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CHARACTER FORMULAS FOR TILTING MODULES OVER KAC-MOODY ALGEBRAS

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Abstract. We show how to express the characters of tilting modules in a (possibly parabolic) category $\mathcal{O}$ over a Kac-Moody algebra in terms of the characters of simple highest weight modules. This settles, in lots of cases, Conjecture 7.2 of *Kazhdan-Lusztig-Polynome and eine Kombinatorik für Kipp-Moduln*, Representation Theory (An electronic Journal of the AMS) (1997), by the author, describing the character of tilting modules for quantum groups at roots of unity.

Introduction

In this article I determine the characters of indecomposable tilting modules in the category $\mathcal{O}$ over an affine Kac-Moody Lie algebra. By an equivalence of categories due to Kazhdan and Lusztig, this leads to character formulas for tilting modules over quantum groups; in particular we prove Conjecture 7.2 from [Soe97] in many cases.

I found the key to the determination of these characters in [Ark96]. There Arkhipov extends Feigin's semi-infinite cohomology and shows in particular, that the category of all modules with a Weyl filtration in positive level is contravariantly equivalent to the analogous category in negative level. Under this equivalence, projective objects have to be transformed into tilting modules; thus the Kazhdan-Lusztig conjectures in positive level lead to character formulas for tilting modules in negative level.

In [Ark96] the contravariant equivalence alluded to above appears as an illustration of a much stronger and deeper semi-infinite duality. I will show in the subsequent sections, how one can get it directly. Then I will discuss the application to tilting modules.

1. The semi-regular bimodule

Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$-graded Lie algebra over the field k with finite dimensional homogeneous pieces, $\dim_k \mathfrak{g}_i < \infty$ for all i.

Definition 1.1. A character $\gamma : \mathfrak{g}_0 \to k$ is called a semi-infinite character for $\mathfrak{g}$ iff we have:

1. As a Lie algebra $\mathfrak{g}$ is generated by $\mathfrak{g}_1$, $\mathfrak{g}_0$ and $\mathfrak{g}_{-1}$.
2. $\gamma([X,Y]) = \mathrm{tr}(\mathrm{ad}X\mathrm{ad}Y : \mathfrak{g}_0 \to \mathfrak{g}_0) \quad \forall X \in \mathfrak{g}_1, Y \in \mathfrak{g}_{-1}$.

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Remark 1.2. Voronov [Vor93] and Arkhipov [Ark96] work without the assumption (1) and rather ask of γ, that $d\gamma$ is the "critical cocycle of $\mathfrak{g}$". The Lie algebras we are interested in however, are generated by $\mathfrak{g}_1, \mathfrak{g}_0$ and $\mathfrak{g}_{-1}$. Since it is easier to treat this case, we restrict our attention to the framework mapped out by the above definition.

We put $\mathfrak{b} = \mathfrak{g}_{\geq 0}$, $\mathfrak{n} = \mathfrak{g}_{<0}$ and denote the enveloping algebras of $\mathfrak{g}, \mathfrak{b}, \mathfrak{n}$ by U, B, N. Certainly U, B and N inherit a $\mathbb{Z}$-grading from the corresponding Lie algebras. We consider for N the graded dual $N^{\circledast} = \bigoplus_i N_i^*$ and make it into an N-bimodule via the prescriptions $(fn)(n_1) = f(nn_1)$, $(nf)(n_1) = f(n_1 n)\ \forall\ f \in N^{\circledast}, n, n_1 \in N$.

Theorem 1.3. *Let $\gamma : \mathfrak{g}_0 \to k$ be a semi-infinite character for $\mathfrak{g}$. Then there exists a $\mathbb{Z}$-graded U-bimodule $S = S_\gamma$ along with an inclusion of $\mathbb{Z}$-graded N-bimodules $\iota : N^{\circledast} \hookrightarrow S$ such that the following hold:*

1. *The map $U \otimes_N N^{\circledast} \to S$, $u \otimes f \mapsto u\iota(f)$ is a bijection.*
2. *The map $N^{\circledast} \otimes_N U \to S$, $f \otimes u \mapsto \iota(f)u$ is a bijection.*
3. *Up to a twist by γ, the inclusion $\iota : N^{\circledast} \hookrightarrow S$ commutes with the adjoint action of $\mathfrak{g}_0$ on both spaces, in formulas $\iota(f \circ \mathrm{ad}H) + (\mathrm{ad}H)\iota(f) = \iota(f)\gamma(H)$ for all $H \in \mathfrak{g}_0$ and $f \in N^{\circledast}$.*

Remarks 1.4.

1. The bimodule S is a semi-infinite analogue of the enveloping algebra, since in analogy to $U \cong B \otimes_k N \cong N \otimes_k B$ we have $S \cong B \otimes_k N^{\circledast} \cong N^{\circledast} \otimes_k B$. We call S the "semi-regular bimodule". In greater generality it is introduced in [Vor93] as "standard semijective module" and in [Ark96] as "semiregular module". The presentation in [Vor93] however, still needs some fixing.
2. The semiregular bimodule S only has homogeneous components of degree ≥ 0.
3. The formula (3) in particular, tells us that $H\iota(\varepsilon) = \iota(\varepsilon)(H + \gamma(H))$ for all $H \in \mathfrak{g}_0$ and $\varepsilon \in N_0^{\circledast}$, in particular for ε the augmentation of N.
4. Going carefully through the proof of the theorem and replacing the comparision of dimension at the end by a more refined argument, we see that it is sufficient to assume $\dim_k \mathfrak{g}_i < \infty$ for $i < 0$.

Proof. Let us start by constructing for an arbitrary character γ of $\mathfrak{g}_0$ a vector space $S = S_\gamma$ with a left and a right action of U.

For any two $\mathbb{Z}$-graded vector spaces M, M', let us define the $\mathbb{Z}$-graded vector space $\mathcal{H}\mathrm{om}_k(M, M')$ with homogeneous components

$$\mathcal{H}\mathrm{om}_k(M, M')_j = \{f \in \mathrm{Hom}_k(M, M') \mid f(M_i) \subset M'_{i+j}\}.$$

For example we have $N^{\circledast} = \mathcal{H}\mathrm{om}_k(N, k)$, if we equip k with the $\mathbb{Z}$-grading $k = k_0$. Now let us consider for an arbitrary character γ of $\mathfrak{g}_0$ the following sequence of isomorphisms of $\mathbb{Z}$-graded vector spaces over k:

$$\mathcal{H}\mathrm{om}_B(U, k_\gamma \otimes_k B) \xrightarrow{\sim} \mathcal{H}\mathrm{om}_k(N, B) \xleftarrow{\sim} N^{\circledast} \otimes_k B \xrightarrow{\sim} N^{\circledast} \otimes_N U.$$

Here $\mathcal{H}\mathrm{om}_B$ denotes the space of all B-homomorphismen in $\mathcal{H}\mathrm{om}_k$, our k_γ is the one-dimensional representation of $\mathfrak{b}$ given by the character $\gamma : \mathfrak{g}_0 \to k$ and the surjection $\mathfrak{b} \twoheadrightarrow \mathfrak{g}_0$, and $k_\gamma \otimes_k B$ is, as a left $\mathfrak{b}$-module, the tensor product of these two representations. The first isomorphism is defined as the restriction to N using the identification $k_\gamma \otimes_k B \xrightarrow{\sim} B$, $1 \otimes b \mapsto b$. The other isomorphisms are obvious. We now put $S_\gamma = N^{\circledast} \otimes_k B$ and define on this space an action of U from the left (resp.

right) by the first two (resp. the last) isomorphisms. Our first goal is to show, that for a semi-infinite character γ the right and the left action of U on S_γ commute. I have to confess that I don't understand the true reason for that and thus have to check by blind calculation.

All our isomorphisms above are compatible with the obvious left action of N and right action of B on our spaces. Thus the left action of N commutes with the right action of U, and similarly the right action of B commutes with the left action U. We thus only have to show that

$$\begin{array}{rcl} H((f\otimes b)Y) & = & (H(f\otimes b))Y, \\ X((f\otimes b)Y) & = & (X(f\otimes b))Y \end{array}$$

for all $H \in \mathfrak{g}_0, X \in \mathfrak{g}_1, Y \in \mathfrak{g}_{-1}, f \in N^{\circledast}, b \in B$. Here we may even assume $b = 1$. Indeed, by right multiplication with $X_1 \in \mathfrak{g}_1$ resp. $H_1 \in \mathfrak{g}_0$ and a short calculation, one easily deduces from our equations with b the analogous equations with bX_1 resp. bH_1. Thus we need only to show

$$\begin{array}{rcl} H((f\otimes 1)Y) & = & (H(f\otimes 1))Y, \\ X((f\otimes 1)Y) & = & (X(f\otimes 1))Y \end{array}$$

for all $H \in \mathfrak{g}_0, X \in \mathfrak{g}_1, Y \in \mathfrak{g}_{-1}, f \in N^{\circledast}$.

So let's calculate. If we denote by $L_Y : N \to N$ the multiplication with $Y \in \mathfrak{g}_{-1}$ from the left, we have $(f \otimes 1)Y = fL_Y \otimes 1$. Also for $H \in \mathfrak{g}_0$ we get from our definitions

$$H(f\otimes 1) = -f(\mathrm{ad}H)\otimes 1 + f\otimes(\gamma(H)+H)$$

where $\mathrm{ad}H : N \to N$ is given by $(\mathrm{ad}H)(n) = Hn - nH$ as usual. Thus we get

$$\begin{array}{rcl} (H(f\otimes 1))Y & = & -f(\mathrm{ad}H)L_Y\otimes 1 + fL_Y\otimes(\gamma(H)+H), \\ & & +fL_{[H,Y]}\otimes 1, \\ H((f\otimes 1)Y) & = & -fL_Y(\mathrm{ad}H)\otimes 1 + fL_Y\otimes(\gamma(H)+H), \end{array}$$

and since $(\mathrm{ad}H)L_Y = L_Y(\mathrm{ad}H) + L_{[H,Y]}$, these expressions still coincide for γ arbitrary.

To determine $X(f \otimes 1)$ we choose a basis $(H_i)_{i\in I}$ of $\mathfrak{g}_0$ and define linear maps $H^i_X, F_X : N \to N$ by

$$nX = Xn + \sum_i H_i H^i_X(n) + F_X(n) \quad \forall n \in N.$$

From our definitions we get

$$X(f\otimes 1) = f\otimes X + \sum_i fH^i_X\otimes(\gamma(H_i)+H_i) + fF_X\otimes 1.$$

We further calculate

$$\begin{array}{rcl} (Yn)X & = & X(Yn) + \sum H_iH^i_X(Yn) + F_X(Yn) \\ Y(nX) & = & YXn + \sum YH_iH^i_X(n) + YF_X(n) \\ & = & XYn + [Y,X]n + \sum H_iYH^i_X(n) \\ & & + \sum[Y,H_i]H^i_X(n) + YF_X(n) \end{array}$$

and with $[Y, X] = \sum c^i_{YX}H_i$ we get the formulas

$$\begin{array}{rcl} H^i_XL_Y & = & L_YH^i_X + c^i_{YX}\mathrm{id}_N \\ F_XL_Y & = & L_YF_X + \sum_i L_{[Y,H_i]}H^i_X. \end{array}$$

Now we get

$$\begin{array}{rcl} X((f\otimes 1)Y) & = & fL_Y\otimes X+\sum_i fL_Y H_X^i\otimes(\gamma(H_i)+H_i) \\ & & +fL_Y F_X\otimes 1 \\ (X(f\otimes 1))Y & = & f\otimes XY+\sum_i fH_X^i\otimes(\gamma(H_i)+H_i)Y+fF_X\otimes Y \\ & = & fL_Y\otimes X+f\otimes[X,Y] \\ & & +\sum_i fH_X^i L_Y\otimes(\gamma(H_i)+H_i) \\ & & +\sum_i fH_X^i L_{[H_i,Y]}\otimes 1 \\ & & +fF_X L_Y\otimes 1 \end{array}$$

and as a condition for $X((f\otimes 1)Y)=(X(f\otimes 1))Y$ we find using our formulas $0=\sum_i c^i_{YX}\cdot\gamma(H_i)+c^i_{[H_i,Y]X}$, in other words $\gamma([X,Y])=\mathrm{tr}(\mathrm{ad}X\mathrm{ad}Y:\mathfrak{g}_0\to\mathfrak{g}_0)$. Thus for every semi-infinite character γ our $S=S_\gamma$ is a U-bimodule. We define $\iota:N^{\circledast}\hookrightarrow S$ by $\iota(f)=f\otimes 1$ and only have to check the properties (1)–(3) from the theorem.

Here (2) and (3) follow directly from the definitions and we only prove (1). Certainly S admits a $\mathbb{Z}$-grading with finite dimensional homogeneous components. By equality of dimensions, it is sufficient to show that $U\otimes_N N^{\circledast}\to S$ is a surjection, hence that $\iota(N^{\circledast})$ already generates S as a left U-module. But our formulas for $X(f\otimes 1)$, $H(f\otimes 1)$ show, that the left U-submodule generated by $\iota(N^{\circledast})$ is stable under the right B-action. □

2. The category of all modules with a finite Verma-flag is its own opposed category

We keep the notations of the preceding section. Let $\gamma:\mathfrak{g}_0\to k$ be a semi-infinite character for $\mathfrak{g}$ and $S=S_\gamma$ the corresponding semi-regular bimodule. Let $\mathcal{M}$ resp. $\mathcal{K}$ denote the categories of all $\mathbb{Z}$-graded representations of $\mathfrak{g}$, which are over N graded free resp. cofree of finite rank, i.e. isomorphic to finite direct sums of maybe grading shifted copies of N resp. $N^{\circledast}$.

Theorem 2.1. *The functor $S\otimes_U$ defines an equivalence of categories $S\otimes_U:\mathcal{M}\xrightarrow{\sim}\mathcal{K}$, such that short exact sequences correspond to short exact sequences.*

Remarks 2.2.
1. The existence of an equivalence $\mathcal{M}\cong\mathcal{K}$ is a result of Arkhipov [Ark96].
2. If $\mathfrak{h}\subset\mathfrak{g}_0$ is an abelian subalgebra such that the adjoint action of $\mathfrak{h}$ on $\mathfrak{n}$ is diagonalizable, then our functor also gives an equivalence $\mathcal{M}_{\mathfrak{h}}\xrightarrow{\sim}\mathcal{K}_{\mathfrak{h}}$ between the categories of all $\mathfrak{h}$-diagonalizable objects of $\mathcal{M}$ resp. $\mathcal{K}$.

Proof. First we deduce from $S\cong N^{\circledast}\otimes_N U$, that $S\otimes_U\cong N^{\circledast}\otimes_N$ indeed gives a functor $T:\mathcal{M}\to\mathcal{K}$, which transforms short exact sequences into short exact sequences. Furthermore multiplication from the right defines an isomorphism $N^{\mathrm{opp}}\to\mathrm{End}_N(N^{\circledast})$, and since we also have $S\cong U\otimes_N N^{\circledast}$ the prescription $\mathrm{Hom}_U(S,\ )\cong\mathrm{Hom}_N(N^{\circledast},\ )$ indeed defines a functor $H:\mathcal{K}\to\mathcal{M}$ making short exact sequences to short exact sequences.

Our functors obviously form an adjoint pair (T,H). To show they are inverse equivalences of categories, we only have to show that for all $M\in\mathcal{M}$ resp. $K\in\mathcal{K}$ the canonical map $M\to HTM$ resp. $THK\to K$ is an isomorphism. But we have

$$\begin{array}{rcl} HTM & = & \mathrm{Hom}_U(S,S\otimes_U M) \\ & = & \mathrm{Hom}_N(N^{\circledast},N^{\circledast}\otimes_N M) \end{array}$$

and for a free N-module M of finite rank certainly the canonical map $M \to \mathrm{Hom}_N(N^{\circledast}, N^{\circledast} \otimes_N M)$ is an isomorphism. Similarly we have

$$\begin{aligned} THK &= S \otimes_U \mathrm{Hom}_U(S, K) \\ &= N^{\circledast} \otimes_N \mathrm{Hom}_N(N^{\circledast}, K) \end{aligned}$$

and for $K = N^{\circledast}$ or, more generally, K cofree of finite rank over N certainly the canonical map $N^{\circledast} \otimes_N \mathrm{Hom}_N(N^{\circledast}, K) \to K$ is an isomorphism. □

In the following corollary the content of the theorem appears most clearly. For a $\mathbb{Z}$-graded space $V = \bigoplus V_i$ let $V^{\circledast} = \mathrm{Hom}_k(V, k)$ denotes its $\mathbb{Z}$-graded dual with homogeneous components $(V^{\circledast})_i = (V_{-i})^*$. If V is a $\mathbb{Z}$-graded representation of $\mathfrak{g}$, the contravariant action $(Xf)(v) = -f(Xv)$ for all $f \in V^{\circledast}$, $X \in \mathfrak{g}$, $v \in V$ makes $V^{\circledast}$ into a $\mathbb{Z}$-graded representation of $\mathfrak{g}$. Let $\mathcal{M}^{\mathrm{opp}}$ denote the opposed category of $\mathcal{M}$.

Corollary 2.3. *The functor $M \mapsto (S \otimes_U M)^{\circledast}$ defines an equivalence of categories $\mathcal{M} \xrightarrow{\sim} \mathcal{M}^{\mathrm{opp}}$, under which short exact sequences correspond to short exact sequences, and such that $U \otimes_B E$ gets mapped to $U \otimes_B (k_{-\gamma} \otimes E^*)$, for every finite dimensional $\mathbb{Z}$-graded representation E of $\mathfrak{g}_0$.*

Proof. Remark that the formulas above define a second left action of $\mathfrak{n}$ on $N^{\circledast}$ that doesn't coincide with the left action from section 1 in general. However, $N^{\circledast}$ with this second $\mathfrak{n}$-action is isomorphic to $N^{\circledast}$ with the first action as a $\mathbb{Z}$-graded $\mathfrak{n}$-module, a possible isomorphism being the transpose of the principal antiautomorphism of N. Hence our functor $V \mapsto V^{\circledast}$ defines an equivalence of categories $\mathcal{K} \xrightarrow{\sim} \mathcal{M}^{\mathrm{opp}}$. The rest of the proof is left to the reader. □

Remarks 2.4.

1. It is not difficult to show that $\mathcal{M}$ consists precisely of those $\mathbb{Z}$-graded $\mathfrak{g}$-modules, which admit a finite filtration with subquotients of the form $U \otimes_B E$ for suitable finite dimensional irreducible $\mathbb{Z}$-graded $\mathfrak{g}_0$-modules E. Therefore we call $\mathcal{M}$ also the category of all finite Verma flag modules.
2. It is also not difficult to show that $\mathcal{M}$ is stable under taking direct summands. More generally an arbitrary direct summand of a graded free N-module of finite rank is itself graded free of finite rank, for any $\mathbb{Z}$-graded algebra N which has no terms of positive degree and whose degree zero part is just the ground field.
3. Under the assumptions of Remark 2.2 (2) our functor also gives an equivalence $\mathcal{M}_{\mathfrak{h}} \xrightarrow{\sim} \mathcal{M}_{\mathfrak{h}}^{\mathrm{opp}}$.

3. Projective objects in $\mathcal{O}$

We now develop some well-known results in great generality and need stronger assumptions than in the first sections. From now on let k be an algebraically closed field of characteristic zero and let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$-graded Lie algebra over k with $\dim_k \mathfrak{g}_i < \infty$ $\forall i \in \mathbb{Z}$ such that $\mathfrak{g}_0$ is reductive and $\mathfrak{g}$ a semisimple $\mathfrak{g}_0$-module for the adjoint action. Then we consider the category $\mathcal{O}$ of all $\mathbb{Z}$-graded $\mathfrak{g}$-modules $M = \bigoplus_{i \in \mathbb{Z}} M_i$, which are locally finite for $\mathfrak{g}_{\geq 0}$ and semisimple for $\mathfrak{g}_0$. For $M, N \in \mathcal{O}$ we thus have $\mathrm{Hom}_{\mathcal{O}}(M, N) = \{f \in \mathrm{Hom}_{\mathfrak{g}}(M, N) \mid f(M_i) \subset N_i \quad \forall i \in \mathbb{Z}\}$.

As before we put $\mathfrak{g}_{\geq 0} = \mathfrak{b}$ and $U(\mathfrak{b}) = B$. Let Λ denote the set of isomorphism classes of irreducible finite dimensional $\mathbb{Z}$-graded $\mathfrak{g}_0$-modules. Such an $E \in \Lambda$ certainly will be concentrated in one degree $|E| \in \mathbb{Z}$, so we have $E = E_{|E|}$. For $E \in \Lambda$ we form the Verma module $\Delta(E) = U \otimes_B E$. Certainly $\Delta(E)$ is an object of $\mathcal{O}$, has a unique simple quotient $L(E)$, and $\{L(E)\}_{E\in\Lambda}$ is a system of representatives for the simple isomorphism classes in $\mathcal{O}$. Dually we form the object $\nabla(E) = \mathcal{H}\mathrm{om}_{\mathfrak{g}_{\leq 0}}(U, E)$ in $\mathcal{O}$ and it is easy to see that $L(E)$ is the smallest non-zero submodule of $\nabla(E)$. More precisely, we consider for $n \in \mathbb{Z}$ in $\mathcal{O}$ the subcategory

$$\mathcal{O}_{\leq n} = \{M \in \mathcal{O} \mid M_i = 0 \text{ if } i > n\}.$$

One shows for arbitrary $E, F \in \Lambda$:

Lemma 3.1. 1. *$\Delta(E)$ is the projective cover of $L(E)$ in $\mathcal{O}_{\leq |E|}$.*
2. *$\nabla(E)$ is the injective hull of $L(E)$ in $\mathcal{O}_{\leq |E|}$.*
3. *$\mathrm{Hom}_{\mathcal{O}}(\Delta(F), \nabla(E)) = 0$ if $F \neq E$. For $F = E$ this space is one-dimensional.*
4. *$\mathrm{Ext}^1_{\mathcal{O}}(\Delta(F), \nabla(E)) = 0$.*

Proof. Left to the reader. □

In general a simple object does not admit a projective cover in $\mathcal{O}$, but only in the truncated categories $\mathcal{O}_{\leq n}$.

Theorem 3.2. 1. *Every simple object $L(E) \in \mathcal{O}_{\leq n}$ admits in $\mathcal{O}_{\leq n}$ a projective cover $P_{\leq n}(E)$, and this projective cover has a finite Δ-flag.*
2. *For $m > n$ the kernel of the surjection $P_{\leq m}(E) \twoheadrightarrow P_{\leq n}(E)$ admits a finite Δ-flag with subquotients of the form $\Delta(F)$ for $m \geq |F| > n$.*
3. *The simple $L(E)$ admits a projective cover $P(E)$ in $\mathcal{O}$ if and only if $P_{\leq n}(E) \cong P_{\leq n+1}(E) \cong \ldots$ for $n \gg 0$, and then we have $P(E) \cong P_{\leq n}(E)$.*

The proof needs some abstract theory.

Lemma 3.3. *Let $\mathcal{A}$ be an abelian category, $p : P \twoheadrightarrow L$ a surjection of an indecomposable projective object onto a simple object. If $E = \mathrm{End}_{\mathcal{A}} P$ is of finite length as a right module over itself, then P is a projective cover of L.*

Proof. Indeed zero and one are the only idempotents of E, since P is indecomposable. The usual arguments via the Fitting decomposition then show, that every element of E is either nilpotent or invertible. Now if $i : U \to P$ is a morphism such that $p \circ i \neq 0$, then by projectivity of P there exists $q : P \to U$ such that $p \circ i \circ q = p$. But then $i \circ q$ is not nilpotent, hence an isomorphism and thus i is surjective. Hence $\ker p$ is the biggest proper subobject of P, that was to be shown. □

Now we prove the theorem.

Proof. (1) For a $\mathbb{Z}$-graded $\mathfrak{b}$-module $K = \bigoplus K_i$ let $\tau_{\leq n} K$ denote the quotient by the submodule of all homogeneous parts of degree $> n$, thus $\tau_{\leq n} K = \bigoplus_{i \leq n} K_i$. For $E \in \Lambda$, then

$$Q = U \otimes_B \tau_{\leq n}(B \otimes_{U(\mathfrak{g}_0)} E)$$

is projective in $\mathcal{O}_{\leq n}$. Indeed, if $\mathrm{Hom}_{\mathfrak{g},\mathbb{Z}}$ denotes the space of all $\mathfrak{g}$-module homomorphisms which are homogeneous of degree zero with respect to the $\mathbb{Z}$-grading, then

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}}(Q, M) &= \mathrm{Hom}_{\mathfrak{g},\mathbb{Z}}(Q, M) \\
&= \mathrm{Hom}_{\mathfrak{b},\mathbb{Z}}(\tau_{\leq n}(B \otimes_{U(\mathfrak{g}_0)} E), M) \\
&= \mathrm{Hom}_{\mathfrak{b},\mathbb{Z}}(B \otimes_{U(\mathfrak{g}_0)} E, M) \\
&= \mathrm{Hom}_{\mathfrak{g}_0,\mathbb{Z}}(E, M)
\end{aligned}$$

for all $M \in \mathcal{O}_{\leq n}$ and $\mathrm{Hom}_{\mathfrak{g}_0,\mathbb{Z}}(E, M)$ is an exact functor in $M \in \mathcal{O}$ by the very definition of this category.

Certainly Q graded free over N of finite rank, and if $n \geq |E|$, there is a surjection $Q \twoheadrightarrow L(E)$. By 3.3 we can take as $P_{\leq n}(E)$ every indecomposable summand of Q which has $L(E)$ as a quotient, and by Remark 2.4 (2) the module $P_{\leq n}(E)$ admits a Δ-flag.

(2) Certainly for $m \geq n$ we have a surjection

$$P_{\leq m}(E) \twoheadrightarrow P_{\leq n}(E),$$

and by the universal properties the kernel of such a surjection has to be a submodule of $P_{\leq m}(E)$ generated by all homogeneous components of degree $> n$. But such a submodule in a graded free N-module is graded free itself (with a basis all vectors of degree $> n$ inside a homogeneous basis of the full module), thus by Remark 2.4 (2) the submodule admits a Δ-flag, too.

(3) If $L(E)$ admits a projective cover $P(E)$ in $\mathcal{O}$, then $P(E)$ is generated by a single vector (indeed by every vector outside the biggest proper submodule), thus $P(E) \in \mathcal{O}_{\leq n}$ for $n \gg 0$, hence $P(E) \cong P_{\leq m}(E)\ \forall m \geq n$.

If, on the other hand, the projective system of the $P_{\leq n}(E)$ stabilizes with an object $P(E)$, we have to show that $P(E)$ is projective in $\mathcal{O}$. Certainly $P(E)$ is generated by one element v. Now let $M \twoheadrightarrow M'$ be a surjection and $f' : P(E) \rightarrow M'$ a morphism, which we want to lift to $f : P(E) \rightarrow M$. Then we choose a preimage $m \in M$ of $m' = f'(v) \in M'$ and consider the surjection $U(\mathfrak{g})m \twoheadrightarrow U(\mathfrak{g})m'$. Now both modules lie in $\mathcal{O}_{\leq n}$ for $n \gg 0$, and since $P(E) \cong P_{\leq n}(E)$ we find the lift we were looking for. $\square$

4. Reciprocity and decompositon of $\mathcal{O}$ into blocks

To formulate the usual reciprocity in full generality we have to introduce multiplicities in full generality.

Definition 4.1. Let $\mathcal{A}$ be an abelian category, $M \in \mathcal{A}$ an object, $L \in \mathcal{A}$ a simple object. The multiplicity $[M : L] \in \mathbb{N} \cup \{\infty\}$ of L in M is the supremum over all (finite) filtrations F of M of the multiplicity of L as a subquotient of the filtration. In formulas

$$[M : L] = \sup_F \#\{i \mid F_i M / F_{i+1} M \cong L\}.$$

This multiplicity is additive, i.e. for every short exact sequence $M' \hookrightarrow M \twoheadrightarrow M''$ we have $[M : L] = [M' : L] + [M'' : L]$. In particular, for $\nabla \in \mathcal{O}_{\leq n}$ and $E \in \Lambda$ we always have $[\nabla : L(E)] = \dim_k \operatorname{Hom}_{\mathcal{O}}(P_{\leq n}(E), \nabla)$. On the other hand, for a module $P \in \mathcal{O}$ with a finite Δ-flag we know by Lemma 3.1 that the multipicity $[P : \Delta(F)]$ of $\Delta(F)$ as a subquotient is $\dim_k \operatorname{Hom}(P, \nabla(F))$. Putting things together we obtain the reciprocity formula

$$[P_{\leq n}(E) : \Delta(F)] = [\nabla(F) : L(E)]$$

for all $E, F \in \Lambda$ and $n \geq \max\{|E|, |F|\}$.

Since it is not a big deal from where we are, I want to discuss the decomposition of $\mathcal{O}$ into blocks, although it is not needed for the main result of this article. Certainly there is a partial order $\geq$ on Λ such that $[\Delta(F) : L(E)] \neq 0 \Rightarrow F \geq E$ and $[\nabla(F) : L(E)] \neq 0 \Rightarrow F \geq E$. From now on let $\geq$ be the smallest such partial order and $\sim$ the equivalence relation generated by it. For every equivalence class $\theta \in \Lambda/\sim$ we consider the category

$$\mathcal{O}_\theta = \{M \in \mathcal{O} \mid [M : L(E)] \neq 0 \Rightarrow E \in \theta\}.$$

Theorem 4.2. *The functor* $\prod_{\theta\in\Lambda/\sim}\mathcal{O}_\theta \to \mathcal{O}$, $(M_\theta)_\theta \mapsto \bigoplus_\theta M_\theta$ *is an equivalence of categories.*

Proof. For $\theta \in \Lambda/\sim$ and $M \in \mathcal{O}$ let $M_\theta \subset M$ denote the submodule generated by all images of morphisms $\varphi : P_{\leq n}(E) \to M$ with $E \in \Lambda$. We show

1. $M_\theta \in \mathcal{O}_\theta$;
2. If $f : M \to N$ is a morphism in $\mathcal{O}$, then $f(M_\theta) \subset N_\theta$;
3. $M = \bigoplus_\theta M_\theta$.

This then proves the theorem. We start with (1). By the definition of our equivalence relation and by the reciprocity formula we have $P_{\leq n}(E) \in \mathcal{O}_\theta$ for all $E \in \theta$. In a short exact sequence in $\mathcal{O}$ the middle term lies in $\mathcal{O}_\theta$ iff both ends do, by the additivity of multiplicities. Thus it will be sufficient to show that $\mathcal{O}_\theta$ is stable under arbitrary direct sums. But every simple subquotient of $\bigoplus_{i\in I} M_i$ is the quotient of a submodule generated by one element, thus occurs also in a finite direct sum. This establishes (1). Now (2) follows from the definitions and from (2) we see that the sum in (3) is direct. We leave it to the reader to show it is all of M. □

Remark 4.3. Let $E \in \Lambda$ be given. If there are only finitely many $F \in \Lambda$ such that $F \geq E$, then $L(E)$ admits by Theorem 3.2 (3) a projective cover $P(E)$ in $\mathcal{O}$.

5. Tilting modules in $\mathcal{O}$

Definition 5.1. By a Δ-flag in an object $M \in \mathcal{O}$ we mean a (possibly infinite) increasing filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots$$

such that $M = \bigcup M_\nu$ and $M_\nu/M_{\nu-1} \cong \Delta(F_\nu)$ for all $\nu \geq 1$, with suitable $F_\nu \in \Lambda$.

Theorem 5.2. *For every* $E \in \Lambda$ *there is a unique up to isomorphism indecomposable object* $T = T(E) \in \mathcal{O}$ *such that:*

1. $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(F), T) = 0$ *for all* $F \in \Lambda$*;*
2. T *admits a* Δ*-flag, starting with* $T_1 \cong \Delta(E)$.

Definition 5.3. This object $T(E)$ is called the tilting module with parameter E.

Remark 5.4. The theorem is a variation of results of [Rin91], who in turn develops results of [Don86] and [CI89] in a general context.

Proof. We start with

Lemma 5.5. 1. *For all* $F, G \in \Lambda$ *both spaces* $\mathrm{Hom}_{\mathcal{O}}(\Delta(F), \Delta(G))$ *and* $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(F), \Delta(G))$ *are of finite dimension.*
2. *For all* $G \in \Lambda$ *and* $i \in \mathbb{Z}$ *there are at most finitely many* $F \in \Lambda$ *such that* $|F| = i$ *and* $\mathrm{Hom}_{\mathcal{O}}(\Delta(F), \Delta(G))$ *or* $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(F), \Delta(G))$ *is not zero.*

Proof. By our assumptions all homogeneous components of $\Delta(G)$ are of finite dimension. This gives (1) and (2) for Hom. Furthermore let n be bigger than $|F|$ and $|G|$. We consider the short exact sequence $\ker \hookrightarrow P_{\leq n}(F) \twoheadrightarrow \Delta(F)$ and get a surjection

$$\mathrm{Hom}_{\mathcal{O}}(\ker, \Delta(G)) \twoheadrightarrow \mathrm{Ext}^1_{\mathcal{O}}(\Delta(F), \Delta(G)).$$

Since here ker admits a finite Δ-flag, we get (1) for Ext^1. Finally all Δ-subquotients of a Δ-flag of $P_{\leq n}(F)$ are of the form $\Delta(H)$, where H is a summand of the $\mathbb{Z}$-graded $\mathfrak{g}_0$-module $(\tau_{\leq n-|F|}U(\mathfrak{g}_{>0})) \otimes_k F$ (and $\mathfrak{g}_0$ acts by the adjoint action on the first factor). For $\mathrm{Hom}_{\mathcal{O}}(\ker, \Delta(G))$ to be different from zero, such an H also has to occur as a summand in the $\mathbb{Z}$-graded $\mathfrak{g}_0$-module $\Delta(G) = U(\mathfrak{g}_{<0}) \otimes_k G$, hence in $(\tau_{\geq |F|-|G|} U(\mathfrak{g}_{<0})) \otimes_k G$.

But the representation theory of reductive Lie algebras tells us, that for given finite dimensional representations U_1, U_2 and G there are up to isomorphism only finitely many simple finite dimensional representations F such that $U_1 \otimes_k F$ and $U_2 \otimes_k G$ admit a common composition factor. $\square$

We show next

Proposition 5.6. *Let $E \in \Lambda$, $m \leq |E|$. There is a unique up to isomorphism indecomposable object $T = T_{\geq m}(E)$ in $\mathcal{O}$ such that:*

1. $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(F), T) = 0$ *for all $F \in \Lambda$ with $|F| \geq m$;*
2. *There is an inclusion $\Delta(E) \hookrightarrow T$, whose cokernel admits a finite Δ-flag such that only subquotients $\Delta(F)$ with $|E| > |F| \geq m$ occur.*

Proof. We start by proving unicity. Let T' be a second object satisfying our conditions. We consider the diagram

$$\begin{array}{ccccc} \Delta(E) & \hookrightarrow & T & \twoheadrightarrow & \text{coker} \\ \| & & & & \\ \Delta(E) & \hookrightarrow & T' & \twoheadrightarrow & \text{coker}'. \end{array}$$

Since by (2) for T' and (1) for T the relevant Ext-group vanishes, we find $\alpha : T' \to T$ making the whole diagram commutative. Similarily we find $\beta : T \to T'$. But $\alpha \circ \beta$ isn't nilpotent, hence an isomorphism since T was indecomposable with $\dim_k(\mathrm{End}_{\mathcal{O}} T) < \infty$. The same holds for $\beta \circ \alpha$, and we deduce $T \cong T'$.

Next we show the existence of $T_{\geq m}(E)$ by induction on m from above. As a basis for our induction we may take $T_{\geq |E|}(E) = \Delta(E)$. Now let $T_{\geq m}(E)$ be constructed already. We then form a sequence $T^{(i)}$ of objects from $\mathcal{O}$ as follows. Start with $T^{(0)} = T_{\geq m}(E)$. If $T^{(i)}$ is already constructed and there is a nonsplit extension $T^{(i)} \hookrightarrow T^{(i+1)} \twoheadrightarrow \Delta(F)$ with $F \in \Lambda$, $|F| = m-1$, take it as $T^{(i+1)}$. Otherwise stop at $T^{(i)}$.

We show that such a sequence $T^{(i)}$ stops with a possible $T_{\geq m-1}(E)$. Indeed by Lemma 5.5 above the number

$$e(T) = \sum_{F \in \Lambda,\ |F| = m-1} \dim_k \mathrm{Ext}^1_{\mathcal{O}}(\Delta(F), T)$$

is finite for all $T \in \mathcal{O}$ with finite Δ-flag. Now if $|F| = |G|$ the Ext-group $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(F), \Delta(G))$ disappears, and this gives us $e(T^{(i+1)}) = e(T^{(i)}) - 1$. Thus our sequence stops with an object $T^{(j)}$ satisfying certainly conditions (1) and (2) for $T_{\geq m-1}(E)$ from the proposition. Instead of proving that $T^{(j)}$ is indeed indecomposable, it is easier to choose an indecomposable summand of $T^{(j)}$, whose homogeneous component of degree $|E|$ doesn't vanish, and this is then the possible $T_{\geq m-1}(E)$ looked for. $\square$

Proposition 5.7. *Let $E \in \Lambda$ and $|E| \geq n > m$. Then there exists an inclusion $T_{\geq n}(E) \hookrightarrow T_{\geq m}(E)$, and the cokernel of every such inclusion admits a Δ-flag with only subquotients $\Delta(F)$ for $n > |F| \geq m$.*

Proof. Consider in $T_{\geq m}(E)$ the submodule T' generated by all homogeneous elements of degree at least n and form a short exact sequence $T' \hookrightarrow T_{\geq m}(E) \twoheadrightarrow \mathrm{koker}$.

Then T' resp. koker admits a Δ-flag, in which there are only subquotients $\Delta(F)$ with $|F| \geq n$ resp. $n > |F| \geq m$. It follows easily that T' satisfies all conditions we put on $T_{\geq n}(E)$ except perhaps indecomposability. Thus $T_{\geq n}(E)$ is a direct summand of T', and we deduce $[T_{\geq n}(E) : \Delta(F)] \leq [T_{\geq m}(E) : \Delta(F)]$. But our inductive construction of $T_{\geq m}(E)$ shows the reverse inequality as well, for $|F| \geq n$. Thus we have equality and can deduce $T' \cong T_{\geq n}(E)$. The proposition follows. □

After these preparations we can at least construct a possible $T = T(E)$ as $T = \lim_{n \to -\infty} T_{\geq n}(E)$. Let us remark right away that every element of End$T$ is either nilpotent or an isomorphism, since T is indecomposable and all its homogeneous components are of finite dimension, so the usual arguments via the Fitting-decomposition work. To prove unicity we use

Lemma 5.8. *In $\mathcal{O}$ there are enough injectives.*

Proof. For $M \in \mathcal{O}$ the biggest $\mathbb{Z}$-graded $\mathfrak{g}$-submodule of $\mathcal{H}\mathrm{om}_{\mathfrak{g}_0}(U, M)$ which lies in $\mathcal{O}$ is an injective object of $\mathcal{O}$ containing M. □

Remark 5.9. By general arguments ([HS71], I.9.2) we can deduce that every simple object $L(E)$ in $\mathcal{O}$ admits an injective hull $I(E)$. One may show in addition that $I(E)$ admits a (possibly infinite) increasing ∇-flag, whose multiplicities are given by the reciprocity formula $[I(E) : \nabla(F)] = [\Delta(F) : L(E)]$. We will neither use nor prove this.

Lemma 5.10. *Let $J \in \mathcal{O}$ be such that* $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(F), J) = 0 \quad \forall F \in \Lambda$. *Then also* $\mathrm{Ext}^1_{\mathcal{O}}(M, J) = 0$ *for all $M \in \mathcal{O}$ with a Δ-flag.*

Proof. Choose a short exact sequence $J \hookrightarrow I \twoheadrightarrow K$ with I injective in $\mathcal{O}$. We get an exact sequence

$$\mathrm{Hom}(M, J) \hookrightarrow \mathrm{Hom}(M, I) \to \mathrm{Hom}(M, K) \twoheadrightarrow \mathrm{Ext}^1(M, J),$$

which starts with an inclusion and ends with a surjection. Let $0 = M_0 \subset M_1 \subset \dots$ be a Δ-flag of M. Then we may rewrite the first three terms of our sequence to

$$\varprojlim \mathrm{Hom}(M_i, J) \hookrightarrow \varprojlim \mathrm{Hom}(M_i, I) \to \varprojlim \mathrm{Hom}(M_i, K),$$

and by our assumption we have for every fixed i a short exact sequence. Using our assumption on J a second time, we see that in addition all maps of the projective system $\mathrm{Hom}(M_i, J)$ are surjections. Therefore by [AM69], 10.2 the projective limit of our short exact sequence is a short exact sequence too and thus $\mathrm{Ext}^1(M, J) = 0$. □

We now show unicity of $T = T(E)$. Let $T' \in \mathcal{O}$ be a second object satisfying all the conditions of the theorem. As before we consider the diagram

$$\begin{array}{ccccc} \Delta(E) & \hookrightarrow & T & \twoheadrightarrow & T/T'_1 \\ \| & & & & \\ \Delta(E) & \hookrightarrow & T' & \twoheadrightarrow & T'/T'_1 \end{array}$$

By the lemma there is a morphism $\alpha : T' \to T$, which makes the diagram commute. Analogously we find $\beta : T \to T'$. Then $\alpha \circ \beta$ is not nilpotent, thus is an automorphism of T. But since also T' was assumed indecomposable, we find that α is an isomorphism. □

Remark 5.11. For the multiplicity of $\Delta(F)$ as a subquotient in a Δ-flag of $M \in \mathcal{O}$ we get by the above lemma the formula $[M : \Delta(F)] = \dim_k \operatorname{Hom}_{\mathcal{O}}(M, \nabla(F))$. In particular, this multiplicity does not depend on the choice of a Δ-flag.

Finally we now obtain the looked-for character formula for tilting modules.

Theorem 5.12. *Let $\gamma : \mathfrak{g}_0 \to k$ be a semi-infinite character for $\mathfrak{g}$. Then for all $E, F \in \Lambda$ we have*

$$[T(E) : \Delta(F)] = [\nabla(k_{-\gamma} \otimes F^*) : L(k_{-\gamma} \otimes E^*)].$$

Proof. If $\mathfrak{h} \subset \mathfrak{g}_0$ is a maximal torus, our $\mathcal{M}_{\mathfrak{h}}$ of Remark 2.2 (2) is precisely the category $\mathcal{O}^{\Delta}$ of all objects of $\mathcal{O}$ with a finite Δ-flag. Our equivalence $\mathcal{O}^{\Delta} \xrightarrow{\sim} (\mathcal{O}^{\Delta})^{\mathrm{opp}}$ from Remark 2.4 (3) has to transform $P_{\leq -n}(k_{-\gamma} \otimes E^*)$ into $T_{\geq n}(E)$ for all $n \leq |E|$, by the very definition of $T_{\geq n}$. Thus for $n \leq |F|$ we get

$$\begin{aligned}[T(E) : \Delta(F)] &= [T_{\geq n}(E) : \Delta(F)] \\ &= [P_{\leq -n}(k_{-\gamma} \otimes E^*) : \Delta(k_{-\gamma} \otimes F^*)] \\ &= [\nabla(k_{-\gamma} \otimes F^*) : L(k_{-\gamma} \otimes E^*],\end{aligned}$$

the last equality by the reciprocity formula. □

6. Projective objects and tilting modules in categories without grading

For the applications which are the goal of this work, it is convenient to hide the $\mathbb{Z}$-grading on the modules. In this section, as in the two preceding ones, let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$-graded Lie-algebra over an algebraically closed field k of characteristic zero such that $\dim_k \mathfrak{g}_i < \infty$ for all i, that $\mathfrak{g}_0$ is reductive, and that $\mathfrak{g}$ is semisimple for $\mathrm{ad}\mathfrak{g}_0$. But we ask in addition that there is an element $\partial \in \mathfrak{g}_0$ such that $[\partial, X] = iX \quad \forall i \in \mathbb{Z}, X \in \mathfrak{g}_i$.

As before put $\mathfrak{b} = \mathfrak{g}_{\geq 0}$ and $B = U(\mathfrak{b})$. Let $\bar{\mathcal{O}}$ denote the category of all $\mathfrak{g}$-modules which are locally finite for $\mathfrak{b}$ and semisimple for $\mathfrak{g}_0$. I want to explain briefly how results for $\bar{\mathcal{O}}$ can be deduced from the analogous results for $\mathcal{O}$. First by assumption ∂ lies in the center of $\mathfrak{g}_0$, thus every $M \in \bar{\mathcal{O}}$ decomposes under ∂ into eigenspaces $M = \bigoplus_{a \in k} M^a$. If we consider for every $a \in k$ the category

$$\mathcal{O}_a = \{M \in \mathcal{O} \mid M^{a+i} = M_i \quad \forall i \in \mathbb{Z}\},$$

we certainly have $\mathcal{O} = \prod_{a \in k} \mathcal{O}_a$. On the other hand, we also have for $\bar{a} \in k/\mathbb{Z}$ the subcategory

$$\bar{\mathcal{O}}_{\bar{a}} = \{M \in \bar{\mathcal{O}} \mid M^b \neq 0 \Rightarrow b \in \bar{a}\}$$

and analogously $\bar{\mathcal{O}} = \prod_{\bar{a} \in k/\mathbb{Z}} \bar{\mathcal{O}}_{\bar{a}}$. But clearly forgetting the $\mathbb{Z}$-grading gives us equivalences $\mathcal{O}_a \xrightarrow{\sim} \bar{\mathcal{O}}_{\bar{a}}$, which we can use to transfer our results from $\mathcal{O}$ to $\bar{\mathcal{O}}$.

To formulate these results for $\bar{\mathcal{O}}$, we need a bit of notation. Let $\bar{\Lambda}$ denote the set of all isomorphism classes of finite dimensional irreducible representations of $\mathfrak{g}_0$. For $E \in \bar{\Lambda}$ we consider in $\bar{\mathcal{O}}$ the Verma module $\Delta(E) = U \otimes_B E$. It has a unique simple quotient $L(E)$. In addition we consider in $\bar{\mathcal{O}}$ for every $E \in \bar{\Lambda}$ the object $\nabla(E) = \operatorname{Hom}_{U(\mathfrak{g}_{\leq 0})}(U, E)^{\mathfrak{g}_0 - fin}$, that is the space of all $\mathfrak{g}_0$-finite vectors in said Hom-space, and $L(E)$ is the socle of $\nabla(E)$. On $\bar{\Lambda}$ let $\leq$ be the smallest partial order such that $[\nabla(E) : L(F)] \neq 0 \Rightarrow E \geq F$ and $[\Delta(E) : L(F)] \neq 0 \Rightarrow E \geq F$. Let $\sim$ denote the equivalence relation on $\bar{\Lambda}$ generated by this partial order. For $\bar{\theta} \in \bar{\Lambda}/\sim$ we put $\bar{\mathcal{O}}_{\bar{\theta}} = \{M \in \bar{\mathcal{O}} \mid [M : L(E)] \neq 0 \Rightarrow E \in \bar{\theta}\}$.

Our theorems of the preceding sections translate into the following.

Theorem 6.1. *The functor* $(M_{\bar\theta})_{\bar\theta} \mapsto \bigoplus_{\bar\theta} M_{\bar\theta}$ *gives an equivalence of categories*

$$\prod_{\bar\theta \in \bar\Lambda/\sim} \bar{\mathcal{O}}_{\bar\theta} \xrightarrow{\sim} \bar{\mathcal{O}}.$$

Remark 6.2. This generalizes results of [DGK82] and [RCW82].

Definition 6.3. A tilting module with parameter $E \in \bar\Lambda$ is an indecomposable object $T = T(E) \in \bar{\mathcal{O}}$ such that

1. T admits a Δ-flag starting with $T_1 \cong \Delta(E)$.
2. $\mathrm{Ext}^1_{\bar{\mathcal{O}}}(\Delta(F), T) = 0 \quad \forall F \in \bar\Lambda$.

Theorem 6.4. *For every* $E \in \bar\Lambda$ *there exists in* $\bar{\mathcal{O}}$ *a tilting module* $T(E)$ *with parameter* E*. It is unique up to isomorphism.*

Remarks 6.5. 1. [Pol91] Remark 5.9 shows that there are enough injectives in $\bar{\mathcal{O}}$, that an injective hull $I(E)$ of $L(E)$ admits a ∇-flag, and that the multiplicities in such a ∇-flag are given by the reciprocity formula

$$[I(E) : \nabla(F)] = [\Delta(F) : L(E)] \quad \forall E, F \in \bar\Lambda.$$

2. In addition Remark 4.3 shows that $L(E)$ admits a projective cover in $\bar{\mathcal{O}}$ if there are but finitely many $F \in \bar\Lambda$ such that $F \geq E$. For this projective cover we then have analogously $[P(E) : \Delta(F)] = [\nabla(F) : L(E)] \quad \forall E, F \in \bar\Lambda$.

From now on let $\gamma : \mathfrak{g}_0 \to k$ be a semi-infinite character of $\mathfrak{g}$, so in particular $\mathfrak{g}$ is generated by $\mathfrak{g}_1, \mathfrak{g}_0$ and $\mathfrak{g}_{-1}$. Let $\bar{\mathcal{O}}^\Delta$ denote the category of all objects in $\bar{\mathcal{O}}$ with a (finite) Δ-flag.

Theorem 6.6. *There is an equivalence of categories* $\bar{\mathcal{O}}^\Delta \to (\bar{\mathcal{O}}^\Delta)^{\mathrm{opp}}$ *such that short exact sequences correspond to short exact sequences, and such that* $\Delta(E)$ *gets transformed into* $\Delta(k_{-\gamma} \otimes E^*)$*, for all* $E \in \bar\Lambda$*.*

If $L(E)$ in $\bar{\mathcal{O}}$ admits a projective cover $P(E)$, then $P(E)$ gets transformed into $T(k_{-\gamma} \otimes E^*)$ under such an equivalence. But in any case we have

Theorem 6.7. *The character of our tilting modules is given by*

$$[T(E) : \Delta(F)] = [\nabla(k_{-\gamma} \otimes F^*) : L(k_{-\gamma} \otimes E^*)] \quad \forall E, F \in \bar\Lambda.$$

7. The case of Kac-Moody algebras

Let $\mathfrak{g}$ be a Kac-Moody algebra, $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra, $\Pi \subset \mathfrak{h}^*$ the simple roots. For $\alpha \in \Pi$ the weight spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ in $\mathfrak{g}$ generate a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ and we let $\alpha^\vee \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ be the vector characterized by $\langle \alpha, \alpha^\vee \rangle = 2$.

Let us consider first on $\mathfrak{g}$ the $\mathbb{Z}$-grading with $\mathfrak{g}_0 = \mathfrak{h}$ and $\mathfrak{g}_1 = \bigoplus_{\alpha \in \Pi} \mathfrak{g}_\alpha$. Since the simple roots are linearly independent, there exists $\partial \in \mathfrak{h}$ such that $\langle \alpha, \partial \rangle = 1 \quad \forall \alpha \in \Pi$, thus $[\partial, X] = iX \quad \forall X \in \mathfrak{g}_i$, $i \in \mathbb{Z}$.

Lemma 7.1 ([Ark96])**.** *Let* $\rho \in \mathfrak{h}^*$ *be such that* $\langle \rho, \alpha^\vee \rangle = 1$ $\forall \alpha \in \Pi$*. Then* 2ρ *is a semi-infinite character for* $\mathfrak{g}$*.*

Proof. We have to show $2\rho([X, Y]) = \mathrm{tr}(\mathrm{ad} X \mathrm{ad} Y : \mathfrak{h} \to \mathfrak{h})$ for all $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\beta}$ and $\alpha, \beta \in \Pi$. If $\alpha \neq \beta$ both sides vanish, if $\alpha = \beta$ the equality follows from the definitions. ☐

In our case certainly $\bar{\Lambda} = \mathfrak{h}^*$ and $[\Delta(\lambda) : L(\mu)] = [\nabla(\lambda) : L(\mu)] \quad \forall \lambda, \mu \in \mathfrak{h}^*$. For $\alpha \in \Pi$ let $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ denote the involution $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$. The subgroup $\mathcal{W} \subset \mathrm{Aut}\mathfrak{h}^*$ generated by the s_α is called the Weyl group. If we put $\mathcal{S} = \{s_\alpha \mid \alpha \in \Pi\}$, then $(\mathcal{W}, \mathcal{S})$ is a Coxeter system. We define the dot-actions of $\mathcal{W}$ on $\mathfrak{h}^*$ by the formula $x \cdot \lambda = x(\lambda + \rho) - \rho$. This action does not depend on the choice of ρ. In [Kas90] Kashiwara establishes a conjecture of Deodhar, Gabber and Kac [DGK82] to the effect, that the Kazhdan-Lusztig-polynomials $P_{x,y}$ for $(\mathcal{W}, \mathcal{S})$ give Jordan-Hölder multiplicities for $\mathfrak{g}$.

Theorem 7.2 ([Kas90])**.** *Let $\mathfrak{g}$ be symmetrizable. Let $\lambda \in \mathfrak{h}^*$ be such that $\langle \lambda + \rho, \alpha^\vee \rangle \in \{1, 2, \dots\} \quad \forall \alpha \in \Pi$. Then $[\Delta(x \cdot \lambda) : L(\nu)] = 0$ for $\nu \notin \mathcal{W} \cdot \lambda$ and*

$$[\Delta(x \cdot \lambda) : L(y \cdot \lambda)] = P_{x,y}(1) \quad \forall x, y \in \mathcal{W}.$$

Now theorem 6.7 gives easily the generalization of the main results of [CI89] to symmetrizable Kac-Moody algebras.

Corollary 7.3. *Suppose $\mathfrak{g}$ is symmetrizable. Let $\mu \in \mathfrak{h}^*$ be such that $\langle \mu + \rho, \alpha^\vee \rangle \in \{-1, -2, \dots\} \quad \forall \alpha \in \Pi$. Then $[T(y \cdot \mu) : \Delta(\nu)] = 0$ for $\nu \notin \mathcal{W} \cdot \mu$ and*

$$[T(y \cdot \mu) : \Delta(x \cdot \mu)] = P_{x,y}(1) \quad \forall x, y \in \mathcal{W}.$$

Let us consider also the parabolic case. Let $\Pi_f \subset \Pi$ be a set of simple roots such that the $\mathfrak{g}_\alpha$ with $\pm\alpha \in \Pi_f$ generate a finite dimensional (necessarily semisimple) subalgebra of $\mathfrak{g}$. Then $\mathfrak{g}$ also admits a $\mathbb{Z}$-grading such that $\mathfrak{h} \subset \mathfrak{g}_0$, $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$ if $\pm\alpha \in \Pi_f$, and $\mathfrak{g}_\alpha \subset \mathfrak{g}_1$ if $\alpha \in \Pi - \Pi_f$, and for this $\mathbb{Z}$-grading our conditions from the beginning of section 6 are satisfied as well. Let $\rho_f \in \mathfrak{h}^*$ denote the half sum of positive roots of $\mathfrak{g}_0$.

Lemma 7.4. *There is a character $\gamma = \gamma_f : \mathfrak{g}_0 \to \mathbb{C}$ which coincides on $\mathfrak{h}$ with $2\rho - 2\rho_f$. Every such character γ is a semi-infinite character for $\mathfrak{g}$.*

Proof. Certainly $\mathfrak{g}$ is generated by $\mathfrak{g}_1$, $\mathfrak{g}_0$ and $\mathfrak{g}_{-1}$, for our new grading as well. We have $\langle \rho, \beta^\vee \rangle = \langle \rho_f, \beta^\vee \rangle = 1 \quad \forall \beta \in \Pi_f$, thus $2\rho - 2\rho_f$ disappears on $[\mathfrak{g}_0, \mathfrak{g}_0] \cap \mathfrak{h}$ and can indeed be extended to a character γ of $\mathfrak{g}_0$. The only thing left to show is the formula

$$\gamma([X, Y]) = \mathrm{tr}(\mathrm{ad}X \mathrm{ad}Y : \mathfrak{g}_0 \to \mathfrak{g}_0) \quad \forall X \in \mathfrak{g}_1, Y \in \mathfrak{g}_{-1}.$$

If this holds for fixed $X \in \mathfrak{g}_1$ with arbitrary $Y \in \mathfrak{g}_{-1}$, then it also holds for $[A, X]$ with arbitrary $Y \in \mathfrak{g}_{-1}$ and $A \in \mathfrak{g}_0$. We leave this verification to the reader and then only have to check that

$$\gamma([X_\alpha, X_\beta]) = \mathrm{tr}(\mathrm{ad}X_\alpha \mathrm{ad}X_\beta : \mathfrak{g}_0 \to \mathfrak{g}_0)$$

for $\alpha \in \Pi - \Pi_f$, β a root of $\mathfrak{g}$ with $\mathfrak{g}_\beta \subset \mathfrak{g}_{-1}$, and $X_\beta \in \mathfrak{g}_\beta$. If $\beta \neq -\alpha$ both sides of our equation vanish, and we are left with the case $\beta = -\alpha$. We then put $X_\beta = Y_\alpha$ and decompose $\mathfrak{g}_0 = \mathfrak{n}_0^+ \oplus \mathfrak{h} \oplus \mathfrak{n}_0^-$ under the adjoint action of $\mathfrak{h}$. All three summands are stable under $\mathrm{ad}X_\alpha \mathrm{ad}Y_\alpha$. Furthermore $[Y_\alpha, \mathfrak{n}_0^+] = 0 = [X_\alpha, \mathfrak{n}_0^-]$, since the weight of these brackets doesn't belong to the root system of $\mathfrak{g}$. Without restriction suppose $[X_\alpha, Y_\alpha] = \alpha^\vee$. We get

$$\begin{aligned} \mathrm{tr}(\mathrm{ad}X_\alpha \mathrm{ad}Y_\alpha : \mathfrak{g}_0 \to \mathfrak{g}_0) &= \mathrm{tr}(\mathrm{ad}X_\alpha \mathrm{ad}Y_\alpha : \mathfrak{h} \to \mathfrak{h}) \\ &\quad + \mathrm{tr}(\mathrm{ad}(\alpha^\vee) : \mathfrak{n}_0^- \to \mathfrak{n}_0^-) \\ &= 2 - 2\rho_f(\alpha^\vee) \\ &= \langle 2\rho - 2\rho_f, \alpha^\vee \rangle. \end{aligned}$$

□

In the parabolic case we can identify $\bar{\Lambda}$ with the set

$$\mathfrak{h}_f^* = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{N} \quad \forall \alpha \in \Pi_f\},$$

by associating to $E \in \bar{\Lambda}$ its highest weight $\lambda(E) \in \mathfrak{h}^*$. For $\lambda = \lambda(E) \in \mathfrak{h}_f^*$ we also write $\Delta^f(\lambda)$ instead of $\Delta(E)$ and call this object the parabolic Verma module of highest weight λ. Analogously we define $\nabla^f(\lambda)$, $T^f(\lambda)$ and $L^f(\lambda)$. However, we will never use $L^f(\lambda)$ since clearly $L^f(\lambda) = L(\lambda)$.

Certainly we have $[\nabla^f(\lambda) : L(\mu)] = [\Delta^f(\lambda) : L(\mu)] \quad \forall \lambda, \mu \in \mathfrak{h}_f^*$. To calculate these multiplicities we take $\lambda = \lambda(E) \in \mathfrak{h}_f^*$ and consider for the $\mathfrak{g}_0$-module E following [BGG75] the resolution

$$0 \to M_r \to \ldots \to M_1 \to M_0 \to E \to 0.$$

So here $M_i = \bigoplus_{l(z)=i} \Delta_f(z \cdot \lambda)$ where $\Delta_f(\mu)$ denotes the Verma for $\mathfrak{g}_0$ with highest weight μ and the sum runs over all elements z of length i of the Weyl group $\mathcal{W}_f = \langle s_\alpha \mid \alpha \in \Pi_f \rangle$ of $\mathfrak{g}_0$. (Remark that $z(\lambda + \rho_f) - \rho_f = z(\lambda + \rho) - \rho \quad \forall z \in \mathcal{W}_f$, since $\langle \rho, \alpha^\vee \rangle = 1 = \langle \rho_f, \alpha^\vee \rangle \quad \forall \alpha \in \Pi_f$.)

Applying the functor $U \otimes_B$ to this exact sequence, we get an exact sequence

$$0 \to U \otimes_B M_r \to \ldots \to U \otimes_B M_0 \to \Delta^f(\lambda) \to 0$$

with $U \otimes_B M_i = \bigoplus_{l(z)=i} \Delta(z \cdot \lambda)$. This implies

$$[\Delta^f(\lambda) : L(\nu)] = \sum_{z \in \mathcal{W}_f} (-1)^{l(z)} [\Delta(z \cdot \lambda) : L(\nu)]$$

for all $\lambda \in \mathfrak{h}_f^*$, $\nu \in \mathfrak{h}^*$. (In this argument the BGG-resolution can be replaced by the Weyl character formula, if one takes the time to introduce suitable Grothendieck groups.) Now let $\mathcal{W}^f \subset \mathcal{W}$ denote the set of shortest representatives for the right cosets of $\mathcal{W}_f$. Thus multiplication gives a bijection $\mathcal{W}_f \times \mathcal{W}^f \xrightarrow{\sim} \mathcal{W}$. If $\mathbb{N}\Pi$ denotes the set of all linear combinations of simple roots with integral nonnegative coefficients, we can describe $\mathcal{W}^f$ alternatively as

$$\mathcal{W}^f = \{x \in \mathcal{W} \mid x^{-1}\Pi_f \subset \mathbb{N}\Pi\}.$$

In particular, for $\lambda \in \mathfrak{h}^*$ with $\langle \lambda, \alpha^\vee \rangle \in \mathbb{N} \quad \forall \alpha \in \Pi$ we have $x \cdot \lambda \in \mathfrak{h}_f^* \quad \forall x \in \mathcal{W}^f$.

For $x, y \in \mathcal{W}^f$ let $n_{x,y} \in \mathbb{Z}[v]$ denote one of the two corresponding parabolic Kazhdan-Lusztig polynomials in [Deo87], in the normalization and notation of [Soe97].

Proposition 7.5. *Let $\mathfrak{g}$ be symmetrizable. Let $\lambda \in \mathfrak{h}^*$ be such that $\langle \lambda, \alpha^\vee \rangle \in \mathbb{N} \quad \forall \alpha \in \Pi$. Then $[\Delta^f(x \cdot \lambda) : L(\nu)] = 0$ for $\nu \notin \mathcal{W}^f \cdot \lambda$ and*

$$[\Delta^f(x \cdot \lambda) : L(y \cdot \lambda)] = n_{x,y}(1) \quad \forall x, y \in \mathcal{W}^f.$$

Proof. By Deodhar [Deo87, Soe97] our parabolic polynomials satisfy $n_{x,y}(1) = \sum_{z \in \mathcal{W}_f} (-1)^{l(z)} P_{zx,y}(1)$. □

This gives a character formula for certain parabolic tilting modules $T^f(\lambda)$. More precisely, let $w_f \in \mathcal{W}_f$ be the longest element. We define a new action of $\mathcal{W}$ on $\mathfrak{h}^*$ by $x \centerdot \lambda = (w_f x w_f) \cdot \lambda$.

Corollary 7.6. *Let $\mathfrak{g}$ be symmetrizable. Suppose $\mu \in \mathfrak{h}^*$ is such that $\langle -w_f\mu - 2\rho + 2\rho_f, \alpha^\vee \rangle \in \mathbb{N} \quad \forall \alpha \in \Pi$. Then $[T^f(y \centerdot \mu) : \Delta^f(\nu)] = 0$ for $\nu \notin \mathcal{W}^f \centerdot \mu$ and*

$$[T^f(y \centerdot \mu) : \Delta^f(x \centerdot \mu)] = n_{x,y}(1) \quad \forall x, y \in \mathcal{W}^f.$$

Proof. We only have to calculate. First remark $\rho - \rho_f = w_f(\rho - \rho_f) = w_f\rho + \rho_f$, thus $w_f\rho = \rho - 2\rho_f$. Furthermore $-w_f\lambda(E)$ is the highest weight of E^*. Let $\gamma = 2\rho - 2\rho_f$ be as above. If $x \centerdot \mu$ is the highest weight of E, then the highest weight of $k_{-\gamma} \otimes E^*$ is precisely $x \cdot \lambda$ for $\lambda = -w_f\mu - 2\rho + 2\rho_f$. We leave the verification to the reader. the corollary follows, in particular we get $y \centerdot \mu \in \mathfrak{h}_f^* \quad \forall y \in \mathcal{W}^f$. □

8. Translation into results for quantum groups

For the application to quantum groups at roots of unity the important case is to take $\mathfrak{g}$ the affinization of a simple complex Lie algebra $\grave{\mathfrak{g}}$ and Π_f the set of all simple roots except the "affine root". To introduce the necessary notation I will start with some recollections concerning the affinization of a simple complex Lie algebra $\grave{\mathfrak{g}}$.

One starts with the loop algebra $\mathcal{L}\grave{\mathfrak{g}} = \grave{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$. Its elements will be understood as maps of the circle S^1 to $\grave{\mathfrak{g}}$ and written

$$X = X(t) = \sum X_i t^i.$$

Let $\grave{\kappa} : \grave{\mathfrak{g}} \times \grave{\mathfrak{g}} \to \mathbb{C}$ be the Killing form. We consider the central extension $\tilde{\mathfrak{g}} = \mathcal{L}\grave{\mathfrak{g}} \oplus \mathbb{C}z$ of $\mathcal{L}\grave{\mathfrak{g}}$ with bracket

$$[X + aK, Y + bK] = [X, Y] - \operatorname{res}(\grave{\kappa}(X, dY))z$$

for all $X, Y \in \mathcal{L}\grave{\mathfrak{g}}$ and $a, b \in \mathbb{C}$, where we understand $d(\sum Y_i t^i) = \sum i Y_i t^{i-1} dt$ and $\operatorname{res}(\sum a_i t^i dt) = a_{-1}$. In fact we should not prefer the Killing form in this construction, but rather interpret $\mathbb{C}z$ as the dual of the one-dimensional space of all $\grave{\mathfrak{g}}$-invariant bilinear forms on $\grave{\mathfrak{g}}$.

Finally we extend the derivation $\partial = t\frac{\partial}{\partial t}$ of $\mathcal{L}\grave{\mathfrak{g}}$ by the rule $\partial(z) = 0$ to a derivation of $\tilde{\mathfrak{g}}$, and adjoining this derivation to $\tilde{\mathfrak{g}}$ we get the so-called affinization $\mathfrak{g} = \tilde{\mathfrak{g}} \oplus \mathbb{C}\partial$ of $\grave{\mathfrak{g}}$ with bracket

$$[X + a\partial, Y + b\partial] = [X, Y] + a\partial(Y) - b\partial(X)$$

for all $X, Y \in \tilde{\mathfrak{g}}$ and $a, b \in \mathbb{C}$.

The center $\mathfrak{z}$ of $\mathfrak{g}$ is the line $\mathbb{C}z$. We equip $\mathfrak{g}$ with the $\mathbb{Z}$-grading $\mathfrak{g}_0 = \grave{\mathfrak{g}} \oplus \mathbb{C}z \oplus \mathbb{C}\partial$, $\mathfrak{g}_i = \mathfrak{g} \otimes t^i$ if $i \neq 0$. Thus $\mathfrak{g}_i$ is the eigenspace of $\operatorname{ad}\partial$ with eigenvalue i, and the semi-infinite characters $\gamma : \mathfrak{g}_0 \to \mathbb{C}$ are precisely the linear maps $\gamma : \mathfrak{g}_0 \to \mathbb{C}$ such that $\gamma(\grave{\mathfrak{g}}) = 0$ and $\gamma(z) = 1$. The Lie algebra $\mathfrak{g}$ is an affine Kac-Moody algebra, and the grading corresponds to a choice $\Pi_f \subset \Pi$ such that $\Pi - \Pi_f$ consists of a single element α_0.

In their articles [KL93, KL94] Kazhdan and Lusztig consider for $c \in \mathbb{C}$ a certain subcategory given by suitable finitness conditions in the category $\tilde{\mathcal{O}}(z = c)$ of all $\tilde{\mathfrak{g}}$-modules, that are killed by $(z - c)$ and on which $\tilde{\mathfrak{g}}_{\geq 0}$ acts locally finitely and $\tilde{\mathfrak{g}}_{>0}$ locally nilpotently. We want to translate our results for $\bar{\mathcal{O}}$ into results for $\tilde{\mathcal{O}}(z = c)$ and need results of [Kac90].

For every $M \in \bar{\mathcal{O}}$ the Casimir Ω gives a locally finite endomorphism $\Omega = \Omega_M \in \operatorname{End}_{\mathfrak{g}} M$, and for every morphism $g : M \to N$ we have $g \circ \Omega_M = \Omega_N \circ g$. For $a \in \mathbb{C}$ let $\bar{\mathcal{O}}(\Omega \simeq a)$ denote the subcategory of all objects of $\bar{\mathcal{O}}$, on which $(\Omega - a)$ acts locally nilpotently.

Proposition 8.1 ([Pol91])**.** *Let $c \in \mathbb{C}$, $c \neq -1/2$. Then forgetting the action of ∂ gives an equivalence of categories*

$$\bar{\mathcal{O}}(\Omega \simeq 0, z = c) \xrightarrow{\sim} \tilde{\mathcal{O}}(z = c).$$

Remarks 8.2. 1. The case $z = -1/2$ is precisely the "critical level". Instead of z Kac uses the "canonical" central element $K = 2hz$ with $h = h^\vee$ the Coxeter number.

2. Under the assumptions of the proposition forgetting the action of ∂ gives more generally equivalences

$$\bar{\mathcal{O}}(\Omega \simeq a, z = c) \xrightarrow{\sim} \tilde{\mathcal{O}}(z = c)$$

for all $a \in \mathbb{C}$. Indeed the eigenvalue of Ω can be adjusted to every given number if we change the action of ∂ by an additive constant; see [Kac90], 12.8.3.

Proof. Given a representation M of $\tilde{\mathfrak{g}}$ such that every $m \in M$ is killed by $\mathring{\mathfrak{g}} \otimes t^i$ for $i \gg 0$, one defines as in [Kac90], 12.8.4 the zeroth Sugawara operator $T_0 : M \to M$. If z acts by a scalar $c \neq -1/2$ on M, the formula $\partial = -(2h(2c+1))^{-1}T_0$ then determines a (functorial) extension of the action of $\tilde{\mathfrak{g}}$ on M to an action of $\mathfrak{g}$; see [Kac90], Corollary 12.8 (a). In case M was already a representation of $\mathfrak{g}$, we have by [Kac90], 12.8.5 the relation

$$T_0 = -2h(2c+1)\partial + \Omega.$$

Now we define an inverse to the functor of the Proposition. Certainly T_0 acts locally finitely on $M \in \tilde{\mathcal{O}}(z = c)$. So if we extend our $\tilde{\mathfrak{g}}$-action on M by $\partial = -(2h(2c+1))^{-1}T_0$ to a $\mathfrak{g}$-action, ∂ acts locally finitely and Ω acts as zero. If we now define a new action of ∂ on M as the semisimplification of the first action, we obtain a new action of $\mathfrak{g}$ on M, and one sees easily that this gives a functor

$$\tilde{\mathcal{O}}(z = c) \to \bar{\mathcal{O}}(\Omega \simeq 0, z = c).$$

On the other hand, the action of ∂ on an object of $\bar{\mathcal{O}}(\Omega \simeq 0, z = c)$ has to be the semisimplification of the action of $-(2h(2c+1))^{-1}T_0$, by the relations between T_0, ∂ and Ω recalled above. Thus we really found an inverse to the functor of the proposition. $\square$

Suppose from now on that $\mathring{\mathfrak{g}}$ has type ADE. To simplify we work with the canonical central element $K = 2hz$, thus the critical level is $K = -h$. In this case Kazhdan and Lusztig [KL93, KL94] show for many $l \in \mathbb{C} - \mathbb{Q}_{\leq 0}$ an equivalence of categories

$$\tilde{\mathcal{O}}^e(K = -h - l) \cong U_\zeta\text{-mod}^{e,1}.$$

Here $\tilde{\mathcal{O}}^e$ denotes the category of all finite length objects in $\tilde{\mathcal{O}}$, we put $\zeta = \exp(-\pi i/l)$, and U_ζ-mod^{e,1} means the category of all finite dimensional representations of type 1 of the quantum group with divided powers U_ζ. In particular, for $l \in \mathbb{Z}_{\geq h}$ the block $\mathcal{B} \subset U_\zeta$-mod^{e,1} of the trivial representation of U_ζ is equivalent to the block of $\bar{\mathcal{O}}$ containing $L(\mu)$ with $\mu = -((h+l)/2h)\gamma$. (This holds for every choice of a semi-infinite character $\gamma = 2\rho - 2\rho_f$.)

To prove Conjecture 7.2 from [Soe97] concerning the character of tilting modules for quantum groups at roots of unity $l > 33$, we thus have to check in $\bar{\mathcal{O}}$ the formula

$$[T^f(y \centerdot \mu) : \Delta^f(x \centerdot \mu)] = n_{x,y}(1)$$

for all $x, y \in \mathcal{W}^f$. By Corollary 7.6 it is sufficient to check $\langle -w_f\mu - 2\rho + 2\rho_f, \alpha^\vee \rangle \in \mathbb{N}$ for all $\alpha \in \Pi$.

Now a possible definition of the Coxeter number is $\langle \rho_f, \alpha_0^\vee \rangle = -h + 1$, we thus get $\langle \gamma, \alpha_0^\vee \rangle = 2h$. Furthermore we have $\langle \gamma, \alpha^\vee \rangle = 0 \quad \forall \alpha \in \Pi_f$. Since $w_f\gamma = \gamma$ we get $-w_f\mu - 2\rho + 2\rho_f = ((h+l)/2h)\gamma - \gamma = ((l-h)/2h)\gamma$, thus this weight vanishes on all coroots $\alpha^\vee$ for $\alpha \in \Pi_f$ and takes on $\alpha_0^\vee$ the value $l - h \in \mathbb{N}$.

If $\grave{\mathfrak{g}}$ is not of type ADE, the argument is more or less the same. The only problem is that the Kazhdan-Lusztig-conjectures for affine Lie algebras in positive level are only known for integral weights. Here there are still gaps in the literature to be filled.

How one can get characters for tilting modules "on the walls" from the character formulas in the principal block is explained in [Soe97].

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