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TRANSFER FACTORS FOR LIE ALGEBRAS

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Abstract. Let G be a quasi-split connected reductive group over a local field of characteristic 0, and fix a regular nilpotent element in the Lie algebra $\mathfrak{g}$ of G. A theorem of Kostant then provides a canonical conjugacy class within each stable conjugacy class of regular semisimple elements in $\mathfrak{g}$. Normalized transfer factors take the value 1 on these canonical conjugacy classes.

1. Introduction

Let k_0 be a local field of characteristic 0, let k be an algebraic closure of k_0, and let Γ denote the Galois group of k over k_0. Let G be a connected reductive group over k_0, with Lie algebra $\mathfrak{g}$. We write $\mathfrak{g}(k_0)$ for the set of k_0-rational points on $\mathfrak{g}$. Let $(H, \mathcal{H}, s, \xi)$ be endoscopic data [LS87] for G. In [LS87] Langlands and Shelstad define transfer factors for G relative to $(H, \mathcal{H}, s, \xi)$, and they conjecture that their transfer factors can be used to define a notion of endoscopic induction (analogous to parabolic induction), associating to any stably invariant distribution on $H(k_0)$ an invariant distribution on $G(k_0)$.

Waldspurger [Wal97] has shown that the conjecture of Langlands and Shelstad would follow from another conjecture, known as the fundamental lemma. Waldspurger's method is to study endoscopy on the Lie algebra $\mathfrak{g}$, and he starts by defining transfer factors for Lie algebras, analogous to those of Langlands and Shelstad.

The main result of this paper gives a new way to express these transfer factors for Lie algebras in the case that the group G is quasi-split over k_0, which we now assume. In order to state this result precisely we must specify how we are normalizing our transfer factors. Therefore we need to fix a k_0-splitting $\mathbf{spl} = (B_0, T, \{X_\alpha\})$ for G. Thus B_0 is a Borel subgroup of G over k_0, T is a maximal k_0-torus in B_0, and $\{X_\alpha\}$ is a collection of simple root vectors $X_\alpha \in \mathfrak{g}_\alpha$, one for each simple root α of T in the Lie algebra of B_0, having the property that $X_{\sigma\alpha} = \sigma(X_\alpha)$ for all $\sigma \in \Gamma$. (As usual, for any root β of T in $\mathfrak{g}$ we write $\mathfrak{g}_\beta$ for the corresponding root subspace of $\mathfrak{g}$.)

Waldspurger's factors are analogous to the transfer factors $\Delta(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G)$ on p. 248 of [LS87] with the factor Δ_{IV} removed. On the quasi-split group G Langlands and Shelstad also define transfer factors $\Delta_0(\gamma_H, \gamma_G)$ (again on p. 248 of [LS87]). These depend on the chosen k_0-splitting $\mathbf{spl}$. The transfer factors $\Delta_0'(X_H, X_G)$ in this paper are complex roots of unity, analogous to $\Delta_0(\gamma_H, \gamma_G)$ with the factor Δ_{IV} removed, and they too depend on the choice of k_0-splitting.

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We write $\mathfrak{b}_0$ for the Lie algebra of the Borel subgroup B_0. For each simple root α we define a root vector $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ by requiring that $[X_\alpha, X_{-\alpha}]$ be the coroot for α, viewed as an element in the Lie algebra of T. We put $X_- := \sum_\alpha X_{-\alpha}$, where α runs over the set of simple roots of T in B_0. Of course X_- lies in $\mathfrak{g}(k_0)$ and depends on the choice of k_0-splitting.

The main result of this paper, Theorem 5.1, is that $\Delta_0'(X_H, X_G) = 1$ whenever X_G lies in $\mathfrak{b}_0(k_0)+X_-$. By results of Kostant [Kos63] every stable conjugacy class of regular semisimple elements in $\mathfrak{g}(k_0)$ meets the set $\mathfrak{b}_0(k_0)+X_-$. Since the values of $\Delta_0'(X_H, X_G)$ and $\Delta_0'(X_H, X_G')$ are related by a simple Galois-cohomological factor whenever X_G and X_G' are stably conjugate, this main result also yields a simple formula (see Corollary 5.2) for $\Delta_0'(X_H, X_G)$ for arbitrary X_G.

The methods used in this paper are variants of ones used in [Lan83], [LS87], [She89]. In particular we rely on a key result of Langlands [Lan83], namely Proposition 5.2. However, since we need this result in a slightly different form, we have included its proof, which is essentially just a rearrangement of the one given by Langlands. The new ingredient in this paper is the connection with Kostant's section. Kostant's section [Kos63] is reviewed in 2.4, following Drinfeld's exposition in a lecture at the IAS in February, 1997.

Throughout this paper we follow the convention that Lie algebras of groups G, B, T are denoted by the corresponding gothic letters $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, and so on. We also use $N_G(T)$ to denote the normalizer in G of T. In the first several sections we work over an algebraically closed field k of characteristic 0; later we work over a local field k_0 with algebraic closure k.

2. Review of Kostant's sections of $\mathfrak{g} \to \mathfrak{t}/W$

2.1. **Basic definitions.** Let G be a connected reductive group over an algebraically closed field k of characteristic 0. We fix a maximal torus T in G and a Borel subgroup B_0 of G containing T; thus $B_0 = TN_0$ where N_0 denotes the unipotent radical of B_0. We write B_∞ for the unique Borel subgroup of G containing T that is opposed to B_0; thus $B_\infty = TN_\infty$ where N_∞ denotes the unipotent radical of B_∞. For any root α of T in G we write $\mathfrak{g}_\alpha$ for the root space of T in $\mathfrak{g}$ corresponding to α. Then $\mathfrak{n}_0 = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$ and $\mathfrak{n}_\infty = \bigoplus_{\alpha<0} \mathfrak{g}_\alpha$. For each simple positive root α we fix a non-zero root vector $X_\alpha \in \mathfrak{g}_\alpha$. The triple $\mathbf{spl} := (B_0, T, \{X_\alpha\})$ is called a *splitting* of G.

Let α be a simple positive root. Let H_α be the coroot for α regarded as an element in $\mathfrak{t}$, and fix $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ by the requirement that $[X_\alpha, X_{-\alpha}] = H_\alpha$. There is a unique homomorphism $\phi_\alpha : SL(2) \to G$ whose differential sends the elements

$$h_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in \mathfrak{sl}(2)$$

to H_α, X_α, $X_{-\alpha}$ respectively.

Let $X_+ := \sum_\alpha X_\alpha$, where the index α runs over all simple positive roots. Let $\mu \in X_*(T)$ be the sum of the positive coroots for T; recall that $\langle \alpha, \mu \rangle = 2$ for every simple positive root α. We now decompose $\mathfrak{g}$ as $\mathfrak{g} = \bigoplus_m \mathfrak{g}(m)$, where $\mathfrak{g}(m)$ is the $(-2m)$-th weight subspace

$$\{Z \in \mathfrak{g} \mid \mathrm{Ad}(\mu(a))Z = a^{-2m} Z \text{ for all } a \in \mathbf{G}_m\}$$

for the adjoint action $\mathrm{Ad}\circ\mu$ of $\mathbf{G}_m$ on $\mathfrak{g}$. For any $d \in \mathbf{Z}$ we put $F^d\mathfrak{g} = \bigoplus_{m\geq d}\mathfrak{g}(m)$. Thus $F^0\mathfrak{g} = \mathfrak{b}_\infty$ and $F^1\mathfrak{g} = \mathfrak{n}_\infty$. Note also that $X_+ \in \mathfrak{g}(-1)$ and hence that $\mathrm{ad}(X_+)(\mathfrak{n}_\infty) \subset \mathfrak{b}_\infty$.

2.2. **Chevalley's Theorem.** We denote by $k[\mathfrak{g}]$ (respectively, $k[\mathfrak{t}]$) the k-algebra of polynomial functions on $\mathfrak{g}$ (respectively, $\mathfrak{t}$). The adjoint action of G on $\mathfrak{g}$ induces an action of G on $k[\mathfrak{g}]$, and the action of the Weyl group W (of T in G) on $\mathfrak{t}$ induces an action of W on $k[\mathfrak{t}]$. A theorem of Chevalley (see [CG97] for example) states that the restriction mapping $k[\mathfrak{g}] \to k[\mathfrak{t}]$ induces an isomorphism $k[\mathfrak{g}]^G \to k[\mathfrak{t}]^W$ from the ring of G-invariants in $k[\mathfrak{g}]$ to the ring of W-invariants in $k[\mathfrak{t}]$. We denote the affine k-variety corresponding to $k[\mathfrak{t}]^W$ by $\mathfrak{t}/W$. The morphism $u : \mathfrak{g} \to \mathfrak{t}/W$ dual to $k[\mathfrak{t}]^W = k[\mathfrak{g}]^G \hookrightarrow k[\mathfrak{g}]$ sends Z to the W-orbit in $\mathfrak{t}$ consisting of elements that are G-conjugate to the semisimple part Z_s of the Jordan decomposition $Z = Z_s + Z_n$ of Z. (Thus Z_s is semisimple, Z_n is nilpotent, and $[Z_s, Z_n] = 0$.)

2.3. **Regular elements of $\mathfrak{g}$.** An element $Z \in \mathfrak{g}$ is said to be *regular* if the dimension of its centralizer in $\mathfrak{g}$ is equal to the dimension of $\mathfrak{t}$. Recall [Kos63] that Z is regular if and only if the nilpotent part Z_n of the Jordan decomposition of Z is a regular element in the centralizer in $\mathfrak{g}$ of the semisimple part Z_s of Z. The map $Z \mapsto Z_s$ induces (see [Kos63]) a bijection from the set of regular $\mathrm{Ad}(G)$-orbits in $\mathfrak{g}$ to the set of semisimple $\mathrm{Ad}(G)$-orbits in $\mathfrak{g}$, and via the map u both sets of orbits can be identified with $\mathfrak{t}/W$.

2.4. **Kostant's section.** Kostant proved that every element in the affine subspace $\mathfrak{b}_\infty + X_+$ of $\mathfrak{g}$ is regular (see Lemma 10 in [Kos63]). For any $H \in \mathfrak{t} \subset \mathfrak{b}_\infty$, the semisimple part of $H + X_+$ is conjugate to H (see Lemma 11 in [Kos63]), and hence $\mathfrak{t} + X_+$ meets every regular $\mathrm{Ad}(G)$-orbit in $\mathfrak{g}$.

Now let $\mathfrak{a}$ be any linear subspace of $\mathfrak{b}_\infty$ that is complementary to $\mathrm{ad}(X_+)(\mathfrak{n}_\infty)$ and stable under the action $\mathrm{Ad}\circ\mu$ of $\mathbf{G}_m$ (with μ as in 2.1). Then Kostant proved (see Remark 19′ in [Kos63]) that $\mathfrak{a} + X_+$ meets every regular $\mathrm{Ad}(G)$-orbit exactly once, and that the composition of the closed embedding $\mathfrak{a} + X_+ \hookrightarrow \mathfrak{g}$ and the morphism $u : \mathfrak{g} \to \mathfrak{t}/W$ is an isomorphism $\mathfrak{a} + X_+ \to \mathfrak{t}/W$ of algebraic varieties. Moreover Kostant proved (see Proposition 19 in [Kos63]) that for any $Z \in \mathfrak{t} + X_+$ there exists a unique element $n(Z) \in N_\infty$ such that $\mathrm{Ad}(n(Z))(Z) \in \mathfrak{a} + X_+$, and that the map $Z \to n(Z)$ is a morphism of algebraic varieties from $\mathfrak{t} + X_+$ to N_∞. But in fact the method of proof of Kostant's Proposition 19 shows more, namely that for any $Z \in \mathfrak{b}_\infty + X_+$ there exists a unique element $n(Z) \in N_\infty$ such that $\mathrm{Ad}(n(Z))(Z) \in \mathfrak{a} + X_+$, and that the map $Z \to n(Z)$ is a morphism of algebraic varieties from $\mathfrak{b}_\infty + X_+$ to N_∞. It follows that the map $(n, Y) \mapsto \mathrm{Ad}(n)(Y)$ is an isomorphism of varieties from $N_\infty \times (\mathfrak{a} + X_+)$ to $\mathfrak{b}_\infty + X_+$.

The results discussed above can be summarized as follows. Every element of $\mathfrak{b}_\infty + X_+$ is regular. The group N_∞ acts on $\mathfrak{b}_\infty + X_+$ (by the adjoint action), and the morphism $\mathfrak{b}_\infty + X_+ \to \mathfrak{t}/W$ (obtained by restricting u to $\mathfrak{b}_\infty + X_+$) is a principal N_∞-bundle. Moreover, this principal bundle admits sections and is therefore trivial.

3. Review of the variety S of stars

3.1. **Notation.** In this section G is again a connected reductive group over an algebraically closed field k of characteristic 0. We fix a maximal torus T in G. We write $X_*(T)$ for the lattice of cocharacters of T, and we write $X_*(T)_{\mathbf{R}}$ for the real vector space obtained from $X_*(T)$ by extending scalars from $\mathbf{Z}$ to $\mathbf{R}$. We write $\mathcal{C}$ for the set of Weyl chambers in $X_*(T)_{\mathbf{R}}$. For any Weyl chamber $C \in \mathcal{C}$ we write B_C for the corresponding Borel subgroup of G containing T. Thus any root of T appearing in the unipotent radical of B_C takes positive values on the chamber C. For any pair C, D of adjacent Weyl chambers, we let $P_{C,D}$ be the unique parabolic subgroup of G that contains both B_C and B_D and whose Levi component has semisimple rank 1. Thus the Lie algebra of $P_{C,D}$ is the direct sum of the Lie algebra of B_C and the root space for the unique root of T that is negative for C and positive for D.

We denote by W the Weyl group of T in G. The group W acts on the left of $X_*(T)$ and $\mathcal{C}$.

3.2. **Definition of the variety of stars.** We now review the definition of the variety S of stars, introduced by Langlands in [Lan83]. Let $\mathcal{B}$ be the flag variety of G. Thus elements of $\mathcal{B}$ are Borel subgroups in G, and G acts on the left of $\mathcal{B}$ by conjugation. Consider a map $\mathcal{C} \to \mathcal{B}$, which we think of as a collection $(B(C))_{C \in \mathcal{C}}$ of Borel subgroups $B(C)$ of G indexed by the set of chambers $C \in \mathcal{C}$. We say that $(B(C))_{C \in \mathcal{C}}$ is a *star* if for every pair C, D of adjacent chambers there exists $g \in G$ such that $gB(C)g^{-1} = B_C$ and $gB(D)g^{-1} \subset P_{C,D}$ (equivalently, for every such pair C, D either $B(C) = B(D)$ or there exists $g \in G$ such that conjugation by g carries the pair $(B(C), B(D))$ into (B_C, B_D)). The set of stars is a Zariski closed subset of the Cartesian product $\mathcal{B} \times \cdots \times \mathcal{B}$, where the factors in the product are indexed by $\mathcal{C}$. Thus the set S of stars is a projective algebraic variety, and the diagonal left action of G on $\mathcal{B} \times \cdots \times \mathcal{B}$ preserves the subset of stars, so that G acts on the left of S. There is also an obvious right action of W on S: an element $w \in W$ sends a star $(B(C))_{C \in \mathcal{C}}$ to the star $C \mapsto B(w(C))$. The actions of G and W on S commute.

A star is said to be *regular* if $B(C) \neq B(D)$ for every pair C, D of adjacent chambers. The set S^0 of regular stars is a Zariski open subset of S, and it is preserved by the actions of G and W. There is an obvious base-point $s_0 \in S^0$, namely the regular star $C \mapsto B_C$. The action of G on S^0 is transitive, and the stabilizer in G of the base-point is T, so that $S^0 \simeq G/T$ as (G, W)-varieties.

3.3. **Some rational functions $z(C, \beta)$ on S.** At this point we need to fix a splitting $(B_0, T, \{X_\alpha\})$ (see 2.1) whose torus component is the torus T we fixed at the beginning of this section. As in 2.1 we write $B_0 = TN_0$ and $B_\infty = TN_\infty$, where B_∞ is the unique Borel subgroup containing T and opposite to B_0. The group B_∞ has a unique open orbit on $\mathcal{B}$, namely the big cell consisting of all Borel subgroups $B \in \mathcal{B}$ opposed to B_∞. The Borel subgroup B_0 lies in the big cell, and the group N_∞ acts simply transitively on the big cell, so that every Borel subgroup B opposed to B_∞ can be written as $B = nB_0n^{-1}$ for a unique element $n \in N_\infty$.

Following Langlands [Lan83], we let $S(B_\infty)$ denote the Zariski open subset of S consisting of all stars $C \mapsto B(C)$ such that $B(C)$ is opposed to B_∞ for all $C \in \mathcal{C}$. For each pair (C, β) consisting of a chamber $C \in \mathcal{C}$ and a root β of T that is simple relative to C (that is, β is positive on C and its kernel is a wall of C), Langlands [Lan83] defines a regular function $z(C, \beta)$ on $S(B_\infty)$. This regular function $z(C, \beta)$ depends on the choice of splitting $(B_0, T, \{X_\alpha\})$. Let w be the unique element in W such that $C = w(C_0)$, where C_0 is the unique chamber in $\mathcal{C}$ such that $B_{C_0} = B_0$, and put $\alpha := w^{-1}(\beta)$; note that α is simple relative to B_0. The value of the regular function $z(C, \beta)$ at a star $(B(C))$ in $S(B_\infty)$ is defined to be the unique element z in the field k such that

$$nB(D)n^{-1} = \exp(-zX_{-\alpha})B_0 \exp(zX_{-\alpha}), \tag{3.1}$$

where $D \in \mathcal{C}$ is the unique chamber adjacent to C across the wall defined by β, and where $n \in N_\infty$ is uniquely determined by the condition $nB(C)n^{-1} = B_0$.

It is obvious that the right W-action on S preserves the subset $S(B_\infty)$. Moreover, the W-action permutes the regular functions $z(C, \beta)$. More precisely, for $w \in W$ we have

$$z(C, \beta) \circ r_w = z(w(C), w(\beta)), \tag{3.2}$$

where $r_w : S(B_\infty) \to S(B_\infty)$ denotes the right action of w.

The left action of $B_\infty = TN_\infty$ on S preserves the subset $S(B_\infty)$. The regular functions $z(C, \beta)$ on $S(B_\infty)$ are invariant under N_∞ and transform as follows under T. Let $t \in T$ and let $l_t : S(B_\infty) \to S(B_\infty)$ denote the left action of t. We use (C, β) to define a B_0-simple root α as before, so that there exists $w \in W$ carrying (C_0, α) into (C, β). Then the transformation law is

$$z(C, \beta) \circ l_t = \alpha(t)^{-1} z(C, \beta). \tag{3.3}$$

We should note that Langlands only defines $z(C, \beta)$ on a certain cross-section for the N_∞-action on $S(B_\infty)$, but since there is no essential difference between a regular function on this cross-section and an N_∞-invariant regular function on $S(B_\infty)$, we have retained the notation used by Langlands for our extended functions.

3.4. **Coordinates on $N_\infty \backslash S(B_\infty)$.** Let $C \mapsto B(C)$ be a star in $S(B_\infty)$. Thus there exists a unique element $n \in N_\infty$ such that $B(C_0) = nB_0n^{-1}$. Moreover for each element $w \in W$ there is a unique element $n_w \in N_\infty$ such that $B(wC_0) = nn_wB_0(nn_w)^{-1}$. The elements n_w can be easily expressed [Lan83] in terms of the values of the regular functions $z(C, \beta)$ at the given star. Indeed, let $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_p}$ be a reduced decomposition for w, where, as usual, we write s_α for the simple reflection associated to the simple root α. For $i = 1, \dots, p$ put $C_i := s_{\alpha_1} \dots s_{\alpha_i} C_0$ and $\beta_i = s_{\alpha_1} \dots s_{\alpha_{i-1}} \alpha_i$. Thus for $i = 1, \dots, p$ the chambers C_{i-1}, C_i are adjacent and separated by β_i. For $i = 1, \dots, p$ let z_i denote the value of the function $z(C_{i-1}, \beta_i)$ on the given star. Then it follows immediately from the definition of the regular functions $z(C_{i-1}, \beta_i)$ that

$$n_w = \exp(-z_1 X_{-\alpha_1}) \exp(-z_2 X_{-\alpha_2}) \dots \exp(-z_p X_{-\alpha_p}). \tag{3.4}$$

3.5. **Refined Bruhat decomposition.** We will soon need the following two well-known statements (part of Bruhat theory). The first statement is that the inclusion of $N_G(T)$ in G induces a bijection from $N_G(T)$ to $N_0 \backslash G / N_0$. The second statement is that $N_0 x_1 N_0 \cdot N_0 x_2 N_0 = N_0 x_1 x_2 N_0$ whenever $x_1, x_2 \in N_G(T)$ satisfy the condition that the length of the image of $x_1 x_2$ in W is the sum of the lengths of the images of x_1, x_2.

3.6. **Analysis of the W-action on $S(B_\infty)$.** Now let $C \mapsto B(C)$ be a regular star in $S(B_\infty)$. Denote this star by s and as usual denote the standard star $C \mapsto B_C$ by s_0. We claim that there exists a unique element $g \in N_\infty N_0$ such that $s = gs_0$. Indeed, let g_1 be any element in G such that $s = g_1 s_0$. Since $B(C_0)$ lies in the big cell, there exists $n \in N_\infty$ such that $g_1 \in nB_0$. Modifying the element g_1 on the right by a suitable element in T, we see that there exists $u \in N_0$ such that $s = nus_0$. This proves the existence of g; its uniqueness is obvious. The uniqueness of the factorization $g = nu$ is also obvious. Note that the element n agrees with the element denoted by n in 3.4.

Recall that there is right action of W on $S(B_\infty)$. Let $w \in W$. Then $s' := sw$ is another regular star in $S(B_\infty)$, so there exists unique $g' = n'u' \in N_\infty N_0$ such that $s' = g's_0$. Let $x \in G$ be defined by the equation $g' = gx$. It is obvious that $x \in N_G(T)$ and that $x \mapsto w$ under the canonical surjection $N_G(T) \to W$, and it is also obvious that x depends only on w and the N_∞-orbit of s. Thus one would expect to be able to express x in terms of w and the values of the regular functions $z(C, \beta)$ on s. Such a formula is implicit in [Lan83] (see the proof of Proposition 5.2 of that paper), and is stated explicitly in the next lemma.

As in 3.4 we choose a reduced decomposition $w = s_{\alpha_1} \dots s_{\alpha_p}$ and use it and the star s to define scalars $z_1, \dots, z_p$. For any simple root α we put

$$\dot{s}_\alpha := \phi_\alpha \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{3.5}$$

where $\phi_\alpha : SL(2) \to G$ is the homomorphism defined in 2.1; of course $\dot{s}_\alpha$ lies in $N_G(T)$ and maps to the simple reflection s_α in W.

Lemma 3.1. *The element x defined above is given by*

$$x = (\alpha_1^\vee(z_1)^{-1}\dot{s}_{\alpha_1})(\alpha_2^\vee(z_2)^{-1}\dot{s}_{\alpha_2}) \dots (\alpha_p^\vee(z_p)^{-1}\dot{s}_{\alpha_p}).$$

Proof. The star s' is equal to $C \mapsto B'(C) := B(w(C))$. On the one hand,

$$B'(C_0) = n'B_0(n')^{-1}.$$

On the other hand,

$$\begin{aligned} B'(C_0) &= B(w(C_0)) \\ &= nn_wB_0(nn_w)^{-1}, \end{aligned}$$

with $n_w \in N_\infty$ as in 3.4 (relative to the star s). Therefore $n^{-1}n' = n_w$, and hence

$$x = g^{-1}g' = u^{-1}n^{-1}n'u' \in N_0n_wN_0. \tag{3.6}$$

A simple calculation in $SL(2)$ shows that for any simple root α

$$N_0 \exp(-zX_{-\alpha})N_0 = N_0\alpha^\vee(z)^{-1}\dot{s}_\alpha N_0.$$

Combining this with equation (3.4) and the second statement in section 3.5, we find that

$$N_0n_wN_0 = N_0yN_0, \tag{3.7}$$

where

$$y = (\alpha_1^\vee(z_1)^{-1}\dot{s}_{\alpha_1})(\alpha_2^\vee(z_2)^{-1}\dot{s}_{\alpha_2}) \dots (\alpha_p^\vee(z_p)^{-1}\dot{s}_{\alpha_p}).$$

Comparing (3.6) and (3.7), we see that $N_0xN_0 = N_0yN_0$, and it then follows from the first statement in section 3.5 that $x = y$, as we needed to show. ☐

3.7. **Some 1-cocycles in $N_G(T)$.** Now suppose that G, T, B_0, B_∞ are defined over a subfield k_0 of k such that k is algebraic over k_0. Thus k is an algebraic closure of k_0, and we put $\Gamma := \mathrm{Gal}(k/k_0)$. We write $\mathfrak{g}(k_0)$ for the subset of $\mathfrak{g}$ consisting of k_0-rational points. Let $Y \in \mathfrak{g}(k_0)$ and suppose that Y is regular semisimple. We choose an element $H \in \mathfrak{t}$ in the G-conjugacy class of Y. Of course H need not be defined over k_0.

There exists $g \in G$, unique up to right multiplication by T, such that $Y = \mathrm{Ad}(g)(H)$. For $\sigma \in \Gamma$ put $x_\sigma := g^{-1}\sigma(g)$. Then $\sigma \mapsto x_\sigma$ is a 1-cocycle of Γ in $N_G(T)$, and replacing g by gt (for $t \in T$) replaces x_σ by $t^{-1}x_\sigma\sigma(t)$.

The star $s := gs_0$ is well-defined. We now assume that s lies in $S(B_\infty)$. As in 3.6 we normalize g within its coset gT by insisting that g lie in $N_\infty N_0$. The next lemma gives a formula for the 1-cocycle x_σ (obtained from this particular g) in terms of the values of the functions $z(C, \beta)$ on the star s. This lemma is a variant of Proposition 5.2 in [Lan83], and our proof (including that of Lemma 3.1) is a rearrangement of Langlands's.

Lemma 3.2. *Let $\sigma \in \Gamma$ and let w_σ be the image of x_σ in W. Choose a reduced decomposition $w_\sigma = s_{\alpha_1} \dots s_{\alpha_p}$ of w_σ, and as in section* 3.4 *use it and the star s to define scalars $z_1, \dots, z_p$. Then x_σ is given by*

$$x_\sigma = (\alpha_1^\vee(z_1)^{-1}\dot{s}_{\alpha_1})(\alpha_2^\vee(z_2)^{-1}\dot{s}_{\alpha_2}) \dots (\alpha_p^\vee(z_p)^{-1}\dot{s}_{\alpha_p}).$$

Proof. Since N_∞, N_0 are defined over k_0, the element $\sigma(g)$ also lies in $N_\infty N_0$. The lemma now follows from Lemma 3.1, applied to the elements g and $g' = \sigma(g)$. □

4. 1-cocycles coming from Kostant's section

4.1. **Goal.** Our next goal is to calculate the 1-cocycles of section 3.7 for elements Y lying in the image of Kostant's section $v : \mathfrak{t}/W \to \mathfrak{g}$. By Lemma 3.2 what we must do is calculate certain values of the functions $z(C, \beta)$.

4.2. **Values of $z(C, \beta)$ on stars coming from Kostant's section.** Let $H \in \mathfrak{t}'$, where $\mathfrak{t}'$ denotes the set of regular elements in $\mathfrak{t}$. We write $p : \mathfrak{t} \to \mathfrak{t}/W$ for the map dual to the inclusion $k[\mathfrak{t}]^W \hookrightarrow k[\mathfrak{t}]$. Let Y be any element in $\mathfrak{b}_\infty + X_+$ such that $u(Y) = p(H)$ (where $u : \mathfrak{g} \to \mathfrak{t}/W$ is as in 2.2). Recall from section 2.4 that the $\mathrm{Ad}(N_\infty)$-orbit of Y is uniquely determined by H.

Now choose $g \in G$ such that $Y = \mathrm{Ad}(g)(H)$ and define a regular star $s := gs_0$ (as in 3.7). The star s depends only on Y, and the N_∞-orbit of s depends only on H.

Lemma 4.1. *The star s lies in $S(B_\infty)$. The value of the regular function $z(C, \beta)$ on s is $\beta(H)$.*

Proof. The construction $H \mapsto s$ is a well-defined map $\delta : \mathfrak{t}' \to N_\infty \backslash S$. The right action of W on S induces a right action of W on $N_\infty \backslash S$, and we define a right action of W on $\mathfrak{t}'$ by converting the usual left action into a right action:

$$Hw := w^{-1}(H).$$

It is then immediate that our map $\delta : \mathfrak{t}' \to N_\infty \backslash S$ is W-equivariant.

The truth of the two statements of the lemma depends only on the N_∞-orbit of s. Therefore we are free to take $Y = H + X_+$. We write the star s obtained from Y and H as $C \mapsto B(C)$. It is well-known (and easy to prove) that since H is regular, the map $u \mapsto \mathrm{Ad}(u)(H)$ is an isomorphism from N_0 to $H + \mathfrak{n}_0$. Therefore there exists $u \in N_0$ such that $Y = \mathrm{Ad}(u)(H)$. Thus $s = us_0$, and it follows that $B(C_0) = B_0$. Since the map $\delta : \mathfrak{t}' \to N_\infty \backslash S$ is W-equivariant, we conclude that $B(C)$ lies in the N_∞-orbit of B_0 for all chambers C, which proves the first statement of the lemma.

Again using the W-equivariance of δ, and using (3.2) as well, we see that in order to prove the second statement of the lemma, it is enough to prove that for every simple root α the function $z(C_0, \alpha)$ takes the value $\alpha(H)$ on s. So fix a simple root α and let $P_\alpha = M_\alpha N_\alpha$ be the associated parabolic subgroup containing B_0;

thus $\mathfrak{m}_\alpha$ is spanned by $\mathfrak{t}$, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, and N_α is the unipotent radical of P_α. The homomorphism $\phi_\alpha : SL(2) \to G$ factors through M_α.

Note that for any scalar z

$$\mathrm{Ad}(\exp(zX_\alpha))(H + X_+) \in H + X_\alpha + [zX_\alpha, H] + \mathfrak{n}_\alpha.$$

Therefore $\mathrm{Ad}(\exp(\alpha(H)^{-1}X_\alpha))(H + X_+)$ lies in $H + \mathfrak{n}_\alpha$. But (again since H is regular) the map

$$n_\alpha \mapsto \mathrm{Ad}(n_\alpha)(H)$$

is an isomorphism from N_α to $H + \mathfrak{n}_\alpha$. Therefore there exists $n_\alpha \in N_\alpha$ such that

$$H + X_+ = \mathrm{Ad}(\exp(-\alpha(H)^{-1}X_\alpha) \cdot n_\alpha)(H).$$

It follows from this equality that

$$B(s_\alpha(C_0)) = \exp(-\alpha(H)^{-1}X_\alpha) \cdot \dot{s}_\alpha \cdot B_0 \cdot (\exp(-\alpha(H)^{-1}X_\alpha) \cdot \dot{s}_\alpha)^{-1}.$$

A simple calculation in $SL(2)$ then shows that

$$B(s_\alpha(C_0)) = \exp(-\alpha(H)X_{-\alpha}) \cdot B_0 \cdot \exp(\alpha(H)X_{-\alpha}),$$

and this (together with the fact that $B(C_0) = C_0$) shows that the value of $z(C_0, \alpha)$ on s is $\alpha(H)$, as desired. □

4.3. **1-cocycles in $N_G(T)$ coming from Kostant's section.** We return to the situation of 3.7. We assume further that the family $\{X_\alpha\}$ is defined over k_0, in the sense that $\sigma(X_\alpha) = X_{\sigma(\alpha)}$ for all $\sigma \in \Gamma$. Thus $(B_0, T, \{X_\alpha\})$ is a k_0-splitting in the terminology of [LS87]. Moreover the element $X_+ = \sum_\alpha X_\alpha$ lies in $\mathfrak{g}(k_0)$.

To define Kostant's section we need to choose a complementary subspace $\mathfrak{a}$ as in 2.4. Of course we may assume that $\mathfrak{a}$ is defined over k_0. Then $\mathfrak{a} + X_+$ is defined over k_0.

Now let $H \in \mathfrak{t}'$ and assume that the image $p(H)$ of H in $\mathfrak{t}/W$ is k_0-rational. Let Y be any k_0-rational element of $\mathfrak{b}_\infty + X_+$ such that $u(Y) = p(H)$. It is clear from the discussion above that such a k_0-rational element exists, and it follows from 2.4 that the $\mathrm{Ad}(N_\infty(k_0))$-orbit of Y is uniquely determined by H. The first statement of Lemma 4.1 implies that the star s defined in 3.7 (for H, Y as above) lies in $S(B_\infty)$. Therefore there exists a unique element $g \in N_\infty N_0$ such that $Y = \mathrm{Ad}(g)(H)$. As in 3.7 we define a 1-cocycle x_σ of Γ in $N_G(T)$ by $x_\sigma = g^{-1}\sigma(g)$. Note that this 1-cocycle is independent of the choice of Y; it depends only on H and our chosen k_0-splitting $\mathbf{spl} = (B_0, T, \{X_\alpha\})$.

Lemma 4.2. *Let $\sigma \in \Gamma$ and let w_σ be the image of x_σ in W. Choose a reduced decomposition $w_\sigma = s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_p}$ of w_σ and for $i = 1, \dots, p$ put $\beta_i = s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_{i-1}}(\alpha_i)$. Then*

$$x_\sigma = \left(\prod_{i=1}^{p} \beta_i^\vee(\beta_i(H))\right)^{-1} \cdot \dot{w}_\sigma,$$

where $\dot{w}_\sigma = \dot{s}_{\alpha_1}\dots\dot{s}_{\alpha_p}$.

Proof. This follows immediately from Lemmas 3.2 and 4.1. □

However the 1-cocycle we really need is a variant of x_σ. We replace $\mathfrak{b}_\infty + X_+$ by $\mathfrak{b}_0 + X_-$, where $X_- := \sum_\alpha X_{-\alpha}$, the sum ranging over simple roots α. In this context Kostant's theory provides an isomorphism (over k_0) from the quotient of $\mathfrak{b}_0 + X_-$ by the adjoint action of N_0 to the space $\mathfrak{t}/W$.

Let $H \in \mathfrak{t}'$ be as above and let Z be any k_0-rational element of $\mathfrak{b}_0 + X_-$ such that $u(Z) = p(H)$. Let h be any element of G such that $Z = \mathrm{Ad}(h)(H)$. Define a 1-cocycle y_σ of Γ in $N_G(T)$ by $y_\sigma = h^{-1}\sigma(h)$. Replacing h by ht, for $t \in T$, has the effect of replacing y_σ by $t^{-1}y_\sigma\sigma(t)$. Thus the 1-cocycle y_σ is well-defined up to such equivalences.

Lemma 4.3. *For a suitable choice of h, the* 1*-cocycle y_σ is given by the following formula. Let $\sigma \in \Gamma$ and let w_σ be the image of y_σ in W. Choose a reduced decomposition $w_\sigma = s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_p}$ of w_σ and for $i = 1, \dots, p$ put $\beta_i = s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_{i-1}}(\alpha_i)$. Then y_σ is given by*

$$y_\sigma = \left(\prod_{i=1}^{p} \beta_i^\vee(\beta_i(H))\right) \cdot \dot{w}_\sigma,$$

where $\dot{w}_\sigma = \dot{s}_{\alpha_1}\dots\dot{s}_{\alpha_p}$.

Proof. Let θ be the automorphism of G that induces -1 on $\mathfrak{t}$ and exchanges X_α and $X_{-\alpha}$ for all simple roots α. It is clear that θ is defined over k_0 and that its square is the identity. Since θ exchanges $\mathfrak{b}_\infty + X_+$ and $\mathfrak{b}_0 + X_-$, it is clear that we get a suitable 1-cocycle y_σ by applying θ to the 1-cocycle x_σ associated to $\theta(H) = -H$. Note that $\theta(\dot{s}_\alpha) = \alpha^\vee(-1)\dot{s}_\alpha$ for any simple root α (a simple calculation in $SL(2)$). The desired result then follows from the previous lemma. ∎

5. Transfer factors $\Delta_0'(X_H, X_G)$ on $\mathfrak{g}$

5.1. **Notation.** As in 3.7 we assume that G is quasi-split over k_0, and we fix a k_0-splitting $\mathbf{spl} = (B_0, T, \{X_\alpha\})$ for G. We assume further that k_0 is a local field (of characteristic 0). Let $(H, \mathcal{H}, s, \xi)$ be endoscopic data for G (see [LS87]).

5.2. **General discussion of transfer factors.** Following Waldspurger [Wal97] we consider transfer factors on $\mathfrak{g}$ analogous to the transfer factors on G defined by Langlands and Shelstad [LS87]. Waldspurger's factors are analogous to the transfer factors $\Delta(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G)$ on p. 248 of [LS87] with the factor Δ_{IV} removed.

On the quasi-split group G Langlands and Shelstad also define transfer factors $\Delta_0(\gamma_H, \gamma_G)$ (again on p. 248 of [LS87]). These depend on the chosen k_0-splitting **spl**. The transfer factors $\Delta_0'(X_H, X_G)$ in this paper are analogous to $\Delta_0(\gamma_H, \gamma_G)$ with the factor Δ_{IV} removed.

5.3. *a***-data.** Let T_H be a maximal torus in H. There is a canonical G-conjugacy class of embeddings $T_H \to G$, and this G-conjugacy class contains members that are defined over k_0. Fix such a k_0-embedding $T_H \to G$, and let T_G denote the image of T_H in G, a maximal torus of G, defined over k_0. We identify T_H with T_G, so that the set R_H of roots of T_H in H becomes a subset of the set R_G of roots of T_G in G.

Recall from [LS87], 2.2 that a-data for T_G consists of elements $a_\alpha \in k^\times$, one for each $\alpha \in R_G$, satisfying

1. $a_{\sigma\alpha} = \sigma(a_\alpha)$ for all $\alpha \in R_G$ and all $\sigma \in \Gamma$,
2. $a_{-\alpha} = -a_\alpha$ for all $\alpha \in R_G$.

We now choose a-data for T_G.

Let G_{sc} denote the simply connected cover of the derived group of G, and let T_G^{sc} denote the inverse image of T_G under the canonical homomorphism from G_{sc} to G. Recall that Langlands and Shelstad (see 2.3 in [LS87]) define an invariant

$\lambda(T_G^{\mathrm{sc}}) \in H^1(k_0, T_G^{\mathrm{sc}})$, which depends on **spl** and the a-data $\{a_\alpha\}$, as well as T_G. We denote by $\lambda(T_G) \in H^1(k_0, T_G)$ the image of $\lambda(T_G^{\mathrm{sc}})$ under the map induced by the canonical homomorphism $T_G^{\mathrm{sc}} \to T_G$.

5.4. **χ-data.** For any root $\alpha \in R_G$ we let k_α (respectively, $k_{\pm\alpha}$) denote the field of definition of α (respectively, the set $\{\alpha, -\alpha\}$). Thus $k_0 \subset k_{\pm\alpha} \subset k_\alpha \subset k$ and $[k_\alpha : k_{\pm\alpha}]$ is 1 or 2. As in [LS87], if $[k_\alpha : k_{\pm\alpha}] = 2$, we say that α (and its Γ-orbit in R_G) is *symmetric*, and we let χ_α denote the quadratic character on $k_{\pm\alpha}^\times$ associated to the quadratic extension $k_\alpha/k_{\pm\alpha}$ by local classfield theory. Transfer factors on $\mathfrak{g}$ are simpler than those on G, in that it is not necessary to extend χ_α to a quasi-character on $k_\alpha^\times$. Consequently, when defining transfer factors on $\mathfrak{g}$, it is irrelevant whether or not $\mathcal{H}$ is isomorphic to ${}^L H$.

5.5. **Definition of $\Delta_0'(X_H, X_G)$.** Let $X_H \in \mathfrak{t}_H$ and assume that its image X_G in $\mathfrak{t}_G$ is regular. We are going to define our transfer factor $\Delta_0'(X_H, X_G)$ by

$$\Delta_0'(X_H, X_G) = \Delta_{\mathrm{I}}(X_H, X_G)\Delta_{\mathrm{II}}(X_H, X_G),$$

where the factors $\Delta_{\mathrm{I}}(X_H, X_G)$, $\Delta_{\mathrm{II}}(X_H, X_G)$, to be defined below, are analogous to the factors $\Delta_{\mathrm{I}}(\gamma_H, \gamma_G)$, $\Delta_{\mathrm{II}}(\gamma_H, \gamma_G)$ of [LS87].

5.6. **Definition of $\Delta_{\mathrm{I}}(X_H, X_G)$.** Let $\hat{T}_G$ denote the complex torus dual to T_G (Langlands duality). The element s appearing in our endoscopic data is a Γ-fixed element in the center of the Langlands dual group $\hat{H}$ of H, and thus can be regarded as a Γ-fixed element $\mathbf{s}_{T_G}$ of $\hat{T}_H = \hat{T}_G$. There is a Tate-Nakayama pairing (see [Kot86] for example)

$$\langle \cdot, \cdot \rangle : H^1(k_0, T_G) \times \hat{T}_G^\Gamma \to \mathbf{C}^\times,$$

where $\hat{T}_G^\Gamma$ denotes the group of fixed points of Γ in $\hat{T}_G$. We define Δ_{I} by

$$\Delta_{\mathrm{I}}(X_H, X_G) := \langle \lambda(T_G), \mathbf{s}_{T_G} \rangle.$$

5.7. **Definition of $\Delta_{\mathrm{II}}(X_H, X_G)$.** We define Δ_{II} by

$$\Delta_{\mathrm{II}}(X_H, X_G) := \prod_\alpha \chi_\alpha \left(\frac{\alpha(X_G)}{a_\alpha} \right),$$

where the product is taken over a set of representatives for the symmetric orbits of Γ in the set $R_G \setminus R_H$.

5.8. **Discussion of $\Delta_0'(X_H, X_G)$.** It is immediate from (the proof of) Lemma 3.2.C of [LS87] that $\Delta_0'(X_H, X_G)$ is independent of the choice of a-data. Thus $\Delta_0'(X_H, X_G)$ depends only on the choice of k_0-splitting **spl**.

Put $\Delta_0'(\gamma_H, \gamma_G) := \Delta_0(\gamma_H, \gamma_G) \cdot \Delta_{\mathrm{IV}}(\gamma_H, \gamma_G)^{-1}$, with Δ_0 and Δ_{IV} as in [LS87]. It is easy to see that for X_H sufficiently close to 0

$$\Delta_0'(X_H, X_G) = \Delta_0'(\exp(X_H), \exp(X_G)).$$

Moreover it is obvious that

$$\Delta_0'(a^2 X_H, a^2 X_G) = \Delta_0'(X_H, X_G)$$

for all $a \in k_0^\times$. These two properties characterize the transfer factors Δ_0' on $\mathfrak{g}$.

Now suppose that $X'_G \in \mathfrak{g}(k_0)$ is stably conjugate to X_G, so that there exists $h \in G$ such that $\mathrm{Ad}(h)(X'_G) = X_G$. Then $\sigma \mapsto h\sigma(h)^{-1}$ is a 1-cocycle of Γ in T_G whose class we denote by $\mathrm{inv}(X_G, X'_G)$. Then

$$\Delta'_0(X_H, X'_G) = \Delta'_0(X_H, X_G) \cdot \langle \mathrm{inv}(X_G, X'_G), s_{T_G}\rangle^{-1}. \tag{5.1}$$

This follows from Lemmas 3.2.B and 3.4.A of [LS87], or rather from their (easy) Lie algebra analogs.

5.9. **Main result.** As above we use our fixed splitting **spl** to define transfer factors $\Delta'_0(X_H, X_G)$ on $\mathfrak{g}$ and to define an element $X_- \in \mathfrak{g}(k_0)$.

Theorem 5.1. *The transfer factor $\Delta'_0(X_H, X_G)$ is equal to 1 whenever X_G lies in the set of k_0-rational elements in $\mathfrak{b}_0 + X_-$.*

Proof. Note that $a_\alpha := \alpha(X_G)$ is a valid choice of a-data for T_G. With this choice of a-data it is obvious that $\Delta_{\mathrm{II}}(X_H, X_G) = 1$, and what we must show is that $\Delta_{\mathrm{I}}(X_H, X_G) = 1$. Since this must be true for all endoscopic data, we must show that $\lambda(T_G^{\mathrm{sc}}) = 1$ for this particular choice of a-data. It is harmless to assume that $G = G_{\mathrm{sc}}$, and we do so in order to simplify notation. Choose $h \in G$ and $H \in \mathfrak{t}$ such that $X_G = \mathrm{Ad}(h)(H)$. (Since the endoscopic data are now irrelevant, it should cause no confusion to use our usual convention of denoting elements of $\mathfrak{t}$ by H.) Thus $hTh^{-1} = T_G$ and we use the inner automorphism $x \mapsto hxh^{-1}$ to identify T with T_G over k. As in 4.3 $y_\sigma := h^{-1}\sigma(h)$ is a 1-cocycle of Γ in $N_G(T)$, and we denote by w_σ the image of y_σ in the Weyl group W. On p. 231 of [LS87] Langlands and Shelstad define a 1-cocycle

$$m_\sigma := \left(\prod_\beta \beta^\vee(a_\beta)\right) \cdot \dot{w}_\sigma$$

of Γ in $N_G(T)$, where $\dot{w}_\sigma$ is defined as in Lemma 4.3, and where the product is taken over all positive roots β of T in G such that $w_\sigma^{-1}(\beta)$ is negative. We are using our identification of T and T_G over k to view β as a root of T_G, so that a_β is defined. For our particular choice of a-data, we have $a_\beta = \beta(H)$, and therefore m_σ is exactly the same as the 1-cocycle appearing in Lemma 4.3. It follows from Lemma 4.3 that by choosing h correctly within its coset hT, we may assume that

$$h^{-1}\sigma(h) = m_\sigma. \tag{5.2}$$

Langlands and Shelstad now define a 1-cocycle of Γ in T_G by $\sigma \mapsto hm_\sigma\sigma(h)^{-1}$, and they define $\lambda(T_G^{\mathrm{sc}})$ to be the class of this 1-cocycle. Since (5.2) shows that this 1-cocycle is trivial, we see that $\lambda(T_G^{\mathrm{sc}})$ is trivial, as desired. □

Corollary 5.2. *The transfer factor $\Delta'_0(X_H, X_G)$ is equal to $\langle \mathrm{inv}(X_G, X'_G), s_{T_G}\rangle$, where X'_G is any k_0-rational element in $\mathfrak{b}_0 + X_-$ that is stably conjugate to X_G.*

Proof. This follows from the theorem together with (5.1). □

References

[CG97] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, Boston, 1997. MR **98i:**22021

[Kos63] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404. MR **28:**1252

[Kot86] R. Kottwitz, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), 365–399. MR **88d:**22027

[Lan83] R. P. Langlands, *Orbital integrals on forms of $SL(3)$, I*, Amer. J. Math. **105** (1983), 465–506. MR **86d:**22012

[LS87] R. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), 219–271. MR **89c:**11172

[She89] D. Shelstad, *A formula for regular unipotent germs*, Astérisque **171–172** (1989), 275–277. MR **91b:**22012

[Wal97] J.-L. Waldspurger, *Le lemme fondamental implique le transfert*, Compositio Math. **105** (1997), 153–236. MR **98h:**22023

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