MODULE EXTENSIONS OVER CLASSICAL
LIE SUPERALGEBRAS

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ABSTRACT. We study certain filtrations of indecomposable injective modules over classical Lie superalgebras, applying a general approach for noetherian rings developed by Brown, Jategaonkar, Lenagan, and Warfield. To indicate the consequences of our analysis, suppose that $\mathfrak{g}$ is a complex classical simple Lie superalgebra and that E is an indecomposable injective $\mathfrak{g}$-module with nonzero (and so necessarily simple) socle L. (Recall that every essential extension of L, and in particular every nonsplit extension of L by a simple module, can be formed from $\mathfrak{g}$-subfactors of E.) A direct transposition of the Lie algebra theory to this setting is impossible. However, we are able to present a finite upper bound, easily calculated and dependent only on $\mathfrak{g}$, for the number of isomorphism classes of simple highest weight $\mathfrak{g}$-modules appearing as $\mathfrak{g}$-subfactors of E.

1. INTRODUCTION

This paper is about prime ideals, indecomposable injective modules, and noetherian ring extensions—building upon [4, 12, 16, 17], [19], [32]. But our work is primarily motivated by, and applicable to, basic questions arising in the representation theory of classical Lie superalgebras.

1.1.   To place our study in context, let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Weyl group W. It is well known, within the corresponding category $\mathcal{O}$, that the Ext (i.e., Ext^1) groups of simple modules can be computed using Kazhdan-Lusztig polynomials [1, 7, 10]. Moreover, calculating Ext within the category of all $\mathfrak{g}$-modules, the following is elementary: Let $\lambda, \lambda' \in \mathfrak{h}^*$. If $L(\lambda)$ and $L(\lambda')$ are simple highest weight $\mathfrak{g}$-modules for which $\mathrm{Ext}\,(L(\lambda), L(\lambda')) \neq 0$, then λ and λ' occur in the same W-orbit in $\mathfrak{h}^*$. Of course, there are no nonsplit extensions of finite dimensional $\mathfrak{g}$-modules.

Generalizing further, let E be the injective hull of $L(\lambda)$, and recall that every essential $\mathfrak{g}$-extension of $L(\lambda)$, and in particular every nonsplit extension of $L(\lambda)$ by a simple $\mathfrak{g}$-module, can be formed from $\mathfrak{g}$-subfactors of E. It follows easily, if $L(\lambda')$ is isomorphic to a simple subfactor of E, that λ and λ' are contained within a single W-orbit; see (5.2). In comparison, if $\mathfrak{g}$ is solvable then the injective hull of a simple finite (and hence one-)dimensional $\mathfrak{g}$-module can have infinitely many pairwise nonisomorphic simple $\mathfrak{g}$-subfactors (cf. [4, §6], recalling that winding-automorphism orbits of prime ideals in this case are either infinite or singletons).

1.2. Now, and for the remainder of this section, let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a complex classical simple Lie superalgebra. (Only in this section, "$\mathfrak{g}$-module" will mean "$\mathbb{Z}_2$-graded $\mathfrak{g}$-module.") The above theory does not smoothly carry over to this setting, as is dramatically illustrated by the following example of Musson [26, §4]: Over $sl(2,1)$ there exists an infinite series $L_1, L_2, \ldots$ of finite dimensional simple modules, no two of which are isomorphic, such that each $\mathrm{Ext}_{sl(2,1)}(L_i, L_{i+1}) \neq 0$. In particular, Kazhdan-Lusztig theory (cf. [29]) does not appear to apply. The extension theory for simple highest weight $\mathfrak{g}$-modules (in the sense, e.g., of [25, 27]— or see (6.4)) is still largely unknown.

To demonstrate the impact of our study on these issues, let E be an indecomposable injective $\mathfrak{g}$-module with nonzero socle (i.e., E is the injective hull of a simple $\mathfrak{g}$-module), and let X be the set of simple highest weight $\mathfrak{g}$-subfactors (up to isomorphism) of E. In view of Musson's example, and the Lie algebra case, it is natural to ask whether X can be infinite. We calculate a finite upper bound— easily determined from $\mathfrak{g}$ and not dependent on E—for the cardinality of X; see (6.7). More precise bounds, concerning nonsplit extensions of simple highest weight modules, are also given; see (6.5). Our bounds do not appear to be very sharp, as they significantly overestimate matters in the case when $\mathfrak{g} = \mathfrak{g}_0$.

To more fully describe the results of this paper, in the language of noetherian ring theory, let $\mathbf{V}$ be the enveloping algebra of $\mathfrak{g}$. The set of *fundamental primes* of an injective $\mathbf{V}$-module was defined in [17], following [4, 12]; see (2.6). Now let P be a prime ideal of $\mathbf{V}$, and let $\mathrm{Fund}(P)$ denote the union of the sets of fundamental primes of indecomposable injective $\mathbf{V}$-modules with associated prime P. Our main theorem establishes a finite upper bound—easily calculated from $\mathfrak{g}$ and not dependent on P—for the cardinality of $\mathrm{Fund}(P)$; see (6.3). Sharper bounds are found for the number of prime ideals "linked" to P by a nonsplit extension of prime submodules; see (2.3ii) and (6.3).

Our general approach is similar to [16], studying "lying over"-type properties of fundamental primes in ring extensions. Also, all of our results for $\mathfrak{g}$ depend on the longstanding and well-known theory for reductive Lie algebras.

2. Noetherian modules

All of the results discussed in this section should be considered already known, even in cases when they have not previously appeared explicitly in the literature; in such instances proofs are included for completeness. Some of the definitions introduced are nonstandard but are especially convenient for our later analysis. The reader is referred to [9] or [23] for general background information on noetherian rings.

2.1. Throughout this section, R will denote a noetherian ring. The set of prime ideals of R will be denoted $\mathrm{Spec}\, R$ and the set of left primitive ideals of R will be denoted $\mathrm{Prim}\, R$. The category of R-modules will be referred to as $\mathrm{Mod}\, R$, and Ext will denote Ext^1. The annihilator in R of a left R-module M will be denoted $\mathrm{ann}_R M$, and the annihilator in R of a right R-module N will be denoted $\mathrm{ann}\, N_R$. Unless otherwise designated, "module" will mean "left module."

2.2. (Following [9].) Let M be a nonzero R-module.

(i) If Q is a (necessarily prime) ideal of R maximal among annihilators of nonzero R-submodules of M, then $\mathrm{ann}\, Q_M = \{m \in M \mid Q.m = 0\}$ is termed an *affiliated*

submodule of M. A series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of R-submodules of M is called an *affiliated series* for M if each M_i/M_{i-1} is an affiliated submodule of M/M_{i-1}, for $1 \leq i \leq n$; the prime ideals $\mathrm{ann}_R(M_i/M_{i-1})$ are the *affiliated primes* of the series. More generally, a prime ideal of R that is an affiliated prime of some affiliated series of M is referred to as an affiliated prime of M.

(ii) If S is a ring and M is an R-S-bimodule, then the affiliated series in (i) consists of R-S-sub-bimodules of M; if M has finite length as an R-S-bimodule, then the set of affiliated primes of M (viewed as a left R-module) coincides with the set of left annihilators of R-S-bimodule composition factors of M.

(iii) If a prime ideal Q of R is the annihilator of a submodule of M, then Q is an *annihilator prime* of M. If $Q = \mathrm{ann}_R M' = \mathrm{ann}_R M$ for all nonzero R-submodules M' of M, and $M \neq 0$, then M is said to be a *prime*, or more precisely, a *Q-prime* R-module. The annihilator of a prime module is necessarily a prime ideal, and an annihilator of a prime R-submodule of M is termed an *associated prime* of M. Note that the annihilator of an affiliated submodule of M is an associated prime. Also, every nonzero module over a noetherian ring contains a prime submodule.

(iv) If M is uniform (i.e., does not contain a direct sum of two nonzero submodules), then it has exactly one associated prime ideal, which is also the unique ideal of R maximal among annihilators of nonzero submodules of M. If M has finite uniform dimension (i.e., contains no direct sum of infinitely many nonzero submodules), then M can be embedded into a finite direct sum of uniform R-module factors of M such that the associated prime of each uniform factor is an associated prime of M. One way to verify this last statement is to recall that the injective hull of M, which is an essential extension of M (i.e., M intersects every nonzero R-submodule of its injective hull nontrivially), will be a finite direct sum of uniform injective R-modules; see, for example, [9, Chapter 4] for more details. Also, recall that an injective R-module is indecomposable if and only if it is uniform.

2.3.   Let Q_α and Q_β be prime ideals of R.

(i) Let A and B be prime noetherian rings, and suppose that C is a nonzero A-B-bimodule finitely generated and torsion free on each side; we say that C is a *bond* from A to B. (A bimodule finitely generated on each side, over prime noetherian rings, is torsion free on each side if and only if it is faithful and prime on each side; see, e.g., [12, 5.1.1].) It follows, for example, from [9, 7.16] that if A is right primitive, then B is right primitive, and if B is left primitive, then A is left primitive. Now suppose there are ideals $I \subset J$ of R such that $\mathrm{ann}_R(I/J) = Q_\alpha$, such that $\mathrm{ann}(I/J)_R = Q_\beta$, and such that I/J is a bond from R/Q_α to R/Q_β. Then we say that I/J is an *ideal link* from Q_α to Q_β and write $Q_\alpha \approx Q_\beta$. (But "$\approx$" is not necessarily an equivalence relation.) If $Q_\alpha Q_\beta \subseteq I \subset J = Q_\alpha \cap Q_\beta$, then we say that there is a *link* from Q_α to Q_β and write $Q_\alpha \rightsquigarrow Q_\beta$. Occasionally, for emphasis, we will refer to links as *direct links*. If there exist prime ideals $Q_1, \ldots, Q_m$ and a sequence of links $Q_\alpha \rightsquigarrow Q_1 \rightsquigarrow \cdots \rightsquigarrow Q_m \rightsquigarrow Q_\beta$, then we write $Q_\alpha \rightsquigarrow \cdots \rightsquigarrow Q_\beta$. (When writing $Q_\alpha \rightsquigarrow \cdots \rightsquigarrow Q_\beta$, we include the possibility that $Q_\alpha = Q_\beta$ and no link exists from Q_α to itself.) In the resulting graph structure on $\mathrm{Spec}\,R$, the connected component containing a prime ideal Q is termed the *clique* of Q and denoted $\mathrm{Clique}(Q)$.

(ii) We will say that a finitely generated uniform R-module M is an *external link* from Q_α to Q_β, and write $Q_\alpha \overset{\text{ext}}{\rightsquigarrow} Q_\beta$, provided that the following condition holds: There exists a short exact sequence

$$0 \to M_\alpha \to M \to M_\beta \to 0$$

of uniform R-modules such that $0 \subset M_\alpha \subset M$ is an affiliated series, such that $Q_\alpha = \text{ann}_R M_\alpha$, such that $Q_\beta = \text{ann}_R M_\beta$, and such that $\text{ann}_R M' = \text{ann}_R M$ for all R-submodules M' of M not contained in M_α. When there are prime ideals $Q_1, \ldots, Q_m$ and a sequence of external links $Q_\alpha \overset{\text{ext}}{\rightsquigarrow} Q_1 \overset{\text{ext}}{\rightsquigarrow} \cdots \overset{\text{ext}}{\rightsquigarrow} Q_m \overset{\text{ext}}{\rightsquigarrow} Q_\beta$, we write $Q_\alpha \overset{\text{ext}}{\rightsquigarrow} \cdots \overset{\text{ext}}{\rightsquigarrow} Q_\beta$. (As before, $Q_\alpha \overset{\text{ext}}{\rightsquigarrow} \cdots \overset{\text{ext}}{\rightsquigarrow} Q_\alpha$ whether or not $Q_\alpha \overset{\text{ext}}{\rightsquigarrow} Q_\alpha$.) With respect to this graph structure on $\text{Spec } R$, the connected component containing a prime ideal Q is termed the *external clique* of Q and denoted ext-Clique(Q).

(iii) (Cf. [12, 6.1.3].) Let M be an external link from Q_α to Q_β, and set $I = \text{ann}_R M$. Then Jategaonkar's Main Lemma (as stated, e.g., in [9, 11.1]) asserts that one of the following two cases must occur: (1) $Q_\alpha \cap Q_\beta / I$ is a link from Q_α to Q_β, or (2) $Q_\beta = I \subsetneq Q_\alpha$.

It follows from [3, 2.3ii] (cf. [9, 11.2]) that if $Q_\alpha \rightsquigarrow Q_\beta$, then $Q_\alpha \overset{\text{ext}}{\rightsquigarrow} Q_\beta$.

(iv) The condition in (ii) can be weakened somewhat. Let N be a uniform R-module, and suppose that Q_α is the unique associated prime ideal of N. As noted previously, Q_α is the unique maximal annihilator in R of nonzero submodules of N. Set $N_\alpha = \text{ann}(Q_\alpha)_N$. Suppose further that $0 \subset N_\alpha \subset N$ is an affiliated series, and that $Q_\beta = \text{ann } N/N_\alpha$. Setting I equal to the ideal of R maximal among annihilators of submodules not containing N_α, we see that $M = \text{ann } I_N$ is an external link from Q_α to Q_β.

(v) If for all external links between prime ideals of R, only case (1) in (iii) holds, then R satisfies the *(left) strong second layer condition*.

(vi) Suppose that R satisfies the strong second layer condition and that $Q_\alpha \approx Q_\beta$. Then $Q_\alpha \rightsquigarrow \cdots \rightsquigarrow Q_\beta$; see [12, 8.2.4].

Although the following lemma is probably well known, we do not know of a suitable reference. The proof is adapted from [12, 7.1.2], which applies to the case when R satisfies the strong second layer condition.

2.4. Lemma. *Let M be a finitely generated nonzero R-module, and let X denote the union of the external cliques of the associated primes of M. Then there exist $Q_1, \ldots, Q_t \in X$ such that Q_i is the annihilator of a prime R-module subfactor of M, for $1 \leq i \leq t$, and such that $Q_1 \cdots Q_t M = 0$. If M has finite length, then $\{Q_1, \ldots, Q_t\}$ can be chosen to be the set of annihilators of composition factors of M.*

Proof. By noetherian induction, we may assume that the conclusion of the lemma holds for all of the proper R-module factors of M. Now suppose that M is not uniform. As noted in (2.2iv), M embeds into a finite direct sum of proper uniform R-module factors of M, each of whose associated primes is an associated prime of M; the lemma follows in this case. Therefore, we may assume that M is uniform. Let Q denote the unique associated prime of M, and let $N = \text{ann}_M Q$. If M has finite length, then Q is the annihilator of a simple submodule of N. If Q' is any associated prime of M/N, then by (2.3iv) we can find a submodule of (the uniform module) M providing an external link from Q to Q'. Therefore, since the conclusion of the lemma holds for M/N, the desired conclusion also holds for M. $\square$

The following is certainly well known, but as above we do not know of an appropriate reference.

2.5. Lemma. *Suppose that Q_α and Q_β are prime ideals of R such that $Q_\alpha \overset{ext}{\rightsquigarrow} Q_\beta$. Then $Q_\alpha \cap Z = Q_\beta \cap Z$, where Z is the center of R.*

Proof. Let M be an external link from Q_α to Q_β, as in (2.3ii). When $Q_\alpha \rightsquigarrow Q_\beta$, the conclusion of the lemma follows immediately. Therefore, we may assume without loss of generality that $Q_\alpha \supsetneq Q_\beta = \operatorname{ann}_R M$, as in (2.3iii), and we may further assume without loss that $Q_\beta = 0$. (Note, however, that the image in R/Q_β of the original Z may be smaller than the center of R/Q_β.) Now let $I = Q_\alpha \cap Z$, and set $J = RI$. By [9, 11.13], for example, J is an AR-ideal (see, e.g., [9, p. 190]). Thus $J^n M = 0$ for some positive integer n, by [9, 11.11b], since J annihilates a nonzero submodule of M, and since M is uniform. Hence $I^n M = 0$. But M is a faithful Z-module, and Z is prime. Consequently, $I = J = 0$, and the lemma follows. $\qquad\square$

2.6.   We now briefly review the construction and basic properties of the fundamental series of an R-module, following [17] (cf. [4], [12, Chapter 9]). To start, assume that R is an algebra of finite Gelfand-Kirillov dimension (GK-dimension) over a field k (cf. [15] or [23] for definitions and background), that the GK-dimension of every R-module with finite GK-dimension is an integer, and that if $0 \to L \to M \to N \to 0$ is an exact sequence of R-modules, then the GK-dimension of M is equal to the maximum of the GK-dimensions of L and N. (The preceding assumptions are summarized by saying that R has *exact integer GK-dimension*.)

(i) Let M be a nonzero R-module. Say that M is *(α-)annihilator homogeneous* for some integer α if $\operatorname{GKdim}(R/\operatorname{ann}_R N) = \alpha$ for all nonzero finitely generated submodules N of M. If X is a set of semiprime ideals of R, then the intersection of the ideals in any nonempty subset of X is referred to as an *X-semiprime* ideal of R.

(ii) Assume that M is an annihilator homogeneous nonzero R-module, and set $F_0(M) = 0$. For $j = 1, 2, \ldots$, set $X_j(M)$ equal to the set of associated primes of $M/F_{j-1}(M)$, and set

$$F_j(M) = \{m \in M \mid Im \in F_{j-1}(M) \text{ for some } X_j(M)\text{-semiprime ideal } I \text{ of } R\}.$$

The series $F_0(M) \subset F_1(M) \subset \cdots$ is the *fundamental series* for M. In [17, 3.2] it is shown that $M = \bigcup_j F_j(M)$. The set of *fundamental primes* of M, denoted $\operatorname{Fund}(M)$, is defined as $\bigcup_j X_j(M)$.

(iii) Now let M be an arbitrary (i.e., not necessarily annihilator homogeneous) nonzero R-module, and let $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ be the distinct integers arising as the GK-dimensions of $R/\operatorname{ann}_R N$, for finitely generated submodules N of M. Set $G_0(M) = 0$, and for $1 \le i \le n$ set $G_i(M)$ equal to

$$\{m \in M \mid m \in G_{i-1}(M) \text{ or } \operatorname{GKdim}(R/\operatorname{ann}_R[Rm + G_{i-1}(M)/G_{i-1}(M)]) = \alpha_i\}.$$

In [17, 4.2] it is proved that

$$G_0(M) \subsetneq G_1(M) \subsetneq \cdots \subsetneq G_n(M) = M,$$

and that each $G_i(M)/G_{i-1}(M)$ is an α_i-annihilator homogeneous R-module. Set

$$\operatorname{Fund}_i(M) = \operatorname{Fund}(G_i(M)/G_{i-1}(M)),$$

and define the *fundamental primes* of M, denoted $\operatorname{Fund}(M)$, to be $\bigcup_i \operatorname{Fund}_i(M)$.

(iv) Continue to let M be an arbitrary nonzero R-module, and let Q be an annihilator prime of M. In [17, 4.4] it is proved that there exist submodules H and J of M such that $QH = 0$ and such that H/J is Q-prime. Choosing $h \in H \setminus J$, we see that $Q = \operatorname{ann}_R R.h$. Consequently, the set of annihilator primes of M is equal to the set of prime annihilators of finitely generated submodules of M.

2.7. (i) Let A and B denote algebras of finite GK-dimension, and let C denote an A-B-bimodule finitely generated on each side. Then $\operatorname{GKdim}(A/\operatorname{ann}_A C) = \operatorname{GKdim}(_A C) = \operatorname{GKdim}(C_B) = \operatorname{GKdim}(B/\operatorname{ann} C_B)$; see, for example, [15, 5.3]. Consequently, if A is a subalgebra of B, and if B is finitely generated on either the right or left as an A-module, then $\operatorname{GKdim}(A) = \operatorname{GKdim}(B)$.

(ii) Suppose that R has finite GK-dimension. Let Q_α and Q_β be prime ideals of R. If $Q_\alpha \subseteq Q_\beta$, then $Q_\alpha = Q_\beta$ if and only if $\operatorname{GKdim}(R/Q_\alpha) = \operatorname{GKdim}(R/Q_\beta)$; see [15, 3.16].

2.8. Assume that R has exact integer GK-dimension. Let E be an injective R-module, and let Q_β be a prime ideal of R. It is proved in [17, 5.4] (noting (2.7i)) that Q_β is a fundamental prime of E if and only if there exists an annihilator prime Q_α of E, and an ideal link I/J from Q_α to Q_β, such that $\operatorname{GKdim}(R/Q_\alpha) = \operatorname{GKdim}(R/J) = \operatorname{GKdim}(R/Q_\beta)$.

2.9. Suppose that R has finite GK-dimension. Let Q_α and Q_β be prime ideals of R.

(i) In view of (2.8), we will say that there is a *strong* ideal link from Q_α to Q_β if there is an ideal link I/J such that $\operatorname{GKdim}(R/Q_\alpha) = \operatorname{GKdim}(R/J) = \operatorname{GKdim}(R/Q_\beta)$.

(ii) Suppose that there is an ideal link I/J from Q_α to Q_β and that $\operatorname{GKdim}(R/Q_\alpha) = \operatorname{GKdim}(R/Q_\beta) = \operatorname{GKdim}(R)$. Since J is contained in both Q_α and Q_β, we see that $\operatorname{GKdim}(R) \geq \operatorname{GKdim}(R/J) \geq \operatorname{GKdim}(R/Q_\alpha) = \operatorname{GKdim}(R)$. Therefore, I/J is a strong ideal link.

(iii) Now assume that R has exact integer GK-dimension. It follows from the remarks after [17, 5.4], when there exists a strong ideal link from Q_α to Q_β, that $Q_\alpha \rightsquigarrow \cdots \rightsquigarrow Q_\beta$. Hence if E is an injective R-module, and Q_β is a fundamental prime of E, then there exists an annihilator prime Q_α of E such that $Q_\alpha \rightsquigarrow \cdots \rightsquigarrow Q_\beta$.

2.10. Assume that R has exact integer GK-dimension, and suppose that $(Q_\alpha \cap Q_\beta)/J$ is a link from Q_α to Q_β. It follows from (2.7i) and, for example, [15, 5.7] that $\operatorname{GKdim}(R/Q_\alpha) = \operatorname{GKdim}(R/J) = \operatorname{GKdim}(R/Q_\beta)$. We see in this case that a direct link is a strong ideal link.

2.11. Lemma. *Assume that R has exact integer GK-dimension, that M is a uniform R-module, that Q_α is the unique associated prime of M, and that Q_β is an annihilator prime of M. Then $Q_\alpha \stackrel{ext}{\rightsquigarrow} \cdots \stackrel{ext}{\rightsquigarrow} Q_\beta$.*

Proof. By (2.6iv), Q_β is the annihilator of a finitely generated submodule of M; this submodule is uniform and its unique associated prime is Q_α. Therefore, we can reduce to the case where $Q_\beta = 0$ and where M is a finitely generated faithful R-module. However, by (2.4), there exist prime ideals $Q_1, \ldots, Q_t$ in the external clique of Q_α such that $Q_1 \cdots Q_t M = 0$. Thus $Q_1 \cdots Q_t = 0$, and so $Q_i = 0$ for some $1 \leq i \leq t$. The lemma follows. $\square$

2.12.   Continue to assume that R has exact integer GK-dimension, and let Q be a prime ideal of R.

(i) Define $\mathrm{Fund}(Q)$, the set of *fundamental primes* of Q, to be the union of the sets of fundamental primes of indecomposable (equivalently, uniform) injective R-modules for which Q is the unique associated prime. It follows from (2.9iii) and (2.11) that $\mathrm{Fund}(Q) \subseteq \mathrm{ext\text{-}Clique}(Q)$.

(ii) If M is a finite length uniform R-module with associated primitive ideal Q, then it is not hard to verify that the annihilators of the composition factors of M are all contained in $\mathrm{Fund}(Q)$.

(iii) Suppose that M is a uniform R-module with associated prime Q, that L is an arbitrary simple R-module subfactor of M, and that $\mathrm{ann}_R L = Q_\beta$. We may embed M into a uniform injective R-module E, and the filtration of E described in (2.6) is exhaustive. Hence there exists $Q_\alpha \in \mathrm{Fund}(Q) \subseteq \mathrm{ext\text{-}Clique}(Q)$ such that $Q_\alpha \subseteq Q_\beta$.

3. Noetherian superalgebras

We now briefly review some generalities, relevant to the study of Lie superaglebras, from the theory of $\mathbb{Z}_2$-graded rings.

3.1.   (i) Throughout this section, k will denote a field not of characteristic two, and R will denote a noetherian k-algebra. We will further assume, throughout, that $R = R_0 \oplus R_1$ is $\mathbb{Z}_2$-graded (i.e., R is an associative superalgebra).

(i) Henceforth, we will use the term *graded* to mean "$\mathbb{Z}_2$-graded" and "homogeneous with respect to the $\mathbb{Z}_2$-grading." If $X = X_0 \oplus X_1$ is a ($\mathbb{Z}_2$-)graded k-vector space, then the elements of X_0 will be called *even* and the elements of X_1 will be called *odd*. Degree-preserving linear maps between graded k-vector spaces will be termed *graded*. However, a k-vector space should be assumed graded only when explicitly specified. *Ungraded* will mean "not necessarily graded."

(ii) Let $X = X_0 \oplus X_1$ be a graded k-vector space. There is an automorphism σ of X, obviously of order two, mapping each even element to itself and each odd element to its additive inverse. Also, X_0 is identical to X^σ, the subspace of σ-invariants. Now let Y be an ungraded subspace of X. The unique maximum graded subspace of X contained in Y will be termed $\mathrm{gr}\, Y$, and it is easy to see that Y is graded if and only if $\sigma(Y) = Y$. Therefore, $\mathrm{gr}\, Y = Y \cap \sigma(Y)$.

(iii) Applied to R, σ is a k-algebra automorphism. Therefore, R_0 is noetherian and R is finitely generated on each side as an R_0-module [24, 1.12, 5.9]. Applied to a graded R-module, σ permutes the ungraded submodules.

(iv) The category of graded left R-modules—with graded morphisms—will be denoted $\mathrm{Grmod}\, R$.

3.2.   (i) We will say that a graded R-module M is *graded-simple* provided the only nonzero graded submodule of M is itself (i.e., M is a simple object in $\mathrm{Grmod}\, R$), a condition that occurs if and only if M_0 and M_1 are simple R_0-modules. Consequently, a graded R-module N has finite length as an ungraded R-module if and and only if N has finite length in $\mathrm{Grmod}\, R$.

(ii) The annihilators of graded-simple R-modules are the *(left) graded-primitive* ideals, and the set of graded-primitive ideals of R will be denoted $\mathrm{GrPrim}\, R$.

(iii) Assume that M is a graded-simple R-module with annihilator p. The R_0-module structure of M ensures that its length as an ungraded R-module is no

greater than two. Moreover, if N is an ungraded simple R-submodule of M, and $\text{ann}_R N = P$, then $M = N + \sigma(N)$, and $p = \text{gr}(P) = P \cap \sigma(P)$.

(iv) Let L be a maximal ungraded left ideal of R. Set $P = \text{ann}_R(R/L)$, and set $M = R/L \cap \sigma(L)$. Then $M \subseteq (R/L) \oplus (R/\sigma(L))$ has length no greater than two, as either an ungraded or graded R-module, and $p = \text{gr}(P) = \text{ann}_R M$. It is now straightforward to check that p is graded-primitive.

(v) We see from (iii) and (iv) that a graded ideal p of R is graded-primitive if and only if there exists a primitive ideal P of R for which $p = \text{gr}(P) = P \cap \sigma(P)$.

3.3. (i) A graded ideal of R is *graded-prime* when it contains no product of strictly larger graded ideals, and a graded ideal is *graded-semiprime* when it is the intersection of graded-prime ideals. Also, R will be termed *graded-prime* (resp. *graded-semiprime*) when the zero ideal of R is graded-prime (resp. graded-semiprime). It is proved as follows that an ideal p of R is graded-prime if and only if there exists a prime ideal P of R such that $p = \text{gr}(P) = P \cap \sigma(P)$: First, when P is a prime ideal of R, it is straightforward to check that $\text{gr}(P)$ is graded-prime. Conversely, let p be a graded prime ideal of R. Since R is noetherian there exist prime ideals $P_1, \ldots, P_t$ of R, each containing p, such that $P_1 \cdots P_t \subseteq p$. But $\text{gr}(P_1) \cdots \text{gr}(P_t) \subseteq p$, and each of the $\text{gr}(P_i)$ contains p. Thus $p = \text{gr}(P_i)$ for some $1 \leq i \leq t$.

(ii) If P is a prime ideal of R, then $P \cap R_0 = \sigma(P) \cap R_0 = \text{gr}(P) \cap R_0 = P^\sigma$ is a semiprime ideal of R_0; see, for example, [24, 1.5].

(iii) Suppose that R is graded-semiprime. Then R_0 is semiprime by (ii). Let X denote the Ore set of regular elements of R_0. By [24, 5.3], X is an Ore set of regular elements of R, and the left Ore localization $X^{-1}R$ is equal to the Goldie quotient ring of R. Because X consists only of even elements of R, the grading on R extends naturally to a grading of $X^{-1}R$.

Now let M be a graded R-module, and let Y denote the Ore set of regular elements of R. Because $X^{-1}R$ is the Goldie quotient ring of R, a submodule N of M is X-torsion (i.e., torsion with respect to X) if and only if N is torsion (i.e., Y-torsion). Since X consists only of even elements, the torsion submodule of M is therefore graded. Also, M is torsion free if and only if M is X-torsion free.

(iv) Continue to assume that R is graded-semiprime. Let $M = M_0 \oplus M_1$ be a graded R-R-bimodule (i.e., $R_i M_j$ and $M_j R_i$ are both contained in M_{i+j}). By (iii), the left torsion submodule T of M is a graded bimodule, and by [12, 5.1.1], T is the largest left unfaithful graded R-R-sub-bimodule of M. We may therefore define a *graded-bond* between graded-prime rings A and B to be a graded bimodule finitely generated and torsion free on each side. *Graded-ideal-links* and *graded(-direct-)links* can be similarly defined.

(v) Suppose that R is graded-prime. Then its Goldie quotient ring is a graded-simple (i.e., containing no proper graded ideals) Artinian ring. Now suppose that A is a graded-simple artinian ($\mathbb{Z}_2$-)graded ring. By (i), A is either simple artinian or equal to a direct product of two isomorphic simple artinian rings. In the first case, A is an $m \times m$ matrix ring over a division ring and we say that m is the *graded-rank* of A. In the second case, A is a direct product of two $n \times n$ matrix rings over division rings, and we say that n is the *graded-rank* of A. The *graded-Goldie-rank* of R is the graded-rank of its Goldie quotient ring.

3.4. Assume that R has exact integer GK-dimension (see (2.6)) and that M is a graded R-module.

(i) Let I be an ideal of R. Since R has exact integer GK-dimension, $\mathrm{GKdim}(R/I)$ $= \mathrm{GKdim}(R/\sigma(I)) = \mathrm{GKdim}[R/(I \cap \sigma(I))] = \mathrm{GKdim}(R/\mathrm{gr}(I))$, by [15, 5.7]. If $m \in M$ and $I = \mathrm{ann}_R Rm$, then $\sigma(I) = \mathrm{ann}_R R\sigma(m)$.

(ii) Let α be the least positive integer occurring among $\mathrm{GKdim}_R(N)$ for finitely generated nonzero R-submodules N of M. It follows from (i) that $G(M) = \{m \in M \mid \mathrm{GKdim}(R/\mathrm{ann}_R Rm) = \alpha\}$ is a σ-stable R-submodule of M, and so $G(M)$ is a graded submodule of M. In particular, $G(M)$ is equal to the sum of homogeneous $m \in M$ such that $\mathrm{GKdim}(R/\mathrm{ann}_R Rm) = \alpha$.

(iii) Let $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ be the integers occurring among $\mathrm{GKdim}(R/\mathrm{ann}_R N)$ for finitely generated nonzero submodules N of M. By (ii), each of the submodules in the series $0 = G_0(M) \subset G_1(M) \subset \cdots \subset G_n(M) = M$, as defined in (2.6iii), is graded.

(iv) Suppose that $\mathrm{GKdim}(R/\mathrm{ann}_R L) = \alpha$ for all finitely generated nonzero graded R-submodules L of M; we will say in this case that M is *(α-)graded-annihilator-homogenous*. Let N be a finitely generated nonzero ungraded submodule of M, and suppose that $\mathrm{ann}_R N = I$. Then $\alpha = \mathrm{GKdim}[R/\mathrm{ann}(N + \sigma(N))] = \mathrm{GKdim}[R/\mathrm{gr}(I)] = \mathrm{GKdim}(R/I)$, by (i). Hence M is α-annihilator homogeneous (see (2.6i)).

(v) We see from (iv) that a graded R-module is α-graded-annihilator homogeneous if and only if it is α-annihilator homogeneous. For $1 \leq i \leq n$, the graded subfactors $G_i(M)/G_{i-1}(M)$ of (iii) are α_i-graded-annihilator-homogeneous, by (2.6iii).

3.5.   The following duality principle of Cohen and Montgomery [5] allows us to apply ungraded noetherian ring theory to $\mathrm{Grmod}\, R$. Set $\widehat{R} = R\#k[\mathbb{Z}_2]^*$ as in [5, §1]. When viewed as either a left or right module over its k-subalgebra R, $\widehat{R}$ is free of rank two. Hence $\widehat{R}$ is noetherian, and if R has finite GK-dimension, then so does $\widehat{R}$. Furthermore, if R has exact integer GK-dimension, then so does $\widehat{R}$, by [22, 1.6].

In [5, 2.2] it is proved that the categories $\mathrm{Grmod}\, R$ and $\mathrm{Mod}\, \widehat{R}$ are isomorphic.

3.6.   (i) In view of (3.2)–(3.5), we now claim that the results mentioned in §2, and their prerequisites, are valid if the modules and bimodules involved are replaced with the corresponding graded objects. (The terms "dimension over k" and "GK-dimension" retain their usual ungraded definitions even when applied to graded modules.) A formal proof of this assertion will be omitted.

(ii) We will continue to use the prefix *graded-* to denote the graded analogue of an ungraded term.

(iii) The existence of a graded-external-link from a graded-prime ideal p_α to a graded-prime ideal p_β will be denoted $p_\alpha \overset{\mathrm{ext}}{\leadsto} p_\beta$, and a graded-link from p_α to p_β will be denoted $p_\alpha \leadsto p_\beta$. When R has exact integer GK-dimension, $\mathrm{GrFund}(M)$ will refer to the fundamental graded-primes of a graded R-module M, and if p is a graded-prime ideal of R we will use $\mathrm{GrFund}(p)$ to denote the union of the sets of the fundamental graded-primes of graded-uniform graded-injective R-modules.

(iv) Note that the graded-injective hull of a graded R-module M will be a maximal graded-essential extension of M.

4. Extensions of rings

This section is devoted to developing some preparatory results concerning ring extensions finitely generated on one side.

4.1. Throughout this section S will denote a noetherian algebra of finite GK-dimension over a field k, and R will denote a noetherian subalgebra of S such that $S = s_1R + \cdots + s_mR$ for some $s_1, \ldots, s_m \in S$. Further suppose, throughout, that P is a prime ideal of S and that Q is a prime ideal of R.

(i) Suppose that Q is minimal over $P \cap R$; we say that P *lies over* Q and that Q *lies under* P. Since R is noetherian, at least one and at most finitely many prime ideals of R lie under P. It follows from [18, 1.1], for example, that there is a bond from S/P to R/Q. Consequently, by (2.7i), $\mathrm{GKdim}(R/Q) = \mathrm{GKdim}(S/P)$. Of course, it is easy to see that P has finite codimension (in S) if and only if Q has finite codimension (in R). From (2.3i) it follows that if P is right primitive, then Q is right primitive, and if Q is left primitive, then P is left primitive.

(ii) If P lies over Q, and if Q is a left annihilator prime of S/P, then we say that P *lies directly over* Q and that Q *lies directly under* P. (Cf. [8, 5.1].) There exists at least one prime ideal of R lying directly under P (see, e.g., [21, 2.6i]).

4.2. Let F denote the Goldie quotient ring of R/Q. Because S is finitely generated as a right R-module, $S \otimes_R F$ has finite length as a right F-module. It is proved in [21, 2.4] that P lies directly over Q if and only if P is the left annihilator in S of an irreducible S-F-bimodule factor of $S \otimes_R F$.

The proof of the following relies on the Joseph-Small Additivity Principle (cf. [2, 13, 31]).

4.3. Proposition. (i) *P lies directly over at most m prime ideals of R.* (ii) *Q lies directly under at most m prime ideals of S.*

Proof. (i) Let $Q_1, \ldots, Q_s$ be prime ideals of R lying directly under P. For $1 \le i \le s$, let F_i denote the Goldie quotient ring of R/Q_i. It follows from (4.2) that P is the left annihilator of an irreducible S-F_i-bimodule factor of $S \otimes_R F_i$. By [9, 7.23], a version of the Additivity Principle,

$$\mathrm{rank}(S/P) \le \mathrm{length}(S \otimes_R F_i)_{F_i} \le m \cdot \mathrm{rank}(R/Q_i),$$

where $\mathrm{rank}()$ denotes Goldie rank and $\mathrm{length}()_{F_i}$ denotes composition length as a right F_i-module. However, it now follows from the standard version of the Additivity Principle (see, e.g., [23, 4.5.4]) that

$$\mathrm{rank}(R/Q_1) + \cdots + \mathrm{rank}(R/Q_s) \le \mathrm{rank}(S/P)$$
$$\le m \cdot \min\left\{\mathrm{rank}(R/Q_1), \ldots, \mathrm{rank}(R/Q_s)\right\}.$$

Hence $s \le m$.

(ii) Suppose that $P_1, \ldots, P_t$ are prime ideals of S lying directly over Q. Another application of the Additivity Principle (this time see, e.g., [9, 7.26]) ensures that $\mathrm{rank}(R/Q) \le \mathrm{rank}(S/P_j)$, for $1 \le j \le t$. Let F denote the Goldie quotient ring of R/Q; it follows from (4.2) that P_j is the annihilator of an irreducible S-F-bimodule factor of $S \otimes_R F$. Therefore, again using [9, 7.23], we see that

$$\mathrm{rank}(S/P_1) + \cdots + \mathrm{rank}(S/P_t) \le m \cdot \min\left\{\mathrm{rank}(S/P_1), \ldots, \mathrm{rank}(S/P_t)\right\},$$

and so $t \le m$. □

The following is analogous to [20, 3.4].

4.4. Proposition. *Suppose that P lies over Q. Then there exists a prime ideal Q' of R such that Q' lies directly under P and such that there is a strong ideal link from Q' to Q.*

Proof. We may assume, without loss of generality, that $P = 0$. We first find a prime ideal Q' of R such that Q' is an annihilator prime of S and such that there is an ideal link from Q' to Q. Let X denote the set of elements of R regular modulo its nilradical. Because S has finite GK-dimension and is finitely generated as a right R-module, it follows from [2, 2.2] that X consists of regular elements of S and is a right Ore set in both R and S. Consequently, the embedding of R into S extends an embedding of the right quotient ring RX^{-1} into SX^{-1}. Note that RX^{-1} is artinian by Small's Theorem (see, e.g., [9, 10.9]), that SX^{-1} is finitely generated as a right RX^{-1}-module, and that SX^{-1} is the Goldie quotient ring of S. Set $A = RX^{-1}$, $B = SX^{-1}$, and $K = QX^{-1}$.

As noted in (2.2iv), we can embed A (as a left A-module) into a finite direct sum of uniform left A-module factors of A such that the unique associated prime of each uniform factor is an associated prime of A; let $L_1, \ldots, L_s$ denote these prime ideals. By [16, 2.3], there now exist prime ideals $K_1, \ldots, K_t$ such that $0 = K_1 K_2 \cdots K_t A = K_1 K_2 \cdots K_t$ and such that for each $1 \le j \le t$ there exists $1 \le i \le s$ for which $L_i \approx K_j$. However, since $K_1 \cdots K_t \subseteq K$, and since K is a minimal prime ideal of A, it follows that $K = K_j$ for some $1 \le j \le t$. In particular, we have found an annihilator prime L, of the left A-module B, such that $L \approx K$. Setting $Q' = L \cap R$, we see that Q' is an annihilator prime of R for which there exists an ideal link from Q' to Q.

By (2.7i) and (4.1i), $\mathrm{GKdim}(R) = \mathrm{GKdim}(S) = \mathrm{GKdim}(R/Q) = \mathrm{GKdim}(R/Q')$. Hence there exists a strong ideal link from Q' to Q, by (2.9ii). The lemma follows. ☐

The next lemma is another consequence of the Additivity Principle.

4.5. Lemma. *Assume that S is prime.*

(i) *Let I be an ideal of R not contained within any minimal prime ideal of R. Then there exists a nonzero ideal of S contained within SI. If S is finitely generated as a left R-module, then there exists a nonzero ideal of S contained within IS.*

(ii) *Let M be a faithful prime S-module, and let I be an ideal of R such that $IN = 0$ for some nonzero R-submodule N of M. Then I is contained within a minimal prime ideal of R. Consequently, if the prime ideal Q of R is an annihilator prime of M (viewed as an R-module), then Q is a minimal prime ideal of R.*

Proof. (i) See [21, 2.1].

(ii) If I is not contained within any minimal prime ideal of R, then, by (i), there exists a nonzero ideal J of S such that $J(SN) = JN \subseteq SIN = 0$, a contradiction to the choice of M. ☐

The following is similar to [19, 2.5].

4.6. Lemma. *Assume that S is prime and finitely generated as both a left and right R-module. Let M be a finitely generated, uniform, faithful S-module, and assume that P_α is the unique associated prime of M. Then there exist prime ideals Q_α and Q_β of R such that P_α lies over Q_α, such that Q_β is a minimal prime ideal of R, and such that $Q_\alpha \overset{ext}{\leadsto} \cdots \overset{ext}{\leadsto} Q_\beta$.*

Proof. As observed in (2.2iv), P_α is the unique maximal annihilator, in S, of nonzero S-submodules of M. Set $N = \mathrm{ann}(P_\alpha)_M$. It follows from (4.5ii) that if Q is an annihilator prime of N, viewed as a left R-module, then P_α lies over Q. Next, choose L to be an R-submodule of M maximal such that $L \cap N = 0$, and set

$K = M/L$. The R-module embedding $N \hookrightarrow K$ is essential, forcing every associated prime, in R, of K to also be an associated prime of N. In particular, P_α lies over each associated prime of K. Now let X denote the union of the external cliques of the associated primes of K. By (2.4), there exist $Q_1, \ldots, Q_t \in X$ such that $Q_1 \cdots Q_t K = 0$. Set $I = Q_1 \cdots Q_t$. Then $ISM = IM \subseteq L$.

Suppose that I is not contained within a minimal prime ideal of R. Then by (4.5i), there exists a nonzero ideal J of S contained within IS. Note that JM is nonzero because M is faithful, and that JM is then an S-submodule of M contained in L. Hence $JM \cap N = 0$, contradicting the uniformity of M as an S-module. Thus I is contained in a minimal prime ideal of R, and one of the $Q_1, \ldots, Q_t$ must be a minimal prime ideal of R. The lemma follows. $\qquad\square$

4.7. Lemma. *Suppose that S has exact integer GK-dimension and that S is finitely generated as both a left and right R-module. Let M be a uniform S-module. Let P_α be the unique associated prime, in S, of M, and suppose that P_β is an annihilator prime, in S, of M. Then there exist prime ideals Q_α and Q_β of R such that Q_α lies under P_α, such that Q_β lies under P_β, and such that $Q_\alpha \overset{ext}{\rightsquigarrow} \cdots \overset{ext}{\rightsquigarrow} Q_\beta$.*

Proof. First, we may assume without loss of generality that $P_\beta = \mathrm{ann}_S M$. Therefore, we can reduce to the case where $P_\beta = 0$ and M is faithful as an S-module. By (2.6iv), we may further assume that M is finitely generated as an S-module. The lemma now follows from (4.6). $\qquad\square$

4.8. Proposition. *Let T be a noetherian k-algebra, containing S as a subalgebra, such that $T = t_1 S + \cdots + t_\ell S$. Let w be a positive integer. Assume there exist, for every prime ideal Q_α of R, at most w prime ideals Q_β such that either Q_α is strongly ideal linked to Q_β or Q_β is strongly ideal linked to Q_α. Let J be a prime ideal of T.*

   (i) *There exist at most mw prime ideals of S lying over Q.*
  (ii) *There exist at most mw prime ideals of R lying under P.*
 (iii) *There exist at most ℓmw prime ideals of T lying over P.*
 (iv) *There exist at most $\ell m^2 w$ prime ideals of S lying under J.*

Proof. Parts (i) and (ii) follow from (4.3) and (4.4).

(iii) Suppose that J lies over P and that P lies over Q. Then $J \cap R \subseteq Q$. It follows from (4.1i) that the GK-dimensions of T/J, S/P, and R/Q all coincide. Therefore, Q must be minimal over $J \cap R$, by (2.7ii), and so J lies over Q. It follows from (i) that at most ℓmw prime ideals of T can lie over Q, and (iii) follows.

(iv) Suppose that P lies under J and directly over Q; as noted in (4.1ii), P lies directly over at least one prime ideal of R. As in the proof of (iii), Q lies under J. By (ii), there are no more than ℓmw prime ideals of R lying under J. By (4.3i), there are at most m prime ideals of S lying directly over Q. Part (iv) follows. $\qquad\square$

The following is similar, for example, to [32, 6.6].

4.9. Lemma. *Let I/J be a strong ideal link from the prime ideal Q_α to the prime ideal Q_β of R. Further suppose that $SJ \cap R = J$ (e.g., suppose that S is free as a right R-module). Then there exist prime ideals P_α and P_β of S such that P_α lies over Q_α, such that P_β lies over Q_β, and such there is a strong ideal link from P_α to P_β.*

Proof. Note that $\text{ann}(S/SJ)_R = J$, and let $K = \text{ann}_S(S/SJ)$. By assumption, $K \cap R \subseteq J$, and so we may suppose without loss of generality that $K = 0$. By (2.7i) and the assumption that I/J is a strong ideal link,

$$\text{GKdim}(R) = \text{GKdim}(S) = \text{GKdim}(R/J) = \text{GKdim}(R/Q_\alpha) = \text{GKdim}(R/Q_\beta).$$

Since $\text{GKdim}(R/Q_\alpha) = \text{GKdim}(R/Q_\beta) = \text{GKdim}(R)$, we see from (2.7ii) that Q_α and Q_β are minimal prime ideals of R. It now follows from [32, 6.3i] that there exist prime ideals P_α and P_β of S such that P_α lies over Q_α, such that P_β lies over Q_β, and such that there is an ideal link L/M from P_α to P_β. Since $\text{GKdim}(S/P_\alpha) = \text{GKdim}(S/P_\beta) = \text{GKdim}(R/Q_\beta) = \text{GKdim}(S)$, it follows from (2.9ii) that there is a strong ideal link from P_α to P_β. $\qquad\square$

5. Extensions of enveloping algebras

In this section we establish bounds on the sizes of sets of fundamental primes, in a setting that includes finite extensions of enveloping algebras of reductive Lie algebras.

5.1.   Let k be a field, and let R be a noetherian k-algebra of exact integer GK-dimension (see (2.6)). Throughout this section, S will denote a (necessarily noetherian) k-algebra, containing R as a subalgebra, such that S is free of rank ℓ as a left R-module and is generated as a right R-module by no more than m elements. Further assume, throughout, that every external clique (see (2.3ii)) of prime ideals of R has cardinality no greater than some fixed positive integer w.

5.2.   For a general example fitting the description of R in (5.1), suppose that k is an algebraically closed field of characteristic zero, that $\mathfrak{a}$ is a (finite dimensional) reductive k-Lie algebra, and that W is the associated Weyl group. Recall that the enveloping algebra of a finite dimensional Lie algebra is noetherian.

(i) Setting $U = U(\mathfrak{a})$, and Z equal to the center of U, it follows from [30, Theorem 1] that the sets of prime ideals of U having common intersection with Z correspond exactly with suitably defined W-orbits in the prime spectrum of the symmetric algebra $S(\mathfrak{h}^*)$. In particular, by (2.5), the external cliques in $\text{Spec}\, U$ have cardinality no greater than $|W|$.

(ii) It is shown, for example, in [15, Chapter 6] that enveloping algebras of finite dimensional Lie algebras, over any field, have exact integer GK-dimension. So suppose, for the moment, that R is the enveloping algebra of a finite dimensional Lie algebra. Recall, if Q is a prime ideal of R, that Q is left primitive if and only if it is right primitive. It follows, when P is a prime ideal of S, that P is left primitive if and only if it is right primitive (see, e.g., [18]).

(iii) Recall that the (left, or equivalently, right) primitive ideals of U (as defined in (i)) are precisely the prime ideals intersecting to maximal ideals of Z, that the minimal primitive ideals of U are the ideals of the form $U\chi$ for maximal ideals χ of Z, and that the set of annihilators of Verma modules over U coincides with the set of minimal primitive ideals of U. The Verma modules are uniform U-modules of finite length, and from Duflo's Theorem it follows that every primitive ideal is the annihilator of a simple factor of a Verma module. (See [6] or [11] for details.) It now follows from (2.4) that the external clique in $\text{Spec}\, U$ of a primitive ideal P of U is exactly the set of primitive ideals whose intersection with Z equals $P \cap Z$.

(iv) It follows from (iii) that if Q_α and Q_β are primitive ideals of U such that $Q_\alpha \subseteq Q_\beta$, then Q_α and Q_β are in the same external clique. Suppose that M is a

uniform U-module with associated prime Q, that Q is primitive, and that L is a simple U-module factor of M. It now follows from (2.12iii) that the annihilator in U of L is in the external clique of Q.

5.3.   Let $T = M_\ell(R)$. By [22, 1.6], both S and T have exact integer GK-dimension. In particular, any pair of strongly ideal linked prime ideals of R, S, or T is contained within a common clique, by (2.9iii). As noted in [19, 7.1], there is a k-algebra embedding of S into T such that T is a free right S-module of rank ℓ. Of course, every external clique of prime ideals of T has cardinality no greater than w.

5.4. Lemma. *Let P be a prime ideal of S.*

(i) *Let C denote the union of the external cliques of the prime ideals of R lying under P, and let C' denote the set of prime ideals in S lying over ideals in C. Then $|C'| \leq m^2 w$.*

(ii) *Let D denote the union of the external cliques of the prime ideals of T lying over P, and let D' denote the set of prime ideals in S lying under ideals in D. Then $|D'| \leq \ell^2 m^3 w^3$.*

Proof. (i) By (4.3ii) and (4.4), $|C| \leq mw$. By (4.4), C' is equal to the set of prime ideals of S lying directly over ideals in C, and so from (4.3i) it follows that $|C'| \leq m^2 w$.

(ii) By (4.8iii), $|D| \leq \ell m w^2$, and so it follows from (4.8iv) that $|D'| \leq \ell^2 m^3 w^3$.   □

Fundamental series and primes (cf. [4, 12, 17]) are discussed in (2.6) and (2.12).

5.5. Theorem. *If P is a prime ideal of S, then $|\mathrm{Fund}(P)| \leq \ell^2 m^5 w^4$.*

Proof. First, let A denote the set of annihilator primes of uniform S-modules with associated prime P. Set C equal to the union of the external cliques of prime ideals of R lying under P, and let C' denote the set of prime ideals of S lying over ideals in C. By (4.7), $A \subseteq C'$, and so $|A| \leq m^2 w$, by (5.4i).

Second, let P_α be a prime ideal of S, and let B denote the set of prime ideals P_β of S for which there is a strong ideal link from P_α to P_β. Let D be the union of the external cliques of the prime ideals of T lying over P_α, and let D' denote the set of prime ideals of S lying under prime ideals in D. By (2.12i) and (4.9), $B \subseteq D'$. Thus $|B| \leq \ell^2 m^3 w^3$, by (5.4ii).

It follows from (2.8) that $|\mathrm{Fund}(P)| \leq |A|\,|B|$. The theorem follows.   □

5.6.   A bound similar to, and sharper than, that of (5.5) can be established for external links (see (2.3)) and can in turn be applied to extensions of simple modules, as follows: Suppose that $0 \to L_\alpha \to M \to L_\beta \to 0$ is a nonsplit extension of simple modules, that $P_\alpha = \mathrm{ann}_S L_\alpha$, and that $P_\beta = \mathrm{ann}_S L_\beta$; then M is an external link from P_α to P_β.

5.7. Theorem. *Let P be a prime ideal of S.*

(i) *There exist at most $\ell^2 m^3 w^3$ prime ideals P' of S such that $P \rightsquigarrow P'$.*

(ii) *There exist at most $m^2 w$ prime ideals P' of S such that $P \overset{ext}{\rightsquigarrow} P'$ but $P \not\rightsquigarrow P'$.*

(iii) *There exist at most $m^2 w + \ell^2 m^3 w^3$ prime ideals P' of S such that $P \overset{ext}{\rightsquigarrow} P'$.*

Proof. (i) By (2.10) and (4.9), if $P \rightsquigarrow P'$ then there exist prime ideals J and J' of T such that J lies over P, such that J' lies over P', and such that there is a strong ideal link from J to J'. Part (i) now follows from (2.12i) and (5.4ii).

(ii) Let M be an external link from P to P', and suppose that $P \not\rightsquigarrow P'$. By (2.3iii), $P' = \mathrm{ann}_S M$, and so by (4.6), or (4.7), there exist prime ideals Q and Q' of R such that P lies over Q, P' lies over Q', and $Q \overset{\mathrm{ext}}{\rightsquigarrow} \cdots \overset{\mathrm{ext}}{\rightsquigarrow} Q'$. Part (ii) now follows from (5.4i).

(iii) This part now follows from (i), (ii), and (2.3iii). $\qquad\square$

5.8. Assume that $R = U(\mathfrak{a})$, for $\mathfrak{a}$ as in (5.2).

(i) Distinct primitive ideals with finite codimension in R cannot have equal intersection with its center, and the analogous statement also holds true for T. Hence, there are no ideal links in either R or T between distinct finite codimensional primitive ideals. It now follows, for example, from (4.9) that linked, primitive, finite codimensional ideals in S lie under a common finite codimensional primitive ideal of T. Moreover, there is a nonsplit extension of finite dimensional simple S-modules if and only if there exists a direct link between their corresponding annihilators.

(ii) Suppose that J is a finite codimensional primitive ideal of T, that P is a finite codimensional primitive ideal of S, and that Q is finite codimensional primitive ideal of R. By (i) and (4.4), Q lies under J (resp. P) if and only if Q lies directly under J (resp. P). Therefore, by (4.3ii) there exist at most ℓm prime ideals of R lying under J, and by (4.3i) there exist at most m prime ideals of S lying over Q. Thus, as in the proof of (4.8), there exist at most ℓm^2 prime ideals of S lying under J.

(iii) Let L be an arbitrary finite dimensional simple S-module, and let X denote the set of finite dimensional simple S-modules L' for which either $\mathrm{Ext}_S(L, L') \neq 0$ or $\mathrm{Ext}_S(L', L) \neq 0$. It follows from (i) and (ii) that $|X| \leq \ell m^2$.

(iv) Let P_α and P_β be prime ideals of S, and suppose that M is an external link from P_α to P_β. We can show as follows that P_α is primitive if and only if P_β is primitive. First, if $P_\alpha \rightsquigarrow P_\beta$, then P_α is primitive if and only if P_β is primitive, by (2.3i) and (5.2ii). We may therefore assume that $P_\alpha \not\rightsquigarrow P_\beta$. By (2.3iii) $P_\beta = \mathrm{ann}_S M$, and by (4.7) it follows that there exist prime ideals Q_α and Q_β of R such that P_α lies over Q_α, such that P_β lies over Q_β, and such that $Q_\alpha \overset{\mathrm{ext}}{\rightsquigarrow} \cdots \overset{\mathrm{ext}}{\rightsquigarrow} Q_\beta$. It now follows from (5.2iii) and (4.1i) that P_α is primitive if and only if P_β is primitive.

5.9. Retaining the assumptions of (5.1), suppose further that the characteristic of k is not equal to two, that S is ($\mathbb{Z}_2$-)graded, that $R = R_0$ is a trivially graded subalgebra of S_0, and that the free left R-module basis for S is comprised of homogeneous elements. The algebra T is then also graded, and R and S are graded subalgebras of T. Following (3.6), the results in this section remain valid when the modules and bimodules involved are replaced with their graded counterparts.

6. Classical Lie superalgebras

We now apply the results of the preceding sections to the representation theory of Lie superalgebras. The reader is referred to [14] or [28] for background information.

6.1. (i) Let k be an algebraically closed field of characteristic zero. Assume throughout this section that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a finite dimensional k-Lie superalgebra, that $\mathbf{V}$ is the associative enveloping algebra of $\mathfrak{g}$, and that $\mathbf{U}$ is the enveloping algebra of the finite dimensional Lie algebra $\mathfrak{g}_0$.

(ii) It follows from the Poincare-Birkhoff-Witt Theorem for Lie superalgebras that $\mathbf{V}$ contains $\mathbf{U}$ as a subalgebra and that $\mathbf{V}$ is free of rank $2^{\dim \mathfrak{g}_1}$ as both a right

and left $\mathbf{U}$-module (and that the right and left bases can be chosen to coincide). Hence $\mathbf{V}$ is noetherian. As remarked in (5.3), $\mathbf{U}$ and $\mathbf{V}$ have exact integer GK-dimension.

(iii) The ($\mathbb{Z}_2$-)grading on $\mathfrak{g}$ extends to a grading of $\mathbf{V}$, and $\mathbf{U} \subseteq \mathbf{V}_0$. Furthermore, the free left and right common $\mathbf{U}$-basis for $\mathbf{V}$ can be chosen to consist of homogeneous elements. In particular, the remarks in (5.9) apply.

6.2. (i) It follows from (5.2ii) that a prime ideal of $\mathbf{V}$ is left primitive if and only if it is right primitive.

(ii) Assume that $\mathfrak{g}$ is simple (i.e., it contains no nonzero graded ideals other than itself). If $\mathfrak{g}_1$ is completely reducible as an ad $\mathfrak{g}_0$-module—a condition that occurs if and only if $\mathfrak{g}_0$ is reductive (see, e.g., [28, Theorem 1, p. 101])—then $\mathfrak{g}$ is called *classical*.

(iii) Suppose that $\mathfrak{g}_0$ is reductive (with $\mathfrak{g}$ not necessarily simple), and let P be a (left or right) primitive ideal of $\mathbf{V}$. It follows from (5.8iv) that the external clique, in Spec $\mathbf{V}$, of P consists only of primitive ideals, and so it follows from (2.12i) that the prime ideals in Fund(P) are all primitive. Similarly, if p is a graded-primitive ideal of $\mathbf{V}$, then the graded-prime ideals in GrFund(p) are graded-primitive.

The next theorem summarizes the immediate consequences of (5.2), (5.5), (5.7iii), and (5.8iii). Passage to the graded case follows from (3.6) and (5.9).

6.3. Theorem. *Assume that $\mathfrak{g}_0$ is reductive (e.g., $\mathfrak{g}$ is classical simple) and that the Weyl group associated to $\mathfrak{g}_0$ has cardinality w. Set $\ell = 2^{\dim \mathfrak{g}_1}$. Let P be a prime ideal of $\mathbf{V}$, and let p be a graded-prime ideal of $\mathbf{V}$.*

(i) $|\operatorname{Fund}(P)| \leq \ell^7 w^4$.

(i′) $|\operatorname{GrFund}(p)| \leq \ell^7 w^4$.

(ii) *There exist at most $\ell^2 w + \ell^5 w^3$ prime ideals P' of $\mathbf{V}$ such that $P \overset{ext}{\rightsquigarrow} P'$.*

(ii′) *There exist at most $\ell^2 w + \ell^5 w^3$ graded-prime ideals p' of $\mathbf{V}$ such that $p \overset{ext}{\rightsquigarrow} p'$.*

(iii) *Let L be a simple finite dimensional $\mathbf{V}$-module. Then there are at most ℓ^3 finite dimensional simple $\mathbf{V}$-modules L' for which $\operatorname{Ext}(L, L')$ or $\operatorname{Ext}(L', L)$, calculated in $\operatorname{Mod} \mathbf{V}$, is not equal to zero.*

(iii′) *Let L be a graded-simple finite dimensional $\mathbf{V}$-module. Then there are at most ℓ^3 finite dimensional graded-simple $\mathbf{V}$-modules L' for which $\operatorname{Ext}(L, L')$ or $\operatorname{Ext}(L', L)$, calculated in $\operatorname{Grmod} \mathbf{V}$, is not equal to zero.*

6.4. We now turn to highest weight modules, following [25] (cf. [27]). Assume that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is classical simple, that W is the Weyl group of $\mathfrak{g}_0$, that $w = |W|$, and that $\ell = 2^{\dim \mathfrak{g}_1}$.

(i) To start, fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ as in [25, 1.1]. Then $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$ is a triangular decomposition of $\mathfrak{g}_0$.

(ii) Set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. For each $\lambda \in \mathfrak{h}_0^*$ there is a unique finite dimensional graded-simple $\mathfrak{b}$-module K_λ such that $\mathfrak{n}^+.K_\lambda = 0$, and such that $h.x = \lambda(h).x$ for all $h \in \mathfrak{h}_0$ and $x \in K_\lambda$. Moreover, all of the finite dimensional graded-simple $\mathfrak{b}$-modules are obtained, up to isomorphism, in this fashion. (See [14, §5.2], [25, §1.1].)

(iii) Fix $\lambda \in \mathfrak{h}_0^*$, and set $\widetilde{M}(\lambda) = \mathbf{V} \otimes_{U(\mathfrak{b})} K_\lambda$. Then $\widetilde{M}(\lambda)$ has finite length as a $\mathbf{V}$-module and has a unique maximal graded $\mathbf{V}$-submodule; the resulting graded-simple factor module is denoted $\widetilde{L}(\lambda)$.

(iv) Extending Duflo's Theorem (see, e.g., [11, 7.4]), Musson proved that the map $\mathfrak{h}_0^* \to \operatorname{GrPrim} \mathbf{V}$, sending $\lambda \mapsto \operatorname{ann}_{\mathbf{V}} \widetilde{L}(\lambda)$, is surjective [25, §2]. Moreover, the fibers of the preceding map all have cardinality no greater than ℓw; see [21, 3.2].

In view of (5.6), the next proposition follows immediately from (6.3ii$'$) and (6.4iv).

6.5. Proposition. *Retain the notation and assumptions of* (6.4). *Let* $\lambda \in \mathfrak{h}_0^*$. *Then there are at most* $\ell^3 w^2 + \ell^6 w^4$ *linear forms* $\lambda' \in \mathfrak{h}_0^*$ *for which*

$$\mathrm{Ext}\left(\widetilde{L}(\lambda), \widetilde{L}(\lambda')\right) \quad or \quad \mathrm{Ext}\left(\widetilde{L}(\lambda'), \widetilde{L}(\lambda)\right),$$

calculated in $\mathrm{Grmod}\,\mathbf{V}$, *is not equal to zero.*

We conclude our study by considering graded-essential extensions of graded-simple highest weight modules.

6.6. Lemma. *Retain the notation and assumptions of* (6.4), *and let* P *be a primitive ideal of* $\mathbf{V}$. *Then there exist at most* $\ell^2 w^3$ *primitive ideals* P' *of* $\mathbf{V}$ *such that* $P \subseteq P'$. *If* p *is a graded-primitive ideal of* $\mathbf{V}$ *then there exist at most* $\ell^2 w^3$ *graded-primitive ideals* p' *of* $\mathbf{V}$ *such that* $p \subseteq p'$.

Proof. Suppose that $Q_1, \dots, Q_t$ are the primitive (see (4.1i)) ideals of $\mathbf{U}$ lying under P. By (4.8ii), $t \leq \ell w$. Now suppose that P' is a primitive ideal of $\mathbf{V}$ containing P, and suppose that P lies over the primitive ideal Q' of $\mathbf{U}$. Then Q' contains Q_i, for some $1 \leq i \leq t$, since there will be a power of $Q_1 \cdots Q_t$ contained inside $P \cap \mathbf{U} \subseteq P' \cap \mathbf{U} \subseteq Q'$. However, there are at most w primitive ideals of $\mathbf{U}$ containing Q_i, and at most ℓw prime ideals of $\mathbf{V}$ lying over Q', by (4.8i). The ungraded part of the lemma now follows, and the graded case is proved similarly. $\square$

6.7. Theorem. *Retain the notation and assumptions of* (6.4). *Let* M *be a graded-uniform* $\mathbf{V}$-*module whose unique associated graded-prime ideal is graded-primitive (e.g., let* M *be a graded-indecomposable graded-injective* $\mathbf{V}$-*module with a nonzero graded-socle). Then there exist at most* $\ell^{10} w^8$ *linear forms* $\lambda \in \mathfrak{h}_0^*$ *such that* $\widetilde{L}(\lambda)$ *is isomorphic to a graded-simple* $\mathbf{V}$-*module subfactor of* M.

Proof. Suppose that $\widetilde{L}(\lambda)$ is a graded-simple $\mathfrak{g}$-subfactor of M, for some $\lambda \in \mathfrak{h}_0^*$, and set $p_\lambda = \mathrm{ann}_{\mathbf{V}}\, \widetilde{L}(\lambda)$. By (2.12iii), p_λ contains some graded-prime ideal $p \in G = \mathrm{GrFund}(M)$. By (6.3i$'$), $|G| \leq \ell^7 w^4$. But p is graded-primitive, by (6.2iii), and so there exist at most $\ell^2 w^3$ graded-primitive ideals of $\mathbf{V}$ containing p, by (6.6). However, as noted in (6.4iv), there exist at most ℓw graded-simple heighest weight modules with a given graded-primitive annihilator in $\mathbf{V}$. The theorem follows. $\square$

6.8. *Remark.* (i) I do not know the sharpness, in general, of the various bounds established in this section. Of course, when $\mathfrak{g} = \mathfrak{g}_0$ they significantly overestimate the situation.

(ii) Musson's example [26, §4], described in (1.2), shows that $\mathbf{V}$ may have infinite cliques (see (2.3)) of finite codimensional graded-primitive ideals.

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