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A GENERALIZATION OF SPRINGER THEORY USING NEARBY CYCLES

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Abstract. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $f : \mathfrak{g} \to G\backslash\backslash\mathfrak{g}$ the adjoint quotient map. Springer theory of Weyl group representations can be seen as the study of the singularities of f.

In this paper, we give a generalization of Springer theory to visible, polar representations. It is a class of rational representations of reductive groups over $\mathbb{C}$, for which the invariant theory works by analogy with the adjoint representations. Let $G \,|\, V$ be such a representation, $f : V \to G\backslash\backslash V$ the quotient map, and P the sheaf of nearby cycles of f. We show that the Fourier transform of P is an intersection homology sheaf on V^*.

Associated to $G \,|\, V$, there is a finite complex reflection group W, called the Weyl group of $G \,|\, V$. We describe the endomorphism ring $\mathrm{End}(P)$ as a deformation of the group algebra $\mathbb{C}[W]$.

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1. Introduction

Let $f : \mathbb{C}^d \to \mathbb{C}$ be a non-constant polynomial. Fix a point $x \in E = f^{-1}(0)$. The *Milnor fiber* of f at x is defined by $F_{f,x} = f^{-1}(\epsilon) \cap \mathrm{B}_{x,\delta}$, where $\mathrm{B}_{x,\delta}$ is the δ-ball around x, and we assume that $0 < |\epsilon| \ll \delta \ll 1$. Since the publication of Milnor's seminal work [Mi], the cohomology of $F_{f,x}$ has been a central object in the study of the singularities of f.

In the language of sheaf theory, the cohomology groups $H^*(F_{f,x})$, for all $x \in E$, fit together to form a constructible, bounded complex P of sheaves on E, called the *sheaf of nearby cycles* of f (i.e., $H^*(F_{f,x})$ is the stalk of P at x). The complex P is a perverse sheaf ([GM2], [KS]).

There are two kinds of questions one may ask about the singularities of f. First, one may be interested in the cohomology of the Milnor fibers $F_{f,x}$ and of other related spaces, as well as in the various maps between such groups. The most important among such maps is the monodromy transformation $\mu_x : H^*(F_{f,x}) \to H^*(F_{f,x})$, arising from the dependence of the Milnor fiber on the choice of the small number $\epsilon \in \mathbb{C}^*$. Much of the work in singularity theory since the publication of [Mi] has centered around such concrete geometric questions.

On the other hand, one may ask: what is the structure of P as an object in the abelian, artinian category of perverse sheaves on E (see [BBD])? In particular, what are the simple constituents of P? Is P semisimple? What are the extensions involved in building P up from its simple constituents? What is the endomorphism ring $\mathrm{End}(P)$? How does the monodromy transformation $\mu \in \mathrm{Aut}(P)$ act?

These two kinds of questions are intimately related. The structure of P as a perverse sheaf is the "glue" that ties together the various local geometric invariants of f at different points. To the author's knowledge, the only general theorem about the structure of P is the deep result of Ofer Gabber (see [BB]) describing the filtration of P arising from the action of the monodromy. Gabber's theorem is a semisimplicity assertion about certain subquotients of P.

The definition of the nearby cycles sheaf P can be extended to the setting of a dominant map $f : \mathbb{C}^d \to \mathbb{C}^r$. In order to assure the desired properties of P, one needs to impose some technical conditions on f (see Section 2.2 below). On the other hand, the resulting theory is richer, because instead of a single monodromy transformation of P, we have an action of the whole fundamental group of the set of regular values of f near zero.

In this paper, we consider the nearby cycles for a certain special class of maps $f : \mathbb{C}^d \to \mathbb{C}^r$, whose components are given by homogeneous polynomials. This class arises naturally from the invariant theory of reductive group actions, and our motivation for studying it comes from the Springer theory of Weyl group representations.

Springer theory (see [BM1], [M], [S3], and Section 2.1 below) can be seen as the study of the singularities of the adjoint quotient map $f : \mathfrak{g} \to G\backslash\backslash\mathfrak{g} = \mathrm{Spec}\,\mathbb{C}[\mathfrak{g}^*]^{\mathrm{G}}$, associated to a complex semisimple Lie algebra. For this map, both the categorical structure of P and its local invariants are well understood. The sheaf P is semisimple. The monodromy group acting on P is the braid group B_W of the Weyl group W of $\mathfrak{g}$; but the monodromy action factors through W. Moreover, this monodromy action gives an isomorphism $\mathbb{C}[W] \cong \mathrm{End}(P)$. Thus, Springer theory gives a relation between the singularities of the nilcone $\mathcal{N} = f^{-1}(0) \subset \mathfrak{g}$ and the representations of the Weyl group.

Dadok and Kac [DK] introduced a class of rational representations $G \,|\, V$ (G reductive/$\mathbb{C}$) for which the invariant theory works by analogy with the adjoint representations. They call this the class of *polar representations*; it includes the adjoint representations, as well as many of the classical invariant problems of linear algebra (see Section 3.2 for examples). For any polar representation $G \,|\, V$, the quotient $G\backslash\backslash V$ is isomorphic to a vector space. In this paper, we study the singularities of the quotient map $f : V \to G\backslash\backslash V$, giving a generalization of Springer theory to polar representations satisfying a mild additional hypothesis.

Our main result (Theorem 3.1, part (i)) is the following. For a polar representation $G \,|\, V$, assume that the fiber $E = f^{-1}(0)$ consists of finitely many G-orbits (this condition is called *visibility*). Then the nearby cycles sheaf P of the quotient map f satisfies

$$\mathcal{F}\, P \cong \mathcal{IC}((V^*)^{rs}, \mathcal{L}), \tag{1}$$

where $\mathcal{F}$ is the geometric Fourier transform functor, and the right-hand side is an intersection homology sheaf on V^*. (See [KS] for a definition of the Fourier transform, and [GM1] for a discussion of intersection homology.) The content of this is that the sheaf P is completely encoded in the single local system $\mathcal{L}$ on a certain locus in the dual space V^*. In the case of Springer theory, this result is due independently to Ginzburg [Gi] and to Hotta-Kashiwara [HK] (see also [Br]). The proof of (1) draws on a general result of the author about the specialization of an affine variety to the asymptotic cone (see [Gr1] and Section 2.2.2).

Further assertions of Theorem 3.1 describe the holonomy of the local system $\mathcal{L}$ and the monodromy action on P of the appropriate fundamental group. It turns out that the semisimplicity observed in Springer theory does not extend to this generalization. Associated to each polar representation $G \,|\, V$, there is a finite complex reflection group W, called the Weyl group of $G \,|\, V$ (it is not, in general, the Weyl group of G). We have $\dim \operatorname{End}(P) = |W|$, but the algebra $\operatorname{End}(P)$ is not, in general, isomorphic to $\mathbb{C}[W]$. Instead, it is given as a kind of a Hecke algebra associated to W.

We should note that our Fourier transform description of P, while well suited to the study of the endomorphism ring and of the monodromy action, falls short of giving the complete structure of P as a perverse sheaf. This is because intersection homology is not an exact functor from local systems to perverse sheaves.

An important special case of this theory, which includes the adjoint representations, arises in the following way. Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be an involutive automorphism of a complex semisimple Lie algebra, and $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ the eigenspace decomposition for θ. Then the adjoint form G_+ of the Lie algebra $\mathfrak{g}_+$ acts on the symmetric space $\mathfrak{g}_-$. Orbits and invariants of the representation $G_+ \,|\, \mathfrak{g}_-$ were studied by Kostant and Rallis in [KR]. This representation is polar and visible. In Theorem 6.1, we explicitly compute the endomorphism ring $\operatorname{End}(P)$ in this case. It is given as a “hybrid” of the group algebra $\mathbb{C}[W]$ and the Hecke algebra of W (which is a Coxeter group) specialized at $q = -1$.

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2. Background

Notations. We will say *sheaf* to mean *complex of sheaves* throughout; all our sheaves will be sheaves of $\mathbb{C}$ vector spaces. Given a map $g : X \to Y$, the symbols g_*, $g_!$ will always denote the *derived* push-forward functors. All perverse sheaves and intersection homology will be taken with respect to the middle perversity (see [GM1], [BBD]); we use the shift conventions of [BBD]. Given a sheaf A on X, and a pair of closed subspaces $Z \subset Y \subset X$, we will write $\mathbb{H}^k(Y, Z; A)$ for the hypercohomology group $\mathbb{H}^k(j_!\, i^* A)$, where $i : Y \setminus Z \to X$ and $j : Y \setminus Z \to Y$ are the inclusion maps. We call $\mathbb{H}^k(Y, Z; A)$ the relative hypercohomology of A.

For an analytic function $f : M \to \mathbb{C}$, we use the notation of [KS, Chapter 8.6] for the nearby and vanishing cycles functors ψ_f, ϕ_f. Let A be a sheaf on M or $M \setminus f^{-1}(0)$. We denote by $\mu : \psi_f A \to \psi_f A$ the monodromy transformation of the nearby cycles. It is the counter-clockwise monodromy for the family of sheaves $\psi_{f/\tau} A$, parametrized by the circle $\{\tau \in \mathbb{C} \,|\, |\tau| = 1\}$.

When V is a $\mathbb{C}$ vector space, we denote by $\mathcal{P}_{\mathbb{C}^*}(V)$ the category of $\mathbb{C}^*$-conic perverse sheaves on V, and by $\mathcal{F} : \mathcal{P}_{\mathbb{C}^*}(V) \to \mathcal{P}_{\mathbb{C}^*}(V^*)$ the (shifted) Fourier transform functor. In the notation of [KS, Chapter 3.7], we have $\mathcal{F} P = P^\wedge[\dim V]$. To avoid cumbersome notation, we will use the following shorthand: if P is a conic perverse sheaf on a closed, conic subvariety $X \subset V$, and $j_X : X \to V$ is the inclusion, then we write $\mathcal{F} P$ instead of $\mathcal{F} \circ (j_X)_* P$. When E is a G-space, we denote by $\mathcal{P}_G(E)$ the category of G-equivariant perverse sheaves on E. The symbol dim, without a subscript, will always denote the *complex* dimension.

2.1. **Springer theory.** In this section we give a brief summary of the Springer theory of Weyl group representations. Springer theory (see [BM1], [S3]) is concerned with exhibiting the Weyl group W of a complex semisimple Lie algebra $\mathfrak{g}$ as the symmetry group of a certain perverse sheaf P on the nilcone $\mathcal{N} \subset \mathfrak{g}$. As we mentioned in the introduction, P is the nearby cycles sheaf for the adjoint quotient map $f : \mathfrak{g} \to G\backslash\backslash\mathfrak{g}$ (here G is the adjoint form of $\mathfrak{g}$). However, this is not the original definition of P, and not the one used in the literature to prove its properties.

We now describe the construction due to Lusztig [Lu1] and Borho-MacPherson [BM1] of the sheaf P and the W-action on it. Consider the Grothendieck simultaneous resolution diagram:

$$\begin{tikzcd}
\tilde{\mathcal{N}} \arrow[r, hook] \arrow[d, "p"] & \tilde{\mathfrak{g}} \arrow[r, "\tilde{f}"] \arrow[d, "q"] & \mathfrak{t} \arrow[d, "f|_{\mathfrak{t}}"] \\
\mathcal{N} \arrow[r, hook, "j"] & \mathfrak{g} \arrow[r, "f"] & W\backslash\backslash\mathfrak{t}
\end{tikzcd}$$

Here $\tilde{\mathfrak{g}}$ is the variety of pairs $(x, \mathfrak{b})$, where $\mathfrak{b} \subset \mathfrak{g}$ is a Borel subalgebra, and $x \in \mathfrak{b}$. This variety is smooth; it is a vector bundle over the flag variety $\mathcal{B}$ of $\mathfrak{g}$. The map f is given by the invariants of the adjoint action $G \,|\, \mathfrak{g}$; the target space is identified with the quotient of a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ by the action of the Weyl group. The map $\tilde{f}$ is the natural morphism making the diagram commute. Finally, $\tilde{\mathcal{N}} = \{\, (x, \mathfrak{b}) \in \tilde{\mathfrak{g}} \mid x \in \mathcal{N} \,\}$.

Note that the map q is a finite cover over the set $\mathfrak{g}^{rs}$ of regular semisimple elements in $\mathfrak{g}$. Given a point $x \in \mathfrak{g}^{rs}$, the fiber $q^{-1}(x)$ can be identified with the Weyl group W. Such an identification requires a choice of a positive Weyl chamber for the Cartan subalgebra $\mathfrak{g}_x \subset \mathfrak{g}$. The Weyl group W, therefore, acts as the deck transformations of the covering $q^{rs} : \tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$ (where $\tilde{\mathfrak{g}}^{rs} = q^{-1}(\mathfrak{g}^{rs})$, and q^{rs} is the restriction of q).

The diagram above is a simultaneous resolution in the following sense. For any point $t \in \mathfrak{t}$, the restriction $q_t : \tilde{f}^{-1}(t) \to f^{-1}(f(t))$ is a resolution of singularities. In particular, the map $p = q_0$ is a resolution of singularities of the nilcone.

We now consider the push-forward sheaves

$$P = p_* \, \mathbb{C}_{\tilde{\mathcal{N}}} \, [\dim \mathcal{N}] \;\text{ and }\; Q = q_* \, \mathbb{C}_{\tilde{\mathfrak{g}}} \, [\dim \mathfrak{g}].$$

Note that $P = j^* \, Q \, [-r]$, where $r = \dim \mathfrak{t}$.

Proposition 2.1 ([Lu1])**.** *The map q is small and the map p is semi-small in the sense of* [GM1]. *Consequently, the sheaf P is perverse, and the sheaf Q is the intersection cohomology extension $\mathcal{IC}(\mathfrak{g}^{rs}, \mathcal{L})$ of the local system $\mathcal{L} = q^{rs}_* \, \mathbb{C}_{\tilde{\mathfrak{g}}^{rs}}$ on $\mathfrak{g}^{rs}$.*

The Weyl group action on $\tilde{\mathfrak{g}}^{rs}$ by deck transformations produces an action on the local system $\mathcal{L}$, and by the functoriality of intersection cohomology on the sheaf Q. In fact, we have $\mathrm{End}(\mathcal{L}) = \mathrm{End}(Q) = \mathbb{C}[W]$, the group algebra of W. Using the relation $P = j^* \, Q \, [-r]$, we obtain a W-action on P as well. This construction of the W-action is due to Lusztig [Lu1]. Originally, the representations of W on the stalks of P, i.e., on the cohomology of the varieties $p^{-1}(x)$ ($x \in \mathcal{N}$), were constructed by Springer ([S1], [S2]). Other constructions of the same representations were given by Slodowy [Sl] and Kazhdan-Lusztig [KL]. The construction of Slodowy is essentially the nearby cycles definition, which is the starting point for this paper. Hotta in [Ho] showed that all of the different constructions agree.

Theorem 2.2 ([BM1])**.**
(i) P is a semisimple perverse sheaf.
(ii) The action of W on P gives an isomorphism $\mathbb{C}[W] \cong \mathrm{End}(P)$.

Borho and MacPherson deduce the first part of Theorem 2.2 from the decomposition theorem of Beilinson, Bernstein, Deligne, and Gabber [BBD], and use a counting argument to prove the second part. A different proof, using the Fourier transform, was later given by Ginzburg [Gi] and Hotta-Kashiwara [HK] (see also [Br]). We will use the Killing form on $\mathfrak{g}$ to identify $\mathfrak{g}$ with its dual, and to regard the Fourier transform on $\mathfrak{g}$ as a functor $\mathcal{F} : \mathcal{P}_{\mathbb{C}^*}(\mathfrak{g}) \to \mathcal{P}_{\mathbb{C}^*}(\mathfrak{g})$.

Theorem 2.3 ([Gi], [HK])**.**
(i) There is an isomorphism $\mathcal{F} \, P \cong Q$.
(ii) The Weyl group W acts on each side of this isomorphism. These two actions differ by the sign character of W (i.e., the character that sends each simple reflection $\sigma \in W$ to -1.)

Theorem 2.3 implies Theorem 2.2, because the functor $\mathcal{F}$ is an equivalence of categories.

2.2. **Nearby cycles.**

2.2.1. *Families over a large base.* The sheaf P constructed in the previous section can be defined in a different way, namely, as the nearby cycles of the adjoint quotient map f. One usually considers the nearby cycles functor for an analytic function $M \to \mathbb{C}$ (see [GM2, Section 6], [KS, Chapter 8.6]). Proposition 2.4 below provides a technical basis for discussing the nearby cycles for a sufficiently nice map $M \to \mathbb{C}^r$.

Let M be a connected complex analytic manifold of dimension d, let $\mathcal{U}$ be a neighborhood of zero in $\mathbb{C}^r$, and let $f : M \to \mathcal{U}$ be an analytic map onto $\mathcal{U}$. Let $\mathcal{U}^{sing} \subset \mathcal{U}$ be the closure of the set of non-regular values of f. We assume that $\mathcal{U}^{sing}$ is a proper analytic subvariety of $\mathcal{U}$. Let $\mathcal{U}^{reg} = \mathcal{U} \setminus \mathcal{U}^{sing}$, and note that the preimage $M^\circ = f^{-1}(\mathcal{U}^{reg}) \subset M$ is a manifold. We assume that there is an analytic Whitney stratification $\mathcal{S}$ of the fiber $E = f^{-1}(0)$, such that Thom's A_f condition holds for the pair (M°, S), for every stratum $S \in \mathcal{S}$. Recall that the A_f condition says that for any sequence of points $x_i \subset M^\circ$, converging to a point $y \in S$, if there exists a limit

$$\Delta = \lim_{i\to\infty} T_{x_i} f^{-1}(f(x_i)) \subset T_y M,$$

then $\Delta \supset T_y S$. If $r = 1$, such a stratification $\mathcal{S}$ can always be found. For $r > 1$, the existence of $\mathcal{S}$ is an actual restriction on f. It implies, in particular, that $\dim E = d - r$. We refer the reader to [Hi] for a detailed discussion of the A_f condition.

Let U be a neighborhood of zero in $\mathbb{C}$, and $\gamma : U \to \mathcal{U}$ be an embedded analytic arc, such that $\gamma(0) = 0$, and $\gamma(z) \in \mathcal{U}^{reg}$ for $z \neq 0$. Form the fiber product $M_\gamma = M \times_{\mathcal{U}} U$, and let $f_\gamma : M_\gamma \to U$ be the projection map. We may consider the nearby cycles sheaf $P_{f_\gamma} = \psi_{f_\gamma} \mathbb{C}_{M_\gamma} [d - r]$.

Proposition 2.4. *(i) The sheaves P_{f_γ} for different γ are all isomorphic. We may therefore omit the subscript γ, and call the sheaf $P_f = P_{f_\gamma}$ the nearby cycles of f. It is a perverse sheaf on E, constructible with respect to $\mathcal{S}$.*

(ii) The local fundamental group $\pi_1(\mathcal{U}^{reg} \cap \mathrm{B}_\epsilon)$, where $\mathrm{B}_\epsilon \subset \mathcal{U}$ is a small ball around the origin, acts on P_f by monodromy. We denote this action by

$$\mu : \pi_1(\mathcal{U}^{reg} \cap \mathrm{B}_\epsilon) \to \mathrm{Aut}(P_f).$$

(iii) The sheaf P_f is Verdier self-dual.

Proof. We refer the reader to [GM2, Section 6] for a proof that each P_{f_γ} is a perverse sheaf constructible with respect to $\mathcal{S}$. The self-duality of P_{f_γ} follows from the fact that the nearby cycles functor ψ_{f_γ} commutes with Verdier duality.

For the other assertions of the proposition, choose a point $x \in E$, and let $S \subset E$ be the stratum containing x. Fix a normal slice $N \subset M$ to S through the point x. Also, fix a pair of numbers $0 < \epsilon \ll \delta \ll 1$ (chosen to be small in decreasing order). For each regular value $\lambda \in \mathcal{U}^{reg}$ with $\|\lambda\| < \epsilon$, consider the Milnor fiber cohomology group

$$H^*(f^{-1}(\lambda) \cap N \cap \mathrm{B}_{x,\delta}),$$

where $\mathrm{B}_{x,\delta} \subset M$ is the δ-ball around x (we fix a Hermitian metric on M for this). It is enough to show that this Milnor fiber cohomology varies as a local system in λ. This can be done by adapting the argument in [Lê1, Section 1]. ∎

Returning to the notation of the previous section, it is easy to check that the adjoint quotient map $f : \mathfrak{g} \to W \backslash\backslash \mathfrak{t}$ and the orbit stratification of the nilcone $\mathcal{N}$ satisfy the hypotheses of Proposition 2.4. Note that the set of regular values of f equals $f(\mathfrak{t}^{reg})$, where $\mathfrak{t}^{reg} = \mathfrak{t} \cap \mathfrak{g}^{rs}$. The fundamental group $\pi_1(f(\mathfrak{t}^{reg}) \cap \mathrm{B}_\epsilon)$ is the braid group B_W associated to W. Theorem 2.5 below is a consequence of the properties of the simultaneous resolution (see [Sl] for an early treatment of these ideas).

Theorem 2.5 ([Ho], [M]).
(i) We have $P \cong P_f$.
(ii) The monodromy action $\mu : B_W \to \mathrm{Aut}(P_f)$ factors through the natural homomorphism $B_W \to W$, producing an action of W on P_f. The isomorphism of part (i) agrees with the W actions on both sides.

Thus, Springer theory can be viewed as the study of the nearby cycles of the adjoint quotient map.

2.2.2. *Specialization to the asymptotic cone.* In this section we summarize the results of [Gr1] on which this paper relies.

Let $V \cong \mathbb{C}^d$ be a complex vector space, and $X \subset V$ be an irreducible, smooth, closed subvariety. We denote by $\bar{V}$ the standard projective compactification of V, and by $\bar{X}$ the closure of X in $\bar{V}$. Set $V^\infty = \bar{V} \setminus V$, and $X^\infty = \bar{X} \cap V^\infty$. The asymptotic cone $\mathrm{As}(X) \subset V$ is defined as the affine cone over X^∞.

Another way to define $\mathrm{As}(X)$ is as follows. Let $\tilde{X}^\circ = \{(\lambda, \tilde{x}) \in \mathbb{C}^* \times V \mid \tilde{x} \in \lambda X\}$, and $\tilde{X}$ be the closure of $\tilde{X}^\circ$ in $\mathbb{C} \times V$. Write $\tilde{f} : \tilde{X} \to \mathbb{C}$ for the projection on the first factor. Then $\mathrm{As}(X) = \tilde{f}^{-1}(0)$. This definition shows that the asymptotic cone is naturally equipped with a nearby cycles sheaf $P = P_X := \psi_{\tilde{f}} \mathbb{C}_{\tilde{X}}[n]$, where $n = \dim X$. The sheaf P is $\mathbb{C}^*$-conic.

Given a Hermitian inner product on V, we may consider for any $x \in X$, the angle $\angle(x, T_x X) \in [0, \pi/2]$ between the vector $x \in V$ and the subspace $T_x X \subset T_x V \cong V$. The variety X is said to be *transverse to infinity* if for some (equivalently for any) inner product on V, there exists a constant $k > 0$, such that for any $x \in X$, we have

$$\angle(x, T_x X) < \frac{k}{\|x\|} \, .$$

Theorem 2.6 ([Gr1, Theorem 1.1]). *Assume $X \subset V$ is transverse to infinity, and let P be the nearby cycles sheaf on $\mathrm{As}(X)$. Let $T^*_X V \subset T^* V \cong V \times V^*$ be the conormal bundle to X, and $p_2 : V \times V^* \to V^*$ be the projection. Let $Y = \overline{p_2(T^*_X V)}$; it is an irreducible cone in V^*. Then we have*

$$\mathcal{F}\, P \cong \mathcal{IC}(Y^\circ, \mathcal{L}),$$

where $\mathcal{F}$ is the geometric Fourier transform functor, $\mathcal{L}$ is a local system on some Zariski open subset Y° of Y, and the right-hand side is the intersection homology extension of $\mathcal{L}$.

We will need some auxiliary facts describing the stalks of $\mathcal{F}\, P$.

Lemma 2.7 ([Gr1, Proposition 3.3]). *In the situation of Theorem 2.6, fix $l \in V^*$, and let $\xi = \mathrm{Re}\,(l)$. Also fix large positive numbers $1 \ll \xi_0 \ll \eta_0$. Then we have*

$$H^i_l\,(\mathcal{F}\, P) \cong H^{i+d+n}\,(\{\, x \in X \mid \xi(x) \le \xi_0 \,\},\, \{\, \|x\| \ge \eta_0 \,\};\, \mathbb{C}).$$

Assume now $l \neq 0$. Let $\Delta \subset V$ be the kernel of l, and $L \subset V$ be any line complementary to Δ. We have $V = \Delta \oplus L$. Take the standard projective compactification $\bar{\Delta}$ of Δ, and let $\hat{V} = \hat{V}_l = \bar{\Delta} \times L$. It is not hard to check that the space $\hat{V}$ is canonically independent of the choice of the line L. Note that $l : V \to \mathbb{C}$ extends to a proper algebraic function $\hat{l} : \hat{V} \to \mathbb{C}$. Let $\hat{X} = \hat{X}_l$ be the closure of X in $\hat{V}$, and $j : X \to \hat{X}$ be the inclusion map. Set $\hat{X}^\infty = \hat{X} \setminus X$.

Lemma 2.8 ([Gr1, Proposition 3.7])**.** *For $l \neq 0$, the statement of Lemma 2.7 can be modified as follows:*

$$H^i_l\,(\mathcal{F}\,P) \cong \mathbb{H}^{i+d}\,(\{\, x \in \hat{X} \,|\, \hat{\xi}(x) \leq \xi_0 \,\},\, \{\, \|\hat{l}(x)\| \geq 2\xi_0 \,\};\; j_!\,\mathbb{C}_X\,[n]),$$

where $\xi_0 \gg 1$, and $\hat{\xi}(x) = \operatorname{Re}(\hat{l}(x))$.

Let $\mathcal{S}$ be an algebraic Whitney stratification of $\mathrm{As}(X)$, written $\mathrm{As}(X) = \bigcup_{S \in \mathcal{S}} S$, satisfying the following three conditions.

(i) $\mathcal{S}$ is conical, i.e., each $S \in \mathcal{S}$ is $\mathbb{C}^*$-invariant.

(ii) Thom's $\mathrm{A}_{\tilde{f}}$ condition holds for the pair $(\tilde{X}^\circ, S)$, for each $S \in \mathcal{S}$.

(iii) Let $\mathcal{S}^\circ = \mathcal{S} \setminus \{\{0\}\}$. For $S \in \mathcal{S}^\circ$, let $S^\infty \subset X^\infty$ be the projectivization of S. Then the decomposition $\bar{X} = X \cup \bigcup_{S \in \mathcal{S}^\circ} S^\infty$ is a Whitney stratification.

The existence of such an $\mathcal{S}$ follows from the general results of stratification theory. Let $a(\mathcal{S}, l)$ be the dimension of the critical locus of $l\,|_{\mathrm{As}(X)}$ with respect to $\mathcal{S}$.

Lemma 2.9 ([Gr1, Section 3.4])**.** *There exists a stratification $\hat{\mathcal{X}}$ of $\hat{X}$ with the following property. Write Z for the critical locus of $\hat{l}\,|_{\hat{X}}$ with respect to $\hat{\mathcal{X}}$. Then*

$$\dim\, Z \cap \hat{X}^\infty < a(\mathcal{S}, l).$$

(We set $\dim \emptyset = -1$*.)*

2.3. **Polar representations.**

2.3.1. *A Summary of the results of Dadok and Kac.* Dadok and Kac [DK] introduced and studied the class of polar representations. Their motivation was to find a class of representations of reductive groups over $\mathbb{C}$, whose invariant theory works by analogy with the adjoint representations. In this section, following [DK], we will recall the definition and the main properties of polar representations.

Let $G\,|\,V$ be a rational linear representation of a connected reductive Lie group over $\mathbb{C}$ in a finite-dimensional vector space. A vector $v \in V$ is called *semisimple* if the orbit $G \cdot v$ is closed, and *nilpotent* if the closure $\overline{G \cdot v}$ contains zero. We say that $v \in V$ is *regular semisimple* if $G \cdot v$ is closed and of maximal dimension among all closed orbits. We will write V^s (V^{rs}) for the set of all semisimple (regular semisimple) vectors in V. The representation $G\,|\,V$ is called *stable* if $\dim G \cdot v \geq \dim G \cdot x$, for any regular semisimple $v \in V$ and any $x \in V$.

For a semisimple vector $v \in V$, define a subspace

$$c_v = \{\, x \in V \mid \mathfrak{g} \cdot x \subset \mathfrak{g} \cdot v \,\},$$

where $\mathfrak{g}$ is the Lie algebra of G. The orbits through c_v thus have "parallel" tangent spaces. The representation $G\,|\,V$ is called *polar* if for some semisimple $v \in V$, we have $\dim c_v = \dim \mathbb{C}[V^*]^G$; in this case, c_v is called a *Cartan subspace*. The prototype of all polar representations (responsible for the name "polar") is the action of the circle S^1 on the plane $\mathbb{R}^2$ by rotations. The class of polar representations includes the adjoint representations (a Cartan subspace for an adjoint representation is just a Cartan subalgebra) and the representations arising from symmetric spaces, studied by Kostant and Rallis in [KR] (see Section 6 below). It is, in fact, much larger. Every representation with $\dim \mathbb{C}[V^*]^G = 1$ is automatically polar. Dadok and Kac [DK] give a complete list of all polar representations of simple G. Their classification includes many of the classical invariant problems of linear algebra. Other examples of polar representations can be found in [K]. Note that a polar representation need not be stable.

Before describing the invariant theory of a polar representation, we recall the notion of a Shephard-Todd group (or a complex reflection group). Let c be a complex vector space. An element $g \in GL(c)$ is called a *complex reflection* if in some basis it is given by the matrix $\operatorname{diag}(e^{2\pi i/n}, 1, \ldots, 1)$, for some integer $n > 1$. A finite subgroup $W \subset GL(c)$ is called a *Shephard-Todd group* if it is generated by complex reflections. Shephard-Todd groups are a natural generalization of Coxeter groups; we refer the reader to [ST] for a discussion of their properties. Let $W \subset GL(c)$ be a Shephard-Todd group, and g be a complex reflection in W. Fix a basis of c in which $g = \operatorname{diag}(e^{2\pi i/n}, 1, \ldots, 1)$. We will say that g is a *primitive reflection* if there is no integer $n' > n$ such that $\operatorname{diag}(e^{2\pi i/n'}, 1, \ldots, 1) \in W$.

Theorem 2.10 ([DK]). *Let $G \,|\, V$ be a polar representation of a connected reductive group.*

(i) All Cartan subspaces of V are G-conjugate.

(ii) Fix a Cartan subspace $c \subset V$. Every vector $v \in c$ is semisimple, and every closed orbit passes through c.

(iii) Let $N_G(c)$ and $Z_G(c)$ be, respectively, the normalizer and the stabilizer of c. The quotient $W = N_G(c)/Z_G(c)$ is a Shephard-Todd group acting on c; it is called the Weyl group *of the representation $G \,|\, V$ (note that, in general, W is not the Weyl group of G).*

(iv) Restriction to c gives an isomorphism of invariant rings $\mathbb{C}[V^*]^G \cong \mathbb{C}[c^*]^W$; this invariant ring is free, generated by homogeneous polynomials. We denote by Q the categorical quotient $G \backslash\backslash V \cong W \backslash\backslash c$. As a variety, Q is isomorphic to a vector space of the same dimension as c.

(v) Let $f : V \to Q$ be the quotient map. Denote by $c^{reg} \subset c$ the set of regular points of $f|_c$. It is the complement of the union of the reflection hyperplanes in c. Then the set $Q^{reg} \subset Q$ of regular values of f is equal to $f(c^{reg})$.

We write $V^\circ = f^{-1}(Q^{reg})$.

Remark 2.11. It is easy to see that $V^s \cap V^\circ \subset V^{rs}$. Dadok and Kac conjecture that, in fact, $V^s \cap V^\circ = V^{rs}$ [DK, p. 521, Conjecture 2].

We will call the fundamental group $\pi_1(Q^{reg})$ the *braid group* of W, and denote it by B_W.

Proposition 2.12 ([DK]). *Let $G \,|\, V$ be a polar representation, as in Theorem 2.10, and $c \subset V$ be a Cartan subspace. Let G_c be the identity component of the stabilizer $Z_G(c)$.*

(i) There exists a compact form $K \subset G$, and a K-invariant inner product $\langle \, , \rangle$ on V, such that each $v \in c$ is of minimal length in its G-orbit.

(ii) The group G_c is reductive, and $K_c = G_c \cap K$ is a compact form of G_c.

(iii) We have a G_c-invariant orthogonal decomposition

$$V = c \oplus U_c \oplus \mathfrak{g} \cdot c,$$

with $\dim\, G_c \backslash\backslash U_c = 0$.

(iv) The representation $G \,|\, V$ is stable if and only if $U_c = 0$.

Let $d = \dim V$, $d_0 = d - \dim U_c$, and $r = \dim c = \dim Q$. Note that V^{rs} is irreducible and smooth of dimension d_0.

The rank of a polar representation $G \,|\, V$ is defined by

$$\operatorname{rank} G \,|\, V = \dim c/V^G,$$

where $V^G \subset V$ is the fixed points of G. Note that V^G is contained in every Cartan subspace of V. Dadok and Kac [DK, Theorem 2.12] show that any polar representation has a decomposition into polar representations of rank one, analogous to the root space decomposition of a semisimple Lie algebra. We will return to this result in Section 5.

2.3.2. *Geometry of the general fiber of a polar quotient map.* We now deduce from the results of [DK] some corollaries about the conormal geometry of the general fiber of a polar quotient map. First, we need to discuss the dual of a polar representation. We continue with the situation of Theorem 2.10 and Proposition 2.12.

Proposition 2.13. *(i) The dual representation $G \,|\, V^*$ is also polar, with the same Weyl group W. Given a Cartan subspace $c \subset V$, as in Proposition 2.12, a Cartan subspace for $G \,|\, V^*$ is given by*

$$c^* = (\mathfrak{g} \cdot c \oplus U_c)^\perp \subset V^*.$$

This establishes a one-to-one correspondence between the Cartan subspaces of V and V^*. Let $\langle\, , \rangle_{V^*}$ be the Hermitian metric on V^* induced by $\langle\, , \rangle$. Then every $l \in c^*$ is of minimal length in its G-orbit.

(ii) Let $l \in c^*$ be regular semisimple. Then the set $\{ v \in V \mid l\,|_{\mathfrak{g}\cdot v} = 0 \}$ equals $c \oplus U_c$.

Proof. The dual representation $G \,|\, V^*$ differs from $G \,|\, V$ by an automorphism of G (this is a general fact about representations of connected reductive groups). Thus, $G \,|\, V^*$ is also polar with a Cartan subspace of the same dimension, and the same Weyl group as $G \,|\, V$.

Take any $l \in c^*$. It follows from Proposition 2.12 that $\langle \mathfrak{g} \cdot l, l \rangle_{V^*} = 0$. By a theorem of Kempf and Ness ([KN], [DK, Theorem 1.1]), this implies that l is semisimple and of minimal length in its orbit. The rest of the proof is an exercise in linear algebra using Proposition 2.12. □

Corollary 2.14. *(i) The set $(V^*)^{rs}$ of regular semisimple points in V^* is an irreducible algebraic manifold of dimension d_0. We have $(V^*)^s \subset \overline{(V^*)^{rs}}$.*

(ii) The subspace $U_c \subset V$ of Proposition 2.12 is canonically independent of the choice of the compact form K, and the inner product $\langle\, , \rangle$. □

Pick a regular value $\lambda \in Q^{reg}$, and let $F = f^{-1}(\lambda) \subset V$. Let $\mathcal{O} \subset F$ be the unique closed G-orbit in F, and let $v \in c \cap \mathcal{O}$.

Proposition 2.15. *(i) There is a unique G-equivariant algebraic map $\phi : F \to \mathcal{O}$, such that $\phi(v + u) = v$, for any $u \in U_c$.*

(ii) The map $\phi : F \to \mathcal{O}$ is a complex algebraic vector bundle of rank $d - d_0$.

(iii) Fix any $x \in \mathcal{O}$. The tangent spaces T_yF, for all $y \in \phi^{-1}(x)$, are parallel.

Proof. Let $\psi : G \times U_c \to V$ be the map $\psi : (g, u) \mapsto g\,(u + v)$. By part (iii) of Proposition 2.12, we have $\operatorname{Im}\psi \subset F$. It is not hard to see that for small $u \in U_c$, the differential $d_{(1,u)}\psi$ is surjective onto the tangent space $T_{u+v}F$, which is parallel to $U_c \oplus \mathfrak{g} \cdot c$. It follows that $\operatorname{Im}\psi$ contains a neighborhood of v in F. Since $\mathcal{O}$ is the unique closed G-orbit in F, we have $\operatorname{Im}\psi = F$. It also follows that $T_{u+v}F$ is parallel to $U_c \oplus \mathfrak{g} \cdot c$, for all $u \in U_c$. It is now not hard to check that setting $\phi(\psi(g, u)) = g\,v$ gives the required definition. $\square$

We can now describe the conormal geometry of F. Consider the conormal bundle $\Lambda_F = T^*_F V \subset T^*V = V \times V^*$, and let $p_2 : \Lambda_F \to V^*$ be the projection map.

Corollary 2.16. *(i) The image $p_2(\Lambda_F) = (V^*)^s$.*

(ii) For any $l \in (V^*)^{rs}$, the critical points of the restriction $l|_F$ form a non-degenerate (Morse-Bott) critical manifold, which is a union of $|W|$ fibers of ϕ. In particular, $l|_F$ is a (complex) Morse function if and only if $G\,|\,V$ is stable.

Proof. Part (i) follows from the G-equivariance of the map p_2, and Propositions 2.13, 2.15. For part (ii) we may assume that $l \in c^*$. Then the critical points of $l\,|_F$ is the set $\phi^{-1}(W \cdot v)$. The G-equivariance of p_2 and Proposition 2.13 imply that the restriction of p_2 to $p_2^{-1}(V^*)^{rs}$ is a smooth submersion. The Morse-Bott property of $l\,|_F$ is simply a restatement of that fact. $\square$

Proposition 2.17. *The variety $F \subset V$ is transverse to infinity in the sense of Section 2.2.2, i.e., using the inner product of Proposition 2.12, we can find a $k > 0$, such that*

$$\angle(x, T_x F) < \frac{k}{\|x\|},$$

for any $x \in F$.

Proof. We will show that we can take $k = \|\,v\,\|$. Let $x \in F$ and $l \in V^*$ be a pair satisfying $(x, l) \in \Lambda_F \subset T^*V = V \times V^*$. Then it is enough to show that $|l(x)| \le \|\,v\,\| \cdot \|\,l\,\|$. By Proposition 2.15, we may write $x = g\,(v + u)$ and $l = g\,l'$, where $g \in G$, $u \in U_c$, and $l' \in c^*$. Then we have $l(x) = l'(v + u) = l'(v)$, and $|l(x)| = |l'(v)| \le \|\,v\,\| \cdot \|\,l'\,\| \le \|\,v\,\| \cdot \|\,l\,\|$, where the last inequality follows from the fact that l' is of minimal length in its orbit (Proposition 2.13). $\square$

3. The main theorem

3.1. **Statement of the theorem.** Let $G\,|\,V$ be a polar representation, as in Theorem 2.10. Assume that either rank $G\,|\,V = 1$, or $G\,|\,V$ is *visible*, i.e., there are finitely many nilpotent orbits in V. This assumption holds for the representations discussed in [KR] and for all of the infinite series listed in [DK] and [K]. As before, we write $f : V \to Q$ for the quotient map, and $E = f^{-1}(0)$ for the zero fiber. The purpose of the additional restriction on $G\,|\,V$ is to ensure that there exists a G-invariant conical stratification $\mathcal{S}$ of E, such that the A_f condition holds for the pair (V°, S), for every $S \in \mathcal{S}$. Indeed, if $G\,|\,V$ is visible, we can take $\mathcal{S}$ to be the orbit stratification. If, instead, rank $G\,|\,V = 1$, we can readily reduce to the case $\dim Q = 1$, then use the general result about the existence of A_f stratifications for functions [Hi, p. 248, Corollary 1].

We are now in the situation of Proposition 2.4, and therefore may consider the nearby cycles sheaf $P = P_f \in \mathcal{P}_G(E)$. In order to fix the up-to-isomorphism ambiguity in the definition of P, we fix a regular value $\lambda \in Q^{reg}$, let $F = f^{-1}(\lambda)$, and identify P with the sheaf P_F given by the specialization of F to $\mathrm{As}(F) = E$, as in Section 2.2.2. This corresponds to specializing along the path $\gamma : z \mapsto f(zF)$. We also have the monodromy action $\mu : B_W = \pi_1(Q^{reg}, \lambda) \to \mathrm{Aut}(P)$. Our main result is Theorem 3.1 below. It gives a description of the pair (P, μ) analogous to Theorem 2.3 in Springer theory. Recall from Corollary 2.14 that the set $(V^*)^{rs}$ of regular semisimple points in V^* is an irreducible algebraic manifold.

Theorem 3.1. *Let $G \,|\, V$ be, as above, a polar representation which is of rank one or visible. Let $c \subset V$ be a Cartan subspace, and W be the Weyl group.*

(i) We have

$$\mathcal{F}\, P \cong \mathcal{IC}((V^*)^{rs}, \mathcal{L}),$$

where the right-hand side is an intersection homology sheaf with coefficients in a local system $\mathcal{L}$ on $(V^*)^{rs}$ of rank $|W|$.

(ii) Write $A = \mathrm{End}(P)$, the endomorphisms of P as a perverse sheaf (forgetting the G-equivariant structure). We have dim $A = |W|$. *The monodromy action μ gives a surjection $\mathbb{C}[B_W] \to A$.*

(iii) Let $\sigma \in W$ be a primitive reflection of order n_σ. It gives rise to an element $\hat{\sigma} \in B_W$, represented by a loop going counter-clockwise around the image $f(c_\sigma)$ of the hyperplane $c_\sigma \subset c$ fixed by σ. The minimal polynomial R_σ of $\mu(\hat{\sigma}) \in A$ has integer coefficients and is of degree n_σ (we normalize R_σ to have leading coefficient 1).

(iv) Fix a basepoint $l \in (V^*)^{rs}$. By part (i), we have an action

$$A = \mathrm{End}(P) \cong \mathrm{End}(\mathcal{L}) \to \mathrm{End}(\mathcal{L}_l)$$

of A on the fiber $\mathcal{L}_l$. There is an identification $\chi : \mathcal{L}_l \cong A$, such that this action is given by left multiplication.

(v) There is a semigroup homomorphism $\rho : \pi_1((V^*)^{rs}, l) \to A^\circ$, the opposite of the algebra A, such that the holonomy of $\mathcal{L}$ is given as ρ followed by the right multiplication action of A° on $\mathcal{L}_l \overset{\chi}{\cong} A$. The homomorphism ρ gives a surjection $\mathbb{C}[\pi_1((V^*)^{rs}, l)] \to A^\circ$.

Remark 3.2. Dadok and Kac conjecture [DK, p. 521, Conjecture 4] that every visible representation is polar.

Remark 3.3. The support $\mathrm{supp}\,(\mathcal{F}\, P) = \overline{(V^*)^{rs}}$ is equal to all V^* if and only if the action $G \,|\, V$ is stable (cf. Corollary 2.14).

Remark 3.4. It is natural to expect from claims (ii) and (iii) of Theorem 3.1 that the algebra A is equal to the quotient $\mathbb{C}[B_W]/\langle\, R_\sigma(\hat{\sigma})\,\rangle$, where σ runs over all the primitive reflections in W. This is always true when W is a Coxeter group. In fact, it is the case in all of the examples known to the author.

Remark 3.5. We will give some additional information about the polynomials R_σ in Section 5.

Remark 3.6. One may ask whether it is always true, say, for a homogeneous polynomial $f : \mathbb{C}^d \to \mathbb{C}$ that the Fourier transform $\mathcal{F}\, P_f$ of the nearby cycles of f is an intersection homology sheaf. The answer is "no." A counterexample is given by $f(x, y, z) = x^2 y + y^2 z$.

3.2. **Examples.**

Example 3.7. Quadrics. Let $G = SO_n$ act on $V = \mathbb{C}^n$ by the standard representation. Any non-isotropic line $c \subset V$ can serve as a Cartan subspace. The Weyl group $W = \mathbb{Z}/2\mathbb{Z}$, and the invariant map $f : V \to \mathbb{C}$ is just the standard quadratic invariant of SO_n. The algebra A of Theorem 3.1 is given by $A = \mathbb{C}[z]/(z-1)^2$, if n is even, and $A = \mathbb{C}[z]/(z^2-1)$, if n is odd.

Example 3.8. Normal Crossings. Consider the action of the torus $G = (\mathbb{C}^*)^{n-1}$ on $V = \mathbb{C}^n$, given by

$$(t_1, \dots, t_{n-1}) : (x_1, \dots, x_n) \mapsto (t_1 x_1,\, t_1^{-1} t_2 x_2,\, t_2^{-1} t_3 x_3,\, \dots,\, t_{n-1}^{-1} x_n).$$

A Cartan subspace for this action is given by $c = \{\, x_1 = \dots = x_n \,\} \subset V$, the Weyl group $W = \mathbb{Z}/n\mathbb{Z}$, and the invariant map $f : V \to \mathbb{C}$ is just the product $f : (x_1, \dots, x_n) \mapsto x_1 \cdot \ldots \cdot x_n$. The algebra A of Theorem 3.1 is given by $A = \mathbb{C}[z]/(z-1)^n$, functions on the n-th order neighborhood of a point.

Example 3.9. The determinant (see [BG]). Let $G = SL_n$ act on

$$V = Mat\,(n \times n; \mathbb{C})$$

by left multiplication. A Cartan subspace $c \subset V$ for this action is given by the scalar matrices, the Weyl group $W = \mathbb{Z}/n\mathbb{Z}$, and the invariant map $f : V \to \mathbb{C}$ is the determinant. As in the previous example, the algebra $A = \mathbb{C}[z]/(z-1)^n$.

Example 3.10. Symmetric matrices (see [BG]). Let $G = SL_n$ act on the space V of symmetric $n \times n$ matrices by $g : x \mapsto gxg^t$. A Cartan subspace $c \subset V$ is again given by the scalar matrices, the Weyl group $W = \mathbb{Z}/n\mathbb{Z}$, and the invariant map $f : V \to \mathbb{C}$ is again the determinant. However, the algebra A is now given by $A = \mathbb{C}[z]/(z-1)^{\lceil n/2 \rceil}(z+1)^{\lfloor n/2 \rfloor}$.

Example 3.11. A "real analog" of Springer theory for SL_n (see [Gr2] and Section 6 below). Let $G = SO_n$ act on the space V of symmetric $n \times n$ matrices by conjugation. A Cartan subspace $c \subset V$ is given by the diagonal matrices. The Weyl group $W = \Sigma_n$, the symmetric group on n letters. The invariant map $f : V \to \mathbb{C}^{n-1}$ is given by sending a matrix $x \in V$ to the characteristic polynomial $\mathrm{char}\,(x)$. The algebra $A = \mathcal{H}_{-1}(\Sigma_n)$, the Hecke algebra of Σ_n specialized at $q = -1$.

Example 3.12. Maps from an orthogonal to a symplectic vector space. Let (U_1, ν) be an orthogonal vector space of dimension $2n+1$ (ν is a non-degenerate quadratic form), and (U_2, ω) a symplectic vector space of dimension $2n$. Set $V = \mathrm{Hom}_{\mathbb{C}}\,(U_1, U_2)$. The group $G = Sp_{2n} \times SO_{2n+1}$ acts on V by left-right multiplication. This action is polar, with a Cartan subspace of dimension n. The Weyl group $W \subset GL_n$ is the semi-direct product of $(\mathbb{Z}/4\mathbb{Z})^n \subset GL_n$, acting by diagonal matrices of fourth roots of unity, and the symmetric group $\Sigma_n \subset GL_n$, acting by permutation matrices. Let $\sigma \in \Sigma_n \subset W$ be any simple reflection, and let $\tau = \mathrm{diag}\,(i, 1, \dots, 1) \in (\mathbb{Z}/4\mathbb{Z})^n \subset W$. The algebra A is given by $A = \mathbb{C}[B_W]/\langle(\sigma^2 - 1),\, (\tau^2 - 1)^2\rangle$.

In all of the above examples, the representation $G\,|\,V$ is stable. Here is an example where it is not.

Example 3.13. An evaluation map. Let $U \cong \mathbb{C}^{2n}$. Set $V = \Lambda^2 U^* \oplus U \oplus U$. The linear group $G = GL(U)$ acts on V. The only invariant is the evaluation map $f : (\omega, u_1, u_2) \mapsto \omega(u_1, u_2)$. A non-zero vector $(\omega, u_1, u_2) \in V$ is semisimple, if and only if ω is of rank 2 and $\omega(u_1, u_2) \neq 0$. The Weyl group $W = \mathbb{Z}/3\mathbb{Z}$, and the algebra $A = \mathbb{C}[z]/(z-1)^3$.

3.3. **A remark on the degree of generality.** The reason we assume in Theorem 3.1 that $G \,|\, V$ is of rank one or visible, is to insure that the A_f condition holds and the sheaf P is well defined. If we dropped this assumption, we could still pick a $\lambda \in Q^{reg}$, and consider a sheaf P_λ coming from the specialization of $F = f^{-1}(\lambda)$ to the asymptotic cone. Using the techniques of this paper and some algebro-geometric generalities, we could then show that the sheaves P_λ for *generic* λ are isomorphic, and that they form a local system over some open set $Q^\circ \subset Q^{reg}$. Furthermore, all of the assertions of Theorem 3.1 could be given meaning and proved in this context. This is somewhat unsatisfactory, since we would have to use the full power of our methods just to show that there is a well posed question. Conjecture 3.14 below implies that the subset Q° is a phantom, and that nearby cycles are well defined for any polar representation.

Conjecture 3.14. *Let $G \,|\, V$ be a polar representation, $f : V \to Q$ be the quotient map, and $E = f^{-1}(0)$. Then there exists an A_f stratification of E (see Section 2.2.1).*

If this conjecture is true, all our results extend automatically to an arbitrary polar representation.

4. Proof of the main theorem

4.1. **A preliminary lemma.** We begin with a preliminary lemma. Recall the G-invariant conical stratification $\mathcal{S}$ of E discussed at the beginning of Section 3. By passing if necessary to a refinement, we may assume that $\mathcal{S}$ satisfies conditions (i)–(iii) of Section 2.2.2, with $X = F$. Indeed, conditions (i) and (ii) already hold without a refinement. To satisfy condition (iii), we use the general fact that any finite, algebraic, G-invariant decomposition of a G-variety (in our case, $\bar{F}$) can be refined to a G-invariant Whitney stratification.

Lemma 4.1. *Let $l \in (V^*)^{rs}$, and let $a(\mathcal{S}, l)$ be the dimension of the critical locus of $l\,|_E$ with respect to $\mathcal{S}$, as in Lemma 2.9. Then $a(\mathcal{S}, l) \leq d - d_0$, where $d = \dim V$, and $d_0 = \dim V^{rs}$.*

Proof. Let $\Lambda_E \subset T^*V = V \times V^*$ be the conormal variety to the stratification $\mathcal{S}$, and $p_2 : V \times V^* \to V^*$ be the projection. Using Proposition 2.13 and the G-invariance of $\mathcal{S}$, it is not hard to show that dim $p_2^{-1}(l) \cap \Lambda_E$ is independent of l for $l \in (V^*)^{rs}$. Therefore,

$$a(\mathcal{S}, l) + \dim (V^*)^{rs} \leq \dim \Lambda_E = d.$$

Together with Corollary 2.14, this proves the lemma. $\square$

4.2. **The stable case.** In this section we prove Theorem 3.1 in the case when the representation $G \,|\, V$ is stable. Then, in Section 4.3, we will indicate how to modify the argument in the nonstable case.

Assuming $G \,|\, V$ is stable, choose a basepoint $l \in (V^*)^{rs}$. Note that $(V^*)^{rs}$ is open in V^*. We may assume that $l \in c^*$, the Cartan subspace of Proposition 2.13. By Proposition 2.17, all the constructions of Section 2.2.2 apply to $X = F$. As in that section, we consider the compactification $\hat{F}$ of F relative to l, and denote by Z the set of critical points of the restriction $\hat{l}\,|_{\hat{F}}$. By Lemmas 2.9, 4.1, and Corollary

2.16, we have $Z \subset c \subset V$. Furthermore, Z is a single W-orbit in c, and each critical point in Z is Morse. Fix a point $e_0 \in Z$, and write $Z = \{e_w\}_{w \in W}$, where $e_w = w\, e_0$.

Recall the description of the stalk $H_l^*(\mathcal{F}\, P)$ in Lemma 2.8. Since dim $Z = 0$, we have $H_l^i(\mathcal{F}\, P) = 0$, unless $i = -d$. Furthermore, by duality

$$H_l^{-d}(\mathcal{F}\, P) \cong H_{d-r}(F, \{\, \xi(y) \geq \xi_0 \,\};\ \mathbb{C}),$$

where $\xi_0 > |\, l(e)\,|$ for all $e \in Z$.

We will need a standard Picard-Lefschetz construction of classes in the relative homology group above (see, for example, [BG, Section 7.2]). Let $e \in Z$ be a critical point. Fix a smooth path $\gamma : [0,1] \to \mathbb{C}$ such that

(i) $\gamma(0) = l(e)$, and $\gamma(1) = \xi_0$;
(ii) $\gamma(t) \notin l(Z)$, for $t > 0$;
(iii) $\gamma(t_1) \neq \gamma(t_2)$, for $t_1 \neq t_2$;
(iv) $\gamma'(t) \neq 0$, for $t \in [0,1]$.

Let $\mathcal{H}_e : T_eF \to \mathbb{C}$ be the Hessian of $l\,|_F$ at e, and let $T_e\,[\gamma] \subset T_eF$ be the positive eigenspace of the (non-degenerate) real quadratic form

$$\mathrm{Re}\,(\,\mathcal{H}_e \,/\, \gamma'(0)\,) : T_eF \to \mathbb{R}.$$

Note that $\dim_{\mathbb{R}} T_e\,[\gamma] = d - r$. Fix an orientation $\mathcal{O}$ of $T_e\,[\gamma]$. The triple $(e, \gamma, \mathcal{O})$ defines a homology class

$$u = u(e, \gamma, \mathcal{O}) \in H_{d-r}(F, \{\, \xi(y) \geq \xi_0 \,\};\ \mathbb{C}).$$

Namely, the class u is represented by an embedded $(d-r)$-disc

$$\kappa : (\mathrm{D}^{d-r}, \partial\, \mathrm{D}^{d-r}) \to (F, \{\, \xi(y) \geq \xi_0 \,\}),$$

such that the image of κ projects onto the image of γ, is tangent to $T_e\,[\gamma]$ at e, and does not pass through any point of Z except e. The sign of u is given by the orientation $\mathcal{O}$. It is a standard fact that $u \neq 0$.

We now fix a path $\gamma_0 : [0,1] \to \mathbb{C}$, satisfying conditions (i)–(iv) above for $e = e_0$, and an orientation $\mathcal{O}_0$ of the space $T_{e_0}[\gamma_0]$. Let $u_0 = u(e_0, \gamma_0, \mathcal{O}_0)$; we will use the same symbol for the corresponding element of $H_l^{-d}(\mathcal{F}\, P)$.

Lemma 4.2. *The image of $u_0 \in H_l^{-d}(\mathcal{F}\, P)$ under the monodromy action of B_W generates the stalk $H_l^{-d}(\mathcal{F}\, P)$ as a vector space. We have* dim $H_l^{-d}(\mathcal{F}\, P) = |W|$.

Proof. For each $w \in W$, pick a lift $b_w \in B_W$. By a standard Picard-Lefschetz theory argument, we have

$$\mu(b_w)\, u_0 = u\,(e_w, \gamma_w, \mathcal{O}_w),$$

where γ_w is some path satisfying conditions (i)–(iv) for $e = e_w$, and $\mathcal{O}_w$ is an orientation of $T_{e_w}[\gamma_w]$. Since $\{e_w\}_{w \in W}$ are the only critical points of the function $\hat{l}\,|_{\hat{F}}$, the elements $\{\mu(b_w)\, u_0\}_{w \in W}$ form a basis of $H_l^{-d}(\mathcal{F}\, P)$. □

Consider now the restriction of $\mathcal{F}\, P$ to the manifold $(V^*)^{rs}$. By Lemma 4.2 and the preceding arguments, it is a perverse sheaf whose stalks only live in degree $-d$ and all have rank $|W|$. Therefore, it is a rank $|W|$ local system with a degree shift. Together with Theorem 2.6, this proves part (i) of Theorem 3.1. The local system $\mathcal{L}$ of Theorem 3.1 is given by $\mathcal{L} = \mathcal{F}\, P\,|_{(V^*)^{rs}}\,[-d]$. Let $h : \pi_1((V^*)^{rs}, l) \to \mathrm{End}(\mathcal{L}_l)$ be the holonomy of $\mathcal{L}$.

Lemma 4.3. *The image of $u_0 \in \mathcal{L}_l$ under the action h generates the stalk $\mathcal{L}_l$ as a vector space.*

Proof. The proof of this is analogous to the proof of Lemma 4.2. Let $Q' = G\backslash\backslash V^*$, and $f' : V^* \to Q'$ be the quotient map. Let $(Q')^{reg} \subset Q'$ be the set of regular values of f'. Set $(V^*)^\circ = f'^{\,-1}((Q')^{reg})$; it is a Zariski open subset of $(V^*)^{rs}$ (see Remark 2.11). We may assume without loss of generality that $l \in (V^*)^\circ$. Let $\mathcal{L}^\circ$ be the restriction of the local system $\mathcal{L}$ to $(V^*)^\circ$, and let $h^\circ : \pi_1((V^*)^\circ, l) \to \mathrm{End}(\mathcal{L}_l)$ be the holonomy of $\mathcal{L}^\circ$. It is enough to show that the image of u_0 under h° generates the stalk $\mathcal{L}_l$.

Since we assume that G is connected, the push-forward homomorphism $f'_* : \pi_1((V^*)^\circ, l) \to \pi_1((Q')^{reg}, f'(l))$ is surjective. By Proposition 2.13, we have an isomorphism $\eta : \pi_1((Q')^{reg}, f'(l)) \cong B_W$ (it is not canonical). The composition $\eta \circ f'_*$, followed by the natural map $B_W \to W$, gives a surjection $\theta : \pi_1((V^*)^\circ, l) \to W$. For each $w \in W$, choose a lift $\alpha_w \in \theta^{-1}(w)$. Then a Picard-Lefschetz theory argument similar to the proof of Lemma 4.2 shows that the elements $\{\, h^\circ(\alpha_w)\, u_0 \,\}_{w \in W}$ form a basis of $\mathcal{L}_l$. □

Claims (ii), (iv), and (v) of Theorem 3.1 follow easily from Lemmas 4.2 and 4.3. We define the vector space map $\chi : A \to \mathcal{L}_l$ of claim (iv) by $\chi : a \mapsto a\, u_0$. By Lemma 4.2, χ is surjective. On the other hand, the action of any $a \in A$ on $\mathcal{L}_l$ must commute with the holonomy representation h. Therefore, Lemma 4.3 implies that χ is injective. This proves that χ is a vector space isomorphism. Claim (ii) now follows from Lemma 4.2; claim (iv) from the definition of χ; and claim (v) from Lemma 4.3, and the fact that every endomorphism of A commuting with the left action of A on itself is of the form $a \mapsto a\, a'$, for some $a' \in A$ (this is true for any associative algebra with unit).

It remains to prove claim (iii) of the theorem. The idea is to use a regular value $\lambda \in Q^{reg}$ which is very near the image $f(c_\sigma)$, then apply a standard carousel argument (see [Lê2] and [BG, Section 7.3]).

Pick a point $v_1 \in c_\sigma \subset c$ which is not fixed by any element of W, other than the powers of σ. The orbit $W \cdot v_1$ consists of $|W|/n_\sigma$ points. Choose a point $v \in c^{reg}$ which is very near v_1. The orbit $W \cdot v$ consists of $|W|$ points which are grouped into $|W|/n_\sigma$ clusters. Each cluster consists of n_σ points surrounding a point in $W \cdot v_1$, and the action of $\sigma \in W$ cyclically permutes the points in each cluster. We may assume that the fiber $F = f^{-1}(\lambda)$ passes through v, so that $Z = W \cdot v$. We may also assume that the choice of $l \in c^* \cap (V^*)^{rs}$ is sufficiently generic, so that the image $l(Z)$ appears in the $\mathbb{C}$-plane as $|W|/n_\sigma$ disjoint clusters, each consisting of n_σ points arranged in a circle (see Figure 1).

Choose a path $\gamma_0 : [0,1] \to \mathbb{C}$, satisfying conditions (i)–(iv) above for $e = v$, which does not "come near" any of the clusters in $l(Z)$, other than the one containing $l(v)$. Fix an orientation $\mathcal{O}_0$ of the space $T_v\,[\gamma]$. Let $u_0 = u(v, \gamma_0, \mathcal{O}_0)$. The element $\hat\sigma \in B_W = \pi_1(Q^{reg}, \lambda)$ is represented by a loop going once counter-clockwise around the image $f(c_\sigma)$, which stays in a small neighborhood of the critical value $f(v_1)$. Then, by a standard carousel argument, the vectors $\{\, \mu(\hat\sigma)^k\, u_0 \,\}_{k=0}^{n_\sigma - 1}$ are linearly independent, and we have

$$\mu(\hat\sigma)^{n_\sigma}\, u_0 = \pm u_0 + \sum_{k=1}^{n_\sigma - 1} g_k \cdot \mu(\hat\sigma)^k\, u_0,$$

where the $\{g_k\}$ are some integers. Together with Lemma 4.3, this implies claim (iii) of Theorem 3.1.

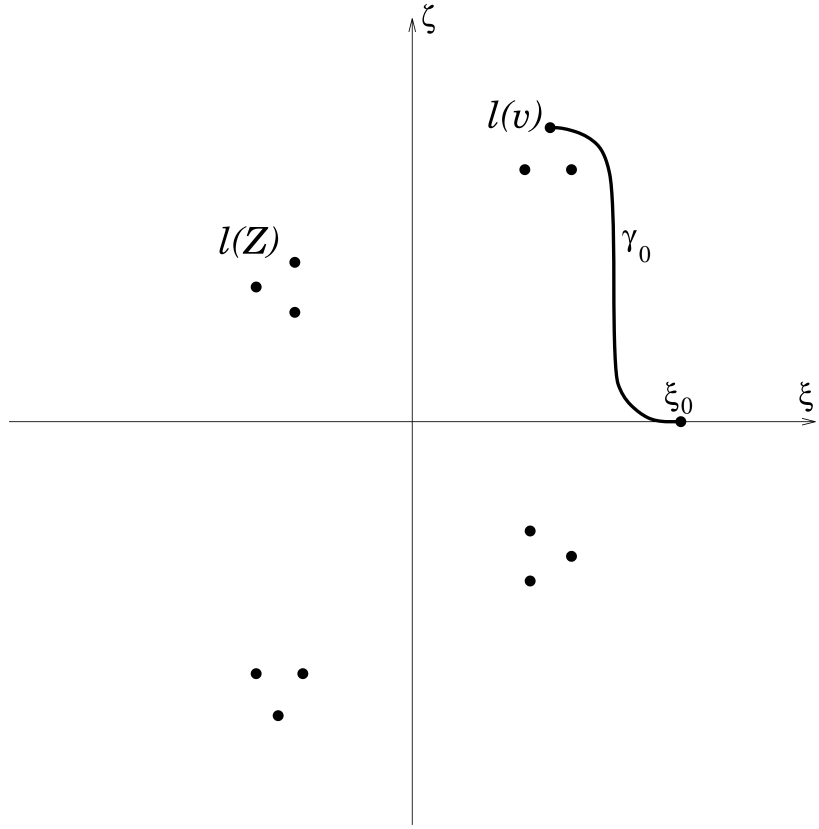

FIGURE 1

4.3. **The nonstable case.** The proof of Theorem 3.1 in the case when $G\,|\,V$ is not stable is analogous to the argument of Section 4.2, with Morse critical points of l replaced by Morse-Bott critical manifolds (see part (ii) of Corollary 2.16). We briefly indicate the changes that have to be made in the nonstable case.

As before, we pick a basepoint $l \in (V^*)^{rs}$, lying in the Cartan subspace c^* of Proposition 2.13, and write Z for the critical points of the restriction $\hat{l}\,|_{\hat{F}}$. Fix a point $e \in (F \cap c) \subset Z$, and a path $\gamma : [0,1] \to \mathbb{C}$, satisfying conditions (i)–(iv) of Section 4.2. The Hessian $\mathcal{H}_e : T_eF \to \mathbb{C}$ of $l\,|_F$ at e will now be degenerate. However, we may still consider the positive eigenspace $T_e\,[\gamma] \subset T_eF$ of the real quadratic form $\mathrm{Re}\,(\,\mathcal{H}_e\,/\,\gamma'(0)\,)$. By Corollary 2.16, we have $\dim_{\mathbb{R}} T_e\,[\gamma] = d_0 - r$. Fix an orientation $\mathcal{O}$ of $T_e\,[\gamma]$. The triple $(e, \gamma, \mathcal{O})$ defines a Picard-Lefschetz class

$$u = u(e, \gamma, \mathcal{O}) \in H_{d_0 - r}\,(F,\, \{\, \xi(y) \geq \xi_0 \,\};\, \mathbb{C}),$$

where ξ_0 is large, exactly as before. We may regard u as an element of $H_l^{-d_0}\,(\mathcal{F}\,P)$.

The main distinction from the stable case is that there is no a priori geometric reason for u to be non-zero. This is because the Morse-Bott manifold containing e is non-compact. However, we may use Lemmas 2.8, 2.9, 4.1, and an argument as in Lemma 4.2, to show that u generates the stalk $H_l^{-d_0}\,(\mathcal{F}\,P)$ under the monodromy action of B_W. Thus, if we had $u = 0$, it would follow that $H_l^{-d_0}\,(\mathcal{F}\,P) = 0$, and by Theorem 2.6, that $P = 0$. This contradiction shows that $u \neq 0$. The rest of the proof is exactly as in the stable case.

5. Rank one representations and the polynomials R_σ

As we mentioned in Section 2, there is an analog of root space decomposition for polar representation. Namely, any polar representation has a decomposition into polar representations of rank one. In this section, we use these rank one representations to give an interpretation (see Theorem 5.2) of the polynomials R_σ appearing in part (iii) of Theorem 3.1. This interpretation will be used in Section 6 to analyze the polar representations arising from symmetric spaces.

Let $G\,|\,V$ be a polar representation, $\sigma \in W$ be a primitive reflection of order n_σ, and $c_\sigma \subset c$ be the hyperplane fixed by σ (cf. Theorem 3.1). Denote by c_σ° the set of all $v \in c_\sigma$ such that the stabilizer $Z_W(v)$ is generated by σ; it is an open subset of the hyperplane c_σ. Let $\mathfrak{g}_\sigma \subset \mathfrak{g}$ be the stabilizer of c_σ, and $G_\sigma \subset G$ the adjoint form of $\mathfrak{g}_\sigma$. In terms of Proposition 2.12, let

$$V_\sigma = c \oplus U_c \oplus \mathfrak{g}_\sigma \cdot c \subset V.$$

If $\mathfrak{g}_\sigma \cdot c = 0$, we say that σ is *of global type*; otherwise, we say that σ is *of local type.*

Proposition 5.1 ([DK]). *(i) The representation $G\,|\,V$ restricts to a representation $G_\sigma\,|\,V_\sigma$, which is polar with Cartan subspace c. The rank of $G_\sigma\,|\,V_\sigma$ is equal to zero if σ is of global type, and to one if σ is of local type.*

(ii) Choose a point $v_1 \in c_\sigma^\circ$. Let $\mathfrak{g}_{v_1} \subset \mathfrak{g}$ be the stabilizer of v_1. Then we have $\mathfrak{g}_{v_1} = \mathfrak{g}_\sigma$, $\mathfrak{g} \cdot v_1 = \mathfrak{g} \cdot c_\sigma$, and $V = V_\sigma \oplus \mathfrak{g} \cdot v_1$.

Proposition 5.1 implies that σ is of local type if and only if $c_\sigma \cap V^{rs} = \emptyset$. Dadok and Kac conjecture (cf. Remark 2.11) that σ is always of local type. We do not know of any counterexamples, but allow for the possibility that they exist. The "root space decomposition" theorem of [DK] involves the representations $G_\sigma\,|\,V_\sigma$, for all primitive reflections σ of local type.

The Weyl group W_σ of $G_\sigma\,|\,V_\sigma$ is naturally a subgroup of W. Moreover, it must be a cyclic group, generated by σ^{m_σ} for some $m_\sigma \in \mathbb{Z}_+$, a divisor of n_σ. Note that when σ is of global type, we have $m_\sigma = n_\sigma$ and $W_\sigma = \{1\}$. Let $f_\sigma : V_\sigma \to Q_\sigma = G_\sigma\backslash\backslash V_\sigma$ be the quotient map, $E_\sigma = f_\sigma^{-1}(0)$ be the zero fiber, and $P_\sigma \in \mathcal{P}_{G_\sigma}(E_\sigma)$ be the nearby cycles of f_σ. Let B_σ be the braid group of W_σ. If σ is of local type, then $B_\sigma \cong \mathbb{Z}$, and we have a monodromy transformation $\mu_\sigma : P_\sigma \to P_\sigma$, given by the action of the counter-clockwise generator of B_σ. If σ is of global type, then $B_\sigma = \{1\}$, and we set $\mu_\sigma = \mathrm{Id} : P_\sigma \to P_\sigma$. Let $\tilde{R}_\sigma$ be the minimal polynomial of $\mu_\sigma \in \mathrm{End}(P_\sigma)$. It is a monic polynomial with integer coefficients of degree n_σ/m_σ. Note that if σ is of global type, then $\tilde{R}_\sigma(z) = z - 1$. By Theorem 3.1, we have $\mathrm{End}(P_\sigma) \cong \mathbb{C}[z]/\tilde{R}_\sigma(z)$.

Theorem 5.2. *Let $G\,|\,V$ be as in Theorem 3.1, and $G_\sigma\,|\,V_\sigma$ be as in Proposition 5.1. Then the polynomial R_σ in part (iii) of Theorem 3.1 is given by $R_\sigma(z) = \tilde{R}_\sigma(z^{m_\sigma})$.*

Proof. Choose a point $v_1 \in c_\sigma$, as in part (ii) of Proposition 5.1. Let $\lambda_1 = f(v_1) \in Q$. Choose a regular value $\lambda \in Q^{reg}$ very near λ_1. The idea of this proof is to break up the specialization of λ to 0 into two steps: first specialize λ to λ_1, then specialize λ_1 to 0. In order to compute the polynomial R_σ, we will only need to understand the first step.

Let $\mathrm{D} = \{\, z \in \mathbb{C} \mid |z| < 2 \,\}$ and $\gamma_1 : \mathrm{D} \to Q$ be an embedded holomorphic arc such that

(i) $\gamma_1(0) = \lambda_1$;
(ii) $\gamma_1(1) = \lambda$;
(iii) $\gamma_1(\mathrm{D} \setminus 0) \subset Q^{reg}$;
(iv) γ_1 is transverse to the image $f(c_\sigma)$.

Form the fiber product $V_1 = V \times_Q \mathrm{D}$, and let $f_1 : V_1 \to \mathrm{D}$ be the projection map. Let $E_1 = f^{-1}(\lambda_1)$. We have $\dim E_1 = \dim E = d - r$. Let $P_1 = \psi_{f_1} \mathbb{C}_{V_1}[d-r]$; we have $P_1 \in \mathcal{P}_G(E_1)$. Write $\mu_1 : P_1 \to P_1$ for the monodromy transformation of P_1.

Claim: There is a functor $\Psi : \mathcal{P}_G(E_1) \to \mathcal{P}_G(E)$ such that $\Psi\, P_1 = P$ and $\Psi(\mu_1) = \mu(\hat{\sigma})$.

To prove the claim, define an arc $\gamma_2 : \mathrm{D} \to Q$ by $\gamma_2 : z \mapsto f(z\, v_1)$. Form $V_2 = V \times_Q \mathrm{D}$, and let $f_2 : V_2 \to \mathrm{D}$ be the projection. We have a map $\pi : V_2 \setminus E \to E_1$, defined by $\pi(z\, e,\, z) = e$, for $z \in \mathrm{D} \setminus 0$, $e \in E_1$. Set $\Psi = \psi_{f_2} \circ \pi^*$. The claim follows form the definition of nearby cycles.

Because of the claim, it will suffice to show that $\tilde{R}_\sigma(\mu_1^{m_\sigma}) = 0$. For this, consider the inclusion $j_\sigma : V_\sigma \to V$ given by $j_\sigma : v \mapsto v + v_1$. By Proposition 5.1, j_σ exhibits V_σ as a G_σ-invariant normal slice to the orbit $G \cdot v_1$. Consider the perverse restriction to the normal slice functor ${}^p j_\sigma^* = j_\sigma^*[d_\sigma - d] : \mathcal{P}_G(E_1) \to \mathcal{P}_{G_\sigma}(E_\sigma)$, where $d_\sigma = \dim V_\sigma$. It is not hard to check that ${}^p j_\sigma^*$ is injective on morphisms. Set $P_2 = {}^p j_\sigma^*\, P_1$, and $\mu_2 = {}^p j_\sigma^*(\mu_1) : P_2 \to P_2$. Then it will suffice to show that

$$\tilde{R}_\sigma(\mu_2^{m_\sigma}) = 0. \tag{2}$$

Let $g_\sigma : Q_\sigma \to Q$ be the natural map. Note that $f \circ j_\sigma = g_\sigma \circ f_\sigma$. Note also that near the point $f_\sigma(v_1)$, the map g_σ is an m_σ-fold cover, ramified along the smooth hypersurface $f_\sigma(c_\sigma)$. It follows that $P_2 \cong \bigoplus_{k=1}^{m_\sigma} P_\sigma$, and that μ_2 is given by the matrix:

$$\begin{pmatrix} 0 & 0 & \dots & 0 & \mu_\sigma \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

This implies (2). $\square$

Remark 5.3. Our technique of first specializing λ to λ_1, then specializing λ_1 to 0 has a counterpart in the resolution approach to Springer theory. Namely, it corresponds to Borho and MacPherson's idea of first "forgetting the complete flag partially," then "forgetting the partial flag completely" (see [BM2, p. 28]).

Corollary 5.4. *All the roots of R_σ are roots of unity.*

Proof. This follows form Theorem 5.2 and the quasi-unipotence of the monodromy transformation μ_σ (see, for example, [Lê2]). $\square$

6. Symmetric spaces

Both Springer theory and Example 3.11 above are special cases of representations arising from symmetric spaces studied by Kostant and Rallis in [KR]. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and $\theta : \mathfrak{g} \to \mathfrak{g}$ an involutive automorphism. Let $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ be the eigenspace decomposition for θ, so that $\theta|_{\mathfrak{g}^\pm} = \pm 1$. Then $\mathfrak{g}^+$ is a Lie algebra, and the adjoint form G^+ of $\mathfrak{g}^+$ acts on the symmetric space $\mathfrak{g}^-$ by conjugation. By the results of [KR], the representation $G^+ \,|\, \mathfrak{g}^-$ is polar and visible. Theorem 6.1 below gives a recipe for computing the algebra A of Theorem 3.1 for this representation.

Let $\mathfrak{g}_{\mathbb{R}}$ be a real form of $\mathfrak{g}$ with a Cartan decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}}^+ \oplus \mathfrak{g}_{\mathbb{R}}^-$, such that $\mathfrak{g}^\pm$ is the complexification of $\mathfrak{g}_{\mathbb{R}}^\pm$ (see [OV] for a discussion of real forms of complex semisimple Lie algebras). Let $c_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}^-$ be a maximal abelian subalgebra. Then the complexification $c \subset \mathfrak{g}^-$ of $c_{\mathbb{R}}$ is a Cartan subspace for $G^+ \,|\, \mathfrak{g}^-$. The Weyl group W of this representation is just the small Weyl group associated to $\mathfrak{g}_{\mathbb{R}}$. It is a Coxeter group acting on $c_{\mathbb{R}}$ with its Euclidean structure induced by the Killing form on $\mathfrak{g}_{\mathbb{R}}$.

Fix a Weyl chamber $C \subset c_{\mathbb{R}}$. Let $\{\, \sigma_i \,\}_{i=1}^r$ ($r = \dim_{\mathbb{R}} c_{\mathbb{R}}$) be the reflections in the walls of C; they give a set of generators for W. Fix a basepoint $b \in C$. Consider the braid group $B_W = \pi_1(Q^{reg}, f(b))$ (we use the notation of Theorem 2.10). Let $\hat{\sigma}_i \in B_W$ be the element represented by the f-image of a path from b to $\sigma_i(b)$ in c^{reg}, which is almost straight but passes counter-clockwise half-way around the hyperplane $c_{\sigma_i} \subset c$ fixed by σ_i. The braid group B_W is generated by the elements $\{\, \hat{\sigma}_i \,\}_{i=1}^r$.

To each reflection σ_i we associate a number $s(i)$ as follows. Let $\mathfrak{g}_c^+ \subset \mathfrak{g}^+$ be the stabilizer of c, and let $\mathfrak{g}_i^+ \subset \mathfrak{g}^+$ be the stabilizer of c_{σ_i}. Then we set

$$s(i) = \dim\, \mathfrak{g}_i^+ - \dim\, \mathfrak{g}_c^+.$$

(This is just half the sum of the dimensions of the root spaces in $\mathfrak{g}_{\mathbb{R}}$ corresponding to σ_i.)

The Killing form on $\mathfrak{g}$ restricts to a non-degenerate G^+-invariant quadratic form on $\mathfrak{g}^-$. We will use it to identify $\mathfrak{g}^-$ with $(\mathfrak{g}^-)^*$, and to regard the Fourier transform on $\mathfrak{g}^-$ as a functor $\mathcal{F} : \mathcal{P}_{\mathbb{C}^*}(\mathfrak{g}^-) \to \mathcal{P}_{\mathbb{C}^*}(\mathfrak{g}^-)$. The set of regular semisimple vectors in $\mathfrak{g}^-$ is given by $(\mathfrak{g}^-)^{rs} = f^{-1}(Q^{reg})$.

Theorem 6.1. *Let $G^+ \,|\, \mathfrak{g}^-$ be the representation arising from an involution θ as above. Then, in terms of Theorem 3.1, we have*

(i) The endomorphism algebra $A = \mathrm{End}(P)$ is given by

$$A = \mathbb{C}[B_W]/\langle\, (\hat{\sigma}_i - 1)(\hat{\sigma}_i + (-1)^{s(i)})\, \rangle_{i=1}^r. \tag{3}$$

(ii) Take $l = b$ as a basepoint for $(\mathfrak{g}^-)^{rs}$. The homomorphism $\rho : \pi_1((\mathfrak{g}^-)^{rs}, b) \to A^\circ$ of part (v) of Theorem 3.1 is given by $\rho = \alpha \circ f_*$, where $f_* : \pi_1((\mathfrak{g}^-)^{rs}, b) \to \pi_1(Q^{reg}, f(b)) = B_W$ is the push-forward by f, and $\alpha : B_W \to A^\circ$ is defined by $\alpha : \hat{\sigma}_i \mapsto (-1)^{s(i)-1} \cdot \hat{\sigma}_i$.

Proof. Because W is a Coxeter group, the braid group quotient in the right-hand side of (3) has the correct dimension, namely the order of W. Thus, for part (i), it is enough to show that for $\sigma = \sigma_i$, we have $R_\sigma(z) = (z-1)(z + (-1)^{s(i)})$. This is an easy application of Theorem 5.2. It follows from the root space decomposition for $\mathfrak{g}_{\mathbb{R}}$ that, in terms of Proposition 5.1, we have $G_\sigma = SO(s(i)+1)$, $V_\sigma = \mathbb{C}^{s(i)+r}$, and G_σ acts by the standard representation on the first $s(i)+1$ coordinates. The computation of R_σ is now given by Theorem 5.2 and Example 3.7.

Part (ii) of the theorem is proved by a Picard-Lefschetz argument as in Section 5 (cf. Lemma 4.3). ∎

Remark 6.2. Note that A is isomorphic to the group algebra $\mathbb{C}[W]$, when $s(i)$ is even for all i, and to the Hecke algebra $\mathcal{H}_{-1}(W)$ specialized at $q = -1$, when $s(i)$

is odd for all i. In general, it is a "hybrid" of the two. Group algebra deformations of this kind were considered by Lusztig in [Lu2].

In the case when all the $s(i)$ are even, we can conclude from Theorem 3.1 that the sheaf P is semisimple. One example of this is the symmetric space $\mathfrak{sl}_{2n}/\mathfrak{sp}_{2n}$. In this example, the Weyl group is Σ_n, and nilpotent orbits are parametrized by the partitions of n. It is shown in [Gr2] that, in fact, this symmetric space gives a complete analog of the Springer correspondence for SL_n.

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