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ADMISSIBLE NILPOTENT COADJOINT ORBITS OF p-ADIC REDUCTIVE LIE GROUPS

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ABSTRACT. The orbit method conjectures a close relationship between the set of irreducible unitary representations of a Lie group G, and *admissible* coadjoint orbits in the dual of the Lie algebra. We define admissibility for nilpotent coadjoint orbits of p-adic reductive Lie groups, and compute the set of admissible orbits for a range of examples. We find that for unitary, symplectic, orthogonal, general linear and special linear groups over p-adic fields, the admissible nilpotent orbits coincide with the so-called *special* orbits defined by Lusztig and Spaltenstein in connection with the Springer correspondence.

1. INTRODUCTION

A fundamental problem in the theory of Lie groups over local fields is the classification and construction of the irreducible unitary representations of a Lie group G. One promising approach to this problem is known as the orbit method.

The orbit method seeks to associate irreducible unitary representations of G to coadjoint orbits in the dual of the Lie algebra of G. The former are algebraic objects; the latter have rich geometric structure. The goal is then to use this geometry to construct the associated representation(s), a process referred to as geometric quantization.

Its first true success was achieved for nilpotent groups by Kirillov [K] over the reals, and later by Moore [Mo] over the p-adics. They proved that for a simply connected, connected nilpotent group G, the orbit method yields a bijective correspondence from the set of coadjoint orbits to the unitary dual. When G is no longer assumed to be simply connected, the correspondence exists and is surjective; however, its domain consists only of those orbits which are suitably "integral" (*cf.* Section 4).

It is known in general that only a subset of the coadjoint orbits will arise in an orbit correspondence. Defining this subset has been, and continues to be, the subject of much research. As the orbit method came to be applied to other classes of groups — in particular, to the abelian, compact, and type I solvable Lie groups — it became clear that the answer must be closely tied to integrality. Yet integrality alone is not a sufficient criterion for the crucial case of reductive groups: all nilpotent orbits are integral, but not all nilpotent orbits have associated representations. One

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example of this phenomenon is the minimal nilpotent orbit of the real symplectic group $Sp(2n,\mathbb{R})$ for $n>1$. The only irreducible representations associated to this orbit are ones of the two-fold covering group of the symplectic group (the *metaplectic group*); there are no representations of $Sp(2n,\mathbb{R})$ itself associated to this orbit [V1, Ch. 10].

In 1980 Duflo [D] proposed for real Lie groups a refinement of integrality — one which gives, in the reductive case, a more restrictive condition on nilpotent orbits. His criterion is called *admissibility*, and he conjectures it should pick out exactly those orbits that do arise in the orbit correspondence. The orbit of G through $f \in \mathfrak{g}^*$ is admissible if there is a unitary representation τ of the metaplectic double cover of G^f (the stabilizer subgroup) whose differential coincides with if. This criterion determines τ on the topological (not algebraic) connected component of the identity.

A suitable interpretation of Duflo's criterion in the p-adic case involves considering the character τ over an open subgroup G^f_0 containing the unipotent elements of G^f. We define this subgroup in Section 2, and discuss admissibility in Section 4.

There is an abundance of evidence supporting the validity of this criterion for real Lie groups. For one, the orbit method correspondence — as proven for nilpotent, compact or solvable Lie groups — can be re-interpreted using admissibility in place of integrality, though the new orbit correspondence may no longer be strictly bijective [V2]. For real reductive Lie groups, the orbit method should be implemented by an inductive construction, following the Jordan decomposition of $f \in \mathfrak{g}^* \cong \mathfrak{g}$; the "building block" representations are to be those representations (of smaller reductive groups) associated to the finitely many admissible nilpotent orbits. The case of semisimple orbits is understood: those giving rise to representations (via parabolic and cohomological induction) are exactly the admissible ones [V1]. There are also partial results along these lines for the nilpotent orbits. It is known that if a representation is associated to a "nice" nilpotent orbit, then that orbit must be admissible [V3, Thm 8.7]. Conversely, in "good" cases, the admissibility datum determines the K-types of the associated representation [V5, Prop 7.9, Thm 8.13]. In general, however, the question of quantizing nilpotent orbits remains open.

In the present work, we wish to understand Duflo's admissibility criterion in the setting of p-adic Lie groups. More precisely, we prove the following theorem (Theorem 5.8 and Theorem 6.2). Recall the notion of special orbits, defined by Lusztig and Spaltenstein in connection with the Springer correspondence [Lu1, Sp].

Theorem. *Let F be a p-adic field. Let G be a reductive Lie group over F. Then*

1. *if $G = SL(n,F)$ or $G = GL(n,F)$, then all orbits are admissible; in particular, the admissible orbits coincide with the special orbits;*
2. *if $G = Sp(2n,F)$, $G = O(V)$ (an orthogonal group) or $G = SO(V)$ (a special orthogonal group), then the admissible orbits coincide with the special orbits;*
3. *if $G = U(V)$ (a unitary group) or $G = SU(V)$ (a special unitary group), then the admissible orbits coincide with the special orbits.*

This result ties in well with Lusztig's work with representations of split reductive Lie groups over finite fields [Lu2], where the "building block" *unipotent* representations are associated to two-sided cells of the Weyl group — or, via the Springer correspondence, with special nilpotent orbits of the group. The Theorem also supports the notion of unipotent representations arising in Arthur's conjectures vis-a-vis the Langlands' program [Ar]: these unipotent representations are associated to nilpotent orbits of the dual group, or, via duality [Sp, III, 10.3, 10.6], to stable special nilpotent orbits of the group.

Finally, it is conjectured that in the p-adic case, the admissibility datum (the orbit together with the representation τ) should determine the leading coefficient in the Harish-Chandra-Howe character expansion of a representation attached to that datum. Mœglin and Waldspurger [M] have shown, for classical p-adic groups, that the leading coefficient(s) must correspond to special nilpotent orbits.

Torasso [T] has recently constructed the minimal representations (those associated to admissible minimal nilpotent orbits) for all real and p-adic reductive Lie groups of rank greater than 2. In particular, for the p-adic case, he has proven this conjecture for the minimal nilpotent orbit. Minimal representations have previously been constructed in many various cases by Flicker, Kazhdan and Savin [KS], [Ka], [FKS], [S].

The problem of determining the admissible nilpotent orbits in the real case was addressed by Schwarz [Sch] in 1988. He proved that the special orbits coincide with the admissible orbits for all real classical groups except the (nonsplit) unitary groups. In this last case, all orbits are special, but not all orbits are admissible. His results on admissibility can be deduced from the techniques used here, though his own methods were quite different.

In her thesis, the author also determined the admissible nilpotent orbits of the split p-adic group H of type G_2. In that case, all special orbits are admissible, the 8-dimensional orbit is neither admissible nor special, but the minimal orbit, which is not special, *is* admissible. Savin has constructed, for certain p-adic fields, a representation of a three-fold central extension of H associated to this orbit. Torasso's construction, which is quite different, provides *a priori* only a representation of the covering group given by an amalgamated sum of the two maximal standard parabolic subgroups of H. In neither case, however, do they produce a representation of H itself. (Note that it is clear from the definition that an orbit admissible for one group remains admissible for any covering group.) For the *real* adjoint group H of type G_2, it is also true that the minimal orbit is admissible non-special, and in that case there is a minimal representation of H [V4].

The paper is organized as follows. In Section 2, we fix our notation and terminology, and recall the notion of the Hilbert symbol and exponential map for p-adic fields. In Section 3, we present the metaplectic group following [LV] and [P]. We then consider the metaplectic covers of certain interesting subgroups of the symplectic group $Sp(W)$ and give criteria for when these covers split.

In Section 4, we begin our discussion of admissibility. We define a criterion for the admissible nilpotent coadjoint orbits following Duflo [D] and Lion and Perrin [LP]. We then proceed to special cases. In Section 5 we consider the symplectic, orthogonal and unitary groups, strongly imitating the techniques used by Mœglin in [M] for the classical groups. We proceed to the case of the special and general linear groups in Section 6.

2. Background: p-adic fields

By a *local field* we mean a locally compact, nondiscrete (commutative) field. The archimedean local fields are $\mathbb{R}$ and $\mathbb{C}$, the fields of real and complex numbers, respectively.

A nonarchimedean local field of characteristic zero is called a *p-adic field*. It is a finite algebraic extension of $\mathbb{Q}_p$, the p-adic completion of the field of rational numbers for some prime p. The nonarchimedean local fields of positive characteristic are the fields $\mathbb{F}_q((t))$, power series in one variable over the field with q elements. Although we are primarily interested in p-adic fields, we will often state results in the broader context of all local fields.

We adopt the following notation for a nonarchimedean local field F:

F^***:** the multiplicative group of F;

val, val$_F$: the unique discrete valuation on F which gives a surjective map of F^* onto $\mathbb{Z}$, and sends 0 to ∞;

$\mathfrak{R}$**:** $= \{a \in F \mid \mathrm{val}(a) \geq 0\}$, the integer ring of F;

$\mathfrak{p}, \mathfrak{p}_F$**:** $= \{a \in F \mid \mathrm{val}(a) \geq 1\}$, the maximal ideal in $\mathfrak{R}$;

q**:** cardinality of the *residue field* $\mathfrak{R}/\mathfrak{p}$;

p**:** characteristic of the residue field, also called the *residual characteristic* of F;

e_F**:** $= \mathrm{val}_F(p)$, the degree of ramification of F over $\mathbb{Q}_p$.

In general, we will denote our base field by F. A quadratic extension of F is denoted $E = F(\omega)$, where $\omega \notin F$, but $\omega^2 \in F$. The norm over F of an element $x + \omega y$ of E is defined by $\mathrm{N}_{E/F}(x + \omega y) = x^2 - \omega^2 y^2 \in F$.

Set $F^{*2} = \{a^2 \mid a \in F^*\}$; this is a subgroup of the multiplicative group of F. When F is a p-adic field with p odd, F^*/F^{*2} has four elements, whereas it has 2^{n+2} elements when F is a degree n extension of $\mathbb{Q}_2$.

2.1. **The Hilbert symbol.** Let $(a/b)_F$ denote the (2-)*Hilbert symbol* of two nonzero elements a and b in F. It is defined to be 1 if a is the norm of an element of $F(\sqrt{b})$, and -1 otherwise. The function $(\cdot/-1)_F \colon F^* \to \{\pm 1\}$ arises in the description of the metaplectic group.

We recall some of the properties of the Hilbert symbol (see, for example, [N, III.5]).

Lemma 2.1. *Let F be a local field and a, b, c elements of F^*. The Hilbert symbol satisfies*

$$(ab/c)_F = (a/c)_F \cdot (b/c)_F, \quad (a/b)_F = (b/a)_F, \quad \text{and} \quad (a/a)_F = (a/-1)_F.$$

Moreover, $(a/b)_F = 1$ for all $b \in F^*$ if and only if $a \in F^{*2}$. In particular, we have

1. $(a/b)_{\mathbb{C}} = 1$ *for all* $a, b \in \mathbb{C}^*$;
2. $(a/-1)_{\mathbb{R}} = \mathrm{sign}(a)$, *for all* $a \in \mathbb{R}^*$;
3. *if F is nonarchimedean, of residual characteristic different from 2, then*

$$(a/-1)_F = (-1)^{\mathrm{val}(a)\,\frac{q-1}{2}} \quad \text{for all } a \in F^*.$$

In particular, $(a/-1)_F = 1$ for every $a \in \mathfrak{R}^*$;

4. *if $F = \mathbb{Q}_2$, then $(-1/-1)_F = -1$, although $(a/-1) = 1$ for any $a \in 1 + \mathfrak{p}^2$.*

This final case of residual characteristic 2 is not well understood in general. We do have the following result, however, which will be useful for our study.

Lemma 2.2 (Fesenko, Vostokov [FV, VII, §4, Ex.6(c)])**.** *Let F be a p-adic field of residual characteristic 2, and let $a, b \in F^*$. If*

$$\mathrm{val}(a-1) + \mathrm{val}(b-1) > 2e_F,$$

then $(a/b)_F = 1$.

2.2. **The exponential map.** Suppose from now on that the characteristic of F is zero. Then one can formally define an exponential map $F \to F^*$ as an infinite power series centered at $0 \in F$. This series converges everywhere for $F = \mathbb{R}$ or $\mathbb{C}$, but for p-adic fields, it converges only on a neighborhood of zero. More precisely, we have the following theorem.

Theorem 2.3 ([N, III.1.2])**.** *Let F be a p-adic field of residual characteristic p. The power series*

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{2.1}$$

and

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

converge uniformly and give mutually inverse isomorphisms (and homeomorphisms)

$$\exp\colon \mathfrak{p}_F^n \to 1 + \mathfrak{p}_F^n \quad \textit{and} \quad \log\colon 1 + \mathfrak{p}_F^n \to \mathfrak{p}_F^n$$

if and only if n is an integer strictly greater than $e_F/(p-1)$.

What is of interest to us here is the exponential map from a Lie algebra to the corresponding group. For linear groups, it is defined by the power series expansion (2.1), where this time we let $x = X$ denote an element of the Lie algebra $\mathfrak{g}$.

Lemma 2.4. *Suppose $G \subset GL(n, F)$ is a linear algebraic group over a p-adic field F, with Lie algebra $\mathfrak{g} \subset End(F^n)$. Then the exponential map defines a continuous map from an open neighborhood of $0 \in \mathfrak{g}$ onto an open neighborhood of $1 \in G$.*

Proof. By standard arguments, we see that the convergence of the exponential map on an element $X \in \mathfrak{gl}(n, F)$ is equivalent to the convergence of the exponential map on the eigenvalues of X. The eigenvalues live in some finite algebraic field extension $\widetilde{F}$ of F. By Theorem 2.3 above, it follows that the exponential map converges uniformly exactly when each eigenvalue λ satisfies

$$\mathrm{val}_{\widetilde{F}}(\lambda) > \frac{e_{\widetilde{F}}}{p-1}.$$

This defines the domain U_e of the exponential map for $GL(n, F)$, which may be viewed as a "tubular neighborhood" of the nilpotent cone in $\mathfrak{gl}(n, F)$. When $G \subset GL(n, F)$ is an algebraic group, then exp is defined on all of $\mathfrak{g} \cap U_e$, and converges to elements of G [Ch, Ch.II, §12] □

Definition 2.5. Let G be a linear algebraic group defined over a local field F, and fix an embedding $G \subset GL(n, F)$. Denote by G_0 the open normal subgroup of G generated by the image of the exponential map.

Remark 2.6. The domain $\mathfrak{g} \cap U_e$ of the exponential map is an $\mathfrak{R}$-submodule of $\mathfrak{g}$ containing all nilpotent elements, but it is not a Lie ring. In many applications it is convenient (or necessary, as in the case of compact p-adic Lie groups) to replace $\mathfrak{g} \cap U_e$ with a smaller $\mathfrak{R}$-lattice which is closed under the Lie commutator. We will not do so here.

When $F = \mathbb{R}$ or $\mathbb{C}$, G_0 coincides with the topological identity component of G. Recall, however, that p-adic groups are totally disconnected, so such a description does not apply. If $\mathbb{G}$ is a simple, simply connected algebraic group defined over

F and isotropic over F, and $G = \mathbb{G}(F)$ is its group of F-points, then $G_0 = G$ by a theorem of Kneser and Tits (see [Pl]). Such groups include $G = SL(n, F)$ and $G = Sp(2n, F)$, for example. For other groups, G_0 is not so easy to identify. We have the following proposition, however, which for example shows that if $G = O(V)$ is an orthogonal group, then $G_0 \subseteq SO(V)$, the special orthogonal group.

Proposition 2.7. *Suppose G is a linear algebraic group defined over a p-adic field F. Then for all $g \in G_0$*

1. $\det g \neq -1$, *and*
2. $(\det g / -1)_F = 1$.

Proof. Recall that for $X \in \mathfrak{g}$ in the domain of convergence of the exponential map, we have $\det(\exp X) = \exp(\mathrm{trace}(X))$. Suppose F has residual characteristic p. Then $-1 = (p-1) + (p-1)p + (p-1)p^2 + \cdots$, which is not in the image of the exponential map on F (Theorem 2.3). The second assertion follows from Lemma 2.1(3) (respectively Lemma 2.2), when $p \neq 2$ (respectively $p = 2$). □

3. The metaplectic group

Let F be a local field, and ψ a nontrivial unramified unitary character of F. Write $\mathbb{C}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. For convenience, not necessity, we assume the characteristic of F is different from 2; see [W] for the more general setting. Let W be a symplectic vector space over F, with symplectic form $\langle , \rangle$. A Lagrangian (subspace) of W is a maximally isotropic subspace with respect to this form. The group of automorphisms of W preserving $\langle , \rangle$ is the symplectic group $Sp(W)$.

3.1. **Metaplectic group.** The metaplectic group arises in [W] as the smallest covering group of the symplectic group to which the projective Segal-Shale-Weil representation lifts as a (true) representation. This cover has been described explicitly in many works, including [LV] and [P], which we follow here.

For any three Lagrangian subspaces l_1, l_2, l_3 of W, let $\tau(l_1, l_2, l_3)$ denote the equivalence class of the quadratic form on $l_1 \oplus l_2 \oplus l_3$ given by

$$(x_1, x_2, x_3) \longmapsto \langle x_1, x_2 \rangle + \langle x_2, x_3 \rangle + \langle x_3, x_1 \rangle$$

(see [LV]). The invariant of this form we need to consider is called the *Weil index*.

The Weil index is a unitary character of the Witt group [LV, A.6] of F, assigning to each nondegenerate quadratic form Q an element $\gamma(Q) \in \mathbb{C}^1$. It is defined by an integral equation in [W, §14], and has been computed explicitly for all local fields in, for example, [P, Appendix]. Although this explicit form is quite complicated for p-adic fields, for $F = \mathbb{R}$, $\gamma(Q)$ is simply $\psi(\frac{1}{8}\text{signature of } Q)$. We abbreviate by $\gamma(a)$ the Weil index of the quadratic form $x \mapsto ax^2$, and extend γ to degenerate quadratic forms by defining $\gamma(Q) = \gamma(Q/\mathrm{rad}(Q))$ and $\gamma(0) = 1$.

Lemma 3.1 (Vergne [LV], Perrin [P], Ranga Rao [RR])**.** *Let W be a symplectic vector space. With respect to a choice l of Lagrangian subspace of W, the 2-cocycle of the Segal-Shale-Weil representation is given by*

$$c_l(g_1, g_2) = \gamma(\tau(l, g_1 l, g_1 g_2 l))$$

for any $g_1, g_2 \in Sp(W)$.

Following [W, §43] and [LV, 1.7.8 and §A.16], we define the *metaplectic group* to be the set

$$Mp(W) = \{(g,t) \in Sp(W) \times \mathbb{C}^1 \mid \Psi(g)^{-1} = t^2\}$$

endowed with the multiplication

$$(g_1,t_1)\cdot(g_2,t_2) = (g_1g_2, t_1t_2\, c_l(g_1,g_2)). \tag{3.1}$$

It is a nontrivial double cover of $Sp(W)$, and is the commutator subgroup of $GMp(W)$, the full $\mathbb{C}^1$ cover defined by (3.1). Here, $\Psi\colon Sp(W) \to \mathbb{C}^1$ is a map which satisfies

$$\Psi(g_1)\Psi(g_2) = c_l(g_1,g_2)^2\,\Psi(g_1g_2). \tag{3.2}$$

It can be described explicitly as follows.

Definition 3.2 ([LV])**.** Let $g \in Sp(W)$, and choose orientations on the vector spaces l and $l \cap gl$; these induce orientations on $l/(l\cap gl)$ and $(gl/(l\cap gl))^*$. If $l \neq gl$, then the isomorphism

$$\begin{array}{rcl}\alpha_{l\mapsto gl}\colon l/(l\cap gl) & \longrightarrow & (gl/(l\cap gl))^* \\ v & \longmapsto & \langle\cdot, v\rangle.\end{array}$$

has a well defined determinant $D(\alpha_{l\mapsto gl})$ modulo F^{*2}. When $l = gl$, set $D(\alpha_{l\mapsto gl}) = \det(g|_l)$, reflecting the extent to which g is orientation-preserving on l.

These values $D(\alpha_{l\mapsto gl})$ depend only on g, and not on the choices made.

Proposition 3.3 (Vergne [LV], Perrin [P])**.** *For any $g \in Sp(W)$, the map*

$$\Psi(g) = (D(\alpha_{l\mapsto gl})/-1)_F\ \gamma(1)^{2(\dim l - \dim(l\cap gl))}$$

satisfies (3.2).

Note that if -1 is a square in F, then $\Psi \equiv 1$, even though the cocycle c_l is nontrivial.

3.2. **Coverings of subgroups.** Suppose we have a homomorphism $i\colon G \to Sp(W)$, where G is Lie group over F. Then we can define the *metaplectic cover* (with respect to i) of G to be the group $G^{Mp} = G^{Mp(W)}$ defined by the commutative diagram

$$\begin{array}{ccc} G^{Mp} & \longrightarrow & Mp(W) \\ \big\downarrow & & \big\downarrow p \\ G & \xrightarrow{\ i\ } & Sp(W). \end{array}$$

More explicitly, it is the subgroup of $G \times Mp(W)$ defined by

$$\{(g,x) \in G \times Mp(W) \mid i(g) = p(x)\}.$$

We say that the metaplectic cover of G (corresponding to W) *splits* if there is a section of this bundle, that is, a homomorphism π of G into $Mp(W)$ satisfying $p \circ \pi = i$.

In this section, we consider various kinds of groups G and maps $i\colon G \to Sp(W)$, with the intent of understanding the factors leading to a splitting of a metaplectic double covering group.

Notation. Let F be $\mathbb{R}$ or a p-adic field. Let W, W_1, W_2 and V be symplectic vector spaces over F. Let G, G_1 and G_2 be Lie groups over F. Denote by ε the nontrivial element in the kernel of the projection map p. Let $Z = Z(W_1, W_2)$ denote the central subgroup $\{(1,1), (\varepsilon, \varepsilon)\}$ of the direct product $Mp(W_1) \times Mp(W_2)$.

Proposition 3.4. *Set $W = W_1 \oplus W_2$. Viewing $Sp(W_1) \times Sp(W_2)$ as a subgroup of $Sp(W)$, we have*

$$(Sp(W_1) \times Sp(W_2))^{Mp(W)} = (Mp(W_1) \times Mp(W_2)) \,/\, Z.$$

Consequently, given a pair of group homomorphisms $i_1 \colon G_1 \to Sp(W_1)$ and $i_2 \colon G_2 \to Sp(W_2)$, the metaplectic cover of $G_1 \times G_2$ induced by the map $i_1 \times i_2$ into $Sp(W)$ splits if and only if each of the metaplectic covers $(G_1)^{Mp(W_1)}$ and $(G_2)^{Mp(W_2)}$ splits.

The situation is slightly different for the case of a single group preserving a decomposition of V into symplectic subspaces. There, two nontrivial metaplectic double covers can induce a trivial cover, in the following sense.

Corollary 3.5. *In the setting of Proposition 3.4, suppose now $G = G_1 = G_2$, and embed G in $G \times G$ along the diagonal as $\delta(g) = (g, g)$.*

1. *Assume the metaplectic cover of G induced by i_2 splits. Then the metaplectic cover of G induced by $(i_1 \times i_2) \circ \delta$ splits if and only if the metaplectic cover of G induced by i_1 splits.*
2. *Suppose $G = Sp(V)$ is itself a symplectic group. If the metaplectic covers of G arising from each of i_1 and i_2 do not split, then the metaplectic cover of G induced by the map $(i_1 \times i_2) \circ \delta$ splits.*

Proof. The first part is straightforward. For the second part, we use the fact that up to inner isomorphism, there is a unique nontrivial two-fold covering group of the symplectic group $G = Sp(V)$ [MVW, Ch2.II.1]. Hence we simply have $G^{Mp(W_1)} \cong G^{Mp(W_2)} \cong Mp(V)$. □

As a special and important application of these last two results, we prove the following proposition. It will be used in Section 5.

Proposition 3.6. *Let W be a symplectic vector space, and U a quadratic vector space. Then $W \otimes U$ is symplectic, and there is a natural map*

$$Sp(W) \longrightarrow Sp(W \otimes U) \tag{3.3}$$

given by $g(w \otimes u) = gw \otimes u$. The metaplectic cover of the symplectic group $Sp(W)$ arising from this map splits if and only if $\dim U$ is even.

Proof. We prove this by induction on the dimension of the quadratic space U. Suppose $\dim U = 1$. Then $W \otimes U \cong W$, since both are symplectic vector spaces of the same dimension. It follows that (3.3) is an isomorphism, and so the metaplectic cover of $Sp(W)$ is just $Mp(W)$, which does not split.

Now suppose we have the result for any dimension less than k. Let U be a k-dimensional quadratic vector space. Choose an orthogonal decomposition $U = U_1 \oplus U_2$, with $\dim U_1 = 1$ and $\dim U_2 = k - 1$. Then $Sp(W)$ preserves the (symplectic) decomposition

$$W \otimes U = (W \otimes U_1) \oplus (W \otimes U_2)\,,$$

and our inductive hypothesis applies to each summand. The result now follows from Corollary 3.5. □

Our final result concerns the case where the image of G lies in a Siegel parabolic subgroup of $Sp(W)$.

Proposition 3.7. *Let W be a symplectic subspace over a p-adic field F, and G a closed linear algebraic subgroup of $Sp(W)$. Suppose G preserves a Lagrangian subspace $l \subset W$. Then the map*

$$\begin{array}{ccc} G_0 & \longrightarrow & Mp(W) \\ g & \longmapsto & (g, 1) \end{array} \tag{3.4}$$

defines a splitting of the metaplectic cover of G over G_0.

Proof. Since G preserves l, the quadratic form $\tau(l, g_1 l, g_1 g_2 l)$ is identically zero for all $g_1, g_2 \in G$. It follows that $c_l(g_1, g_2) = 1$, so (3.4) defines a homomorphism of G_0 into $GMp(W)$. To see that it takes values in $Mp(W)$, first note that $\dim l - \dim(l \cap gl) = \dim l - \dim l = 0$, so the $\gamma(1)$ term in Proposition 3.3 disappears. Moreover, Proposition 2.7 implies that $(\det(g|_l)/-1)_F = 1$ for all $g \in G_0$. Thus we have $\Psi(g) = 1$ for all $g \in G_0$, which completes the proof. □

4. Admissibility of an orbit

Let G be a reductive Lie group defined over a local field F of characteristic zero. Let $\mathfrak{g}$ be the Lie algebra of G. Choose an element f in $\mathfrak{g}^*$, the dual of the Lie algebra, and consider its orbit under the coadjoint action of G. This orbit $G \cdot f$ has a natural symplectic manifold structure, given on the tangent space at the base point f by the Kirillov-Kostant symplectic form

$$\omega(X_1, X_2) = f([X_1, X_2]), \qquad \text{for all } X_1, X_2 \in \mathfrak{g}/\mathfrak{g}^f.$$

The action of G^f on $\mathfrak{g}/\mathfrak{g}^f$ clearly preserves the symplectic form ω, and hence gives rise to a homomorphism (denoted Ad by abuse of notation)

$$Ad\colon G^f \longrightarrow Sp(\mathfrak{g}/\mathfrak{g}^f). \tag{4.1}$$

For our purposes, it is convenient to work in the Lie algebra. Choose some faithful finite-dimensional representation ρ of $\mathfrak{g}$, and define a G-invariant nondegenerate bilinear form B on $\mathfrak{g}$ via

$$B(X_1, X_2) = \operatorname{Tr} \rho(X_1)\rho(X_2) \qquad \text{for all } X_1, X_2 \in \mathfrak{g}.$$

Identify f with its preimage under the isomorphism $X \mapsto B(X, \cdot)$. An element of the Lie algebra is nilpotent (respectively, semisimple) if it acts nilpotently (respectively, semisimply) in any algebraic representation. We transport this notion to $\mathfrak{g}^*$.

From now on, assume f is nilpotent. Then by the Jacobson-Morozov theorem, we can choose a Lie triple in $\mathfrak{g}$ having f as its nilnegative element; that is, we can find an isomorphism

$$\phi\colon \mathfrak{sl}(2, F) \to \mathfrak{g}$$

sending Y to f, where $\{H, X, Y\}$ denotes the usual basis of $\mathfrak{sl}(2, F)$, with Lie bracket given by

$$[H, X] = 2X \qquad [H, Y] = -2Y \qquad [X, Y] = H.$$

To simplify notation, we write ϕ for $\phi(\mathfrak{sl}(2, F))$.

A coadjoint orbit $G \cdot f$ is said to be *integral* if there exists a character χ of G_0^f satisfying $\chi(\exp X) = \psi(f(X))$ for all $X \in \mathfrak{g}^f$ in the domain of the exponential map. Here, $G_0^f = (G^f)_0$ is the subgroup of Definition 2.5, and ψ is the character of F fixed at the beginning of Section 3.

Duflo defined an *admissible* coadjoint orbit of a real Lie group [D, p. 154] using the metaplectic double cover $(G^f)^{Mp}$ of G^f corresponding to the map (4.1). We say an orbit is admissible if there exists a representation τ of $(G^f)_0^{Mp}$ such that $\tau(\varepsilon) = -1$ (that is, τ is a *genuine* representation of the two-fold covering group) and $\tau(\exp X)$ is a multiple of the character $\psi(f(X))$ for all $X \in \mathfrak{g}^f$ in the domain of the exponential map.

If f is nilpotent, then $f|_{\mathfrak{g}^f} \equiv 0$, so the question of admissibility reduces to the following.

Definition 4.1. A nilpotent coadjoint orbit $G \cdot f$ is *admissible* if the metaplectic cover of G^f corresponding to $Sp(\mathfrak{g}/\mathfrak{g}^f)$ splits over the subgroup G_0^f.

Remark 4.2. It is worth comparing this definition to the one used by Torasso [T]. He defines a nilpotent orbit to be admissible if there exists a normal subgroup of finite index in G^f over which the metaplectic cover $(G^f)^{Mp}$ splits. Certainly the open subgroup G_0^f used here need not have finite index in G^f, since the image in F of the exponential map has infinite index in F. Nonetheless, both definitions require open normal subgroups containing the group generated by the unipotent elements of G^f.

The definition adopted here is preferable because it allows the lifting of arbitrary characters from the Lie algebra (as described before Defintion 4.1), which in turn permits an extension of this definition of admissibility to non-nilpotent orbits. Moreover this subgroup G_0^f is readily computable.

It is, however, unnecessary to consider all of G^f to determine admissibility. We can write $G^f = G^\phi \ltimes U(G^f)$, where G^ϕ is the centralizer of ϕ in G, and $U(G^f)$ is the unipotent radical of G^f [C, Prop 5.5.9]. The latter term is a unipotent subgroup, isomorphic to a vector space (its Lie algebra) via the exponential map. It follows that any double cover of $U(G^f)$ must split.

Hence the condition for admissibility can fail only over G_0^ϕ, and we restrict ourselves to considering the diagram

$$\begin{array}{ccc} (G^\phi)^{Mp} & \longrightarrow & Mp(\mathfrak{g}/\mathfrak{g}^f) \\ \downarrow & & \downarrow p \\ G^\phi & \xrightarrow{Ad} & Sp(\mathfrak{g}/\mathfrak{g}^f). \end{array}$$

There is one further reduction to make. Decompose $\mathfrak{g}/\mathfrak{g}^f$ into eigenspaces under $\phi(H)$, the semisimple element of ϕ. Denoting the eigenspace corresponding to eigenvalue i by $(\mathfrak{g}/\mathfrak{g}^f)[i]$, we write

$$\mathfrak{g}/\mathfrak{g}^f = \bigoplus_{i \in \mathbb{Z}} (\mathfrak{g}/\mathfrak{g}^f)[i]. \tag{4.2}$$

In fact, $(\mathfrak{g}/\mathfrak{g}^f)[1] = \mathfrak{g}[1]$ is a symplectic subspace of $\mathfrak{g}/\mathfrak{g}^f$ preserved by G^ϕ.

Remark 4.3. If $\mathfrak{g}[1] = 0$, then necessarily $\mathfrak{g}[2m+1] = 0$ for all integers m, and the orbit $G \cdot f$ is said to be *even*. In the classical Lie algebras, it is equivalent to the condition that in the partition type classification of the orbit (over the algebraic closure $\overline{F}$), all parts have the same parity [CMcG].

Definition 4.4. Call a nilpotent coadjoint orbit $G \cdot f$ *pseudo-admissible* if the double cover of G^ϕ defined by

$$\begin{array}{ccc} (G^\phi)^{Mp} & \longrightarrow & Mp(\mathfrak{g}[1]) \\ \downarrow & & \downarrow p \\ G^\phi & \xrightarrow{Ad} & Sp(\mathfrak{g}[1]) \end{array}$$

splits over G_0^ϕ.

Proposition 4.5. *Suppose G is a reductive Lie group over a p-adic field F. Then a nilpotent orbit $G \cdot f$ is admissible if and only if it is pseudo-admissible.*

Proof. Write $\mathfrak{g}/\mathfrak{g}^f = W \oplus \mathfrak{g}[1]$ where W denotes the sum of the remaining weight spaces in (4.2). Both W and $\mathfrak{g}[1]$ are symplectic subspaces preserved by G^ϕ. The subspace

$$l = \bigoplus_{i>1} (\mathfrak{g}/\mathfrak{g}^f)[i]$$

is a G^ϕ-invariant Lagrangian of W. By Proposition 3.7, the metaplectic cover of G_0^ϕ corresponding to $Sp(W)$ splits. Hence the equivalence of admissibility and pseudo-admissibility follows from Corollary 3.5, applied to the symplectic decomposition $\mathfrak{g}/\mathfrak{g}^f = W \oplus \mathfrak{g}[1]$. □

Thus the question of admissibility is determined on $\mathfrak{g}[1]$. It follows, for example, that even orbits are always admissible.

In subsequent sections, we analyze G^ϕ and $\mathfrak{g}[1]$ for particular choices of G.

5. Symplectic, orthogonal and unitary groups

Let F be $\mathbb{R}$ or a p-adic field. We analyze the following cases together.

Symplectic: Let V be a vector space over F equipped with a symplectic form $\langle , \rangle$. Then G is the symplectic group $Sp(V, F)$.

Orthogonal: Let V be a vector space over F equipped with a (nondegenerate) symmetric bilinear form $\langle , \rangle$. Then let G be either the orthogonal group $O(V, F)$, the group of automorphisms of this form, or the special orthogonal group $SO(V, F)$, consisting of those automorphisms of determinant equal to 1.

Unitary: Choose a quadratic extension $E = F(\omega)$ of F. Let V be a vector space over E equipped with a hermitian form $\langle , \rangle$, and denote by $U(V) \subset GL(V, E)$ the group of automorphisms of this form. Let G be either the unitary group $U(V)$, or its subgroup the special unitary group $SU(V)$.

Notation. If $\pi \colon \mathfrak{sl}(2, F) \to \mathrm{End}(V)$ is a representation, let

$$V[i] = \{v \in V \mid \pi(H)v = iv\}$$

denote its ith weight space under the semisimple element of $\mathfrak{sl}(2, F)$. Let W^m denote the unique (up to equivalence) irreducible representation of $\mathfrak{sl}(2, F)$ of dimension m. If $w_m \in W^m[m-1]$ is a nonzero highest weight vector of W^m, then a basis of the $\mathfrak{sl}(2, F)$-module W^m is

$$\{w_m, Yw_m, \ldots, Y^{m-1}w_m\}, \tag{5.1}$$

where we recall from Section 4 that $Y \in \mathfrak{sl}(2, F)$ is the lowering operator.

Choose a nilpotent orbit $G\cdot f$ and a Lie triple ϕ as in Section 4. Since $\mathfrak{g}$ acts on V, so does the subalgebra $\phi(\mathfrak{sl}(2,F))$. Decompose V into irreducible subrepresentations under ϕ. We can write

$$V = \bigoplus_{m\geq 1} V^m, \tag{5.2}$$

where V^m is the subspace of all copies of W^m in V.

Remark 5.1. This decomposition is strongly related to the partition classification of nilpotent orbits (over $\overline{F}$). If $V^m \neq 0$, then m occurs in the partition for $G\cdot f$, with multiplicity equal to the number of times W^m occurs in V^m.

5.1. **Structure of G^ϕ and $\mathfrak{g}[1]$.** Our first step is to understand these V^m subspaces better. Define $V^{\langle m\rangle}$ to be the G^ϕ-space $\mathrm{Hom}_{\mathfrak{sl}(2,F)}(W^m, V)$, with the action of $g\in G^\phi$ on an element $T\in V^{\langle m\rangle}$ given by $(g\cdot T)(w) = g(T(w))$, for all $w\in W^m$. There is a G^ϕ-equivariant isomorphism

$$\begin{array}{ccc} W^m\otimes V^{\langle m\rangle} & \to & V^m \\ w\otimes T & \mapsto & T(w), \end{array} \tag{5.3}$$

where G^ϕ acts only on the second factor of the tensor product. Note that we may identify $V^{\langle m\rangle}$ with each nontrivial weight space $V^m[i]$, via the isomorphism

$$\begin{array}{ccc} V^{\langle m\rangle} & \to & V^m[i] \\ T & \mapsto & T(Y^{(m-1-i)/2}w_m). \end{array} \tag{5.4}$$

Since the direct sum in (5.2) is orthogonal with respect to the form $\langle,\rangle$ on V, the restriction of $\langle,\rangle$ to each V^m is well defined and nondegenerate.

Recall that there is a nondegenerate bilinear form b_m, unique up to multiplication by a scalar, on each irreducible representation W^m of $\mathfrak{sl}(2,F)$. It is $\mathfrak{sl}(2,F)$-invariant, and has the property that for all $x\in W^m[i]$ and $y\in W^m[j]$, both nonzero,

$$b_m(x,y)\neq 0 \quad \Leftrightarrow \quad i=-j. \tag{5.5}$$

Without loss of generality, we fix a scaling of b_m so that $b_m(w_m, Y^{m-1}w_m)=1$ for each m. (This depends of course on our choice of w_m made earlier.) Note that the form b_m on W^m is orthogonal if m is odd, and symplectic if m is even.

Lemma 5.2. *Define a nondegenerate bilinear form $(,)_m$ on $V^{\langle m\rangle}$ by*

$$(S,T)_m = \langle S(w_m), T(Y^{m-1}w_m)\rangle = \langle S(w_m), \phi(Y)^{m-1}T(w_m)\rangle$$

for all $S,T\in V^{\langle m\rangle}$. Then the form on $W^m\otimes V^{\langle m\rangle}$ defined by

$$(x\otimes S, y\otimes T) = b_m(x,y)\cdot(S,T)_m, \tag{5.6}$$

for $x,y\in W^m$ and $S,T\in V^{\langle m\rangle}$ is equivalent to $\langle,\rangle|_{V^m}$ via the isomorphism (5.3).

Proof. To prove the equivalence of the form defined in (5.6) and the form $\langle,\rangle$, we use (5.3) and compare them explicitly on a spanning set of the form

$$\{Y^k w_m\otimes T \mid 0\leq k\leq m-1,\ T\in V^{\langle m\rangle}\}.$$

□

Remark 5.3. The form $(\,,\,)_m$ is of the same type (symplectic, orthogonal or Hermitian) as $\langle\,,\,\rangle$ if m is odd, and of opposite type if m is even. (In the Hermitian case, take "opposite type" to mean skew-Hermitian.)

Now set $G^{\phi,m} = \mathrm{Aut}(V^{\langle m\rangle}, (\,,\,)_m)$. By the preceding lemma and (5.3), we have a natural isomorphism

$$G^{\phi} = \prod_{m\geq 1} G^{\phi,m} \tag{5.7}$$

for G symplectic, orthogonal or unitary. For the special orthogonal and special unitary groups, G^{ϕ} is the subgroup given by the intersection of the product in (5.7) with G. Note, however, that $G_0^{\phi} = \prod_{m\geq 1} G_0^{\phi,m}$, so for the special orthogonal groups the question of splitting is the same as for the full orthogonal group, since $O(V)_0 \subset SO(V)$. In the case of the special unitary group, we shall see that the metaplectic cover of $(U(V)^{\phi})^{Mp}$ induced by its map into $Sp(\mathfrak{g}[1])$ splits over $U(V)_0^{\phi}$ in all cases (see Section 5.6), and therefore also over the subgroup $SU(V)_0^{\phi}$. We will thus simplify our notation by writing (5.7) in all cases.

5.2. **Decomposition of $\mathfrak{g}[1]$ under the action of G^{ϕ}.** Now that we have a more precise understanding of G^{ϕ} and the decomposition of V under ϕ, we proceed to the main problem of the section, namely, the decomposition of $\mathfrak{g}[1]$ under the adjoint action of G^{ϕ}.

When $G = Sp(V)$ or $G = O(V)$ (or $SO(V)$), $\mathfrak{g}$ is isomorphic to $\bigwedge^2 V$ and $\mathrm{Sym}^2 V$, respectively, via the G-equivariant isomorphisms generated by

$$v\otimes w \pm w\otimes v \longmapsto \langle\cdot, w\rangle v \pm \langle\cdot, v\rangle w. \tag{5.8}$$

A similar result holds for the case $G = U(V)$, as follows. Define the E-vector space $\overline{V}$ to be equal to V as a vector space over F, but having the conjugate E action. We write elements of $\overline{V}$ as $\overline{v}$, so that multiplication by an element λ of E takes the form $\lambda\cdot\overline{v} = \overline{\overline{\lambda}v}$ (where the multiplication on the right is that of V).

Let $\bigwedge(V,\overline{V})$ denote the F-subspace of $V\otimes_E \overline{V}$ spanned by the skew elements $\{v\otimes\overline{w} - w\otimes\overline{v} \mid v,w\in V\}$. It is a vector space over F (not E), and is isomorphic to $\mathfrak{g}$ via the G-equivariant linear map generated by

$$v\otimes\overline{w} - w\otimes\overline{v} \longmapsto \langle\cdot,\overline{w}\rangle v - \langle\cdot,\overline{v}\rangle w. \tag{5.9}$$

The overlines on the right-hand side are unnecessary, and will be omitted from now on.

Under the isomorphisms (5.8) and (5.9), the $+1$ weight space of $\mathfrak{g}$ under $\phi(H)$ can be identified with $\bigwedge^2 V[1]$, $\mathrm{Sym}^2 V[1]$, or $\bigwedge(V,\overline{V})[1]$, respectively. This holds for $SU(V)$ as well, since its Lie algebra differs from that of $U(V)$ only by a central part, which must lie in the *zero* weight space under $\phi(H)$.

Each of these identifications has a nice interpretation in terms of the weight spaces of V itself, as described in the following lemma.

Lemma 5.4. *For each pair (m,m') of positive integers, with m even and m' odd, the space*

$$\mathfrak{g}_{m,m'}[1] = \bigoplus_{i=-m+1}^{m-1} V^m[i]\otimes\overline{V^{m'}[-i+1]}$$

corresponds to a symplectic subspace of $\mathfrak{g}[1]$*. In fact,*

$$\mathfrak{g}[1] \cong \bigoplus_{\substack{m\ even \\ m'\ odd}} \mathfrak{g}_{m,m'}[1], \tag{5.10}$$

and this decomposition is G^ϕ*-equivariant and orthogonal with respect to the symplectic form* ω *on* $\mathfrak{g}[1]$*. The action of* G^ϕ *on* $\mathfrak{g}_{m,m'}[1]$ *is given by the usual action of* $G^{\phi,m} \times G^{\phi,m'}$ *on each of the spaces* $V^m[i] \otimes \overline{V^{m'}[-i+1]}$*.*

The notation $V^m[i] \otimes \overline{V^{m'}[-i+1]}$ is perhaps an awkward compromise: when V is an F-vector space, we mean $V^m[i] \otimes_F V^{m'}[-i+1]$; when V is an E-vector space, we mean $V^m[i] \otimes_E \overline{V^{m'}[-i+1]}$, viewed as an F-vector space by restriction of scalars.

Proof of Lemma 5.4. This is an exercise in understanding the isomorphisms (5.8) and (5.9). First note that the isomorphism (5.10) is well defined, precisely because the fixed parity of m ensures we include each symmetric (respectively, skew-symmetric) pair exactly once. We use a direct computation to prove that the decomposition in the lemma respects the symplectic form, as follows.

Let m, n be even integers, and m', n' odd integers (all positive). For $Z_1 \in \mathfrak{g}_{m,m'}[1]$ and $Z_2 \in \mathfrak{g}_{n,n'}[1]$, the form ω is given by

$$\omega(Z_1, Z_2) = \operatorname{Tr} \phi(Y)[Z_1, Z_2]. \tag{5.11}$$

We readily compute the Lie bracket of Z_1 with Z_2 via $[Z_1, Z_2]v = Z_1Z_2v - Z_2Z_1v$ for all $v \in V$ using (5.8) and (5.9). We give explicit formulas for the case $G = U(V)$; the other cases are similar.

Without loss of generality, we may assume

$$Z_1 = x_1 \otimes \overline{y_1} - y_1 \otimes \overline{x_1},$$
$$Z_2 = x_2 \otimes \overline{y_2} - y_2 \otimes \overline{x_2}$$

for some $x_1 \in V^m[i]$, $x_2 \in V^n[j]$, $y_1 \in V^{m'}[-i+1]$ and $y_2 \in V^{n'}[-j+1]$. Here, m and n are even, so i and j are odd. Then we have

$$\begin{aligned}[Z_1, Z_2]v &= [x_1 \otimes \overline{y_1} - y_1 \otimes \overline{x_1}\ , x_2 \otimes \overline{y_2} - y_2 \otimes \overline{x_2}]v \\ &= \langle v, y_2\rangle\langle x_2, y_1\rangle x_1 - \langle v, y_2\rangle\langle x_2, x_1\rangle y_1 \\ &\quad - \langle v, x_2\rangle\langle y_2, y_1\rangle x_1 + \langle v, x_2\rangle\langle y_2, x_1\rangle y_1 \\ &\quad - \langle v, y_1\rangle\langle x_1, y_2\rangle x_2 + \langle v, y_1\rangle\langle x_1, x_2\rangle y_2 \\ &\quad + \langle v, x_1\rangle\langle y_1, y_2\rangle x_2 - \langle v, x_1\rangle\langle y_1, x_2\rangle y_2.\end{aligned} \tag{5.12}$$

Recall that $\langle V[i], V[j]\rangle \equiv 0$ unless $i = -j$. Hence all of the terms in (5.12) are zero, except in the following cases.

Case $i = -j + 2$**:** Then $\langle y_1, y_2\rangle$ can be nonzero; we have

$$[Z_1, Z_2]v = \langle v, x_1\rangle\langle y_1, y_2\rangle x_2 - \langle v, x_2\rangle\langle y_2, y_1\rangle x_1.$$

Case $i = -j$**:** Then $\langle x_1, x_2\rangle$ can be nonzero; we have

$$[Z_1, Z_2]v = \langle v, y_1\rangle\langle x_1, x_2\rangle y_2 - \langle v, y_2\rangle\langle x_2, x_1\rangle y_1.$$

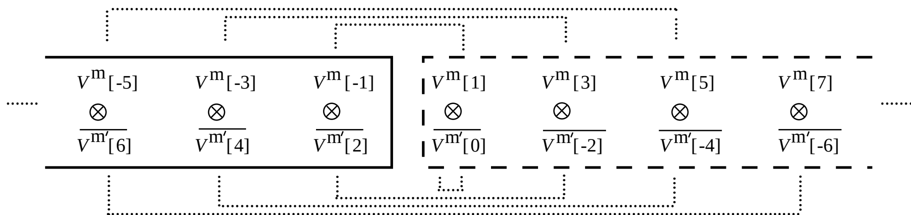

FIGURE 5.1. The symplectic vector space $\mathfrak{g}_{m,m'}[1]$. Dotted lines join spaces which admit a nonzero pairing under ω. Those along the bottom represent pairings for which (5.14) is not identically zero, and those along the top correspond to (5.15). The isotropic space $Y_{m,m'}$ is spanned by all the vector spaces within the thick solid line. The image of δ is a small subspace of the span of the spaces within the thick dashed line.

To compute $\omega(Z_1, Z_2)$ using (5.11), choose an orthogonal basis $\{e_k\}$ of V. Set $\langle e_k, e_k\rangle = \alpha_k$ and recall that

$$\operatorname{Tr}\phi(Y)[Z_1, Z_2] = \sum_k {\alpha_k}^{-1}\langle \phi(Y)[Z_1, Z_2]e_k, e_k\rangle. \tag{5.13}$$

Using the relation $\langle v, w\rangle = \sum_k {\alpha_k}^{-1}\langle v, e_k\rangle\langle e_k, w\rangle$, we deduce the following.

Case $i = -j + 2$**:**

$$\omega(Z_1, Z_2) = \langle \phi(Y)x_2, x_1\rangle\langle y_1, y_2\rangle - \langle \phi(Y)x_1, x_2\rangle\langle y_2, y_1\rangle. \tag{5.14}$$

Case $i = -j$**:**

$$\omega(Z_1, Z_2) = \langle \phi(Y)y_2, y_1\rangle\langle x_1, x_2\rangle - \langle \phi(Y)y_1, y_2\rangle\langle x_2, x_1\rangle. \tag{5.15}$$

Finally, using the fact that $\langle V^m, V^n\rangle \equiv 0$ for any $m \neq n$, we conclude that (5.14) and (5.15) are identically zero unless $m = n$ and $m' = n'$. Thus, the images of $\mathfrak{g}_{m,m'}[1]$ are pairwise othogonal in $\mathfrak{g}[1]$.

The last statement of the lemma follows from the G^ϕ-equivariance of (5.8) and (5.9). □

Refer to Figure 5.1 for a representation of the nonzero pairings within $\mathfrak{g}_{m,m'}[1]$, as determined by equations (5.14) and (5.15). It is clear how the conditions $i = -j+2$ and $i = -j$ arise through the symmetry of the weight spaces about zero.

5.3. **Finding G^ϕ-invariant Lagrangian decompositions.** We now have a decomposition of $\mathfrak{g}[1]$, and may apply the results of Section 3.2 to study the question of admissibility of $G \cdot f$ by studying the metaplectic cover of G_0^ϕ coming from each $\mathfrak{g}_{m,m'}[1]$ individually.

Following Mœglin [M, §1.3], we define

$$Y_{m,m'} = \bigoplus_{i \leq -1} V^m[i] \otimes \overline{V^{m'}[-i+1]}.$$

This is an isotropic subspace of $\mathfrak{g}_{m,m'}[1]$ (*cf.* Figure 5.1).

If $m < m'$, then $\dim_F Y = \frac{1}{2}\dim_F \mathfrak{g}_{m,m'}[1]$, so $Y_{m,m'}$ is a Lagrangian subspace of $\mathfrak{g}_{m,m'}[1]$. An element $(g, g') \in G^{\phi,m} \times G^{\phi,m'}$ preserves $Y_{m,m'}$, and acts on it block-diagonally, with its natural action on each F-vector space $V^m[i] \otimes \overline{V^{m'}[-i+1]}$. We are thus in the setting of Proposition 3.7, and conclude that the metaplectic cover corresponding to $Sp(\mathfrak{g}_{m,m'}[1])$ splits over $G_0^{\phi,m} \times G_0^{\phi,m'}$.

It remains to consider those $\mathfrak{g}_{m,m'}[1]$ with $m > m'$. In these cases, $Y_{m,m'}$ is isotropic but not maximally so. However, it does pair nondegenerately with the isotropic space

$$X_{m,m'} = \bigoplus_{i \geq 3} V^m[i] \otimes \overline{V^{m'}[-i+1]},$$

as one can deduce from Figure 5.1. The leftover piece, $V^m[1] \otimes \overline{V^{m'}[0]}$, although not itself symplectic, is G^ϕ-equivariantly isomorphic to a symplectic vector subspace of $\mathfrak{g}_{m,m'}[1]$, as described in the following lemma.

Lemma 5.5 ([M, §1.3])**.** *Suppose $m > m'$. Define $\delta\colon V^m[1] \otimes \overline{V^{m'}[0]} \to \mathfrak{g}_{m,m'}[1]$ by*

$$\delta\colon x \otimes \overline{y} \mapsto \sum_{k=0}^{(m'-1)/2} (-1)^k \left(\phi(Y)^{-k} x \otimes \phi(Y)^k \overline{y}\right). \tag{5.16}$$

This is a $G^{\phi,m} \times G^{\phi,m'}$-equivariant isomorphism onto an orthogonal complement of $Y_{m,m'} \oplus X_{m,m'}$ in $\mathfrak{g}_{m,m'}[1]$. The form ω, restricted to $\mathfrak{g}_{m,m'}[1]$, pulls back under δ to a multiple of the form $Tr_{E/F}(\overline{(,)_m}\,(,)_{m'})$ on $V^m[1] \otimes \overline{V^{m'}[0]}$.

Proof. It follows immediately from Figure 5.1 that $\omega(\mathrm{Im}(\delta), X_{m,m'}) \equiv 0$. One can also prove that $\omega(\mathrm{Im}(\delta), Y_{m,m'}) \equiv 0$ with an explicit calculation similar to that in the proof of Lemma 5.4; use the G-equivariance of $\langle , \rangle$ to cancel nonzero terms in the alternating sum (5.16).

The final assertion can also be proven using an explicit calculation, which we include here. Note that we use the isomorphism (5.4) to identify $(,)_m$, a form on $V^{\langle m \rangle}$, with a form on $V^m[1]$ (and similarly for m'). We consider the case where $G = U(V)$; the other cases are analogous.

Let $Z_1 = x_1 \otimes \overline{y_1}$ and $Z_2 = x_2 \otimes \overline{y_2}$ be two elements of $V^m[1] \otimes_E \overline{V^{m'}[0]}$. Then

$$\begin{aligned}
\omega(\delta(Z_1), \delta(Z_2)) &= \sum_{k,l=0}^{(m'-1)/2} (-1)^{k+l}\, \omega\left(\phi(Y)^{-k} x_1 \otimes \phi(Y)^k \overline{y_1}\,, \phi(Y)^{-l} x_2 \otimes \phi(Y)^l \overline{y_2}\right) \\
&= \langle \phi(Y) x_2, x_1 \rangle \langle y_1, y_2 \rangle - \langle \phi(Y) x_1, x_2 \rangle \langle y_2, y_1 \rangle \quad \text{by (5.14)} \\
&= \overline{\langle x_1, \phi(Y) x_2 \rangle} \langle y_1, y_2 \rangle + \langle x_1, \phi(Y) x_2 \rangle \overline{\langle y_1, y_2 \rangle} \\
&= \mathrm{Tr}_{E/F}(\overline{\langle x_1, \phi(Y) x_2 \rangle} \langle y_1, y_2 \rangle).
\end{aligned}$$

Using Lemma 5.2 and unraveling the isomorphism (5.4), we see that, up to a factor of ± 1, this last term equals

$$\mathrm{Tr}_{E/F}(\overline{(x_1, x_2)_m}\,(y_1, y_2)_{m'}),$$

as required. □

We now wish to use the results of Section 3.2 with regard to the symplectic decomposition

$$\mathfrak{g}_{m,m'}[1] = (Y_{m,m'} \oplus X_{m,m'}) \;\oplus\; \delta(V^m[1] \otimes \overline{V^{m'}[0]}).$$

The first piece has a G^ϕ-invariant Lagrangian, so we can apply Proposition 3.7 to conclude that the metaplectic cover of $G_0^{\phi,m} \times G_0^{\phi,m'}$ coming from its inclusion into

$Sp(Y_{m,m'} \oplus X_{m,m'})$ splits. Thus to determine the splitting of the metaplectic cover of $G_0^{\phi,m} \times G_0^{\phi,m'}$ induced by all of $\mathfrak{g}_{m,m'}[1]$, it suffices by Corollary 3.5 to compute the covering corresponding to $\delta(V^m[1] \otimes \overline{V^{m'}[0]})$.

This second piece does not admit a polarization invariant under $G^{\phi,m} \times G^{\phi,m'}$, and so the results of Section 3.2 do not apply directly. We must proceed on a case-by-case basis, for each of our groups G.

To reduce notation, we use Lemma 5.5 to identify $V^m[1] \otimes \overline{V^{m'}[0]}$ and its image under δ.

5.4. **Symplectic Group.** Here, $G^{\phi,m}$ is an orthogonal group and $G^{\phi,m'}$ is a symplectic group (m even, m' odd) by Remark 5.3. Let $l \oplus l'$ be any Lagrangian decomposition of $V^{m'}[0]$; then $(V^m[1] \otimes l) \oplus (V^m[1] \otimes l')$ is a Lagrangian decomposition of $V^m[1] \otimes V^{m'}[0]$ preserved by $G^{\phi,m}$. We apply Proposition 3.7, to conclude that the corresponding metaplectic cover of the orthogonal group $G^{\phi,m}$ splits over $G_0^{\phi,m}$ for all $m > m'$.

Since the product (5.7) in G^ϕ is direct, we can consider the case of the splitting over the symplectic group separately by Proposition 3.4. Construct the quadratic vector space

$$U_{m'} = \bigoplus_{\substack{m>m' \\ m \text{ even}}} V^m[1].$$

Then $\delta(U_{m'} \otimes V^{m'}[0])$ is precisely the last remaining piece of $\mathfrak{g}[1]$ on which we need to determine the splitting of the metaplectic cover of $G^{\phi,m'}$; on all other direct summands of $\mathfrak{g}[1]$, either we have already proven that the metaplectic cover splits, or no weight space of $V^{m'}$ occurs and so the action of $G^{\phi,m'}$ is trivial.

Now apply Corollary 3.5 to the symplectic vector space $V^{m'}[0]$ and the orthogonal space $U_{m'}$. Recall that $Sp(V^{m'}[0]) = G^{\phi,m'}$. We conclude that the metaplectic cover of $G^{\phi,m'}$ induced from its action on $\delta(U_{m'} \otimes V^{m'}[0])$ splits if and only if $\dim U_{m'}$ is even. This number $\dim U_{m'}$ is just the number of even parts m greater than m' (counted with multiplicity) in the partition type classification of the orbit (Remark 5.1).

To summarize: we have found that the metaplectic cover $(G^\phi)^{Mp(\mathfrak{g}[1])}$ splits over each of its orthogonal group components $G_0^{\phi,m}$ (m even). Over each of its symplectic group components $G^{\phi,m'}$ (m' odd), we have found that the cover splits if and only if the sum of the multiplicities in V of all the spaces W^m with m even and $m > m'$ is *even*. The number of such m less than m' does not affect the splitting.

5.5. **Orthogonal group.** This case is exactly equivalent to that of a symplectic group. Interchange m and m' everywhere in the discussion, to conclude that the metaplectic cover splits only when the sum of the multiplicities in V of all the spaces $W^{m'}$ with m' odd and $m' < m$ is even for each fixed even m.

5.6. **Unitary group.** Here, $(,)_m$ is a skew-Hermitian form, and $(,)_{m'}$ is Hermitian by Remark 5.3, so each of $G^{\phi,m}$ and $G^{\phi,m'}$ is again a unitary group. The following lemma is adapted from [Pr, §1] to include the case of residual characteristic equal to 2. My thanks to Jeff Adams for pointing out the necessary correction.

Lemma 5.6. *Let F be a p-adic field and E a quadratic extension of F. Let V and W be vector spaces over E equipped, respectively, with a Hermitian form $(,)_V$ and* *a skew-Hermitian form $(,)_W$. The isometry groups $U(V)$ and $U(W)$ form a dual pair in $Sp(V \otimes_E W)$, where $V \otimes_E W$ is viewed as a symplectic vector space over F with symplectic form*

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \frac{1}{2} Tr_{E/F}((v_1, v_2)_V \overline{(w_1, w_2)_W}).$$

Then the metaplectic covers of $U(V)_0$ and $U(W)_0$ induced from $Mp(V \otimes_E W)$ both split.

Proof. It suffices to consider $U(V)$ by symmetry. Harris, Kudla and Sweet computed splittings

$$U(V) \times U(W) \longrightarrow GMp(V \otimes_E W) \tag{5.17}$$

explicitly in [HKS], where GMp denotes the full $\mathbb{C}^1$-cover of $Sp(V \otimes_E W)$ defined by (3.1). This gives a splitting over $SU(V) = SU(V) \times \{1\}$ by restriction. Since $SU(V)$ is equal to its commutator subgroup, (5.17) takes $SU(V)$ into the group of commutators of $GMp(V \otimes_E W)$, which is exactly $Mp(V \otimes_E W)$ [MVW, Ch.2.II.9]. This splitting map is unique.

Next, include $U(1)$ into $U(V)$ via the inclusion of the 1-dimensional Hermitian space into V. We deduce from [Ad, Lemma 1.1] and Proposition 2.7 that the metaplectic cover of $U(1)$ induced by its map into $Sp(V \otimes_E W)$ splits over $U(1)_0$ when $p = 2$, and over all of $U(1)$ when $p \neq 2$.

These two splittings together give a splitting over $U(V)_0 \subseteq SU(V) \rtimes U(1)_0$, since the uniqueness of the splitting over $SU(V)$ implies, in particular, that it will be normalized by the chosen splitting over $U(1)_0$. □

Remark 5.7. Lemma 5.6 fails for $F = \mathbb{R}$: the metaplectic cover of $U(1) = U(1)_0$ does not split in that case.

Therefore the metaplectic cover of $G_0^{\phi,m} \times G_0^{\phi,m'}$ induced by $Sp(V^m[1] \otimes_E \overline{V^{m'}[0]})$ splits for all p-adic fields F. Apply Proposition 3.4 to deduce that the metaplectic cover of G_0^{ϕ} induced from its action on all of $Sp(\mathfrak{g}[1])$ must split.

5.7. **Summary.** In light of Remark 5.1, we have proven the following theorem.

Theorem 5.8. *Suppose F is a p-adic field, and let E be a quadratic extension of F.*

If $G = Sp(V, F)$, then a coadjoint orbit $G \cdot f$ is admissible if and only if, in the partition corresponding to the orbit, the number of even parts (counted with multiplicity) greater than any odd part is even.

If $G = O(V, F)$ or $SO(V, F)$, then a coadjoint orbit $G \cdot f$ is admissible if and only if, in the partition corresponding to the orbit, the number of odd parts (counted with multiplicity) less than any even part is even.

If $G = U(V) \subset GL(V, E)$, or $SU(V)$, then every coadjoint orbit is admissible.

This characterization of the admissible orbits might seem no more than a peculiarity, if not for the notion of *special* nilpotent orbits, as defined by Lusztig and Spaltenstein. One of the descriptions of the set of special orbits in the classical groups is given in terms of the partition-type classification of nilpotent orbits (see [CMcG, §6.3]). We deduce the following corollary.

Corollary 5.9. *Let G and F be as in Theorem 5.8. The admissible orbits under G coincide with the special orbits.*

6. Deciding admissibility for the special and general linear groups

Let F be a p-adic field, and let V be an n-dimensional vector space over F. The *general linear group* $GL(n, F)$ is the group of all automorphisms of V. The *special linear group* $SL(n, F)$ consists of all $g \in GL(n, F)$ with determinant 1. Similarly, we can identify the Lie algebra $\mathfrak{gl}(n, F)$ with $\mathrm{End}(V)$, and $\mathfrak{sl}(n, F)$ with all endomorphisms of trace 0. It will be convenient for us to use the isomorphism $\mathrm{End}(V) = V \otimes V^*$, where $V^* = \mathrm{Hom}_F(V, F)$. The action of a group element g on $f \in V^*$ is given by $(gf)(v) = f(g^{-1}v)$, for all $v \in V$. With respect to these identifications, the adjoint action of $G = GL(n, F)$ (or of $G = SL(n, F)$) is given by $\mathrm{Ad}\, g(v \otimes f) = gv \otimes gf$ for all $v \otimes f \in \mathfrak{g}$.

We determine the admissibility of nilpotent orbits for both groups in this section. When there is no distinction to make, let $(G, \mathfrak{g})$ refer to either pair of Lie group and Lie algebra.

Let $G \cdot f$ be a nilpotent orbit in $\mathfrak{g}$, and let ϕ be a corresponding $\mathfrak{sl}(2, F)$-triple as in Section 4. Then $\phi = \phi(\mathfrak{sl}(2, F))$ is a subalgebra of $\mathfrak{g}$, and so acts on the space V. Decompose V into irreducible subrepresentations under this action, and further into weight spaces under $\phi(H)$, as in Section 5. We have

$$V = \bigoplus_{m \geq 1} V^m \cong \bigoplus_{m \geq 1} \bigoplus_{i=-m+1}^{m-1} V^m[i], \tag{6.1}$$

where V^m is the subspace of all copies of the m-dimensional irreducible representation of $\mathfrak{sl}(2, F)$ in V, and $V^m[i]$ denotes its ith weight space with respect to $\phi(H)$. The group G^ϕ will preserve each $V^m[i]$, since it intertwines the $\mathfrak{sl}(2, F)$-action on V. The space V^* decomposes in the same way, with the action on each component $(V^*)^m[i] \cong (V^m[-i])^*$ given by the adjoint (negative transpose).

The next step is to determine $\mathfrak{g}[1]$. Since any central part of $\mathfrak{gl}(n, F)$ lies in its zero weight space, $\mathfrak{g}[1]$ is the same for both $\mathfrak{gl}(n, F)$ and $\mathfrak{sl}(n, F)$. With respect to the identification of $\mathfrak{gl}(n, F)$ with $V \otimes V^*$, we define, for each pair of positive integers (m, m'), with m even and m' odd, a subalgebra

$$\mathfrak{g}_{m,m'}[1] = \bigoplus_{-m+1 \leq i \leq m-1} \left(V^m[i] \otimes (V^*)^{m'}[-i+1] \;\oplus\; V^{m'}[-i+1] \otimes (V^*)^m[i] \right).$$

Note that the terms $V^{m'}[-i+1]$, $(V^*)^{m'}[-i+1]$ are zero if $-i+1$ is not between $m'-1$ and $-m'+1$.

Lemma 6.1. *The space* $\mathfrak{g}[1]$ *decomposes as*

$$\mathfrak{g}[1] = \bigoplus_{(m,m')} \mathfrak{g}_{m,m'}[1], \tag{6.2}$$

where this sum runs over all pairs (m, m') *as above. This decomposition respects the symplectic form on* $\mathfrak{g}[1]$*, as well as the action of* G^ϕ*; each* $\mathfrak{g}_{m,m'}[1]$ *is a symplectic,* G^ϕ*-invariant subspace of* $\mathfrak{g}[1]$.

Proof. It is clear that (6.2) is an isomorphism of vector spaces. The G^ϕ-equivariance of the decomposition is also clear; G^ϕ, in fact, preserves each of the subspaces $V^m[i]$, for any integer $m > 0$.

What remains to be proven is that, with respect to the symplectic form ω on $\mathfrak{g}[1]$, $\omega(\mathfrak{g}_{m,m'}[1], \mathfrak{g}_{n,n'}[1]) = 0$ whenever $(m, m') \neq (n, n')$ are pairs as above. As in the proof of Lemma 5.4, let $Z_1 \in \mathfrak{g}_{m,m'}[1]$, $Z_2 \in \mathfrak{g}_{n,n'}[1]$ be arbitrary. Without loss of generality, assume we have $Z_1 = v_1 \otimes f_1$ and $Z_2 = v_2 \otimes f_2$, where either $v_1 \otimes f_1 \in V^m[i] \otimes (V^*)^{m'}[-i+1]$ or $v_1 \otimes f_1 \in V^{m'}[-i+1] \otimes (V^*)^m[i]$. Similarly, for $v_2 \otimes f_2$, with m, m', i replaced by n, n', j. To distinguish the many cases, let $m(v_k)$ (respectively $m(f_k)$) denote the dimension of the $\mathfrak{sl}(2,F)$-subspace in which v_k (respectively f_k) lies, for $k = 1, 2$.

Then we compute, for each $w \in V$,

$$\begin{aligned} [Z_1, Z_2]w &= [v_1 \otimes f_1, v_2 \otimes f_2]w \\ &= (v_1 \otimes f_1)(v_2 \otimes f_2)w - (v_2 \otimes f_2)(v_1 \otimes f_1)w \\ &= (v_1 \otimes f_1)(f_2(w)v_2) - (v_2 \otimes f_2)(f_1(w)v_1) \\ &= f_2(w)f_1(v_2)v_1 - f_1(w)f_2(v_1)v_2. \end{aligned}$$

Consequently, using (5.11) and summing over dual bases as in (5.13), we deduce

$$\begin{aligned} \omega(Z_1, Z_2) &= \operatorname{Tr} \phi(Y)[Z_1, Z_2] \\ &= f_2(\phi(Y)v_1)f_1(v_2) - f_1(\phi(Y)v_2)f_2(v_1). \end{aligned}$$

Recalling that $f(v) \neq 0$ only if $m(v) = m(f)$ and the $\mathfrak{sl}(2,F)$-weight of v is the negative of that of f, we immediately conclude that $\omega(v_1 \otimes f_1, v_2 \otimes f_2) = 0$ unless $m(v_1) = m(f_2)$ and $m(v_2) = m(f_1)$, as required. □

Finally, note that each $\mathfrak{g}_{m,m'}[1]$ (m even, m' odd) contains the G^ϕ-invariant Lagrangian given by

$$l_{m,m'} = \bigoplus_{-m+1 \le i \le m-1} V^m[i] \otimes (V^*)^{m'}[-i+1].$$

It follows from Proposition 3.7 that the metaplectic cover of G^ϕ arising from its action on each of the symplectic spaces $\mathfrak{g}_{m,m'}[1]$ splits over G_0^ϕ. By Corollary 3.5, this implies that the entire metaplectic double cover of G_0^ϕ corresponding to $\mathfrak{g}[1]$ splits, and that the orbit $G \cdot f$ is admissible.

We have proven the following theorem.

Theorem 6.2. *Let F be a p-adic field. For the groups $GL(n,F)$ and $SL(n,F)$, every nilpotent coadjoint orbit is admissible.*

All orbits under these groups are special, so we have the following immediate corollary.

Corollary 6.3. *Let F be a p-adic field. For the groups $GL(n,F)$ and $SL(n,F)$, the admissible orbits coincide exactly with the special orbits.*

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