}(\Phi) \cong \mathbb{C}[W_\ell/(W_{\ell_1} \times \cdots \times W_{\ell_n})]$ as a $\mathbb{C}[W_\ell]$-module and thus its dimension is given by $\dim \mathcal{M}(\Phi) = \frac{\ell!}{\ell_1! \cdots \ell_n!}$.

Let $\lambda, \mu \in \mathfrak{t}_n^*$ be such that $\lambda - \mu \in P(V_n^{\otimes \ell})$. Then putting

(2.2.6)
$$\ell_i = \langle \lambda - \mu, \epsilon_i^\vee \rangle_n \in \mathbb{Z}_{\geq 0} \quad (i = 1, \ldots, n),$$

we have $l = \sum_{i=1}^n \ell_i$. Define an ordered sequence

(2.2.7)
$$\Phi_{\lambda,\mu} := ([\langle \mu + \rho, \epsilon_1^\vee \rangle_n, \langle \lambda + \rho, \epsilon_1^\vee \rangle_n - 1], \ldots, [\langle \mu + \rho, \epsilon_n^\vee \rangle_n, \langle \lambda + \rho, \epsilon_n^\vee \rangle_n - 1])$$

of segments. We put

(2.2.8)
$$\mathcal{M}(\lambda, \mu) = \mathcal{M}(\Phi_{\lambda,\mu}), \quad \mathbf{1}_{\lambda,\mu} = \mathbf{1}_{\Phi_{\lambda,\mu}},$$

where $\mathbf{1}_{\Phi_{\lambda,\mu}}$ is as in (2.2.4).

*Remark* 2.2.1. (i) For any ordered sequence $\Phi = (\Delta_1, \ldots, \Delta_n)$ of segments, one can find $\lambda, \mu \in \mathfrak{t}_n^*$ such that $\Phi = \Phi_{\lambda,\mu}$.

(ii) For $\Phi = (\Delta_1, \ldots, \Delta_n)$ and $w \in W_n$, put $w(\Phi) = (\Delta_{w(1)}, \ldots, \Delta_{w(n)})$. Then, for $\Phi_{\lambda,\mu}$ in (2.2.7), we have $w(\Phi_{\lambda,\mu}) = \Phi_{w\circ\lambda, w\circ\mu}$.

Recall that the simple modules of $W_\ell$ are parametrized by unordered partitions of $\ell$ (or Young diagrams of size $\ell$). We let $S_\gamma$ denote the simple $W_\ell$-module corresponding to the partition $\gamma$. Let $[\lambda - \mu]$ denote the unordered partition of $\ell$ obtained from $(\ell_1, \ldots, \ell_n)$ by forgetting the order. As is well-known, it holds that

(2.2.9)
$$\mathcal{M}(\lambda, \mu) \cong S_{[\lambda-\mu]} \oplus \bigoplus_{\beta \rhd [\lambda-\mu]} S_\beta^{\oplus a_\beta},$$

as a $\mathbb{C}[W_\ell]$-module . Here $\rhd$ denotes the dominance order in the set of partitions, and $a_\beta$ are some non-negative integers.

Let $\mathcal{Y}_\ell(n)$ denote the set of Young diagrams of size $\ell$ consisting of at most $n$ rows. We say that an $H_\ell$-module $Y$ is of *level* $n$ if $Y = \bigoplus_{\gamma \in \mathcal{Y}_\ell(n)} S_\gamma^{\oplus a_\gamma}$ for some $a_\gamma \in \mathbb{Z}_{\geq 0}$. The induced module $\mathcal{M}(\lambda, \mu)$ ($\lambda, \mu \in \mathfrak{t}_n^*$) is of level $n$. Of course, any finite-dimensional $H_\ell$-module is of level $\ell$.

2.3. **Zelevinsky's classification of simple modules.** Let $\lambda, \mu \in \mathfrak{t}_n^*$ be such that $\lambda - \mu \in P(V_n^{\otimes \ell})$. Suppose that $\lambda \in D_n$. Then for any $i < j$, we have

$$\langle \lambda + \rho, \epsilon_i^\vee \rangle_n - \langle \lambda + \rho, \epsilon_j^\vee \rangle_n = \langle \lambda + \rho, \alpha_{ij}^\vee \rangle_n \notin \mathbb{Z}_{<0}.$$

This implies that the segment $[\langle \mu + \rho, \epsilon_j^\vee \rangle_n, \langle \lambda + \rho, \epsilon_j^\vee \rangle_n - 1]$ does not *precede* the segment $[\langle \mu + \rho, \epsilon_i^\vee \rangle_n, \langle \lambda + \rho, \epsilon_i^\vee \rangle_n - 1]$ in the sense of [Ze1, 4.1]. Thus the following statement follows from [Ze1, Theorem 6.1-(a)] and [Ro, §5] (see also [Ch2]). (Recall that the representation theory of the degenerate affine Hecke algebra is related to that of the affine Hecke algebra by Lusztig [Lu1].)

**Theorem 2.3.1** ([Ze1, Ro])**.** *Let* $\lambda \in D_n$ *and* $\mu \in \lambda - P(V_n^{\otimes \ell})$.

(i) *In the decomposition* (2.2.9), $S_{[\lambda-\mu]}$ *generates* $\mathcal{M}(\lambda, \mu)$ *over* $H_\ell$.

(ii) *The $H_\ell$-module $\mathcal{M}(\lambda,\mu)$ has the unique simple quotient, which is denoted by $\mathcal{L}(\lambda,\mu)$.*
(iii) *The $\mathcal{L}(\lambda,\mu)$ contains $S_{[\lambda-\mu]}$ with multiplicity one as a $\mathbb{C}[W_\ell]$-module.*

*Remark* 2.3.2. The statement (i) easily follows from (ii) and (iii).

For $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$, the $H_\ell$-module $\mathcal{M}(\lambda,\mu)$ is called a *standard module*.

By Remark 2.2.1 and [Ze1, Theorem 6.1-(b)(c)], we have the following statements:

**Theorem 2.3.3** ([Ze1, Theorem 6.1-(c)])**.** *Any simple $H_\ell$-module of level $n$ is isomorphic to $\mathcal{L}(\lambda,\mu)$ for some $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$.*

For $\eta \in \mathfrak{t}_n^*$, let $W_n[\eta]$ denote the stabilizer of $\eta$:

$$W_n[\eta] = \{w \in W_n \mid w(\eta) = \eta\}, \tag{2.3.1}$$

which is a parabolic subgroup of $W_n$.

**Proposition 2.3.4** ([Ze1, Theorem 6.1-(b)])**.** *Suppose that $\lambda,\mu \in D_n$ and $w,y \in W_n$ satisfy $\lambda - w\circ\mu \in P(V_n^{\otimes \ell})$ and $\lambda - y\circ\mu \in P(V_n^{\otimes \ell})$. Then the following conditions are equivalent*:

(i) $y \in W_n[\lambda+\rho]wW_n[\mu+\rho]$.
(ii) $\mathcal{M}(\lambda, w\circ\mu) \cong \mathcal{M}(\lambda, y\circ\mu)$.
(iii) $\mathcal{L}(\lambda, w\circ\mu) \cong \mathcal{L}(\lambda, y\circ\mu)$.

*Remark* 2.3.5. Let $\lambda,\mu \in D_n$ and $w \in W_n$ such that $\lambda - w\circ\mu \in P(V_n^{\otimes \ell})$. Then the condition (i) in Proposition 2.3.4 implies $\lambda - y\circ\mu \in P(V_n^{\otimes \ell})$. We often use the following fact from Proposition 2.3.4:

$$\mathcal{M}(\lambda, w\circ\mu) \cong \mathcal{M}(\lambda, w^\lambda\circ\mu) \cong \mathcal{M}(\lambda, w^\lambda_\mu\circ\mu), \tag{2.3.2}$$

$$\mathcal{L}(\lambda, w\circ\mu) \cong \mathcal{L}(\lambda, w^\lambda\circ\mu) \cong \mathcal{L}(\lambda, w^\lambda_\mu\circ\mu). \tag{2.3.3}$$

Here $w^\lambda$ (resp. $w^\lambda_\mu$) denotes the unique longest element in $W_n[\lambda+\rho]w$ (resp. $W_n[\lambda+\rho]wW_n[\mu+\rho]$).

## 3. Functors $F_\lambda$

3.1. **Construction.** Let us recall the definition of the functor

$$F_\lambda : \mathcal{O}(\mathfrak{gl}_n) \to \mathcal{R}(H_\ell)$$

introduced in [AS]. Here $\mathcal{R}(H_\ell)$ denotes the category of finite-dimensional representations of $H_\ell$. Let $V_n = \mathbb{C}^n$ denote the vector representation of $\mathfrak{gl}_n$.

**Lemma 3.1.1** ([AS])**.** *For any $X \in \mathcal{O}(\mathfrak{gl}_n)$, there exists a unique homomorphism*

$$\theta : H_\ell \to \mathrm{End}_{\mathfrak{gl}_n}(X \otimes V_n^{\otimes \ell}) \tag{3.1.1}$$

*such that*

$$\theta(s_i) = \Omega_{i\,i+1} \qquad (i = 1,\dots,\ell-1), \tag{3.1.2}$$

$$\theta(\epsilon_i^\vee) = \sum_{0\le j<i} \Omega_{ji} + \frac{n-1}{2} \qquad (i = 1,\dots,\ell), \tag{3.1.3}$$

*where*

(3.1.4)

$$\Omega_{ji} = \sum_{1\le k,m\le n} 1^{\otimes j} \otimes E_{k,m} \otimes 1^{\otimes i-j-1} \otimes E_{m,k} \otimes 1^{\otimes \ell-i} \in \mathrm{End}(X \otimes V_n^{\otimes \ell}).$$

*Remark* 3.1.2. Note that the action of $W_\ell$ given by (3.1.2) is just the natural action of $W_\ell$ on $V_n^{\otimes \ell}$.

Let $\lambda \in D_n$ and $X \in \mathrm{obj}\mathcal{O}(\mathfrak{gl}_n)$. We define

$$F_\lambda(X) = (X \otimes V_n^{\otimes \ell})_\lambda^{[\lambda]} \tag{3.1.5}$$

with an induced $H_\ell$-module structure through the homomorphism $\theta$. We also introduce an $H_\ell$-module structure on $\big((X \otimes V_n^{\otimes \ell})/\mathfrak{n}_n^-(X \otimes V_n^{\otimes \ell})\big)_\lambda$. Then the bijection given in Lemma 1.3.1 gives an $H_\ell$-isomorphism

$$F_\lambda(X) \cong \big((X \otimes V_n^{\otimes \ell})/\mathfrak{n}_n^-(X \otimes V_n^{\otimes \ell})\big)_\lambda . \tag{3.1.6}$$

Obviously $F_\lambda$ defines an exact functor from $\mathcal{O}(\mathfrak{gl}_n)$ to $\mathcal{R}(H_\ell)$.

3.2. **Image of functors.** We extend the definition of $\mathcal{M}(\lambda,\mu)$ for any $\lambda, \mu \in \mathfrak{t}_n^*$ by

$$\mathcal{M}(\lambda,\mu) = 0 \text{ for } \lambda,\mu \in \mathfrak{t}_n^* \text{ such that } \lambda - \mu \notin P(V_n^{\otimes \ell}). \tag{3.2.1}$$

Let $\{u_i\}_{i=1,\dots,n}$ denote the standard basis of $V_n = \mathbb{C}^n$. For $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$, we define an element $u_{\lambda,\mu} \in \big((M(\mu) \otimes V_n^{\otimes \ell}) \,/\, \mathfrak{n}_n^-(M(\mu) \otimes V_n^{\otimes \ell})\big)_\lambda$ as the image of $v_\mu \otimes u_1^{\otimes \ell_1} \otimes \cdots \otimes u_n^{\otimes \ell_n} \in M(\mu) \otimes V_n^{\otimes \ell}$, where $\ell_i = \langle \lambda - \mu, \epsilon_i^\vee \rangle_n$. It was shown in [AS] that there exists an $H_\ell$-homomorphism

$$\mathcal{M}(\lambda,\mu) \to \big(M(\mu) \otimes V_n^{\otimes \ell} \,/\, \mathfrak{n}_n^-(M(\mu) \otimes V_n^{\otimes \ell})\big)_\lambda , \tag{3.2.2}$$

which sends $\mathbf{1}_{\lambda,\mu}$ to $u_{\lambda,\mu}$, and that this is bijective. Combining (3.1.6), we have

**Theorem 3.2.1** ([AS])**.** *For each $\lambda \in D_n$ and $\mu \in \mathfrak{t}_n^*$, there is an isomorphism of $H_\ell$-modules*

$$F_\lambda(M(\mu)) \cong \mathcal{M}(\lambda,\mu).$$

*In particular, the $H_\ell$-module $F_\lambda(M(\mu))$ has the unique simple quotient.*

For $\eta \in \mathfrak{t}_n^*$, put $R_n[\eta] = \{\alpha \in R_n \mid \langle \eta, \alpha^\vee \rangle_n = 0\}$. It is not difficult to see that $R_n[\eta]$ is a root system and its Weyl group is a stabilizer $W_n[\eta]$ of $\eta$: $W_n[\eta] = \langle s_\alpha \mid \alpha \in R_n[\eta] \rangle$.

A proof of the following statement is given in §4.2.

**Theorem 3.2.2.** *Let $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$.*

(i) *If $\mu$ satisfies the condition*

$$\langle \mu + \rho, \alpha^\vee \rangle_n \in \mathbb{Z}_{\le 0} \quad \text{for any } \alpha \in R_n^+ \cap R_n[\lambda + \rho], \tag{3.2.3}$$

*then we have*

$$F_\lambda(L(\mu)) \cong \mathcal{L}(\lambda,\mu), \tag{3.2.4}$$

*where $\mathcal{L}(\lambda,\mu)$ is the unique simple quotient of $\mathcal{M}(\lambda,\mu)$.*

(ii) *If $\mu$ does not satisfy the condition* (3.2.3)*, then we have*

$$F_\lambda(L(\mu)) = 0. \tag{3.2.5}$$

*Remark* 3.2.3. (i) In the case $\ell = n$, Theorem 3.2.2 was proved in [AS] using the Kazhdan-Lusztig type multiplicity formula for $\mathcal{O}(\mathfrak{gl}_n)$ and that for $\mathcal{R}(H_\ell)$ (see §5.2). The proof given in §4.2 does not depend on these multiplicity formulas.

(ii) The $W_n[\lambda+\rho]$ acts on $\lambda - P_n$ by $\mu \mapsto w \circ \mu$ ($\mu \in \lambda - P_n,\ w \in W_n[\lambda+\rho]$) (Remark 2.3.5). By a standard argument (see e.g. [Hu, 1.12]), it can be shown that a fundamental domain for this action on $\lambda - P_n$ is given by

$$A := \{\mu \in \lambda - P_n \mid \langle \mu+\rho, \alpha^\vee\rangle_n \in \mathbb{Z}_{\le 0} \quad \text{for any } \alpha \in R_n^+ \cap R_n[\lambda+\rho]\}. \tag{3.2.6}$$

(iii) We can express $\mu$ in Theorem 3.2.2 as $\mu = w \circ \tilde{\mu}$ with some $w \in W_n$ and $\tilde{\mu} \in D_n$. Then the condition (3.2.3) is equivalent to

$$\mu = w^\lambda \circ \tilde{\mu} \quad \text{or equivalently} \quad \mu = w^\lambda_{\tilde{\mu}} \circ \tilde{\mu}.$$

Here $w^\lambda$ (resp. $w^\lambda_{\tilde{\mu}}$) denotes the unique longest element in the coset $W_n[\lambda+\rho]w$ (resp. $W_n[\lambda+\rho]wW_n[\tilde{\mu}+\rho]$). This can be shown as follows:

By Remark 3.2.3-(ii), it is enough to see that $w^\lambda \circ \tilde{\mu}$ and $w^\lambda_{\tilde{\mu}} \circ \tilde{\mu}$ belong to $A$ for any $w \in W_n$ and $\tilde{\mu} \in D_n$ such that $w \circ \tilde{\mu} \in \lambda - P_n$. (Note that this also implies $w^\lambda \circ \tilde{\mu} = w^\lambda_{\tilde{\mu}} \circ \tilde{\mu}$.) Take any $\alpha \in R_n^+ \cap R_n[\lambda+\rho]$. Note that $w^\lambda \circ \tilde{\mu} \in \lambda - P_n$ and thus $\langle w^\lambda \circ \tilde{\mu} + \rho, \alpha^\vee\rangle_n \in \mathbb{Z}$. Since $s_\alpha \in W_n[\lambda+\rho]$, we have $l(s_\alpha w^\lambda) < l(w^\lambda)$ and thus $(w^\lambda)^{-1}(\alpha) \in -R_n^+$. This implies $\langle w^\lambda \circ \tilde{\mu} + \rho, \alpha^\vee\rangle_n = \langle \tilde{\mu} + \rho, (w^\lambda)^{-1}(\alpha^\vee)\rangle_n \in \mathbb{Z}_{\le 0}$. Hence $w^\lambda \circ \tilde{\mu} \in A$. Similarly, it follows that $w^\lambda_{\tilde{\mu}} \circ \tilde{\mu} \in A$.

From Theorem 2.3.3 and Remark 2.3.5, we have

**Corollary 3.2.4.** *Any finite-dimensional simple $H_\ell$-module of level $n$ is isomorphic to $F_\lambda(L(\mu))$ for some $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$ satisfying* (3.2.3).

## 4. Contravariant forms and the Jantzen filtration

We remark on contravariant bilinear forms on $\mathfrak{gl}_n$-modules and those on $H_\ell$-modules. We relate them via the functor $F_\lambda$. As a consequence, we have a proof of Theorem 3.2.2 (a similar argument can be seen in the theory of Jantzen's translation functors [Ja]). We also prove that the Jantzen filtration on the Verma modules are transformed to the Jantzen filtration on the standard modules.

4.1. **Contravariant forms.** Let $X \in \mathrm{obj}\mathcal{O}(\mathfrak{gl}_n)$. A bilinear form $(\,|\,)_X : X \times X \to \mathbb{C}$ is called a $\mathfrak{gl}_n$-*contravariant form* if

$$(xv|u)_X = (v|\sigma(x)u)_X \quad \text{for all } u, v \in X,\ x \in \mathfrak{gl}_n, \tag{4.1.1}$$

where $\sigma$ is the transposition (§1.1). For $Y \in \mathrm{obj}\mathcal{R}(H_\ell)$, a bilinear form $(\,|\,)_Y : Y \times Y \to \mathbb{C}$ is called an $H_\ell$-*contravariant form* if

$$(xv|u)_Y = (v|\iota(x)u)_Y \quad \text{for all } u, v \in Y,\ x \in H_\ell, \tag{4.1.2}$$

where $\iota$ is given in Lemma 2.1.2.

Let us recall some fundamental facts on contravariant bilinear forms. The following lemma is easily shown.

**Lemma 4.1.1.** *Let $X \in \mathrm{obj}\mathcal{O}(\mathfrak{gl}_n)$ be equipped with a $\mathfrak{gl}_n$-contravariant bilinear form $(\,|\,)_X$. Then we have*

$$X^{[\lambda]} \perp X^{[\mu]} \quad \text{unless} \quad \lambda \in W_n \circ \mu, \tag{4.1.3}$$

$$X_\lambda \perp X_\mu \quad \text{unless} \quad \lambda = \mu. \tag{4.1.4}$$

**Lemma 4.1.2.** (i) *Let $\mu \in \mathfrak{t}_n^*$. A $\mathfrak{gl}_n$-contravariant form on $M(\mu)$ is unique up to constant multiples.*

(ii) *Let $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$. An $H_\ell$-contravariant form on $\mathcal{M}(\lambda,\mu)$ is unique up to constant multiples.*

*Proof.* (i) is well-known. We will prove (ii). Recall the decomposition (2.2.9):

$$\mathcal{M}(\lambda,\mu) \cong S_{[\lambda-\mu]} \oplus \bigoplus_{\beta \triangleright [\lambda-\mu]} S_\beta^{\oplus a_\beta}$$

as a $\mathbb{C}[W_\ell]$-module. Because an $H_\ell$-contravariant form is $W_\ell$-invariant, its restriction to $S_{[\lambda-\mu]}$ is unique up to constant, and we have

$$S_{[\lambda-\mu]} \perp \bigoplus_{\beta \triangleright [\lambda-\mu]} S_\beta^{\oplus a_\beta}. \tag{4.1.5}$$

From Theorem 2.3.1-(i), $S_{[\lambda-\mu]}$ generates $\mathcal{M}(\lambda,\mu)$ over $H_\ell$. Thus the statement follows. □

It is easy to construct a nonzero $\mathfrak{gl}_n$-contravariant form on $M(\mu)$. It is also known that there exists a nonzero contravariant form on $\mathcal{M}(\lambda,\mu)$ (see [Ro, CG] and also Remark 4.2.2). In the rest of this paper, we fix a canonical $\mathfrak{gl}_n$-contravariant form $(\,|\,)_{M(\mu)}$ on $M(\mu)$ by $(v_\mu|v_\mu)_{M(\mu)} = 1$.

**Lemma 4.1.3.** (i) *Let $\mu \in \mathfrak{t}_n^*$ and let $N$ be a unique maximal submodule of $M(\mu)$. Then*

$$N = \mathrm{rad}(\,|\,)_{M(\mu)}, \tag{4.1.6}$$

*where $\mathrm{rad}(\,|\,)_{M(\mu)}$ denotes the radical of $(\,|\,)_{M(\mu)}$.*

(ii) *Let $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$. Let $(\,|\,)_{\mathcal{M}(\lambda,\mu)}$ be a nonzero $H_\ell$-contravariant form on $\mathcal{M}(\lambda,\mu)$ and let $\mathcal{N}$ be a unique maximal submodule of $\mathcal{M}(\lambda,\mu)$. Then we have*

$$\mathcal{N} = \mathrm{rad}(\,|\,)_{\mathcal{M}(\lambda,\mu)}.$$

*Proof.* (i) is well-known. Let us prove (ii). It is obvious that $\mathrm{rad}(\,|\,) \subseteq \mathcal{N}$. Theorem 2.3.1 implies that $\mathcal{N} \subseteq \bigoplus_{\beta \triangleright [\lambda-\mu]} \mathcal{S}_\beta^{\oplus \dashv_\beta}$ with some $a_\beta \in \mathbb{Z}_{\geq 0}$. Thus we have $S_{[\lambda-\mu]} \perp \mathcal{N}$ by (4.1.5). Hence Theorem 2.3.1-(i) implies that $\mathcal{N} \subseteq \mathrm{rad}(\,|\,)_{\mathcal{M}(\lambda,\mu)}$. □

Let $X, Y \in \mathrm{obj}\mathcal{O}(\mathfrak{gl}_n)$ with $\mathfrak{gl}_n$-contravariant forms $(\,|\,)_X$, $(\,|\,)_Y$. Then the tensor product $X \otimes Y$ is equipped with a natural $\mathfrak{gl}_n$-contravariant bilinear form such that $(u \otimes v \mid u' \otimes v')_{X \otimes Y} = (u \mid u')_X\,(v \mid v')_Y$ for $u, u' \in X$ and $v, v' \in Y$. The following simple lemma will play a key role.

**Lemma 4.1.4.** *Let $\lambda \in D_n$. Let $X$ be a highest weight module* (*i.e. a quotient of a Verma module*) *of $\mathfrak{gl}_n$.*

- (i) *The $\mathfrak{gl}_n$-contravariant form on $X \otimes V_n^{\otimes \ell}$ is also $H_\ell$-contravariant, and thus it induces an $H_\ell$-contravariant form on $(X \otimes V_n^{\otimes \ell})_\lambda^{[\lambda]} = F_\lambda(X)$.*
- (ii) *If the $\mathfrak{gl}_n$-contravariant form on $X$ is non-degenerate, then the induced contravariant form on $F_\lambda(X)$ is non-degenerate.*

*Proof.* (i) Put $\Omega = \sum_{1 \leq k,m \leq n} E_{k,m} \otimes E_{m,k}$, where $E_{k,m}$ $(1 \leq k, m \leq n)$ denote the matrix units as in §1.1. Then we have $(\sigma \otimes \sigma)(\Omega) = \Omega$. Thus, for any $u, v \in X \otimes V_n^{\otimes \ell}$, we have $(\Omega_{ij}u|v)_{X\otimes V_n^{\otimes\ell}} = (u|\Omega_{ij}v)_{X\otimes V_n^{\otimes\ell}}$ $(0 \le i,j \le \ell)$, where $\Omega_{ij}$ is as in (3.1.4). This implies

$$(s_iu|v)_{X\otimes V_n^{\otimes\ell}} = (u|s_iv)_{X\otimes V_n^{\otimes\ell}} \quad (i = 1,\dots,\ell-1),$$
$$(\epsilon_i^\vee u|v)_{X\otimes V_n^{\otimes\ell}} = (u|\epsilon_i^\vee v)_{X\otimes V_n^{\otimes\ell}} \quad (i = 1,\dots,\ell),$$

from which the statement follows. (ii) follows from Lemma 4.1.1. □

As a consequence of Lemma 4.1.4-(i), the canonical $\mathfrak{gl}_n$-contravariant form on $M(\mu)$ induces an $H_\ell$-contravariant form on $\mathcal{M}(\lambda,\mu) = F_\lambda(M(\mu))$, which we call the canonical contravariant form on $\mathcal{M}(\lambda,\mu)$. By Lemma 4.1.3-(i), the $\mathfrak{gl}_n$-contravariant form on $L(\mu)$ is non-degenerate, and by Lemma 4.1.4-(ii), it induces a non-degenerate $H_\ell$-contravariant form on $F_\lambda(L(\mu))$. By Lemma 4.1.3-(ii), we have

**Corollary 4.1.5.** *Suppose that $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes\ell})$. Then the $H_\ell$-module $F_\lambda(L(\mu))$ is simple unless it is zero.*

4.2. **Proof of Theorem 3.2.2.** By $F_\lambda(M(\mu)) \cong \mathcal{M}(\lambda,\mu)$ and Corollary 4.1.5, it follows that $F_\lambda(L(\mu))$ is isomorphic to $\mathcal{L}(\lambda,\mu)$ or zero. Hence the proof of Theorem 3.2.2 is reduced to the following lemma:

**Lemma 4.2.1.** *Let $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes\ell})$. Then $F_\lambda(L(\mu)) \neq 0$ if and only if $\mu$ satisfies the condition* (3.2.3).

*Remark* 4.2.2. Lemma 4.2.1 implies that the canonical $\mathfrak{gl}_n$-contravariant form on $M(\mu)$ induces a nonzero $H_\ell$-contravariant form on $F_\lambda(M(\mu))$ if and only if the condition (3.2.3) is satisfied. By Remark 2.3.5 and Remark 3.2.3, it follows that any standard module admits a nonzero $H_\ell$-contravariant form.

*Proof of Lemma 4.2.1.* First we show the "only if" part. Suppose that $\mu$ does not satisfy (3.2.3). Then there exists $\alpha \in R_n^+$ such that $\langle \mu+\rho, \alpha^\vee\rangle \in \mathbb{Z}_{>0}$ and $\langle \lambda+\rho,\alpha^\vee\rangle = 0$. The first inequality implies $M(s_\alpha \circ \mu) \subset M(\mu)$, and the second equality implies $\mathcal{M}(\lambda,\mu) \cong \mathcal{M}(\lambda, s_\alpha\circ\mu)$ (Proposition 2.3.4). Hence we have $F_\lambda(L(\mu)) = 0$, because it is a quotient of $F_\lambda(M(\mu))/F_\lambda(M(s_\alpha\circ\mu)) = 0$.

Let us prove the "if" part. We can write $\mu$ as

$$\mu = w \circ \tilde{\mu},$$

where $w \in W_n$ and $\tilde{\mu} \in D_n$.

Then the condition (3.2.3) implies $\mu = w_{\tilde{\mu}}^\lambda \circ \tilde{\mu}$, where $w_{\tilde{\mu}}^\lambda$ is the longest element in $W_n[\lambda+\rho]w_{\tilde{\mu}}W_n[\tilde{\mu}+\rho]$ (see Remark 3.2.3). In the Grothendieck group of $\mathcal{O}(\mathfrak{gl}_n)$, we write

$$M(w_{\tilde{\mu}}^\lambda \circ \tilde{\mu}) = L(w_{\tilde{\mu}}^\lambda \circ \tilde{\mu}) + \sum_{y_{\tilde{\mu}}} a_{y_{\tilde{\mu}}} L(y_{\tilde{\mu}} \circ \tilde{\mu}). \tag{4.2.1}$$

Here the sum runs over those elements $y_{\tilde{\mu}} \in W_n$ such that $y_{\tilde{\mu}}$ is longest in $y_{\tilde{\mu}}W_n[\mu+\rho]$ and

$$y_{\tilde{\mu}} > w_{\tilde{\mu}}^\lambda. \tag{4.2.2}$$

Applying $F_\lambda$ to (4.2.1) we have

$$\mathcal{M}(\lambda, w_{\tilde{\mu}}^\lambda \circ \tilde{\mu}) = F_\lambda(L(w_{\tilde{\mu}}^\lambda \circ \tilde{\mu})) + \sum_{y_{\tilde{\mu}}} a_{y_{\tilde{\mu}}} F_\lambda(L(y_{\tilde{\mu}} \circ \tilde{\mu})) \tag{4.2.3}$$

in the Grothendieck group of $\mathcal{R}(H_\ell)$. Assuming that $F_\lambda(L(w^\lambda_{\tilde\mu}\circ\tilde\mu)) = 0$, we will deduce a contradiction. Since the multiplicity of $\mathcal{L}(\lambda, w^\lambda_{\tilde\mu}\circ\tilde\mu)$ in $\mathcal{M}(\lambda, w^\lambda_{\tilde\mu}\circ\tilde\mu)$ is nonzero, Corollary 4.1.5 implies

$$\mathcal{L}(\lambda, w^\lambda_{\tilde\mu}\circ\tilde\mu) = F_\lambda(L(y_{\tilde\mu}\circ\tilde\mu)) = \mathcal{L}(\lambda, y_{\tilde\mu}\circ\tilde\mu)$$

for some $y_{\tilde\mu}$. But this implies $y_{\tilde\mu} \in W_n[\lambda+\rho]w^\lambda_{\tilde\mu}W_n[\tilde\mu+\rho]$ by Proposition 2.3.4, and thus we have $l(y_{\tilde\mu}) \le l(w^\lambda_{\tilde\mu})$. This contradicts (4.2.2). □

4.3. **The Jantzen filtrations.** Throughout this subsection, we fix a weight $\delta \in \mathfrak{t}_n^*$. Let $A = \mathbb{C}[\,t\,]_{(t)}$ denote the localization of $\mathbb{C}[\,t\,]$ at the prime ideal $(t)$. We use the notation $\eta^t = \eta + \delta t \in \mathfrak{t}_n^* \otimes A$ for $\eta \in \mathfrak{t}_n^*$.

For $\mu \in \mathfrak{t}_n^*$, let $M(\mu^t)$ be the Verma module of $\mathfrak{gl}_n \otimes A$ with highest weight $\mu^t$:

$$M(\mu^t) = (U(\mathfrak{gl}_n)\otimes A)\otimes_{U(\mathfrak{b}_n^+)\otimes A}(Av_{\mu^t}).$$

The canonical $\mathfrak{gl}_n$-contravariant bilinear form on $M(\mu)$ can be naturally extended to a $\mathfrak{gl}_n\otimes A$-contravariant form $(\,|\,)_{M(\mu^t)}$ on $M(\mu^t)$ (with respect to the anti-involution $\sigma\otimes \mathrm{id}_A$) with values in $A$.

Define

(4.3.1)
$$M(\mu^t)_j = \{v \in M(\mu^t) \mid (v \mid u)_{M(\mu^t)} \in t^jA \text{ for all } u \in M(\mu^t)\}.$$

Putting $M(\mu)_j = M(\mu^t)_j\,/(tM(\mu^t)\cap M(\mu^t)_j)$ we have a filtration

$$M(\mu) = M(\mu)_0 \supseteq M(\mu)_1 \supseteq M(\mu)_2 \supseteq \cdots \tag{4.3.2}$$

by $\mathfrak{gl}_n$-modules called the *Jantzen filtration* [Ja].

Our next aim is to define the Jantzen filtration on the standard module, which was introduced in [Ro]. Let $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$. Analogously to §2.2, we define an $H_\ell\otimes A$-module $\mathcal{M}(\lambda^t,\mu^t)$ by

$$\mathcal{M}(\lambda^t,\mu^t) = (H_\ell\otimes A)\otimes_{H_{\lambda,\mu}\otimes A}(A\mathbf{1}_{\lambda^t,\mu^t}).$$

Put $X = M(\mu^t)\otimes V_n^{\otimes\ell}$, which is equipped with a $\mathfrak{gl}_n\otimes A$-contravariant form $(\,|\,)_X$. Then $\mathfrak{t}_n^*\otimes A$ acts semisimply on $X$ and it follows that

$$X = \bigoplus_{\eta^t\in\mu^t+P_n} X_{\eta^t}, \tag{4.3.3}$$

$$X_{\eta^t}\perp X_{\nu^t} \text{ unless } \mu = \nu. \tag{4.3.4}$$

Let $\chi_{\eta^t} : Z(U(\mathfrak{gl}_n)\otimes A) \to A$ be the infinitesimal character of $M(\eta^t)$. Following [GJ2, 1.8], we define for $\eta \in \mathfrak{t}_n^*$ an ideal $J_{\eta^t}$ of $Z(U(\mathfrak{gl}_n)\otimes A)$ by

$$J_{\eta^t} = \bigcap_{w\in W_n} \operatorname{Ker}\chi_{(w\circ\eta)^t},$$

and define

$$X^{[\eta^t]} = \{v \in X \mid J^k_{\eta^t}v = 0 \text{ for some } k\}. \tag{4.3.5}$$

Obviously $X^{[\eta^t]}$ depends only on the equivalence class $[\eta]$ of $\eta$ with respect to the equivalence relation (1.3.3).

The following lemma will be used later (see Theorem 4.3.5).

**Lemma 4.3.1** ([GJ2, Proposition 1.8.4]). *We have*

$$X = \bigoplus_{[\eta]\in\mathfrak{t}_n^*/\sim} X^{[\eta^t]}, \tag{4.3.6}$$

$$X^{[\eta^t]} \perp X^{[\nu^t]} \text{ unless } [\eta] = [\nu]. \tag{4.3.7}$$

On the $\mathfrak{gl}_n \otimes A$-module $X = M(\mu^t) \otimes V_n^{\otimes \ell}$, we can define an action of $H_\ell \otimes A$ commuting with $\mathfrak{gl}_n \otimes A$ as in Lemma 3.1.1. We define an induced $H_\ell \otimes A$-module structure on the following spaces:

$$(X/\mathfrak{n}_n^- X)_{\lambda^t}, \quad (X^{[\lambda^t]})_{\lambda^t}. \tag{4.3.8}$$

With respect to this action, the natural map

$$(X^{[\lambda^t]})_{\lambda^t} \to (X/\mathfrak{n}_n^- X)_{\lambda^t} \tag{4.3.9}$$

is an $H_\ell \otimes A$-homomorphism.

Similarly to (3.2.2), we can construct an $H_\ell \otimes A$-homomorphism

$$\mathcal{M}(\lambda^t, \mu^t) \to (X/\mathfrak{n}_n^- X)_{\lambda^t}. \tag{4.3.10}$$

The following lemma is elementary.

**Lemma 4.3.2.** *Let $M$ and $N$ be free $A$-modules of finite rank, and let $f : M \to N$ be an $A$-homomorphism. If the specialization*

$$\bar{f} : M/tM \to N/tN$$

*at $t = 0$ is a $\mathbb{C}$-isomorphism, then $f$ is an $A$-isomorphism.*

Using Lemma 4.3.2, we get

**Proposition 4.3.3.** *The $H_\ell \otimes A$-homomorphisms* (4.3.9) *and* (4.3.10) *are bijective*:

$$(X^{[\lambda^t]})_{\lambda^t} \cong (X/\mathfrak{n}_n^- X)_{\lambda^t} \cong \mathcal{M}(\lambda^t, \mu^t). \tag{4.3.11}$$

*Proof.* The specialization of (4.3.9) (resp. (4.3.10)) at $t = 0$ gives the isomorphism in Lemma 1.3.1 (resp. (3.2.2)). Therefore by Lemma 4.3.2, it is enough to show that $(X^{[\lambda^t]})_{\lambda^t}$, $(X/\mathfrak{n}_n^- X)_{\lambda^t}$ and $\mathcal{M}(\lambda^t, \mu^t)$ are all free $A$-modules of finite rank. Obviously they are finitely generated over $A$. It is also clear that $\mathcal{M}(\lambda^t, \mu^t)$ is free. Since $A$ is a principal ideal domain and $X$ is a free $A$-module, its subspace $(X^{[\lambda^t]})_{\lambda^t}$ is a free $A$-module. Finally, let us show that $(X/\mathfrak{n}_n^- X)_{\lambda^t}$ is a free $A$-module. By the isomorphism

(4.3.12)
$$X = M(\mu^t) \otimes V_n^{\otimes \ell} \cong (U(\mathfrak{gl}_n) \otimes A) \otimes_{U(\mathfrak{b}_n^+)\otimes A} (Av_{\mu^t} \otimes V_n^{\otimes \ell})$$

as $U(\mathfrak{gl}_n) \otimes A$-modules, it follows that

$$(X/\mathfrak{n}_n^- X)_{\lambda^t} \cong (V_n^{\otimes \ell})_{\lambda - \mu} \otimes A \tag{4.3.13}$$

as $A$-modules. This is a free $A$-module. □

It follows that the $\mathfrak{gl}_n \otimes A$-contravariant form on $X = M(\mu^t) \otimes V_n^{\otimes \ell}$ is also $H_\ell \otimes A$-contravariant. Through the isomorphism

$$\mathcal{M}(\lambda^t, \mu^t) \cong (X^{[\lambda^t]})_{\lambda^t} \subset X, \tag{4.3.14}$$

we introduce an $A$-valued $H_\ell \otimes A$-contravariant form on $\mathcal{M}(\lambda^t, \mu^t)$.

Assume that $\mu$ satisfies the condition (3.2.3) in Theorem 3.2.2. Then the induced contravariant form is nonzero (since its specialization at $t=0$ is nonzero). Therefore we have a filtration

$$\mathcal{M}(\lambda,\mu)=\mathcal{M}(\lambda,\mu)_0\supseteq\mathcal{M}(\lambda,\mu)_1\supseteq\mathcal{M}(\lambda,\mu)_2\supseteq\cdots \tag{4.3.15}$$

by $H_\ell$-modules, which we call the Jantzen filtration. Recall that any standard module is isomorphic to $\mathcal{M}(\lambda,\mu)$ for some $\lambda\in D_n$ and $\mu\in\lambda-P(V_n^{\otimes\ell})$ satisfying (3.2.3) (Remark 2.3.5).

*Remark* 4.3.4. In [Ro], the deformation direction $\delta$ is restricted by a certain condition. The construction above gives the definition of the Jantzen filtration for an arbitrary direction $\delta$.

**Theorem 4.3.5.** *Suppose that $\lambda\in D_n$ and $\mu\in\lambda-P(V_n^{\otimes\ell})$ satisfy the condition* (3.2.3)*. Then $F_\lambda(M(\mu)_j)=\mathcal{M}(\lambda,\mu)_j$.*

*Proof.* It is easy to check that $F_\lambda(M(\mu)_j)\subseteq\mathcal{M}(\lambda,\mu)_j$. To prove the opposite inclusion, let

$$p: M(\mu^t)\otimes V_n^{\otimes\ell}\to(M(\mu^t)\otimes V_n^{\otimes\ell})_{\lambda^t}^{[\lambda^t]}=\mathcal{M}(\lambda^t,\mu^t)$$

denote the natural projection. Note that $(M(\mu^t)\otimes V_n^{\otimes\ell}))_{\lambda^t}^{[\lambda^t]}\perp\operatorname{Ker}p$ by (4.3.4) and Lemma 4.3.1. Fix any orthonormal basis $\{b_i\}_{i=1}^{n^\ell}$ of $V_n^{\otimes\ell}$ with respect to the $\mathfrak{gl}_n$-contravariant form $(\,|\,)_{V_n^{\otimes\ell}}$.

Take any $u\in\mathcal{M}(\lambda^t,\mu^t)_j\subseteq(M(\mu^t)\otimes V_n^{\otimes\ell}))_{\lambda^t}^{[\lambda^t]}$ and write as $u=\sum_i a_i\otimes b_i$ with $a_i\in M(\mu^t)$. Then for any $v\in M(\mu^t)$ and $k$, we have

$$\begin{aligned}(a_k\mid v)_{M(\mu^t)}&=(u\mid v\otimes b_k)_{M(\mu^t)\otimes V_n^{\otimes\ell}}=(u\mid p(v\otimes b_k))_{M(\mu^t)\otimes V_n^{\otimes\ell}}\\&=(u\mid p(v\otimes b_k))_{(M(\mu^t)\otimes V_n^{\otimes\ell})_{\lambda^t}^{[\lambda^t]}}\in t^jA.\end{aligned}$$

This implies $a_k\in M(\mu^t)_j$ and thus $u\in(M(\mu^t)_j\otimes V_n^{\otimes\ell})_{\lambda^t}^{[\lambda^t]}$. Therefore we have $F_\lambda(M(\mu)_j)\supseteq\mathcal{M}(\lambda,\mu)_j$. □

## 5. Consequences

5.1. **BGG resolution.** Recall the generalization of the BGG resolution for certain simple $\mathfrak{gl}_n$-modules given by Gabber and Joseph [GJ1].

We fix $\mu\in\mathfrak{t}_n^*$ such that $-(\mu+\rho)$ is dominant and regular, i.e. $\langle-(\mu+\rho),\alpha^\vee\rangle_n\notin\mathbb{Z}_{\leq 0}$ for all $\alpha\in R_n^+$. Set $R_n^\mu=\{\alpha\in R_n\mid\langle\mu,\alpha^\vee\rangle_n\in\mathbb{Z}\}$. It is known that $R_n^\mu$ is a root system and its Weyl group coincides with the integral Weyl group

$$W_n^\mu=\{w\in W_n\mid w\circ\mu-\mu\in Q_n\}. \tag{5.1.1}$$

Set $R_n^{\mu+}=R_n^\mu\cap R_n^+$ and let $\Pi_n^\mu$ be the set of simple roots of $R_n^{\mu+}$.

Fix $B\subseteq\Pi_n^\mu$. The length function $l_B$ and the Bruhat order of $W_B$ are defined with respect to the set of simple roots $B$. Let $w_B$ be a unique longest element of $W_B$ with respect to $l_B$. Put $\mu_B=w_B\circ\mu$. Gabber and Joseph constructed the exact sequence

$$0\leftarrow L(\mu_B)\leftarrow C_0\leftarrow C_1\leftarrow\cdots \tag{5.1.2}$$

of $\mathfrak{gl}_n$-modules, where

$$C_i=\bigoplus_{y\in W_B,\ l_B(y)=i}M(y\circ\mu_B).$$

We apply $F_\lambda$ to the sequence (5.1.2). Then Theorem 3.2.1 and Theorem 3.2.2 imply the following:

**Theorem 5.1.1.** *Let $\mu$ and $B$ be as above. Suppose that $\lambda \in D_n \cap (\mu_B + P(V_n^{\otimes \ell}))$ satisfies $\langle \lambda + \rho, \alpha^\vee \rangle \neq 0$ for any $\alpha \in B$. Then there exists an exact sequence*

$$0 \leftarrow \mathcal{L}(\lambda, \mu_B) \leftarrow \mathcal{C}_0 \leftarrow \mathcal{C}_1 \leftarrow \cdots \tag{5.1.3}$$

*of $H_\ell$-modules, where*

$$\mathcal{C}_i = \bigoplus_{y \in W_B,\ l_B(y) = i} \mathcal{M}(\lambda, y \circ \mu_B).$$

*Remark* 5.1.2. In the case $\mu_B \in P_n^+$ and $B = \Pi_\ell$ (the original BGG case [BGG]), the corresponding sequence has been obtained by Cherednik [Ch1] by a different method (see also [Ze4, AST]).

5.2. **Kazhdan-Lusztig formulas.** For a module $M$ and simple module $L$, let $[M : L]$ denote the multiplicity of $L$ in the composition series of $M$.

Recall that $W_n^\mu$ denotes the integral Weyl group of $\mu \in \mathfrak{t}_n^*$ (see (5.1.1)). The following formula is a direct consequence of Theorem 3.2.1 and Theorem 3.2.2.

**Theorem 5.2.1.** *Let $\lambda, \mu \in D_n$ and let $w, y \in W_n^\mu$ such that $\lambda - w \circ \mu, \lambda - y \circ \mu \in P(V_n^{\otimes \ell})$. Then we have*

$$[\mathcal{M}(\lambda, w \circ \mu) : \mathcal{L}(\lambda, y \circ \mu)] = [M(w \circ \mu) : L(y^\lambda \circ \mu)], \tag{5.2.1}$$

*where $y^\lambda$ denotes the longest element in $W_n[\lambda + \rho]y$.*

Let $\lambda, \mu \in D_n$ and $w, y \in W_n^\mu$ be as in Theorem 5.2.1. The equality (5.2.1) has been known through the following two multiplicity formulas:

$$[M(w \circ \mu) : L(y \circ \mu)] \quad = P_{w, y_\mu}(1), \tag{5.2.2}$$

$$[\mathcal{M}(\lambda, w \circ \mu) : \mathcal{L}(\lambda, y \circ \mu)] = P_{w, y_\mu^\lambda}(1). \tag{5.2.3}$$

Here $P_{w,y}(q) \in \mathbb{Z}[q, q^{-1}]$ denotes the Kazhdan-Lusztig polynomial [KL1] of the Hecke algebra associated to $W_n^\mu$ (we put $P_{w,y}(q) = 0$ for $w \not< y$ for convenience), and $y_\mu$ (resp. $y_\mu^\lambda$) denotes the longest element in $yW_n[\mu+\rho]$ (resp. $W_n[\lambda+\rho]yW_n[\mu+\rho]$).

*Remark* 5.2.2. It follows from (5.2.2) and (5.2.3) that $P_{w,y_\mu}(1) = P_{w_\mu, y_\mu}(1)$ and $P_{w, y_\mu^\lambda}(1) = P_{w_\mu, y_\mu^\lambda}(1) = P_{w_\mu^\lambda, y_\mu^\lambda}(1)$. The latter is expressed in terms of the intersection cohomology concerning nilpotent orbits on the quiver variety [Ze3].

The formula (5.2.2) was conjectured by Kazhdan and Lusztig [KL1] and proved by Beilinson and Bernstein [BB1] and Brylinski and Kashiwara [BK]. The formula (5.2.3) was conjectured by Zelevinsky [Ze2] (see also [Ze3]) and proved by Ginzburg [Gi1] (see also [CG]). The theory of perverse sheaves plays an essential role in these proofs.

Theorem 5.2.1 (proved in a purely algebraic way) says that the Kazhdan-Lusztig formula (5.2.2) is equivalent to its degenerate affine Hecke analogue (or its $p$-adic analogue) (5.2.3). The implication (5.2.2)⇒(5.2.3) is obvious. The implication (5.2.3)⇒(5.2.2) is proved as follows. Take any $\mu \in D_n$ and $w, y \in W_n^\mu$. Then we can find $\ell \in \mathbb{Z}_{\geq 2}$ and $\lambda \in D_n^\circ$ such that

$$\lambda - z \circ \mu \in P(V_n^{\otimes \ell}) \text{ for all } z \in W_n^\mu.$$

In this case $F_\lambda(L(z \circ \mu))$ never vanishes and thus it is isomorphic to $\mathcal{L}(\lambda, z \circ \mu)$. Now (5.2.3) implies (5.2.2).

5.3. **Rogawski's conjecture.** Let $\{M(\mu)_j\}_j$ and $\{\mathcal{M}(\lambda,\mu)_j\}_j$ be the Jantzen filtrations defined in §4.3. As a direct consequence of Theorem 3.2.2 and Theorem 4.3.5, we have

**Theorem 5.3.1.** *Let $\lambda, \mu \in D_n$ and $w, y \in W_n^\mu$ (see (5.1.1)) be such that $\lambda - w \circ \mu, \lambda - y \circ \mu \in P(V_n^{\otimes \ell})$. Then we have*

$$[\mathcal{M}(\lambda, w\circ\mu)_j : \mathcal{L}(\lambda, y\circ\mu)] = [M(w^\lambda \circ \mu)_j : L(y^\lambda \circ \mu)], \tag{5.3.1}$$

*where $w^\lambda$ and $y^\lambda$ denote the longest element in $W_n[\lambda+\rho]w$ and $W_n[\lambda+\rho]y$, respectively.*

A priori the Jantzen filtrations depend on the choice of the deformation direction $\delta \in \mathfrak{t}_n^*$. It has been known that the Jantzen filtration on $M(\mu)$ does not depend on the choice of $\delta$ for which $(\,|\,)_{M(\mu^t)}$ is non-degenerate [Ba]. Now Theorem 4.3.5 implies

**Proposition 5.3.2.** *Let $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes\ell})$ satisfy (3.2.3). Then the Jantzen filtration on $\mathcal{M}(\lambda,\mu)$ does not depend on the choice of $\delta$ such that*

$$\langle \delta, \alpha^\vee \rangle_n \neq 0 \quad \textit{for any } \alpha \in R_n^+ \textit{ such that } \langle \mu+\rho, \alpha^\vee\rangle_n \in \mathbb{Z}_{>0}. \tag{5.3.2}$$

*Remark* 5.3.3. For $\lambda$ and $\mu$ as in Proposition 5.3.2, the condition (5.3.2) is equivalent to the condition that the $H_\ell \otimes A$-contravariant form $(\,|\,)_{\mathcal{M}(\lambda^t,\mu^t)}$ is non-degenerate.

We say that the Jantzen filtration $\{M(\mu)_j\}_j$ (or $\{\mathcal{M}(\lambda,\mu)_j\}_j$) is *regular* if the deformation direction $\delta$ satisfies (5.3.2). The following formula was conjectured in [GJ2, GM], and proved in [BB2].

**Theorem 5.3.4** ([BB2])**.** *Let $\mu \in D_n$ and $w, y \in W_n^\mu$. Suppose that $w$ and $y$ are the longest elements in $wW_n[\mu+\rho]$ and $yW_n[\mu+\rho]$, respectively. For the regular Jantzen filtration $\{M(w\circ\mu)_j\}_j$, we have*

$$\sum_{j\in\mathbb{Z}_{\geq 0}} [\mathrm{gr}_j\, M(w\circ\mu) : L(y\circ\mu)] q^{(l_\mu(y)-l_\mu(w)-j)/2} = P_{w,y}(q), \tag{5.3.3}$$

*where $P_{w,y}(q)$ denotes the Kazhdan-Lusztig polynomial of $W_n^\mu$, and $l_\mu$ denotes the length function on $W_n^\mu$.*

Combining with Theorem 5.3.1, the improved Kazhdan-Lusztig formula (5.3.3) implies its degenerate affine Hecke analogue, which was conjectured in [Ro].

**Theorem 5.3.5** (cf. [Gi2, Theorem 2.6.1])**.** *Let $\lambda, \mu \in D_n$ and $w, y \in W_n^\mu$ be such that $\lambda - w\circ\mu,\ \lambda - y\circ\mu \in P(V_n^{\otimes\ell})$. Suppose that $w$ and $y$ are the longest elements in $W_n[\lambda+\rho]wW_n[\mu+\rho]$ and $W_n[\lambda+\rho]yW_n[\mu+\rho]$, respectively. For the regular Jantzen filtration $\{\mathcal{M}(\lambda, w\circ\mu)_j\}_j$, we have*

$$\sum_{j\in\mathbb{Z}_{\geq 0}} [\mathrm{gr}_j\, \mathcal{M}(\lambda, w\circ\mu) : \mathcal{L}(\lambda, y\circ\mu)] q^{(l_\mu(y)-l_\mu(w)-j)/2} = P_{w,y}(q), \tag{5.3.4}$$

*where $P_{w,y}(q)$ denotes the Kazhdan-Lusztig polynomial of $W_n^\mu$, and $l_\mu$ denotes the length function on $W_n^\mu$.*

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