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THE FIVE EXCEPTIONAL SIMPLE LIE SUPERALGEBRAS OF VECTOR FIELDS AND THEIR FOURTEEN REGRADINGS

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ABSTRACT. The five simple exceptional complex Lie superalgebras of vector fields are described. One of them, $\mathfrak{kas}$, is new; the other four are explicitly described for the first time. All nonisomorphic maximal subalgebras of finite codimension of these Lie superalgebras, i.e., all *other* realizations of these Lie superalgebras as Lie superalgebras of vector fields, are also described; there are 14 of them altogether. All of the exceptional Lie superalgebras are obtained with the help of the Cartan prolongation or a generalized prolongation.

INTRODUCTION

V. Kac conjectured [K1] (Theorem 10 and Conjecture 1) that infinite dimensional simple Lie superalgebras of vector fields with polynomial or formal coefficients are only straightforward analogs of the four well-known Cartan series $\mathfrak{vect}(n)$, $\mathfrak{svect}(n)$, $\mathfrak{h}(2n)$ and $\mathfrak{k}(2n+1)$ (of all, divergence-free, hamiltonian and contact vector fields, respectively, realized on the space of dimension indicated). Since superdimension is a pair of numbers, Kac's examples of simple vectorial Lie superalgebras "double" Cartan's list of simple vectorial Lie algebras.

It soon became clear ([L1], [ALSh]) that the actual list of simple vectorial Lie superalgebras "doubles" that of Cartan twice, not once (the nondirect "super" counterparts $\mathfrak{m}$ and $\mathfrak{sm}$ of $\mathfrak{k}$ as well as $\mathfrak{le}$ and $\mathfrak{sle}$ — the counterparts of $\mathfrak{h}$ — were discovered).

Moreover, even the Lie superalgebras of the four well-known series ($\mathfrak{vect}$, $\mathfrak{svect}$, $\mathfrak{h}$ and $\mathfrak{k}$) have, in addition to the dimension of the superspace on which they are usually realized, one more discrete parameter governing their other, *nonstandard*, realizations ([ALSh]). In other words, one Lie superalgebra of vector fields has several (but not too many!) different *nonisomorphic* realizations as a filtered Lie superalgebra.

Furthermore, several of these Lie superalgebras have deformations (see [L2], [Ko1], [Ko2], [L3], [LSh2]). Some of these deformed Lie superalgebras are very interesting due to their applications to physics.

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Next, three exceptional vectorial algebras were discovered [Sh1], followed by a fourth exception [Sh2]. The purpose of this note is to give a more lucid description of these exceptions, and introduce the most remarkable fifth exception ($\mathfrak{kas}$). (For a related construction of Lie superalgebras of string theories cf. [GLS] and [CK].)

In this note the ground field is $\mathbb{C}$. First, we recall the background from Linear Algebra in Superspaces. Then we recall the definition of the main tool in the construction of our examples: the notion of Cartan prolongation and its generalizations (cf. [Sh1]). We also recall some facts from the classification of simple Lie superalgebras of vector fields, cf. [L2] and [L3].

The main result of this paper is the discovery and a description of the five exceptional simple Lie superalgebras of vector fields and their fourteen W-regradings.

Here are the names of the exceptional simple Lie superalgebras, the description in terms of (generalized) Cartan prolongation (for definitions see sec. 0.5), a natural minimal simple ambient and the values r of the regradings (K stands for a grading consistent with parity); these regradings are described in §7; the indeterminates, whose degrees are fixed below, are introduced, respectively: in §5 for $\mathfrak{vle}$, in §3 for $\mathfrak{kas}$, in Appendix 3 for $\mathfrak{mb}$ and $\mathfrak{ksle}$:

1) $\mathfrak{vle}(4|3;r) = (\Pi(\Lambda(3)/\mathbb{C}\cdot 1), \mathfrak{cvect}(0|3))_* \subset \mathfrak{vect}(4|3), \quad r = 0, 1, K$

$r = 0 : \deg y = \deg u_i = \deg \xi_i = 1$

$r = 1 : \deg y = \deg \xi_1 = 0,\ \deg u_2 = \deg u_3 = \deg \xi_2 = \deg \xi_3 = 1,\ \deg u_1 = 2$

$r = K : \deg y = 0, \deg u_i = 2;\ \deg \xi_i = 1$

2) $\mathfrak{vas}(4|4) = (\text{spin}, \mathfrak{as})_* \subset \mathfrak{vect}(4|4)$

3) $\mathfrak{kas} \subset \mathfrak{k}(1|6;r), \quad r = 0(K), 1, 3\xi, 3\eta$

$r = 0(K) : \deg t = 2,\ \deg \eta_i = 1;\ \deg \xi_i = 1;\ \deg_{Lie} = \deg -2$

$r = 1 : \deg \xi_1 = 0,\ \deg \eta_1 = \deg t = 2,$

$\deg \xi_2 = \deg \xi_3 = \deg \eta_2 = \deg \eta_3 = 1;\ \deg_{Lie} = \deg -2$

$r = 3\xi : \deg \xi_i = 0,\ \deg \eta_i = \deg t = 1;\ \deg_{Lie} = \deg -1$

$r = 3\eta : \deg \eta_i = 0, \deg \xi_i = \deg t = 1; \deg_{Lie} = \deg -1$

4) $\mathfrak{mb}(4|5;r) = (\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^m_* \subset \mathfrak{m}(4), \quad r = 0, 1, K$

$r = 0 : \deg \tau = 2,\ \deg u_i = \deg \xi_i = 1 \text{ for } i = 0, 1, 2, 3;\ \deg_{Lie} = \deg -2$

$r = 1 : \deg \tau = \deg \xi_0 = \deg u_1 = 2,\ \deg u_2 = \deg u_3 = \deg \xi_2 = \deg \xi_3 = 1;$

$\deg \xi_1 = \deg u_0 = 0;\ \deg_{Lie} = \deg -2$

$r = K : \deg \tau = \deg \xi_0 = 3,\ \deg u_0 = 0,\ \deg u_i = 2; \deg \xi_i = 1 \text{ for } i > 0;$

$\deg_{Lie} = \deg -3$

5) $\mathfrak{ksle}(9|6;r) = (\mathfrak{hei}(8|6), \mathfrak{svect}_{3,4}(4))^k_* \subset \mathfrak{k}(9|6), \quad r = 0, 2, K$

$r = 0 : \deg t = 2,\ \deg p_i = \deg q_i = \deg \xi_i = \deg \eta_i = 1;\ \deg_{Lie} = \deg -2$

$r = 2 : \deg t = \deg q_3 = \deg q_4 = \deg \eta_1 = 2,$

$\deg q_1 = \deg q_2 = \deg p_1 = \deg p_2 = \deg \eta_2 = \deg \eta_3 = \deg \zeta_2 = \deg \zeta_3 = 1;$

$\deg p_3 = \deg p_4 = \deg \zeta_1 = 0;\ \deg_{Lie} = \deg -2$

$r = K : \deg t = \deg q_i = 2,\ \deg p_i = 0;\ \deg \zeta_i = \deg \eta_i = 1;\ \deg_{Lie} = \deg -2.$

These names reflect the method of construction of these algebras, rather than their own properties. To name and understand these superalgebras adequately, their further interpretation is required. In parentheses stands the superdimension of the superspace of indeterminates on which the algebra is realized by vector fields; this realization is considered as a point of reference for regradings r. Since superdimensions of $\mathfrak{g}$ are distinct for the thirteen exceptional simple vectorial Lie superalgebras $\mathfrak{g}$, it is natural to call them briefly $\mathfrak{e}(5|4), \dots, \mathfrak{e}(5|10)$ (see the table below) except for the first, $\mathfrak{sle}(4|3)$.

More exactly, we consider *simple* filtered Lie superalgebras $\mathcal{L}$ with decreasing filtration of the form

$$\text{(WF)} \qquad \mathcal{L} = \mathcal{L}_{-d} \supset \mathcal{L}_{-d+1} \supset \dots \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \dots$$

of finite *depth* d. The very term "filtered algebra" implies that $[\mathcal{L}_i, \mathcal{L}_j] \subset \mathcal{L}_{i+j}$ and we additionally require that

1) $\mathcal{L}_0$ is a maximal subalgebra of finite codimension;

2) the filtration is *transitive*: for any non-zero $x \in L_k$ for $k \geq 0$, where $L_k = \mathcal{L}_k/\mathcal{L}_{k+1}$, there is $y \in L_{-1}$ such that $[x, y] \neq 0$.

Conditions 1) and 2) manifestly imply that $\dim L_k < \infty$ for all k and the $\mathbb{Z}$-graded Lie superalgebra $L = \bigoplus_{k\geq -d} L_k$ associated with $\mathcal{L}$ grows *polynomially*, i.e., $\dim \bigoplus_{k\leq n} L_k$ grows as a polynomial in n. Such filtrations are called, after [W], *Weisfeiler filtrations*; we will shortly write W-filtrations and call the gradings associated with W-filtrations *W-gradings*. Since such filtered Lie superalgebras $\mathcal{L}$ (and associated with them graded ones, L) can be realized by *vector fields* with formal or polynomial coefficients, we refer to these Lie superalgebras as *vectorial* ones.

Any W-filtration can be considered as a basis of neighborhoods of zero in a topology, so the result can be read as the list of the exceptional simple complete vectorial Lie superalgebras. Thus, from the point of view of classification of the W-filtered complete Lie superalgebras, there are five *families* of exceptional algebras consisting of 14 individual algebras. The algebras inside each family are isomorphic as abstract ones, but are distinct as filtered ones. Here are the corresponding first terms of the graded algebras (cf. sec. 0.8), where the sign $\ni\!\!\!\!+$ (resp. $\in\!\!\!\!+$) denotes the semidirect sum with the subspace or ideal on the left (right) of it:

$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	dim $\mathfrak{g}_-$
$\mathfrak{vle}(4\|3)$	$-$	$\Pi(\Lambda(3)/\mathbb{C}1)$	$\mathfrak{c}(\mathfrak{vect}(0\|3))$	$4\|3$
$\mathfrak{vle}(4\|3;1)$	$\mathbb{C}\cdot 1$	$\mathrm{id}\otimes\Lambda(2)$	$\mathfrak{c}(\mathfrak{sl}(2)\otimes\Lambda(2)\ni\!\!\!\!+ T^{1/2}(\mathfrak{vect}(0\|2))$	$5\|4$
$\mathfrak{vle}(4\|3;K)$	$\mathrm{id}(\mathfrak{sl}(3))$	$\mathrm{id}(\mathfrak{sl}(3))\otimes\mathrm{id}(\mathfrak{sl}(2))\otimes 1$	$\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathbb{C}z$	$3\|6$
$\mathfrak{vas}(4\|4)$	$-$	spin	$\mathfrak{as}$	$4\|4$
$\mathfrak{kas}$	$\mathbb{C}\cdot 1$	id	$\mathfrak{co}(6)$	$1\|6$
$\mathfrak{kas}(;1)$	$\Lambda(1)$	$\mathrm{id}(\mathfrak{sl}(2))\otimes\mathrm{id}(\mathfrak{gl}(2))\otimes\Lambda(1)$	$(\mathfrak{sl}(2)\oplus\mathfrak{gl}(2))\ni\!\!\!\!+\mathfrak{vect}(0\|1)$	$5\|5$
$\mathfrak{kas}(;3\xi)$	$-$	$\Lambda(3)$	$\Lambda(3)\oplus\mathfrak{sl}(1\|3)$	$4\|4$
$\mathfrak{kas}(;3\eta)$	$-$	$Vol_0(0\|3)$	$\mathfrak{c}(\mathfrak{vect}(0\|3))$	$4\|3$
$\mathfrak{mb}(4\|5)$	$\Pi(\mathbb{C}\cdot 1)$	$Vol(0\|3)$	$\mathfrak{c}(\mathfrak{vect}(0\|3))$	$4\|5$
$\mathfrak{mb}(4\|5;1)$	$\Lambda(2)/\mathbb{C}\cdot 1$	$\mathrm{id}\otimes\Lambda(2)$	$\mathfrak{c}(\mathfrak{sl}(2)\otimes\Lambda(2)\ni\!\!\!\!+ T^{1/2}(\mathfrak{vect}(0\|2))$	$5\|6$
$\mathfrak{mb}(4\|5;K)$	$\mathrm{id}(\mathfrak{sl}(3))$	$\mathrm{id}(\mathfrak{sl}(3))\otimes\mathrm{id}(\mathfrak{sl}(2))\otimes 1$	$\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathbb{C}z$	$3\|8$
$\mathfrak{ksle}(9\|6)$	$\mathbb{C}\cdot 1$	$\Pi(T_0^0(\vec{0}))$	$\mathfrak{svect}(0\|4)_{3,4}$	$9\|6$
$\mathfrak{ksle}(9\|6;2)$	$\mathrm{id}(\mathfrak{sl}(3\|1))$	$\mathrm{id}(\mathfrak{sl}(2))\otimes\Lambda(3)$	$(\mathfrak{sl}(2)\otimes\Lambda(3))\ni\!\!\!\!+\mathfrak{sl}(1\|3)$	$11\|9$
$\mathfrak{ksle}(9\|6;K)$	id	$\Lambda^2(\mathrm{id})$	$\mathfrak{sl}(5)$	$5\|10$

Observe that $\mathfrak{mb}(4|5;K)_{-3} \cong \Pi(\mathrm{id}(\mathfrak{sl}(2)))$, whereas in all the other cases $\mathfrak{g}_{-3} = 0$.

Observe that unlike the regradings of the series (see sec. 0.4) where the minimal realization is attained at $r = 0$, some of the exceptional algebras have several minimal realizations.

The word "exceptional" implies that a classification is handy; indeed, for the detailed proof announced in [LSh1] and during a conference in honor of D. Buchsbaum (November 1997, Boston); see [LSh4], [LSh2].

The article is divided into several sections, according to the method of construction. Boring calculations are gathered in the appendices. The statements on simplicity are proved via Kac's criteria; cf. [K1].

Open problems. (1) Give more explicit geometric realizations of the exceptional Lie superalgebras (what structures do they preserve?).

(2) Certain exceptional Lie superalgebra are deformations of (nonsimple) Lie superalgebras whose brackets are easy to describe. In this paper the cocycle is described clumsily, in components of the generating functions. Describe the cocycle (i.e., the bracket itself) in terms of generating functions rather than their components. (An attempt is made in [ShP].)

(3) Find out what our exceptional Lie superalgebras add to the list of simple finite dimensional Lie algebras over an algebraically closed field of characteristic 2 via *Leites' conjecture*, either directly (cf. [L2], [KL]) or via Volichenko algebras ([LSe]).

Remark. The results of this paper and the related contribution to classification of the stringy superalgebras [GLS] (hep-th 9702120) were obtained in Stockholm in June 1996 and delivered at the seminar of E. Ivanov, JINR, Dubna (July, 1996), Voronezh winter school (Jan. 12–18, 1997). This paper was preprinted as hep-th 9702121, new §7 is added to it now; a brief description is also to appear in Russian in *Functionalnyj Analiz i Prilozheniya*.

0. Background

0.1. **Linear algebra in superspaces. Generalities.** Superization has certain subtleties, often disregarded or expressed as in [L], [L3] or [M]; too briefly. We will dwell on them a bit.

A *superspace* is a $\mathbb{Z}/2$-graded space; for a superspace $V = V_{\bar 0} \oplus V_{\bar 1}$ denote by $\Pi(V)$ another copy of the same superspace: with the shifted parity, i.e., $(\Pi(V))_{\bar i} = V_{\bar i + \bar 1}$. The *superdimension* of V is $\dim V = p + q\varepsilon$, where $\varepsilon^2 = 1$ and $p = \dim V_{\bar 0}$, $q = \dim V_{\bar 1}$. (Usually $\dim V$ is expressed as a pair (p, q) or $p|q$; this obscures the fact that $\dim V \otimes W = \dim V \cdot \dim W$ which is clear with the use of ε.)

A superspace structure in V induces the superspace structure in the space $\mathrm{End}(V)$. A *basis of a superspace* is always a basis consisting of *homogeneous* vectors; let $Par = (p_1, \dots, p_{\dim V})$ be an ordered collection of their parities. We call Par the *format* of the basis of V. A square *supermatrix* of format (size) Par is a $\dim V \times \dim V$ matrix whose ith row and ith column are of the same parity p_i. The matrix unit E_{ij} is supposed to be of parity $p_i + p_j$ and the bracket of supermatrices (of the same format) is defined via Sign Rule:

if something of parity p moves past something of parity q the sign $(-1)^{pq}$ accrues; the formulas defined on homogeneous elements are extended to arbitrary ones via linearity.

Examples of application of Sign Rule: setting $[X,Y]=XY-(-1)^{p(X)p(Y)}YX$ we get the notion of the supercommutator and the ensuing notions of the supercommutative superalgebra and the Lie superalgebra (that in addition to superskew-commutativity satisfies the super Jacobi identity, i.e., the Jacobi identity amended with the Sign Rule). The derivation of a superalgebra A is a linear map $D: A \longrightarrow A$ such that it satisfies the Leibniz rule (and Sign rule)

$$D(ab)=D(a)b+(-1)^{p(D)p(a)}aD(b).$$

In particular, let $A=\mathbb{K}[x]$ be the free supercommutative polynomial superalgebra in $x=(x_1,\dots,x_n)$, where the superstructure is determined by the parities of the indeterminates: $p(x_i)=p_i$. Partial derivatives are defined (with the help of super Leibniz Rule) by the formulas

$$\frac{\partial x_i}{\partial x_j}=\delta_{i,j}.$$

Clearly, the collection $\mathfrak{der} A$ of all superdifferentiations of A is a Lie superalgebra whose elements are of the form

$$\sum f_i(x)\frac{\partial}{\partial x_i}.$$

(We do not usually use the sign $\wedge$ for the wedge product of differential forms on supermanifolds: in what follows we assume that the exterior differential is odd and the differential forms constitute a supercommutative superalgebra; however, we sometimes keep using the sign $\wedge$ while working on manifolds in order not to deviate too far from conventional notations.)

Usually, *Par* is of the form $(\bar 0,\dots,\bar 0,\bar 1,\dots,\bar 1)$. Such a format is called *standard*. In this paper we can do without nonstandard formats. But they are vital in the study of systems of simple roots that the reader might be interested in; besides, they are direct analogs of the nonstandard gradings we consider.

The *general linear* Lie superalgebra of all supermatrices of size *Par* is denoted by $\mathfrak{gl}(Par)$; usually, $\mathfrak{gl}(\bar 0,\dots,\bar 0,\bar 1,\dots,\bar 1)$ is abbreviated to $\mathfrak{gl}(\dim V_{\bar 0}|\dim V_{\bar 1})$. Any matrix from $\mathfrak{gl}(Par)$ can be expressed as the sum of its even and odd parts; in the standard format this is the block expression:

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}+\begin{pmatrix}0 & B\\ C & 0\end{pmatrix},\quad p\left(\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}\right)=\bar 0,\ p\left(\begin{pmatrix}0 & B\\ C & 0\end{pmatrix}\right)=\bar 1.$$

The *supertrace* is the map $\mathfrak{gl}(Par)\longrightarrow\mathbb{C}$, $(A_{ij})\mapsto\sum(-1)^{p_i}A_{ii}$. Since $\mathrm{str}[x,y]=0$, the space of supertraceless matrices constitutes the *special linear* Lie subsuperalgebra $\mathfrak{sl}(Par)$.

Lie superalgebras that preserve bilinear forms: two types. To the linear map F of superspaces there corresponds the dual map F^* between the dual superspaces; if A is the supermatrix corresponding to F in a basis of the format *Par*, then to F^* the *supertransposed* matrix A^{st} corresponds:

$$(A^{st})_{ij}=(-1)^{(p_i+p_j)(p_i+p(A))}A_{ji}.$$

The supermatrices $X\in\mathfrak{gl}(Par)$ such that

$$X^{st}B+(-1)^{p(X)p(B)}BX=0\quad\text{for a homogeneous matrix } B\in\mathfrak{gl}(Par)$$

constitute the Lie superalgebra $\mathfrak{aut}(B)$ that preserves the bilinear form on V with matrix B.

Recall that the *supersymmetry* of the homogeneous form ω means that its matrix B satisfies the condition $B^u = B$, where $B^u = \begin{pmatrix} R^t & (-1)^{p(B)}T^t \\ (-1)^{p(B)}S^t & -U^t \end{pmatrix}$ for the matrix $B = \begin{pmatrix} R & S \\ T & U \end{pmatrix}$. Similarly, *skew-supersymmetry* of B means that $B^u = -B$.

Most popular canonical forms of the nondegenerate supersymmetric form are the ones whose supermatrices in the standard format are the following canonical ones, B_{ev} or B'_{ev}:

$$B_{ev}(m|2n) = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where } J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

or

$$B'_{ev}(m|2n) = \begin{pmatrix} \text{antidiag}(1, \dots, 1) & 0 \\ 0 & J_{2n} \end{pmatrix}.$$

The usual notation for $\mathfrak{aut}(B_{ev}(m|2n))$ is $\mathfrak{osp}(m|2n)$ or $\mathfrak{osp}^{sy}(m|2n)$.

Recall that the "upsetting" of forms $u : Bil(V, W) \longrightarrow Bil(W, V)$ becomes for $V = W$ an involution $u : B \mapsto B^u$. This involution separates symmetric and skew-symmetric forms. The passage from V to $\Pi(V)$ sends the supersymmetric forms to superskew-symmetric ones, preserved by the "symplectico-orthogonal" Lie superalgebra $\mathfrak{osp}^{sk}(m|2n)$ which is isomorphic to $\mathfrak{osp}^{sy}(m|2n)$ but has a different matrix realization. We never use notation $\mathfrak{sp}'\mathfrak{o}(2n|m)$ in order not to confuse with the special Poisson superalgebra.

In the standard format the matrix realizations of these algebras are:

$$\mathfrak{osp}(m|2n) = \left\{ \begin{pmatrix} E & Y & X^t \\ X & A & B \\ -Y^t & C & -A^t \end{pmatrix} \right\}; \quad \mathfrak{osp}^{sk}(m|2n) = \left\{ \begin{pmatrix} A & B & X \\ C & -A^t & Y^t \\ Y & -X^t & E \end{pmatrix} \right\},$$

$$\text{where } \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{sp}(2n), \quad E \in \mathfrak{o}(m) \text{ and } {}^t \text{ is the usual transposition.}$$

A nondegenerate supersymmetric odd bilinear form $B_{odd}(n|n)$ can be reduced to the canonical form whose matrix in the standard format is J_{2n}. A canonical form of the superskew odd nondegenerate form in the standard format is $\Pi_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. The usual notation for $\mathfrak{aut}(B_{odd}(Par))$ is $\mathfrak{pe}(Par)$. The passage from V to $\Pi(V)$ sends the supersymmetric forms to superskew-symmetric ones and establishes an isomorphism $\mathfrak{pe}^{sy}(Par) \cong \mathfrak{pe}^{sk}(Par)$. This Lie superalgebra is called, as A. Weil suggested, *periplectic*. In the standard format these superalgebras are shorthanded as in the following formula, where their matrix realizations are also given:

$$\mathfrak{pe}^{sy}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = -B^t, C = C^t \right\};$$

$$\mathfrak{pe}^{sk}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = B^t, C = -C^t \right\}.$$

The *special periplectic* superalgebra is $\mathfrak{spe}(n) = \{X \in \mathfrak{pe}(n) : \text{str}X = 0\}$.

Observe that though the Lie superalgebras $\mathfrak{osp}^{sy}(m|2n)$ and $\mathfrak{pe}^{sk}(2n|m)$, as well as $\mathfrak{pe}^{sy}(n)$ and $\mathfrak{pe}^{sk}(n)$, are isomorphic, the difference between them is sometimes crucial, see Remark 0.6 below.

0.2. **Vectorial Lie superalgebras. The standard realization.** The elements of $\mathcal{L} = \mathfrak{der}\ \mathbb{C}[[u]]$ are considered as vector fields. The Lie algebra $\mathcal{L}$ has only one maximal subalgebra $\mathcal{L}_0$ of finite codimension (consisting of the fields that vanish at the origin). The subalgebra $\mathcal{L}_0$ determines a filtration of $\mathcal{L}$: set

$$\mathcal{L}_{-1} = \mathcal{L} \quad \text{and} \quad \mathcal{L}_i = \{D \in \mathcal{L}_{i-1} : [D, \mathcal{L}] \subset \mathcal{L}_{i-1}\} \text{ for } i \geq 1. \tag{0.2.1}$$

The associated graded Lie algebra $L = \bigoplus_{i\geq -1} L_i$, where $L_i = \mathcal{L}_i/\mathcal{L}_{i+1}$, consists of the vector fields with *polynomial* coefficients.

Suppose $\mathcal{L}_0 \subset \mathcal{L}$ is a maximal subalgebra of finite codimension and $\mathcal{L}_0$ contains no ideals of $\mathcal{L}$. For the Lie algebra $\mathcal{L} = \mathfrak{der}\,\mathbb{C}[u]$ the minimal nontrivial $\mathcal{L}_0$-submodule of $\mathcal{L}$ containing $\mathcal{L}_0$ coincides with $\mathcal{L}$. This is not so for superalgebras; not all subalgebras $\mathcal{L}$ of $\mathfrak{der}\,\mathbb{C}[u, \xi]$ have this property. Let $\mathcal{L}_{-1}$ be a minimal subspace of $\mathcal{L}$ containing $\mathcal{L}_0$, different from $\mathcal{L}_0$ and $\mathcal{L}_0$-invariant. Construct a filtration of $\mathcal{L}$ by setting for $i \geq 1$:

$$\mathcal{L}_{-i-1} = [\mathcal{L}_{-1}, \mathcal{L}_{-i}] + \mathcal{L}_{-i} \quad \text{and} \quad \mathcal{L}_i = \{D \in \mathcal{L}_{i-1} : [D, \mathcal{L}_{-1}] \subset \mathcal{L}_{i-1}\}. \tag{0.2.2}$$

Since the codimension of $\mathcal{L}_0$ is finite, the filtration takes the form

$$\mathcal{L} = \mathcal{L}_{-d} \supset \dots \mathcal{L}_0 \supset \dots \tag{0.2.3}$$

for some d. This d is called the *depth* of $\mathcal{L}$ or of the associated graded Lie superalgebra L.

Considering the subspaces (0.2.3) as the basis of a topology, we can complete the graded or filtered Lie superalgebras L or $\mathcal{L}$; the elements of the completion are the vector fields with formal power series as coefficients. Though the structure of the graded algebras is easier to describe, in applications the completed Lie superalgebras are usually needed.

Unlike Lie algebras, simple vectorial *super*algebras possess *several* maximal subalgebras of finite codimension. We will describe them, together with the corresponding gradings, in sec. 0.4.

1) General algebras. Let $x = (u_1, \dots, u_n, \theta_1, \dots, \theta_m)$, where the u_i are even indeterminates and the θ_j are odd ones. The Lie superalgebra $\mathfrak{vect}(n|m)$ consists of superdifferentiations of $\mathfrak{der}\ \mathbb{C}[x]$; it is called *the general vectorial superalgebra*.

2) Special algebras. The *divergence* of the field $D = \sum\limits_i f_i \frac{\partial}{\partial u_i} + \sum\limits_j g_j \frac{\partial}{\partial \theta_j}$ is the function (in our case: a polynomial, or a series)

$$\mathrm{div} D = \sum_i \frac{\partial f_i}{\partial u_i} + \sum_j (-1)^{p(g_j)} \frac{\partial g_i}{\partial \theta_j}.$$

• The Lie superalgebra $\mathfrak{svect}(n|m) = \{D \in \mathfrak{vect}(n|m) : \mathrm{div} D = 0\}$ is called the *special* or *divergence-free vectorial superalgebra*. The notion of divergence depends on coordinates. Another description of $\mathfrak{svect}$ is as follows:

$$\mathfrak{svect}(n|m) = \{D \in \mathfrak{vect}(n|m) : L_D vol_x = 0\},$$

where vol_x is the volume form with constant coefficients in coordinates x and L_D the Lie derivative with respect to D.

• The Lie superalgebra $\mathfrak{svect}_\lambda(0|m) = \{D \in \mathfrak{vect}(0|m) : \mathrm{div}(1+\lambda\theta_1 \cdot \dots \cdot \theta_m)D = 0\}$ — the deform of $\mathfrak{svect}(0|m)$ — has no particular name and is also called the *deformed special* or *deformed divergence-free vectorial superalgebra*. Clearly, $\mathfrak{svect}_\lambda(0|m) \cong \mathfrak{svect}_\mu(0|m)$ for $\lambda\mu \neq 0$.

Observe that $p(\lambda) \equiv m \pmod 2$, i.e., for odd m the parameter of deformation λ is odd.

Remark. Sometimes we write $\mathfrak{vect}(x)$ or even $\mathfrak{vect}(V)$ if $V = Span(x)$ and use similar notations for the subalgebras of $\mathfrak{vect}$ introduced below. Some algebraists sometimes abbreviate $\mathfrak{vect}(n)$ and $\mathfrak{svect}(n)$ to W_n (in honor of Witt) and S_n, respectively.

3) The algebras that preserve Pfaff equations and differential 2-forms.

• Set $u = (t, p_1, \dots, p_n, q_1, \dots, q_n)$; let

$$\tilde{\alpha}_1 = dt + \sum_{1\leq i\leq n}(p_i dq_i - q_i dp_i) \ + \sum_{1\leq j\leq m}\theta_j d\theta_j \quad \text{and} \quad \tilde{\omega}_0 = d\tilde{\alpha}_1 \ .$$

The form $\tilde{\alpha}_1$ is called *contact*, the form $\tilde{\omega}_0$ is called *symplectic*.

Sometimes it is more convenient to redenote the θ's and set

$$\xi_j = \frac{1}{\sqrt{2}}(\theta_j - i\theta_{r+j}); \quad \eta_j = \frac{1}{\sqrt{2}}(\theta_j + i\theta_{r+j}) \ \text{(here } i^2 = -1) \qquad \text{for } j \leq r = [m/2], \quad \theta = \theta_{2r+1}$$

and in place of $\tilde{\omega}_0$ or $\tilde{\alpha}_1$ take α and $\omega_0 = d\alpha_1$, respectively, where

$$\alpha_1 = dt + \sum_{1\leq i\leq n}(p_i dq_i - q_i dp_i) + \sum_{1\leq j\leq r}(\xi_j d\eta_j + \eta_j d\xi_j) \quad \text{if} \ \ m = 2r$$

$$\alpha_1 = dt + \sum_{1\leq i\leq n}(p_i dq_i - q_i dp_i) + \sum_{1\leq j\leq r}(\xi_j d\eta_j + \eta_j d\xi_j) + \theta d\theta \quad \text{if} \ \ m = 2r+1.$$

The Lie superalgebra that preserves the *Pfaff equation* $\alpha_1 = 0$, i.e., the superalgebra

$$\mathfrak{k}(2n+1|m) = \{D \in \mathfrak{vect}(2n+1|m) : L_D\alpha_1 = f_D\alpha_1\} \qquad \text{for a polynomial } f_D \in \mathbb{C}[t, p, q, \theta]$$

is called the *contact superalgebra*.

The Lie superalgebra $\mathfrak{po}(2n|m)$ that preserves not just the Pfaff equation determined by α_1 but the form itself, i.e.,

$$\mathfrak{po}(2n|m) = \{D \in \mathfrak{k}(2n+1|m) : L_D\alpha_1 = 0\}$$

is called the *Poisson* superalgebra. (A geometric interpretation of the Poisson superalgebra: it is the Lie superalgebra that preserves the connection with form α_1 in the line bundle over a symplectic supermanifold with the symplectic form $d\alpha_1$.)

• Similarly, set $u = q = (q_1, \dots, q_n)$, let $\theta = (\xi_1, \dots, \xi_n; \tau)$ be odd. Set

$$\alpha_0 = d\tau + \sum_i(\xi_i dq_i + q_i d\xi_i), \qquad \omega_1 = d\alpha_0$$

and call these forms the *odd-contact* and, as A. Weil suggested, *periplectic*, respectively.

The Lie superalgebra that preserves the Pfaff equation $\alpha_0 = 0$, i.e., the superalgebra

$$\mathfrak{m}(n) = \{D \in \mathfrak{vect}(n|n+1) : L_D\alpha_0 = f_D \cdot \alpha_0 \ \text{ for a polynomial } \ f_D \in \mathbb{C}[q, \xi, \tau]\}$$

is called the *odd-contact superalgebra*.

The Lie superalgebra

$$\mathfrak{b}(n) = \{D \in \mathfrak{m}(n) : L_D\alpha_0 = 0\}$$

is called the *Buttin* superalgebra ([L3]). (A geometric interpretation of the Buttin superalgebra: it is the Lie superalgebra that preserves the connection with form α_1 in the line bundle of rank ε over a periplectic supermanifold, i.e., the supermanifold with the periplectic form $d\alpha_0$.)

The Lie superalgebras

$$\mathfrak{sm}(n) = \{D \in \mathfrak{m}(n) : \operatorname{div} D = 0\}\ ,\ \mathfrak{sb}(n) = \{D \in \mathfrak{b}(n) : \operatorname{div} D = 0\}$$

are called the *divergence-free* (or *special*) *odd-contact* and *special Buttin* superalgebras, respectively.

Remark. A relation with finite dimensional geometry is as follows. Clearly, $\ker \alpha_1 = \ker \tilde{\alpha}_1$. The restriction of ω_0 to $\ker \alpha_1$ is the orthosymplectic form $B_{ev}(m|2n)$; the restriction of ω_0 to $\ker \tilde{\alpha}_1$ is $B'_{ev}(m|2n)$. Similarly, the restriction of ω_1 to $\ker \alpha_0$ is the periplectic form $B_{odd}(n|n)$.

0.3. **Generating functions.** A laconic way to describe the elements of $\mathfrak{k}$, $\mathfrak{m}$ and their subalgebras is via generating functions.

• Odd form α_1 or $\tilde{\alpha}_1$. For $f \in \mathbb{C}[t, p, q, \theta]$ set:

$$K_f = (2 - E)(f)\frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t}E,$$

where $E = \sum\limits_i y_i \frac{\partial}{\partial y_i}$ (here the y are all the coordinates except t) is the *Euler operator* (which counts the degree with respect to the y), and H_f is the hamiltonian field with Hamiltonian f that preserves $d\tilde{\alpha}_1$:

$$H_f = \sum_{i \leq n}(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}) - (-1)^{p(f)}\left(\sum_{j \leq m}\frac{\partial f}{\partial \theta_j}\frac{\partial}{\partial \theta_j}\right).$$

The choice of the form α_1 instead of $\tilde{\alpha}_1$ only affects the form of H_f that we give for $m = 2k + 1$:

$$H_f = \sum_{i \leq n}(\tfrac{\partial f}{\partial p_i}\tfrac{\partial}{\partial q_i} - \tfrac{\partial f}{\partial q_i}\tfrac{\partial}{\partial p_i}) - (-1)^{p(f)}\left(\sum_{j \leq k}(\tfrac{\partial f}{\partial \xi_j}\tfrac{\partial}{\partial \eta_j} + \tfrac{\partial f}{\partial \eta_j}\tfrac{\partial}{\partial \xi_j}) + \tfrac{\partial f}{\partial \theta}\tfrac{\partial}{\partial \theta}\right).$$

• Even form α_0. For $f \in \mathbb{C}[q, \xi, \tau]$ set:

$$M_f = (2 - E)(f)\frac{\partial}{\partial \tau} - Le_f - (-1)^{p(f)}\frac{\partial f}{\partial \tau}E,$$

where $E = \sum\limits_i y_i \frac{\partial}{\partial y_i}$ (here the y are all the coordinates except τ) is the Euler operator, and

$$Le_f = \sum_{i \leq n}(\frac{\partial f}{\partial q_i}\ \frac{\partial}{\partial \xi_i} + (-1)^{p(f)}\frac{\partial f}{\partial \xi_i}\ \frac{\partial}{\partial q_i}).$$

Since

$$L_{K_f}(\alpha_1) = 2\frac{\partial f}{\partial t}\alpha_1, \qquad L_{M_f}(\alpha_0) = -(-1)^{p(f)}2\frac{\partial f}{\partial \tau}\alpha_0,$$

it follows that $K_f \in \mathfrak{k}(2n + 1|m)$ and $M_f \in \mathfrak{m}(n)$. Observe that

$$p(Le_f) = p(M_f) = p(f) + \bar{1}.$$

• To the supercommutators $[K_f, K_g]$ or $[M_f, M_g]$ there correspond *contact brackets* of the generating functions:

$$[K_f, K_g] = K_{\{f,g\}_{k.b.}}; \qquad [M_f, M_g] = M_{\{f,g\}_{m.b.}}.$$

The explicit formulas for the contact brackets are as follows. Let us first define the brackets on functions that do not depend on t (resp. τ).

The *Poisson bracket* $\{\cdot,\cdot\}_{P.b.}$ (in the realization with the form $\tilde{\omega}_0$) is given by the formula

$$\{f,g\}_{P.b.} = \sum_{i\leq n} (\frac{\partial f}{\partial p_i}\ \frac{\partial g}{\partial q_i} - \ \frac{\partial f}{\partial q_i}\ \frac{\partial g}{\partial p_i}) - (-1)^{p(f)} \sum_{j\leq m}\ \frac{\partial f}{\partial \theta_j}\ \frac{\partial g}{\partial \theta_j} \text{ for } f,g\in\mathbb{C}[p,q,\theta]$$

and in the realization with the form ω_0 for $m = 2k+1$ it is given by the formula

$$\begin{aligned}\{f,g\}_{P.b.} &= \sum_{i\leq n} (\frac{\partial f}{\partial p_i}\ \frac{\partial g}{\partial q_i} - \ \frac{\partial f}{\partial q_i}\ \frac{\partial g}{\partial p_i})\\ &- (-1)^{p(f)} \left(\sum_{j\leq k}(\frac{\partial f}{\partial \xi_j}\ \frac{\partial g}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j}\ \frac{\partial g}{\partial \xi_j}) + \frac{\partial f}{\partial \theta}\ \frac{\partial g}{\partial \theta}\right) \text{ for } f,g\in\mathbb{C}[p,q,\xi,\eta,\theta].\end{aligned}$$

The *Buttin bracket* $\{\cdot,\cdot\}_{B.b.}$ is given by the formula

$$\{f,g\}_{B.b.} = \sum_{i\leq n} (\frac{\partial f}{\partial q_i}\ \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)}\ \frac{\partial f}{\partial \xi_i}\ \frac{\partial g}{\partial q_i}) \text{ for } f,g\in\mathbb{C}[q,\xi].$$

Remark. What we call here the Buttin bracket was discovered in pre-super era by Schouten. Buttin was the first to observe that the Schouten bracket determines a Lie superalgebra; Leites interpreted it in terms of mechanic ([L1]) and Batalin with Vilkovisky later rediscovered this mechanic with much success, see [GPS]. The *Schouten bracket* was originally defined on the superspace of multivector fields on a manifold, i.e., on the superspace $\Gamma(\Lambda^{\bullet}(T(M)))\cong\Lambda^{\bullet}_{\mathcal{F}}(Vect(M))$ of sections of the exterior algebra (over the algebra $\mathcal{F}$ of functions) of the tangent bundle. The explicit formula (in which the hatted slot should be ignored, as usual) of the Schouten bracket is

$$(*)\qquad \begin{aligned}&[X_1\wedge\dots\wedge\dots\wedge X_k, Y_1\wedge\dots\wedge Y_l]\\ &= \textstyle\sum_{i,j}(-1)^{i+j}[X_i,Y_j]\wedge X_1\wedge\dots\wedge\hat{X}_i\wedge\dots\wedge X_k\wedge Y_1\wedge\dots\wedge\hat{Y}_j\wedge\dots\wedge Y_l.\end{aligned}$$

With the help of Sign Rule we easily superize formula $(*)$ for the case when manifold M is replaced with supermanifold $\mathcal{M}$. Let x and ξ be the even and odd coordinates on $\mathcal{M}$. Setting $\theta_i = \Pi(\partial x_i) = \check{x}_i$, $q_j = \Pi(\partial\xi_j) = \check{\xi}_j$ we get an identification of the Schouten bracket of vector fields on $\mathcal{M}$ with the Buttin bracket of functions on the supermanifold $\check{\mathcal{M}}$ whose coordinates are x,ξ and $\check{x}$, $\check{\xi}$; the transformation of x,ξ induces that of the checked coordinates. (Physicists call the checked variables *ghosts*; cf. [GPS].)

In terms of the Poisson and Buttin brackets, respectively, the contact brackets take the form

$$\{f,g\}_{k.b.} = (2-E)(f)\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2-E)(g) - \{f,g\}_{P.b.}$$

and, respectively,

$$\{f,g\}_{m.b.} = (2-E)(f)\frac{\partial g}{\partial \tau} + (-1)^{p(f)}\frac{\partial f}{\partial \tau}(2-E)(g) - \{f,g\}_{B.b.}.$$

The Lie superalgebras of *Hamiltonian fields* (or *Hamiltonian superalgebra*) and its special subalgebra (defined only if $n = 0$) are

$$\mathfrak{h}(2n|m) = \{D \in \mathfrak{vect}(2n|m) : \ L_D\omega_0 = 0\}$$

and

$$\mathfrak{sh}(m) = \{H_f \in \mathfrak{h}(0|m) : \int f vol_\theta = 0\}.$$

Its odd analogues are the Lie superalgebra of Leitesian fields introduced in [L1] and its special subalgebra:

$$\mathfrak{le}(n) = \{D \in \mathfrak{vect}(n|n) : L_D\omega_1 = 0\} \text{ and } \mathfrak{sle}(n) = \{D \in \mathfrak{le}(n) : \mathrm{div} D = 0\}.$$

It is not difficult to prove the following isomorphisms (as superspaces):

$$\begin{aligned}\mathfrak{k}(2n+1|m) &\cong Span(K_f : f \in \mathbb{C}[t, p, q, \xi]);\\ \mathfrak{h}(2n|m) &\cong Span(H_f : f \in \mathbb{C}[p, q, \theta]);\\ \mathfrak{m}(n) &\cong Span(M_f : f \in \mathbb{C}[\tau, q, \xi]);\\ \mathfrak{le}(n) &\cong Span(Le_f : f \in \mathbb{C}[q, \xi]).\end{aligned}$$

Remark. 1) It is obvious that the Lie superalgebras of the series $\mathfrak{vect}$, $\mathfrak{svect}$, $\mathfrak{h}$ and $\mathfrak{po}$ for $n = 0$ are finite dimensional.

2) A Lie superalgebra of the series $\mathfrak{h}$ is the quotient of the Lie superalgebra $\mathfrak{po}$ modulo the one-dimensional center $\mathfrak{z}$ generated by constant functions.

Similarly, $\mathfrak{le}$ and $\mathfrak{sle}$ are the quotients of $\mathfrak{b}$ and $\mathfrak{sb}$, respectively, modulo the one-dimensional (odd) center $\mathfrak{z}$ generated by constant functions.

Set $\mathfrak{spo}(m) = \{K_f \in \mathfrak{po}(0|m) : \int f vol_\xi = 0\}$ and $\mathfrak{sh}(m) = \mathfrak{spo}(m)/\mathbb{C} \cdot K_1$.

Since, as is easy to see ([GLS]), $\mathrm{div} K_f = (2n+2-m)\frac{\partial f}{\partial t}$ for $K_f \in \mathfrak{k}(2n+1|m)$, the divergence-free subalgebra of $\mathfrak{k}(2n+1|m)$ is either the algebra itself (for $m = 2n+2$) or its Poisson subalgebra. Nothing new.

The "odd" counterpart is more interesting. Since

$$\mathrm{div} M_f = (-1)^{p(f)} 2 \left((1-E)\frac{\partial f}{\partial \tau} - \sum_{i \le n} \frac{\partial^2 f}{\partial q_i \partial \xi_i} \right),$$

we can define the divergence-free subalgebra of $\mathfrak{m}(n)$:

$$\mathfrak{sm}(n) = Span \left(M_f \in \mathfrak{m}(n) : (1-E)\frac{\partial f}{\partial \tau} = \sum_{i \le n} \frac{\partial^2 f}{\partial q_i \partial \xi_i} \right).$$

In particular,

$$\mathrm{div} Le_f = (-1)^{p(f)} 2 \sum_{i \le n} \frac{\partial^2 f}{\partial q_i \partial \xi_i}.$$

(The odd analog of the Laplacian, namely, the operator

$$\Delta = \sum_{i \le n} \frac{\partial^2}{\partial q_i \partial \xi_i}$$

on a periplectic supermanifold appeared in physics under the name of *BRST operator*; cf. [GPS]. The divergence-free vector fields from $\mathfrak{sle}(n)$ are generated by *harmonic* functions, i.e., such that $\Delta(f) = 0$.)

Lie superalgebras $\mathfrak{sle}(n)$, $\mathfrak{sb}(n)$ and $\mathfrak{svect}(1|n)$ have ideals $\mathfrak{sle}^\circ(n)$, $\mathfrak{sb}^\circ(n)$ and $\mathfrak{svect}^\circ(n)$ of codimension 1 (more exactly, ε^{n-1}) defined from the exact sequences

$$0 \longrightarrow \mathfrak{sle}^\circ(n) \longrightarrow \mathfrak{sle}(n) \longrightarrow \mathbb{C}\cdot Le_{\xi_1\dots\xi_n} \longrightarrow 0,$$
$$0 \longrightarrow \mathfrak{sb}^\circ(n) \longrightarrow \mathfrak{sb}(n) \longrightarrow \mathbb{C}\cdot M_{\xi_1\dots\xi_n} \longrightarrow 0,$$
$$0 \longrightarrow \mathfrak{svect}^\circ(1|n) \longrightarrow \mathfrak{svect}(1|n) \longrightarrow \mathbb{C}\cdot \xi_1\dots\xi_n\frac{\partial}{\partial t} \longrightarrow 0.$$

0.4. **Nonstandard realizations.** In [LSh4] we classified the nonstandard gradings of the simple vectorial Lie superalgebras. Here are the main points. Clearly, the gradings in the series $\mathfrak{vect}$ induce the gradings in the series $\mathfrak{svect}$, and $\mathfrak{svect}^\circ$; the gradings in $\mathfrak{m}$ induce the gradings in $\mathfrak{sm}$, $\mathfrak{le}$, $\mathfrak{sle}$, $\mathfrak{sle}^\circ$, $\mathfrak{b}$, $\mathfrak{sb}$, $\mathfrak{sb}^\circ$; the gradings in $\mathfrak{k}$ induce the gradings in $\mathfrak{po}$, $\mathfrak{h}$. In what follows we consider $\mathfrak{k}(2n+1|m)$ as preserving the Pfaff equation $\alpha=0$, where

$$\alpha = dt + \sum_{i\le n}(p_i dq_i - q_i dp_i) + \sum_{j\le r}(\xi_j d\eta_j + \eta_j d\xi_j) + \sum_{k\ge m-2r}\theta_k d\theta_k.$$

The standard realizations are marked by $(*)$ and in this case indication to $r=0$ is omitted; note that (bar several exceptions for small m,n) it corresponds to the case of the minimal codimension of $\mathcal{L}_0$. Observe that the Lie superalgebras corresponding to different values of r are isomorphic as abstract Lie superalgebras, but as filtered ones they are distinct.

Lie superalgebra	its $\mathbb{Z}$-grading
$\mathfrak{vect}(n\|m;r)$, $0\le r\le m$	$\deg u_i=\deg\xi_j=1$ for any i,j $(*)$
	$\deg\xi_j=0$ for $1\le j\le r$; $\deg u_i=\deg\xi_{r+s}=1$ for any i,s
$\mathfrak{m}(n;r)$, $0\le r\le n$	$\deg\tau=2$, $\deg q_i=\deg\xi_i=1$ for any i $(*)$
	$\deg\tau=\deg q_i=1$, $\deg\xi_i=0$ for any i
	$\deg\tau=\deg q_i=2$, $\deg\xi_i=0$ for $1\le i\le r<n$; $\deg u_{r+j}=\deg\xi_{r+j}=1$ for any j
$\mathfrak{k}(2n+1\|m;r)$, $0\le r\le[\frac{m}{2}]$	$\deg t=2$, $\deg p_i=\deg q_i=\deg\xi_j=\deg\eta_j=\deg\theta_k=1$ for any i,j,k $(*)$
	$\deg t=\deg\xi_i=2$, $\deg\eta_i=0$ for $1\le i\le r\le[\frac{m}{2}]$; $\deg p_i=\deg q_i=\deg\theta_j=1$ for $j\ge1$ and all i
$\mathfrak{k}(1\|2m;m)$	$\deg t=\deg\xi_i=1$, $\deg\eta_i=0$ for $1\le i\le m$

Observe that $\mathfrak{k}(1|2;1)\cong\mathfrak{vect}(1|1)$ and $\mathfrak{m}(1;1)\cong\mathfrak{vect}(1|1)$ as filtered Lie superalgebras.

The exceptional nonstandard gradings. Observe immediately that though these regradings are not Weisfeiler ones they are used all the time in string theories.

Denote the indeterminates and their respective exceptional degrees as follows (here $\mathfrak{k}(1|2)$ is considered in the realization that preserves the Pfaff equation $\alpha_1=0$):

		$\mathfrak{vect}(1\|1)$	$\mathfrak{k}(1\|2)$	$\mathfrak{m}(1)$
$\mathfrak{vect}(1\|1)$	t,ξ	$1,1$	$2,1$	$1,-1$
$\mathfrak{k}(1\|2)$	t,ξ,η	$1,1,0$	$2,1,1$	$1,2,-1$
$\mathfrak{m}(1)$	τ,q,ξ	$1,1,0$	$1,2,-1$	$2,1,1$

Denote the nonstandard exceptional realizations by indicating the above degrees after a semicolon. In addition to the isomorphisms indicated above we get the following isomorphisms of the filtered Lie superalgebras:

$$\mathfrak{vect}(1|1;2,1)\cong\mathfrak{k}(1|2);\qquad \mathfrak{k}(1|2;1,2,-1)\cong\mathfrak{m}(1);$$
$$\mathfrak{vect}(1|1;1,-1)\cong\mathfrak{m}(1);\qquad \mathfrak{m}(1;1,2,-1)\cong\mathfrak{k}(1|2).$$

0.5. **Cartan prolongs.** We will repeatedly make use of Cartan's prolongation; see [St]. So let me recall the definition and generalize it somewhat. Let $\mathfrak{g}$ be a Lie algebra, V a $\mathfrak{g}$-module, S^i the operator of the ith symmetric power. Set $\mathfrak{g}_{-1}=V$, $\mathfrak{g}_0=\mathfrak{g}$ and define the ith *Cartan prolong* for $i>0$ as

$$\begin{aligned}\mathfrak{g}_i=\{X\in \mathrm{Hom}(\mathfrak{g}_{-1},\mathfrak{g}_{i-1}) : X(v_0)(v_1,v_2,\dots v_i)\\ =X(v_1)(v_0,v_2,\dots,v_i)\ \text{ for any }\ v_i\in\mathfrak{g}_{-1}\}.\end{aligned}$$

Equivalently, let $i: S^{k+1}(\mathfrak{g}_{-1})^*\otimes\mathfrak{g}_{-1}\longrightarrow S^k(\mathfrak{g}_{-1})^*\otimes\mathfrak{g}_{-1}^*\otimes\mathfrak{g}_{-1}$ be a natural embedding and $j: S^k(\mathfrak{g}_{-1})^*\otimes\mathfrak{g}_0\longrightarrow S^k(\mathfrak{g}_{-1})^*\otimes\mathfrak{g}_{-1}^*\otimes\mathfrak{g}_{-1}$ a natural map. Then $\mathfrak{g}_k=i(S^{k+1}(\mathfrak{g}_{-1})^*\otimes\mathfrak{g}_{-1})\cap j(S^k(\mathfrak{g}_{-1})^*\otimes\mathfrak{g}_0)$.

The sum $(\mathfrak{g}_{-1},\mathfrak{g}_0)_*=\bigoplus_{i\geq -1}\mathfrak{g}_i$ is called the *Cartan prolong* (the result of Cartan's *prolongation*) of the pair $(V,\mathfrak{g})$. (In what follows $\cdot$ in superscript denotes, as is now customary, the collection of all degrees, while $*$ is reserved for dualization; in the subscripts we retain the old fashioned $*$ instead of $\cdot$ to avoid too close a contact with the punctuation marks.)

Suppose that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is faithful. Then, clearly,

$$\begin{aligned}(\mathfrak{g}_{-1},\mathfrak{g}_0)_*\subset\mathfrak{vect}(n)=\mathfrak{der}\,\mathbb{C}[x_1,\dots,x_n],\ \text{ where }\ n=\dim\ \mathfrak{g}_{-1}\ \text{ and}\\ \mathfrak{g}_i=\{D\in\mathfrak{vect}(n):\deg D=i,[D,X]\in\mathfrak{g}_{i-1}\text{ for any }X\in\mathfrak{g}_{-1}\}.\end{aligned}$$

It is subject to an easy verification that the Lie algebra structure on $\mathfrak{vect}(n)$ induces same on $(\mathfrak{g}_{-1},\mathfrak{g}_0)_*$.

Of the four simple vectorial Lie algebras, three are Cartan prolongs:

$$\mathfrak{vect}(n)=(\mathrm{id},\mathfrak{gl}(n))_*,\quad \mathfrak{svect}(n)=(\mathrm{id},\mathfrak{sl}(n))_*\quad\text{and}\quad \mathfrak{h}(2n)=(\mathrm{id},\mathfrak{sp}(n))_*.$$

The fourth one — $\mathfrak{k}(2n+1)$ — is also the prolong under a trifle more general construction described as follows.

1) The generalized Cartan prolong. Let $\mathfrak{g}_-=\bigoplus_{-d\leq i\leq -1}\mathfrak{g}_i$ be a nilpotent $\mathbb{Z}$-graded Lie algebra and $\mathfrak{g}_0\subset\mathfrak{der}_0\mathfrak{g}$ a Lie subalgebra of the $\mathbb{Z}$-grading-preserving derivations. For $k>0$ define the kth prolong of the pair $(\mathfrak{g}_-,\mathfrak{g}_0)$ to be

$$\mathfrak{g}_k=(j(S^{\bullet}(\mathfrak{g}_-)^*\otimes\mathfrak{g}_0)\cap i(S^{\bullet}(\mathfrak{g}_-)^*\otimes\mathfrak{g}_-))_k,$$

where the subscript k in the right hand side singles out the component of degree k.

Set $(\mathfrak{g}_-,\mathfrak{g}_0)_*=\bigoplus_{i\geq -d}\mathfrak{g}_i$; then, as is easy to verify, $(\mathfrak{g}_-,\mathfrak{g}_0)_*$ is a Lie algebra.

What is the Lie algebra of contact vector fields in these terms? Denote by $\mathfrak{hei}(2n)$ the Heisenberg Lie algebra: its space is $W\oplus\mathbb{C}\cdot z$, where W is a $2n$-dimensional space endowed with a nondegenerate skew-symmetric bilinear form B and the bracket in $\mathfrak{hei}(2n)$ is given by the following conditions:

$$z\text{ is in the center and }[v,w]=B(v,w)\cdot z\text{ for any }v,w\in W.$$

Clearly, $\mathfrak{k}(2n+1)$ is $(\mathfrak{hei}(2n),\mathfrak{csp}(2n))_*$, where for any $\mathfrak{g}$ we write $\mathfrak{cg}=\mathfrak{g}\oplus\mathbb{C}\cdot z$ or $\mathfrak{c}(\mathfrak{g})$ to denote the trivial central extension with the 1-dimensional even center generated by z.

2) The partial Cartan prolong. The Cartan prolongation $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ starts with nonpositive elements and is completely determined by them. Define *the partial Cartan prolongation* with a part of $\mathfrak{g}_1$. Over $\mathbb{C}$, this construction is a purely super phenomenon but in the study of Lie algebras over fields of prime characteristic it was independently observed.

Take a $\mathfrak{g}_0$-submodule $\mathfrak{h}_1$ in $\mathfrak{g}_1$. Suppose $[\mathfrak{g}_{-1}, \mathfrak{h}_1] = \mathfrak{g}_0$, not a *proper* subalgebra of $\mathfrak{g}_0$. Define the second prolongation of $(\bigoplus_{i\leq 0}\mathfrak{g}_i, \mathfrak{h}_1)$ to be $\mathfrak{h}_2 = \{D \in \mathfrak{g}_2 : [D, \mathfrak{g}_{-1}] \subset \mathfrak{h}_1\}$. The terms $\mathfrak{h}_i$ for $i > 2$ are similarly defined: $\mathfrak{h}_i = \{D \in \mathfrak{g}_i : [D, \mathfrak{g}_{-1}] \subset \mathfrak{h}_{i-1}\}$. Set $\mathfrak{h}_i = \mathfrak{g}_i$ for $i \leq 0$ and $\mathfrak{h}_* = \sum \mathfrak{h}_i$.

Example. $\mathfrak{vect}(1|n;n)$ is a subalgebra of $\mathfrak{k}(1|2n;n)$. The former is obtained as Cartan's prolong of the same nonpositive part as $\mathfrak{k}(1|2n;n)$ and a submodule of $\mathfrak{k}(1|2n;n)_1$. The simple exceptional superalgebra $\mathfrak{kas}$ introduced in §3 is another example.

To see the difference with the conventional Lie algebra case, consider $\mathfrak{vect}(m)_1$. As $\mathfrak{vect}(m)_0 = \mathfrak{gl}(m)$-module, it has two components, one, $\mathfrak{g}_1'$, is isomorphic to $\mathfrak{vect}(m|n)^*_{-1}$, the other one, $\mathfrak{g}_1''$, consists of divergence-free fields and is isomorphic to $\mathfrak{svect}(m)_1$. If we take $\mathfrak{g}_1'$, then its partial Cartan prolongation terminates at once ($\mathfrak{vect}(m)_- \oplus \mathfrak{g}_1' \cong \mathfrak{sl}(m+1)$); whereas $[\mathfrak{vect}(m)_{-1}, \mathfrak{g}_1''] = \mathfrak{sl}(m)$, a proper subalgebra of $\mathfrak{vect}(m)_0$. Clearly, the situation is the same for $\mathfrak{vect}(m|n)$.

0.6. **Lie superalgebras of vector fields as the Cartan prolongs.** The superization of the constructions from sec. 0.5 are straightforward: via Sign Rule. We thus get:

$$\mathfrak{vect}(m|n) = (\mathrm{id}, \mathfrak{gl}(m|n))_*; \quad \mathfrak{svect}(m|n) = (\mathrm{id}, \mathfrak{sl}(m|n))_*;$$
$$\mathfrak{h}(2m|n) = (\mathrm{id}, \mathfrak{osp}^{sk}(m|2n))_*; \mathfrak{le}(n) = (\mathrm{id}, \mathfrak{pe}^{sk}(n))_*; \quad \mathfrak{sle}(n) = (\mathrm{id}, \mathfrak{spe}^{sk}(n))_*.$$

Remark. Observe that the Cartan prolongs $(\mathrm{id}, \mathfrak{osp}^{sy}(m|2n))_*$ and $(\mathrm{id}, \mathfrak{pe}^{sy}(n))_*$ are finite dimensional.

The generalized Cartan prolong of the pair $(\mathfrak{hei}(2n), \mathfrak{sp}(2n))$ described in sec. 0.5 has, after superization, two analogs associated with the contact series $\mathfrak{k}$ and $\mathfrak{m}$, respectively.

• First, we define $\mathfrak{hei}(2n|m)$ on the direct sum of a $(2n, m)$-dimensional superspace W endowed with a nondegenerate skew-symmetric bilinear form and a $(1, 0)$-dimensional space spanned by z.

Clearly, we have $\mathfrak{k}(2n+1|m) = (\mathfrak{hei}(2n|m), \mathfrak{c}(\mathfrak{osp}^{sk}(m|2n)))_*$ and, given $\mathfrak{hei}(2n|m)$ and a subalgebra $\mathfrak{g}$ of $\mathfrak{c}(\mathfrak{osp}^{sk}(m|2n))$, we call $(\mathfrak{hei}(2n|m), \mathfrak{g})_*$ the *k-prolong* of $(W, \mathfrak{g})$, where W is the identity $\mathfrak{c}(\mathfrak{osp}^{sk}(m|2n))$-module.

• The odd analog of $\mathfrak{k}$ is associated with the following odd analog of $\mathfrak{hei}(2n|m)$. Denote by $\mathfrak{ab}(n)$ the *antibracket* Lie superalgebra: its space is $W \oplus \mathbb{C} \cdot z$, where W is an $n|n$-dimensional superspace endowed with a nondegenerate skew-symmetric odd bilinear form B; the bracket in $\mathfrak{ab}(n)$ is given by the following formulas:

$$z \text{ is odd and lies in the center; } [v, w] = B(v, w) \cdot z \text{ for } v, w \in W.$$

Set $\mathfrak{m}(n) = (\mathfrak{ab}(n), \mathfrak{c}(\mathfrak{pe}^{sk}(n)))_*$ and, given $\mathfrak{ab}(n)$ and a subalgebra $\mathfrak{g}$ of $\mathfrak{c}(\mathfrak{pe}^{sk}(n))$, we call $(\mathfrak{ab}(n), \mathfrak{g})_*$ the *m-prolong* of $(W, \mathfrak{g})$, where W is the identity $\mathfrak{c}(\mathfrak{pe}^{sk}(n))$-module.

Generally, given a nondegenerate form B on a superspace W and a superalgebra $\mathfrak{g}$ that preserves B, we refer to the above generalized prolongations as to *mk-prolongation* of the pair $(W, \mathfrak{g})$.

0.7. **Deformations of the Buttin superalgebras and $\mathfrak{vect}(m|n)$-modules.** Here we reproduce a result of Kotchetkoff [Ko1] with some corrections (cf. [Ko2], [L3] and [LSh2]).

To consider the deformations, recall the definition of the $\mathfrak{vect}(m|n)$-module of tensor fields of type V; see [BL]. Let V be the $\mathfrak{gl}(m|n) = \mathfrak{vect}_0(m|n)$-module with the lowest weight λ. Make V into a $\mathfrak{g}_{\geq}$-module, where $\mathfrak{g} = \mathfrak{vect}(m|n)$ and $\mathfrak{g}_{\geq} = \bigoplus_{i\geq 0}\mathfrak{g}_i$, setting $\mathfrak{g}_+ \cdot V = 0$ for $\mathfrak{g}_+ = \bigoplus_{i>0}\mathfrak{g}_i$. Let us realize $\mathfrak{g}$ by vector fields on the $m|n$-dimensional linear complex supermanifold $\mathcal{C}^{m|n}$ with coordinates x. The superspace $T(V) = \mathrm{Hom}_{U(\mathfrak{g}_+)}(U(\mathfrak{g}), V)$ is isomorphic, due to the Poincaré–Birkhoff–Witt theorem, to $\mathbb{C}[[x]] \otimes V$. Its elements have a natural interpretation as formal tensor fields of type V. When $\lambda = (a, \dots, a)$ we will simply write $T(\vec{a})$ instead of $T(\lambda)$.

Example. $T(\vec{0})$ is the superspace of functions; $Vol(m|n) = T(1, \dots, 1; -1, \dots, -1)$ (the semicolon separates the first m coordinates of the weight with respect to the matrix units E_{ii} of $\mathfrak{gl}(m|n)$) is the superspace of *densities* or *volume forms*. We denote the generator of $Vol(m|n)$ corresponding to the ordered set of coordinates x by $vol(x)$ or vol_x. The space of λ-densities is $Vol^\lambda(m|n) = T(\lambda, \dots, \lambda; -\lambda, \dots, -\lambda)$. In particular, $Vol^\lambda(m|0) = T(\vec{\lambda})$ but $Vol^\lambda(0|n) = T(\overrightarrow{-\lambda})$. Over $\mathfrak{vect}(0|n)$, we further set $Vol_0 = \{v \in T(\overrightarrow{-1}) : \int v = 0\}$ and $T_0(\vec{0}) = T_0(\vec{0})/\mathbb{C}\cdot 1$; by definition, $Vol \cong T(\vec{0})$ over $\mathfrak{svect}(0|n)$, so we can set $T_0^0(\vec{0}) = Vol_0/\mathbb{C}\cdot 1$.

As is clear from the definition of the Buttin bracket, there is a regrading (namely, $\mathfrak{b}(n; n)$ given by $\deg\xi_i = 0, \deg q_i = 1$ for all i) under which $\mathfrak{b}(n)$, initially of depth 2, takes the form $\mathfrak{g} = \bigoplus_{i\geq -1}\mathfrak{g}_i$ with $\mathfrak{g}_0 = \mathfrak{vect}(0|n)$ and $\mathfrak{g}_{-1} \cong \Pi(\mathbb{C}[\xi])$.

Let us replace the $\mathfrak{vect}(0|n)$-module $\mathfrak{g}_{-1}$ of functions (with inverted parity) with the module of λ-densities, i.e., set $\mathfrak{g}_{-1} \cong \mathbb{C}[\xi](vol_\xi)^\lambda$, where

$$L_D(vol_\xi)^\lambda = \lambda \mathrm{div} D \cdot vol_\xi^\lambda \quad\text{and}\quad p(vol_\xi)^\lambda = \bar{1}.$$

Then $\mathfrak{b}_\lambda(n; n)$ — the Cartan's prolong $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = (\Pi(Vol(0|n)^\lambda), \mathfrak{vect}(0|n))_*$ — is a deform of $\mathfrak{b}(n; n)$. The collection of these deforms for various $\lambda \in \mathbb{C}$ constitutes a deformation of $\mathfrak{b}(n; n)$; we called it the *main deformation*; see [ALSh]. (Though main, this deformation is not the quantization of the Buttin bracket; for the latter see [Ko1] or [L3].) The deform $\mathfrak{b}_\lambda(n)$ of $\mathfrak{b}(n)$ is the regrading of $\mathfrak{b}_\lambda(n; n)$ inverse to the regrading of $\mathfrak{b}(n)$ into $\mathfrak{b}(n; n)$.

Another description of the main deformation is as follows. Set

$$\mathfrak{b}_{a,b}(n) = \{M_f \in \mathfrak{m}(n) :\ a\ \mathrm{div} M_f = (-1)^{p(f)} 2(a - bn)\frac{\partial f}{\partial \tau}\}. \tag{0.7}$$

It is subject to a direct check that $\mathfrak{b}_{a,b}(n; n) \cong \mathfrak{b}_\lambda(n; n)$ for $\lambda = \frac{2a}{n(a-b)}$. This isomorphism shows that λ actually runs over $\mathbb{CP}^1$, not $\mathbb{C}$. Observe that for $a = nb$, i.e., for $\lambda = \frac{2}{n-1}$, we have $\mathfrak{b}_{nb,b}(n) \cong \mathfrak{sm}(n)$.

As follows from the description of $\mathfrak{vect}(m|n)$-modules ([BL]) and the criteria for simplicity of $\mathbb{Z}$-graded Lie superalgebras ([K1]), the Lie superalgebras $\mathfrak{b}_\lambda(n)$ are simple for $n > 1$ and $\lambda \neq 0,\ -1$. The same criteria also make it clear that the $\mathfrak{b}_\lambda(n)$ are nonisomorphic for distinct λ's. (Notice, that at some values of λ the Lie superalgebras $\mathfrak{b}_\lambda(n)$ have additional deformations distinct from the above. These deformations are partly described in [Ko1], [L3].)

The geometric interpretation of $\mathfrak{b}_{a,b}(n)$ follows from (0.7): this is the Lie superalgebra that preserves $vol^a_{q,\xi,\tau}\alpha_0^{a-bn}$. The meaning of parameters a and b is clear from column $\mathfrak{g}_0$ in row $\mathfrak{b}_{a,b}(n)$ in Table 0.8.

0.8. **Several first terms that determine the Cartan and mk-prolongations.** To facilitate the comparison of various vectorial superalgebras, consider the following table. The central element $z \in \mathfrak{g}_0$ is supposed to be chosen so that it acts on $\mathfrak{g}_k$ as $k \cdot \mathrm{id}|_{\mathfrak{g}_k}$. As in Introduction, the sign $\ni\!\!\!+$ (resp. $\in\!\!\!+$) denotes the semidirect sum with the subspace or ideal on the left (right) of it; $\Lambda(r) = \mathbb{C}[\xi_1, \dots, \xi_r]$ is the Grassmann superalgebra.

$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$
$\mathfrak{vect}(n\|m;r)$	–	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{gl}(n\|m-r)\otimes\Lambda(r) \ni\!\!\!+ \mathfrak{vect}(0\|r)$
$\mathfrak{vect}(1\|m;m)$	–	$\Lambda(m)$	$\Lambda(m) \ni\!\!\!+ \mathfrak{vect}(0\|m)$
$\mathfrak{svect}(n\|m;r)$	–	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{sl}(n\|m-r)\otimes\Lambda(r) \ni\!\!\!+ \mathfrak{vect}(0\|r)$
$\mathfrak{svect}(1\|m;m)$	–	$Vol(0\|m)$	$\mathfrak{vect}(0\|m)$
$\mathfrak{svect}^\circ(1\|m;m)$	–	$Vol_0(0\|m)$	$\mathfrak{vect}(0\|m)$
$\mathfrak{svect}^\circ(1\|2)$	–	$T_0(\vec{0})$	$\mathfrak{vect}(0\|2) \cong \mathfrak{sl}(1\|2)$
$\mathfrak{svect}(2\|1)$	–	$\Pi(T_0(\vec{0}))$	$\mathfrak{vect}(0\|2) \cong \mathfrak{sl}(2\|1)$

$\mathfrak{k}(2n+1\|m;r)$	$\Lambda(r)$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{cosp}(m-2r\|2n)\otimes\Lambda(r) \ni\!\!\!+ \mathfrak{vect}(0\|r)$
$\mathfrak{h}(2n\|m;r)$	$\Lambda(r)/\mathbb{C}\cdot 1$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{osp}(m-2r\|2n)\otimes\Lambda(r) \ni\!\!\!+ \mathfrak{vect}(0\|r)$
$\mathfrak{k}(1\|2m;m)$	–	$\Lambda(m)$	$\Lambda(m) \ni\!\!\!+ \mathfrak{vect}(0\|m)$
$\mathfrak{k}(1\|2m+1;m)$	$\Lambda(m)$	$\Pi(\Lambda(m))$	$\Lambda(m) \ni\!\!\!+ \mathfrak{vect}(0\|m)$

Recall that $\mathfrak{b}_{a,b}(n) \cong \mathfrak{b}_\lambda(n)$ for $\lambda = \frac{2a}{n(a-b)}$; z is the center (unit matrix) in $\mathfrak{g}_0$, d is an outer derivation — the grading operator — of $\mathfrak{g}_0$.

$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$
$\mathfrak{b}_{a,b}(n;r)$	$\Pi(\Lambda(r))$	$\mathrm{id}\otimes\Lambda(r)$	$(\mathfrak{spe}(n-r) \ni\!\!\!+ \mathbb{C}(az+bd))\otimes\Lambda(r) \ni\!\!\!+ \mathfrak{vect}(0\|r)$
$\mathfrak{b}_\lambda(n;n)$	–	$\Pi(Vol^\lambda(0\|n))$	$\mathfrak{vect}(0\|n)$
$\mathfrak{m}(n;r)$	$\Pi(\Lambda(r))$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{cpe}(n-r)\otimes\Lambda(r) \ni\!\!\!+ \mathfrak{vect}(0\|r)$
$\mathfrak{m}(n;n)$	–	$\Pi(\Lambda(n))$	$\Lambda(n) \ni\!\!\!+ \mathfrak{vect}(0\|n)$
$\mathfrak{le}(n;r)$	$\Pi(\Lambda(r))/\mathbb{C}\cdot 1$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{pe}(n-r)\otimes\Lambda(r) \ni\!\!\!+ \mathfrak{vect}(0\|r)$
$\mathfrak{le}(n;n)$	–	$\Pi(T_0(\vec{0}))$	$\mathfrak{vect}(0\|n)$
$\mathfrak{sle}^\circ(n;r)$	$\Pi(\Lambda(r))/\mathbb{C}\cdot 1$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{spe}(n-r)\otimes\Lambda(r) \ni\!\!\!+ \mathfrak{vect}(0\|r)$
$\mathfrak{sle}^\circ(n;n)$	–	$\Pi(T_0(\vec{0}))$	$\mathfrak{svect}(0\|n)$

1. The exceptional Lie superalgebra $\mathfrak{vas}(4|4) = (\mathrm{spin}, \mathfrak{as})_*$

1.1. **A. Sergeev's extension.** Let ω be a nondegenerate superskew-symmetric odd bilinear form on an (n,n)-dimensional superspace V. In the standard basis of V (all the even vectors come first) the canonical matrix of the form ω is $\begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$ and the elements of $\mathfrak{pe}(n) = \mathfrak{aut}(\omega)$ can be represented by supermatrices of the form $\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}$, where $b = b^t,\ c = -c^t$. The Lie superalgebra $\mathfrak{spe}(n)$ is singled out by the requirement that $\mathrm{tr} a = 0$. Setting

$$\deg\left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\right) = -1, \quad \deg\left(\begin{pmatrix} a & 0 \\ 0 & -a^t \end{pmatrix}\right) = 0, \quad \deg\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right) = 1, \tag{1.1}$$

we endow $\mathfrak{pe}(n)$ with a $\mathbb{Z}$-grading. It is known ([K1]) that $\mathfrak{spe}(n) = \mathfrak{pe}(n) \cap \mathfrak{sl}(n|n)$ is a simple Lie superalgebra for $n \geq 3$.

A. Sergeev proved (1977, unpublished) that there exists just one nontrivial central extension of $\mathfrak{spe}(n)$. It exists for $n = 4$ and is denoted by $\mathfrak{as}$. Let us represent an arbitrary element $A \in \mathfrak{as}$ as a pair $A = x + d \cdot z$, where $x \in \mathfrak{spe}(4)$, $d \in \mathbb{C}$ and z is the central element. In the matrix form the bracket in $\mathfrak{as}$ is

$$\begin{aligned} &\left[\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + d \cdot z, \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix} + d' \cdot z\right] \\ &\quad = \left[\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix}\right] - \frac{1}{2}\mathrm{tr}\, c\tilde{c}' \cdot z, \end{aligned} \tag{1.2}$$

where for the skew-symmetric matrix $c_{ij} = E_{ij} - E_{ji}$ we set $\tilde{c}_{ij} = c_{kl}$ for the even permutation $(1234) \mapsto (ijkl)$. Clearly, $\deg z = -2$ with respect to the grading (1.1).

1.2. The Lie superalgebra $\mathfrak{as}$ can also be described with the help of the spinor representation. Consider $\mathfrak{po}(0|6)$, the Lie superalgebra whose superspace is the Grassmann superalgebra $\Lambda(\xi, \eta)$ generated by $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ and the bracket is the Poisson bracket. Recall that $\mathfrak{h}(0|6) = Span(H_f : f \in \Lambda(\xi, \eta))$.

Now, observe that $\mathfrak{spe}(4)$ can be embedded into $\mathfrak{h}(0|6)$. Indeed, setting $\deg \xi_i = \deg \eta_i = 1$ for all i we introduce a $\mathbb{Z}$-grading on $\Lambda(\xi, \eta)$ which, in turn, induces a $\mathbb{Z}$-grading on $\mathfrak{h}(0|6)$ of the form $\mathfrak{h}(0|6) = \bigoplus_{i \geq -1} \mathfrak{h}(0|6)_i$. Since $\mathfrak{sl}(4) \cong \mathfrak{o}(6)$, we can identify $\mathfrak{spe}(4)_0$ with $\mathfrak{h}(0|6)_0$.

It is not difficult to see that the elements of degree -1 in $\mathfrak{spe}(4)$ and $\mathfrak{h}(0|6)$ constitute isomorphic $\mathfrak{sl}(4) \cong \mathfrak{o}(6)$-modules. It is subject to a direct verification that it is possible to embed $\mathfrak{spe}(4)_1$ into $\mathfrak{h}(0|6)_1$.

Sergeev's extension $\mathfrak{as}$ is the result of the restriction on $\mathfrak{spe}(4) \subset \mathfrak{h}(0|6)$ of the cocycle that turns $\mathfrak{h}(0|6)$ into $\mathfrak{po}(0|6)$. The quantization deforms $\mathfrak{po}(0|6)$ into $\mathfrak{gl}(\Lambda(\xi))$; the through maps $T_\lambda : \mathfrak{as} \longrightarrow \mathfrak{po}(0|6) \longrightarrow \mathfrak{gl}(\Lambda(\xi))$ are representations of $\mathfrak{as}$ in the $4|4$-dimensional modules spin_λ *distinct* for distinct values λ of the central element z. (Here $\lambda \in \mathbb{C}$ plays the role of Planck's constant.) The explicit form of T_λ is as follows:

$$T_\lambda : \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + d \cdot z \mapsto \begin{pmatrix} a & b - \lambda\tilde{c} \\ c & -a^t \end{pmatrix} + \lambda d \cdot 1_{4|4}, \tag{1.3}$$

where $1_{4|4}$ is the unit matrix and $\tilde{c}$ is defined after formula (1.2). Clearly, T_λ is an irreducible representation.

1.3. Theorem. 1) *The Cartan prolong* $\mathfrak{f}^\lambda = (\mathrm{spin}_\lambda, \mathfrak{as})_*$ *is infinite dimensional and simple for* $\lambda \neq 0$.

2) $\mathfrak{f}^\lambda \cong \mathfrak{f}^\mu$ *if* $\lambda \cdot \mu \neq 0$.

Observe that though the representations T_λ are distinct for $\lambda \neq 0$, the corresponding Cartan prolongs are isomorphic.

Convention. For brevity, we denote the isomorphic superalgebras $\mathfrak{f}^\lambda = (\mathrm{spin}_\lambda, \mathfrak{as})_*$ for any $\lambda \neq 0$ by $\mathfrak{vas}(4|4) = (\mathrm{spin}, \mathfrak{as})_*$.

Proof. Heading 1) consists of two statements: a) $(\mathrm{spin}, \mathfrak{as})_k \neq 0$ for all $k > 0$; b) the Lie superalgebra $(\mathrm{spin}_\lambda, \mathfrak{as})_*$ has no nontrivial $\mathbb{Z}$-graded ideals.

a) This follows from the fact that the elements $u_i^{k+1}\partial\xi_i$ belong to $(\mathrm{spin}, \mathfrak{as})_k$ for any $k > 0$ and any i (to prove the statement it suffices to consider only one i).

b) Assume the contrary: let $\mathfrak{i} = \bigoplus_{k\geq -1} \mathfrak{i}_k$ be a nonzero ideal of $\mathfrak{h} = (\mathrm{spin}, \mathfrak{as})_*$. Let $x \in \mathfrak{i}_k$ be a nonzero homogeneous element. Since $\mathfrak{g} = \mathfrak{vect}(4|4) \supset (\mathrm{spin}, \mathfrak{as})_*$ is transitive, then the superspace ($k+1$ brackets)

$$[\mathfrak{h}_{-1}, [\mathfrak{h}_{-1}, \dots, [\mathfrak{h}_{-1}, x] \dots] = [\mathfrak{g}_{-1}, [\mathfrak{g}_{-1}, \dots, [\mathfrak{g}_{-1}, x] \dots] \subset \mathfrak{i}_{-1}$$

is a nonzero subspace of $\mathfrak{h}_{-1}$. Since T_λ is irreducible, $\mathfrak{i}_{-1} = \mathfrak{h}_{-1}$. The Jacobi identity implies that $[\mathfrak{i}_{-1}, \mathfrak{h}_1] \subset \mathfrak{i}_0$ is an ideal of $\mathfrak{h}_0$.

But $\mathfrak{h}_0 = \mathfrak{as}$ has only one nontrivial ideal, the center. Since $[\mathfrak{i}_{-1}, \mathfrak{h}_1] = [\mathfrak{h}_{-1}, \mathfrak{h}_1]$ contains elements of the form $u_i\frac{\partial}{\partial\xi_i}$ for any i, which do not belong to the center, it follows that $\mathfrak{i}_0 = \mathfrak{h}_0$. In particular, $\mathfrak{i}_0$ contains the element

$$T_\lambda(z) = -\lambda \sum (u_i\frac{\partial}{\partial u_i} + \xi_i\frac{\partial}{\partial \xi_i}).$$

But $[T_\lambda(z), h] = -\lambda \cdot k \cdot h$ for any $h \in \mathfrak{h}_k$. Hence, $\mathfrak{i} = \mathfrak{h}$ and $\mathfrak{h}$ is simple.

2) follows from the fact that the nonpositive parts of $\mathfrak{f}^\lambda$ and $\mathfrak{f}^\mu$ are isomorphic. □

1.4. **A problem.** For $\lambda = 0$ the representation T_0 is not faithful and $T_0(\mathfrak{as}) = \mathfrak{spe}(4)$. The Cartan prolong of the pair $(\mathrm{id}, \mathfrak{spe}(4))$ is well-known: this is $\mathfrak{sle}(4)$. Recall that we can realize $\mathfrak{le}(n)$ by the generating functions — the elements of $\mathbb{C}[u, \xi]$ — with the *Buttin* bracket. The subalgebra $\mathfrak{sle}(n)$ is generated by harmonic functions, i.e., by functions that satisfy $\Delta(f) = 0$, where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial u_i \partial\xi_i}$. The exceptional Lie superalgebra $(\mathrm{spin}, \mathfrak{as})_*$ is a deform of $\mathfrak{sle}(4) \not\ni \mathbb{C} \cdot \sum(u_i\frac{\partial}{\partial u_i} + \xi_i\frac{\partial}{\partial\xi_i})$. An explicit expression of the corresponding cocycle is desirable: it will enable us to express the bracket in $(\mathrm{spin}, \mathfrak{as})_*$ in terms of harmonic functions (plus one more element).

2. An explicit form of the vector fields from $\mathfrak{vas}(4|4) = (\mathrm{spin}, \mathfrak{as})_* \subset \mathfrak{svect}(4|4)$

Every element $D \in \mathfrak{vect}(4|4)$ is of the form $D = \sum_{i\leq 4}(P_i\frac{\partial}{\partial\xi_i} + Q_i\frac{\partial}{\partial u_i})$, where $P_i, Q_i \in \mathbb{C}[u, \xi]$.

2.1. Lemma. *The homogeneous (with respect to parity) vector field* $D \in \mathfrak{vect}(4|4)$ *belongs to* $(\mathrm{spin}, \mathfrak{as})_*$ *if and only if it satisfies the following system of equations:*

$$\frac{\partial Q_i}{\partial u_j} + (-1)^{p(D)}\frac{\partial P_j}{\partial \xi_i} = 0 \text{ for any } i \neq j; \tag{2.1}$$

$$\frac{\partial Q_i}{\partial u_i} + (-1)^{p(D)}\frac{\partial P_i}{\partial \xi_i} = \frac{1}{2}\sum_{1\leq j\leq 4}\frac{\partial Q_j}{\partial u_j} \text{ for } i = 1, 2, 3, 4; \tag{2.2}$$

$$\frac{\partial Q_i}{\partial \xi_j} + \frac{\partial Q_j}{\partial \xi_i} = 0 \text{ for any } i, j; \tag{2.3}$$

$$\frac{\partial P_i}{\partial u_j} - \frac{\partial P_j}{\partial u_i} = (-1)^{p(D)}\cdot\lambda\cdot(\frac{\partial Q_k}{\partial \xi_l} - \frac{\partial Q_l}{\partial \xi_k}) \tag{2.4}$$
$$\text{for any even permutation } \begin{pmatrix}1 & 2 & 3 & 4\\ i & j & k & l\end{pmatrix}.$$

2.1.1. *Remark.* 1) Observe that the sum of the 4 equations (2.2) yields that $\mathrm{div}D = 0$, i.e., $(\mathrm{spin}, \mathfrak{as})_* \subset \mathfrak{svect}(4|4)$.

2) For $\lambda = 0$ the system (2.1)–(2.4) singles out the superalgebra

$$(\mathfrak{sle}^\circ(4) \ni \mathbb{C}\cdot\sum(u_i\frac{\partial}{\partial u_i} + \xi_i\frac{\partial}{\partial \xi_i})) \ni \mathbb{C}\cdot \mathrm{Le}_{\xi_1\xi_2\xi_3\xi_4}.$$

2.1.2. *Remark.* Actually, any Cartan prolongation is obtained as a solution of some system of differential equations with constant coefficients. For Lie algebras this fact is lucidly explained in [St]. The supercase is absolutely analogous.

Let $\mathfrak{g}_{-1} = V = Span(\frac{\partial}{\partial x_i}, i = 1, \dots, n)$. Then any vector field $D = \sum f_i(x)\partial_{x_i}$ generates a linear operator $L_D : V \longrightarrow \mathfrak{vect}(V)$ — the Lie derivative — $L_D(\frac{\partial}{\partial x_i}) = [D, \frac{\partial}{\partial x_i}]$. This operator is a tensor object determined by the matrix $P(D) = (P_{i,j})$, where $P_{i,j} = (-1)^{p(D)p(x_j)}\frac{\partial f_i}{\partial x_j}$. If $D \in \mathfrak{vect}(V)_0$, then the matrix $P(D)$ is a numerical one and can be singled out from $\mathfrak{gl}(V)$ by a homogeneous linear system. The fact that any operator of the left adjoint action $\mathrm{ad}^l(\frac{\partial}{\partial x_i})$ commutes with its right twin $\mathrm{ad}^r(\frac{\partial}{\partial x_i})$ means that any vector field $D \in \mathfrak{vect}(V)$ belongs to the Cartan prolongation $(V, \mathfrak{g})_*$ if and only if the matrix $P(D)$ is a $\mathfrak{g}$-valued function on V.

Proof. Denote by $\mathfrak{g}^\lambda = \bigoplus_{i\geq -1}\mathfrak{g}_i^\lambda$ for $\lambda \neq 0$ the space of solutions of the system (2.1)–(2.4). Clearly, $\mathfrak{g}_{-1}^\lambda \cong \mathfrak{vect}(4|4)_{-1}$. Let $D \in \mathfrak{g}_0^\lambda$. Then the matrix of the operator D in its action on $\mathfrak{g}_{-1}^\lambda$ is of the form

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}, \text{ where } a_{ij} = \frac{\partial Q_i}{\partial u_j}, \quad b_{ij} = \frac{\partial P_i}{\partial u_j},$$
$$c_{ij} = (-1)^{p(D)}\frac{\partial Q_i}{\partial \xi_j}, \quad d_{ij} = (-1)^{p(D)}\frac{\partial P_i}{\partial \xi_j}.$$

Therefore, equations (2.1) and (2.2) mean that $a + d^t = (\frac{1}{2}\mathrm{tr}a)\cdot 1_4$, equations (2.3) that $c + c^t = 0$, equations (2.4) that $b - b^t = \lambda(\tilde{c} - \tilde{c}^t)$. Set

$$a_0 = a - (\frac{1}{4}\mathrm{tr}a)\cdot 1_4, \quad d_0 = d - (\frac{1}{4}\mathrm{tr}a)\cdot 1_4.$$

Then

$$a_0 + d_0^t = a + d^t - (\frac{1}{2}\mathrm{tr}a)\cdot 1_4 = 0; \quad c + c^t = 0; \quad b - b^t = \lambda(\tilde{c} - \tilde{c}^t). \tag{2.5}$$

Comparing formulas (2.5) with (1.3) we see that $\mathfrak{g}_0^\lambda$ coincides with the image of $\mathfrak{as}$ under T_λ, i.e., with $(\mathrm{spin}, \mathfrak{as})_0$.

Set

$$D_{u_j}(D)=\sum_{i\leq 4}(\frac{\partial P_i}{\partial u_j}\frac{\partial}{\partial \xi_i}+\frac{\partial Q_i}{\partial u_j}\frac{\partial}{\partial u_i})$$

and

$$\tilde{D}_{\xi_j}(D)=(-1)^{p(D)}\sum_{i\leq 4}(\frac{\partial P_i}{\partial \xi_j}\frac{\partial}{\partial \xi_i}+\frac{\partial Q_i}{\partial \xi_j}\frac{\partial}{\partial u_i}).$$

The operators D_{u_j} and $\tilde{D}_{\xi_j}$, clearly, *commute* with the $\mathfrak{g}^{\lambda}_{-1}$-action. Observe that the operators *commute*, not *super*commute.

Since equations (2.1)–(2.4) is a linear combination of only these operators, the definition of Cartan prolongation itself ensures an isomorphism of $(\mathfrak{g}^{\lambda})_n$ with $(\mathfrak{f}^{\lambda})_n$. □

2.2. **The right inverse of Δ on $\mathfrak{sle}^{\circ}$.** Let f be an arbitrary homogeneous with respect to the degree in u and ξ harmonic function, distinct from $\xi_1\dots\xi_n$, i.e., an arbitrary generating function for $\mathfrak{sle}^{\circ}(n)$. Then $f=\Delta(F)$ for some function F (as follows from the computation of the homology of Δ which is an easy exersise; the answer: the homology space $H(\Delta)$ is spanned by $\xi_1\dots\xi_n$). Clearly, F is determined up to an arbitrary harmonic summand. Set $\Phi=\sum u_i\xi_i$. Then

$$\Delta(\Phi f)=(\Delta\Phi)f-\Phi\Delta f-\{\Phi,f\}=\big(n+\deg_u f-\deg_\xi f\big)\,f. \tag{2.6}$$

Define the *right inverse* of Δ by the formula

$$\Delta^{-1}f=\frac{1}{\nu(f)}(\Phi f),\quad\text{where}\quad \nu(f)=n+\deg_u f-\deg_\xi f. \tag{2.7}$$

Since the kernel of Δ is nonzero, Δ has no inverse. Still, Δ^{-1} maps $\mathfrak{sle}^{\circ}(n)$ into $\mathfrak{le}(n)$ without kernel and on $\mathfrak{sle}^{\circ}(n)$ the following formula holds:

$$\Delta(\Delta^{-1}f)=f\quad\text{for } \mathrm{Le}_f\in\mathfrak{sle}^{\circ}(n).$$

2.3. Theorem. *Any vector field $D\in\mathfrak{g}^{\lambda}$ can be represented in the form*

$$D=D_f+cZ,\quad\textit{where}\quad c\in\mathbb{C}\quad\textit{and}\quad Z=\sum_{i\leq 4}(u_i\frac{\partial}{\partial u_i}+\xi_i\frac{\partial}{\partial \xi_i}),$$

where $\mathrm{Le}_f\in\mathfrak{sle}^{\circ}(4)$ *and where (recall that $A_n\subset S_n$ denotes the subgroup of even permutations):*

$$\begin{aligned}&D_f=\mathrm{Le}_f+\lambda\left(-\mathrm{Le}_{\hat f}+2\sum_{1\leq i\leq 4;\ (i,j,k,l)\in A_4}\frac{\partial^3(\Delta^{-1}(f))}{\partial\xi_j\partial\xi_k\partial\xi_l}\frac{\partial}{\partial\xi_i}\right)\\ &\textit{for } \hat f=4\Delta^{-1}\left(\frac{\partial^4(\Delta^{-1}(f))}{\partial\xi_1\partial\xi_2\partial\xi_3\partial\xi_4}\right).\end{aligned} \tag{2.8}$$

For the proof see Appendix 1.

Corollary. 1) *The Lie superalgebra $\mathfrak{g}^{\lambda}$ is a deformation of $\mathfrak{sle}^{\circ}(4)\supplus\mathbb{C}\cdot Z$.*

2) *If $\deg_\xi f\leq 1$, then $D_f=\mathrm{Le}_f$, hence, $\mathfrak{h}=\{c\cdot Z+D_f : \deg_\xi f\leq 1, c\in\mathbb{C}\}$ remains rigid under this deformation.*

3) *Let $\Omega=du_1\wedge du_2\wedge du_3\wedge du_4$ be the volume element on the underlying manifold of the $\mathcal{C}^{4|4}$. Observe that the volume element $vol(u,\xi)$ on the whole $\mathcal{C}^{4|4}$ is invariant with respect to the $\mathfrak{g}^{\lambda}$-action, but Ω is not invariant. It is invariant, however, with respect to the nondeformed subalgebra $\mathfrak{h}$.*

4) *Let $D \in \mathfrak{g}^\lambda$; let L_D be the Lie derivative. Denote by $\nabla = \sum \frac{\partial}{\partial u_i}\frac{\partial}{\partial \xi_i}$ the bivector dual to ω. Observe that the left hand sides of equations (2.1)–(2.4) determine the coefficients of the 2-form $L_D(\omega)$:*
equations (2.1) determine the coefficients of $du_j d\xi_i$;
equations (2.2) determine the coefficients of $du_i d\xi_i$;
equations (2.3) determine the coefficients of $d\xi_j d\xi_i$;
equations (2.4) determine the coefficients of $du_j du_i$.
The right hand sides of (2.2) determine the nonzero coefficients of the form $\frac{1}{2}(\sum \frac{\partial Q_i}{\partial u_i})\omega$, while the right hand sides of (2.4) determine the nonzero coefficients of the form $\lambda(L_D\Omega) * \nabla$, where $*$ is the convolution of tensors.

Therefore, (2.1)–(2.4) can be rewritten in the form

$$L_D\omega = \frac{1}{2}(\sum \frac{\partial Q_i}{\partial u_i})\omega + \lambda(L_D\Omega) * \nabla. \tag{2.9}$$

If we replace coefficient of ω in the right hand side of (2.9) with an arbitrary function $\Psi(u, \xi)$ and add the constraint

$$\operatorname{div}(D) = \sum(\frac{\partial Q_i}{\partial u_i} - (-1)^{p(D)}\frac{\partial P_i}{\partial \xi_i}) = 0, \tag{2.10}$$

then the sum of the four equations (2.2) with (2.10) automatically yields

$$\Psi(u, \xi) = \frac{1}{2}\sum \frac{\partial Q_i}{\partial u_i}.$$

Thus, we can distinguish the Lie superalgebra $\mathfrak{g}^\lambda$ by

$$\begin{cases} L_D\omega & = & \Psi \cdot \omega & +\lambda(L_D\Omega) * \nabla, \\ \operatorname{div}(D) & = & 0. \end{cases} \tag{2.11}$$

3. The exceptional Lie subsuperalgebra $\mathfrak{kas}$ of $\mathfrak{k}(1|6)$

If the operator d that determines a $\mathbb{Z}$-grading of the Lie superalgebra $\mathfrak{g}$ does not belong to $\mathfrak{g}$, we denote the Lie superalgebra $\mathfrak{g} \supplus \mathbb{C} \cdot d$ by $\mathfrak{d}\mathfrak{g}$. Recall also that $\mathfrak{c}(\mathfrak{g})$ or just $\mathfrak{c}\mathfrak{g}$ denotes the trivial 1-dimensional central extension of $\mathfrak{g}$ with the even center.

3.1. The Lie superalgebra $\mathfrak{g} = \mathfrak{k}(1|2n)$ is generated by the functions from $\mathbb{C}[t, \xi, \eta]$, where $\xi = (\xi_1, \dots, \xi_n)$, $\eta = (\eta_1, \dots, \eta_n)$. The standard $\mathbb{Z}$-grading of $\mathfrak{g}$ is induced by the $\mathbb{Z}$-grading of $\mathbb{C}[t, \xi, \eta]$ given by $\deg t = 2$, $\deg \xi_i = \deg \eta_i = 1$; namely, $\deg K_f = \deg f - 2$. Clearly, in this grading $\mathfrak{g}$ is of depth 2. Let us consider the functions that generate several first homogeneous components of $\mathfrak{g} = \bigoplus_{i \geq -2} \mathfrak{g}_i$:

component	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	$\mathfrak{g}_1$
its generators	1	$\Lambda^1(\xi, \eta)$	$\Lambda^2(\xi, \eta) \oplus \mathbb{C} \cdot t$	$\Lambda^3(\xi, \eta) \oplus t\Lambda^1(\xi, \eta)$

As one can prove directly, the component $\mathfrak{g}_1$ generates the whole subalgebra $\mathfrak{g}_+ \subset \mathfrak{g}$ of the elements of positive degree. The component $\mathfrak{g}_1$ splits into two $\mathfrak{g}_0$-modules $\mathfrak{g}_{11} = \Lambda^3$ and $\mathfrak{g}_{12} = t\Lambda^1$. It is obvious that $\mathfrak{g}_{12}$ is always irreducible and the component $\mathfrak{g}_{11}$ is trivial for $n = 1$.

The Cartan prolongations of these components are well-known:

$$(\mathfrak{g}_- \oplus \mathfrak{g}_0, \mathfrak{g}_{11})^{mk}_* \cong \mathfrak{po}(0|2n) \oplus \mathbb{C} \cdot K_t \cong \mathfrak{d}(\mathfrak{po}(0|2n));$$
$$(\mathfrak{g}_- \oplus \mathfrak{g}_0, \mathfrak{g}_{12})^{mk}_* = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{12} \oplus \mathbb{C} \cdot K_{t^2} \cong \mathfrak{osp}(2n|2).$$

Observe a remarkable property of $\mathfrak{k}(1|6)$. For $n>1$ and $n\neq 3$ the component $\mathfrak{g}_{11}$ of $\mathfrak{k}(1|2n)$ is irreducible. For $n=3$ the component splits into the two irreducible dual modules that we will denote by $\mathfrak{g}_{11}^{\xi}$ and $\mathfrak{g}_{11}^{\eta}$. Observe further, that $\mathfrak{g}_0=\mathfrak{o}(6)\cong\mathfrak{sl}(4)$. As $\mathfrak{sl}(4)$-modules, $\mathfrak{g}_{11}^{\xi}$ and $\mathfrak{g}_{11}^{\eta}$ are the symmetric squares $S^2(\mathrm{id})$ and $S^2(\mathrm{id}^*)$ of the identity 4-dimensional representation and its dual, respectively. Explicitly,

$$\begin{aligned}\mathfrak{g}_{11}^{\xi}=Span\,(&\xi_1\xi_2\xi_3,\ \xi_1(\xi_2\eta_2+\xi_3\eta_3),\ \xi_2(\xi_1\eta_1+\xi_3\eta_3),\ \eta_3(\xi_1\eta_1-\xi_2\eta_2),\ \xi_1\eta_2\eta_3,\\ &\xi_3(\xi_1\eta_1+\xi_2\eta_2),\ \eta_2(\xi_1\eta_1-\xi_3\eta_3),\xi_2\eta_1\eta_3,\ \eta_1(\xi_2\eta_2-\xi_3\eta_3),\xi_3\eta_1\eta_2)\end{aligned}$$

and $\mathfrak{g}_{11}^{\eta}$ is obtained from $\mathfrak{g}_{11}^{\xi}$ after the replacement $\xi\longleftrightarrow\eta$.

3.2. Theorem. 1) *The Cartan prolong* $(\mathfrak{g}_-\oplus\mathfrak{g}_0,\mathfrak{g}_{11}^{\xi}\oplus\mathfrak{g}_{12})_*^{mk}$ *is infinite dimensional and simple. It is isomorphic to* $(\mathfrak{g}_-\oplus\mathfrak{g}_0,\mathfrak{g}_{11}^{\eta}\oplus\mathfrak{g}_{12})_*^{mk}$.

2) $(\mathfrak{g}_-\oplus\mathfrak{g}_0,\mathfrak{g}_{11}^{\xi})_*^{mk}\cong(\mathfrak{g}_-\oplus\mathfrak{g}_0,\mathfrak{g}_{11}^{\eta})_*^{mk}\cong\mathfrak{as}\ni\mathbb{C}K_t\cong\mathfrak{d}(\mathfrak{as})$.

Proof. Heading 2) is straightforward; the simplicity in heading 1) follows from Kac's criterion [K1]. To see that the Cartan prolong $(\mathfrak{g}_-\oplus\mathfrak{g}_0,\mathfrak{g}_{11}^{\xi}\oplus\mathfrak{g}_{12})_*^{mk}$ is infinite dimensional we need the following lemma which clinches the proof. □

Lemma. *Denote* $\mathfrak{h}=(\mathfrak{g}_-\oplus\mathfrak{g}_0,\mathfrak{g}_{11}^{\xi}\oplus\mathfrak{g}_{12})_*^{mk}$. *Consider the* $\mathbb{Z}$*-grading of* $\mathfrak{h}$ *induced by the standard grading of* $\mathfrak{k}(1|6)$.

For $k>1$ *the operator* $T_k=(\mathrm{ad}\ K_{t^2})|_{\mathfrak{h}_k}$ *determines an isomorphism of* $\mathfrak{sl}(4)$*-modules* $\mathfrak{h}_k$ *and* $\mathfrak{h}_{k+2}$. *The operator* $T_1=(\mathrm{ad}\ K_{t^2})|_{\mathfrak{g}_{11}^{\xi}}$ *determines an isomorphism of* $\mathfrak{g}_{11}^{\xi}$ *with its image.*

Proof. We easily check that $K_{t^2}\in\mathfrak{h}$ and

$$\mathrm{ad}\ K_{t^2}=2t(t\partial t+E-2),\quad \text{where } E=\sum(\xi_i\frac{\partial}{\partial\xi_i}+\eta_i\frac{\partial}{\partial\eta_i}).$$

Therefore, $\mathrm{Ker}(\mathrm{ad}\ K_{t^2})$ in $\mathfrak{k}(1|6)$ consists of the fields generated by the functions f such that $\deg_t f+\deg_\xi f+\deg_\eta f-2=0$, i.e., $\mathrm{Ker}(\mathrm{ad}\ K_{t^2})\cong\mathfrak{sl}(4)\oplus\mathfrak{g}_{12}\oplus\mathbb{C}K_{t^2}$.

This makes it clear that, first, $\mathrm{ad}\ K_{t^2}$ is $\mathfrak{sl}(4)$-invariant; second, the operators T_k have no kernel for $k>0$. □

We will denote the simple exceptional Lie superalgebra described in heading 1) of Theorem 3.2 by $\mathfrak{kas}$.

4. Prolongs of the Lie superalgebras $\mathfrak{cg}$. The exceptional Cartan prolong $\mathfrak{vle}(4|3)=(\Pi(T_0(\vec{0})),\mathfrak{cvect}(0|3))_*$

In order to construct this exceptional example we have to recall (see sec. 0.7) that on the supermanifold of purely odd dimension the space of volume forms is $T(\overrightarrow{-1})$ and the space of half-densities is $T(\overrightarrow{-1/2})$ (not $T(\overrightarrow{1})$ and $T(\overrightarrow{1/2})$ as on manifolds).

4.1. Let us now describe a construction of several exceptional simple Lie superalgebras. Let $\mathfrak{u}=\mathfrak{vect}(m|n)$, let $\mathfrak{g}=(\mathfrak{u}_{-1},\mathfrak{g}_0)_*$ be a simple Lie subsuperalgebra of $\mathfrak{u}$. Moreover, let there exist an element $D\in\mathfrak{u}_0$ that determines an exterior derivation of $\mathfrak{g}$ and has no kernel on $\mathfrak{u}_+=\bigoplus_{i>0}\mathfrak{u}_i$. Let us study the prolong $\tilde{\mathfrak{g}}=(\mathfrak{g}_{-1},\mathfrak{g}_0\oplus\mathbb{C}D)_*$.

Lemma. *Either* $\tilde{\mathfrak{g}}$ *is simple or* $\tilde{\mathfrak{g}}=\mathfrak{g}\ni\mathbb{C}D$, *where* $\mathfrak{g}=(\mathfrak{u}_{-1},\mathfrak{g}_0)_*$.

Proof. Let I be a nonzero graded ideal of $\tilde{\mathfrak{g}}$. The subsuperspace $(\operatorname{ad}\mathfrak{u}_{-1})^{k+1}a$ of $\mathfrak{u}_{-1}$ is nonzero for any nonzero homogeneous element $a\in\mathfrak{u}_k$ and $k\geq 0$. Since $\mathfrak{g}_{-1}=\mathfrak{u}_{-1}$, the ideal I contains nonzero elements from $\mathfrak{g}_{-1}$; by simplicity of $\mathfrak{g}$ the ideal I contains the whole $\mathfrak{g}$. If, moreover, $[\mathfrak{g}_{-1},\tilde{\mathfrak{g}}_1]=\mathfrak{g}_0$, then by definition of the Cartan prolongation $\tilde{\mathfrak{g}}=\mathfrak{g}\ni\mathbb{C}D$.

If, instead, $[\mathfrak{g}_{-1},\tilde{\mathfrak{g}}_1]=\mathfrak{g}_0\oplus\mathbb{C}D$, then $D\in I$ and since $[D,\mathfrak{u}_+]=\mathfrak{u}_+$, we derive that $I=\mathfrak{g}$. In other words, $\mathfrak{g}$ is simple. □

4.2. **Example 1.** Take $\mathfrak{u}=\mathfrak{vect}(2^{n-1}|2^{n-1}-1)$. Consider $\mathfrak{u}_{-1}$ as $\Pi(T_0(\vec{0}))$ and set $\mathfrak{g}_{-1}=\mathfrak{u}_{-1}$, $\mathfrak{g}_0=\mathfrak{vect}(0|n)$. Clearly, $\mathfrak{g}_{-1}$ is a $\mathfrak{g}_0$-module. Then $(\mathfrak{g}_{-1},\mathfrak{g}_0)_*$ is a simple Lie superalgebra isomorphic to $\mathfrak{le}(n;n)$. The isomorphism is established with the help of a regrading. For the operator D of the exterior derivation of $\mathfrak{le}(n;n)$ we take the grading operator $d\in\mathfrak{m}(n;n)_0\subset\mathfrak{u}_0$, i.e., $\mathfrak{g}_0\oplus\mathbb{C}D\cong\mathfrak{cvect}(0|n)$.

In particular, for $n=2$ we have $\mathfrak{g}_{-1}=\Pi(\xi_1,\xi_2,\xi_1\xi_2)$; $\mathfrak{g}_0=\mathfrak{vect}(0|2)\cong\mathfrak{sl}(2|1)$. Then $\mathfrak{c}(\mathfrak{g}_0)=\mathfrak{gl}(2|1)$ and $(\Pi(T_0(\vec{0})),\mathfrak{cvect}(0|2))_*\cong\mathfrak{vect}(2|1)$.

Theorem. 1) $\mathfrak{vle}(4|3)=(\Pi(T_0(\vec{0})),\mathfrak{cvect}(0|3))_*$ *is a simple Lie superalgebra.*
2) $(\Pi(T_0(\vec{0})),\mathfrak{cvect}(0|n))_*\cong\mathfrak{d}(\mathfrak{le}(n;n))$ *for* $n>3$.

Proof. Thanks to Lemma 4.1 heading 1) follows from the fact that the Cartan prolong $(\Pi(T_0(\vec{0})),\mathfrak{cvect}(0|3))_*$ is bigger than $\mathfrak{sle}^\circ(3;3)\ni\mathbb{C}D$; we will prove this fact in §5. Heading 2) is proved in Appendix 2. □

4.3. To clarify the structure of $\mathfrak{vle}(4|3)=(\Pi(T_0(\vec{0})),\mathfrak{cvect}(0|3))_*$, consider one more construction. Let us describe one wonderful property of $\mathfrak{sle}^\circ(3)$ that singles it out among the $\mathfrak{sle}^\circ(n)$.

In the standard grading of $\mathfrak{g}=\mathfrak{sle}^\circ(3)$ we have $\dim\mathfrak{g}_{-1}=(3,3)$, $\mathfrak{g}_0\cong\mathfrak{spe}(3)$. For the regraded superalgebra $R\mathfrak{g}=\mathfrak{sle}^\circ(3;3)$ we have $\dim R\mathfrak{g}_{-1}=(3,3)$, $R\mathfrak{g}_0=\mathfrak{svect}(0|3)\cong\mathfrak{spe}(3)$. Therefore, for $\mathfrak{sle}^\circ(3)$ and only for it among the $\mathfrak{sle}^\circ(n)$, the regrading R determines a nontrivial automorphism. In terms of generating functions the regrading R is given by the formulas:

1) if $\deg_\xi(f)=0$, then $R(f)=\Delta(f\xi_1\xi_2\xi_3)$;
2) if $\deg_\xi(f)=1$, then $R(f)=f$;
3) if $\deg_\xi(f)=2$, then $R(f)=\frac{\partial^3(\Delta^{-1}f)}{\partial\xi_1\partial\xi_2\partial\xi_3}$ (see (2.6)).

We see that $R^2=SIGN$, where the operator $SIGN$ is defined by the formulas

$$SIGN(D)=(-1)^{p(D)}D \text{ on the vector fields}$$

and

$$SIGN(f)=(-1)^{p(f)+1}f \text{ on the generating functions.}$$

Let now $\mathfrak{g}=\mathfrak{le}(3;3)$ and $i_1:\mathfrak{g}\longrightarrow\mathfrak{u}=\mathfrak{vect}(4|3)$ be the embedding that preserves the standard $\mathbb{Z}$-grading of $\mathfrak{g}$. Let $\mathfrak{h}=\mathfrak{le}(3)$ and $\tilde{\mathfrak{h}}=\mathfrak{sle}^\circ(3)\subset\mathfrak{h}$. Then the map

$$i_2=SIGN\circ i_1R:\tilde{\mathfrak{h}}\longrightarrow\mathfrak{u}$$

is an embedding that preserves the grading of $\tilde{\mathfrak{h}}$.

Observe that $\tilde{\mathfrak{h}}_0=\mathfrak{spe}(3)$ and $\mathfrak{h}_0=\mathfrak{pe}(3)\cong\tilde{\mathfrak{h}}_0\oplus\mathbb{C}(\sum q_i\xi_i)$. On the space $i_2(\mathfrak{h}_{-1})$, the action of $z=-2i_1(\sum q_i\xi_i)+3d$ coincides with the action of $\sum q_i\xi_i$ on $\mathfrak{h}_{-1}$. Therefore, setting $i_2(\sum q_i\xi_i)=z$ we get an embedding $i_2(\mathfrak{h}_{-1}\oplus\mathfrak{h}_0)\longrightarrow\mathfrak{u}$ that can be extended to an embedding of $\mathfrak{h}$ to $\mathfrak{u}$. Under this embedding

$$i_1(\mathfrak{g}_{-1}\oplus\mathfrak{g}_0)+i_2(\mathfrak{h}_{-1}\oplus\mathfrak{h}_0)=\mathfrak{u}_{-1}\oplus(\mathfrak{g}_0\oplus\mathbb{C}d),$$

i.e., the *nondirect* sum of the images of i_1 and i_2 covers the whole nonpositive part of $\mathfrak{vle}(4|3)$.

Thus, there are two distinct embeddings of $\mathfrak{le}(3) \cong \mathfrak{le}(3;3)$ into $(\Pi(T_0(\vec{0})), \mathfrak{cvect}(0|3))_*$:

$$i_1 : \mathfrak{le}(3;3) \longrightarrow \mathfrak{vle}(4|3) \tag{4.1.1}$$

and

$$i_2 : \mathfrak{le}(3) \longrightarrow \mathfrak{vle}(4|3) \tag{4.1.2}$$

such that

$$i_1(\mathfrak{le}(3;3)) + i_2(\mathfrak{le}(3)) = \mathfrak{vle}(4|3)$$

(the sum in the left hand side is *not* a direct one!). As a linear space, $\mathfrak{vle}(4|3)$ is the quotient of $\mathfrak{le}(3;3)) \oplus \mathfrak{le}(3)$ modulo the subspace $V = \{(SIGN \circ Rg \oplus -g) : g \in \mathfrak{sle}^\circ(3)\}$. The map φ defined by the formula

$$\varphi|_{i_1(\mathfrak{le}(3;3))} = SIGN \circ i_2 i_1^{-1}; \qquad \varphi|_{i_2(\mathfrak{le}(3))} = i_1 i_2^{-1}$$

determines a nontrivial automorphism of $\mathfrak{vle}(4|3)$.

5. An explicit form of the vector fields from $\mathfrak{vle}(4|3) = (\Pi(T_0(\vec{0})), \mathfrak{cvect}(0|3))_* \subset \mathfrak{vect}(4|3)$

Here we consider $\mathfrak{vle}(4|3)_0 = \mathfrak{cvect}(0|3)$ and the module $\mathfrak{vle}(4|3)_{-1} = \Pi(T_0(\vec{0}))$ over it. Let us identify this module with the quotient of $\Pi(\Lambda(\eta_1, \eta_2, \eta_3))$ modulo constants and redenote the basis in $\Pi(\Lambda(\eta_1, \eta_2, \eta_3))$ so that this quotient becomes spanned by the partial derivatives with respect to the certain new indeterminates. Namely, we set

$$\Pi(\eta_1\eta_2\eta_3) \mapsto -\frac{\partial}{\partial y}; \quad \Pi(\eta_i) \mapsto -\frac{\partial}{\partial u_i}; \quad \Pi(\frac{\partial \eta_1\eta_2\eta_3}{\partial \eta_i}) \mapsto -\frac{\partial}{\partial \xi_i}.$$

Every element $D \in \mathfrak{vect}(4|3)$ is of the form $D = \sum\limits_{i\leq 3}(P_i\frac{\partial}{\partial \xi_i} + Q_i\frac{\partial}{\partial u_i}) + R\frac{\partial}{\partial y}$, where $P_i, Q_i, R \in \mathbb{C}[y, u, \xi]$.

5.1. Lemma. *Set* $\mathfrak{g}_{-1} = Span(\frac{\partial}{\partial y}; \frac{\partial}{\partial u_i}, \frac{\partial}{\partial \xi_i}$ *for* $i \leq 3)$, $\mathfrak{g}_0 = \mathfrak{cvect}(0|3)$. *The homogeneous (with respect to parity) vector field* $D \in \mathfrak{vect}(4|3)$ *belongs to* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ *if and only if it satisfies the following system of equations:*

$$\frac{\partial Q_i}{\partial u_j} + (-1)^{p(D)}\frac{\partial P_j}{\partial \xi_i} = 0 \text{ for any } i \neq j; \tag{5.1}$$

$$\frac{\partial Q_i}{\partial u_i} + (-1)^{p(D)}\frac{\partial P_i}{\partial \xi_i} = \frac{1}{2}\left(\sum_{1\leq j\leq 3}\frac{\partial Q_j}{\partial u_j} + \frac{\partial R}{\partial y}\right) \text{ for } i = 1, 2, 3; \tag{5.2}$$

$$\frac{\partial Q_i}{\partial \xi_j} + \frac{\partial Q_j}{\partial \xi_i} = 0 \text{ for any } i, j; \text{ in particular } \frac{\partial Q_i}{\partial \xi_i} = 0; \tag{5.3}$$

$$\frac{\partial P_i}{\partial u_j} - \frac{\partial P_j}{\partial u_i} = -(-1)^{p(D)}\frac{\partial R}{\partial \xi_k} \text{ for any even permutation } \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}; \tag{5.4}$$

$$\frac{\partial Q_i}{\partial y} = 0 \text{ for } i = 1, 2, 3; \tag{5.5}$$

$$\frac{\partial P_k}{\partial y} = (-1)^{p(D)}\frac{1}{2}\Big(\frac{\partial Q_i}{\partial \xi_j} - \frac{\partial Q_j}{\partial \xi_i}\Big) \text{ for any even permutation } \begin{pmatrix} 1 & 2 & 3\\ i & j & k\end{pmatrix}. \tag{5.6}$$

Proof is similar to that of Lemma 2.1. □

Remark. The left hand sides of (5.1)–(5.6) determine the coefficients of the 2-form $L_D\omega$, where L_D is the Lie derivative and $\omega = \sum\limits_{1\le i\le 3} du_i d\xi_i$. It is interesting to interpret the right hand sides of these equations.

5.2. Theorem. *Every solution of the system (5.1)–(5.6) is of the form:*

$$\begin{aligned} D = \mathrm{Le}_f + yB_f - (-1)^{p(f)}\left(y\Delta(f) + y^2\frac{\partial^3 f}{\partial\xi_1\partial\xi_2\partial\xi_3}\right)\frac{\partial}{\partial y}\\ + B_g - (-1)^{p(g)}\left(\Delta(g) + 2y\frac{\partial^3 g}{\partial\xi_1\partial\xi_2\partial\xi_3}\right)\frac{\partial}{\partial y},\end{aligned} \tag{5.7}$$

where $f, g\in\mathbb{C}[u,\xi]$ are arbitrary and the operator B_f is given by the formula:

$$B_f = \frac{\partial^2 f}{\partial\xi_2\partial\xi_3}\frac{\partial}{\partial\xi_1} + \frac{\partial^2 f}{\partial\xi_3\partial\xi_1}\frac{\partial}{\partial\xi_2} + \frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\frac{\partial}{\partial\xi_3}. \tag{5.8}$$

Proof is similar to that of Theorem 2.3; see Appendix 1 and [ShP]. □

Formula (5.7) makes it possible to explicitly express the two embeddings (4.1.1)

$$i_1, i_2 : \mathfrak{le}(3) \longrightarrow \mathfrak{vle}(4|3).$$

The first embedding i_1 preserves the grading of $\mathfrak{le}(3;3)$, cf. 0.4. I do not know any compact general formula for i_1 and can only determine it component-wise (mind that A_3 in the first line of the following displayed formula is the group of even permutations):

(5.9)

$$\begin{aligned} i_1(\mathrm{Le}_{f(u)}) &= \mathrm{Le}_{\sum\frac{\partial f}{\partial u_i}\xi_j\xi_k - yf}, \text{ where } y \text{ is treated as a parameter}\\ &\quad\text{and } (i,j,k)\in A_3;\\ i_1(\mathrm{Le}_{\sum f_i(u)\xi_i}) &= \mathrm{Le}_f - \varphi(u)\sum\xi_i\frac{\partial}{\partial\xi_i} + (-\varphi(u)y + \Delta(\varphi(u)\xi_1\xi_2\xi_3))\frac{\partial}{\partial y},\\ &\quad\text{where } \varphi(u) = \Delta(f);\\ i_1\left(\mathrm{Le}_{\sum\limits_{1\le i\le 3;\ (i,j,k)\in A_3}\psi_i(u)\xi_k\xi_l}\right) &= B_f - \Delta(f)\frac{\partial}{\partial y}, \text{ where } B_f \text{ is given by (5.8)};\\ i_1(\mathrm{Le}_{\psi(u)\xi_1\xi_2\xi_3}) &= -\psi(u)\frac{\partial}{\partial y}.\end{aligned}$$

The second embedding i_2 preserves the standard grading of $\mathfrak{le}(3)$. It is given by the formulas

$$i_2(\mathrm{Le}_f) = \mathrm{Le}_f + yB_f - (-1)^{p(f)}\left(y\Delta(f) + y^2\frac{\partial^3 f}{\partial\xi_1\partial\xi_2\partial\xi_3}\right)\frac{\partial}{\partial y}. \tag{5.10}$$

6. The exceptional simple Lie superalgebras of depth 2: $\mathfrak{mb}(4|5) = (\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^m_*$ and $\mathfrak{ksle}(9|6) = (\mathfrak{hei}(8|6), \mathfrak{svect}(4)_{3,4})^k_*$

Two more examples of exceptional simple Lie superalgebras are obtained with the help of a construction that generalizes the constructions from §4 to Lie superalgebras of depth 2. Let $\mathfrak{u} = \bigoplus_{i\ge -2}\mathfrak{u}_i$ be either $\mathfrak{m}(n)$ or $\mathfrak{k}(2m+1|n)$; let $\mathfrak{g} = (\mathfrak{u}_-, \mathfrak{g}_0)_*$

be a subalgebra of $\mathfrak{u}$ such that the subspace $\mathfrak{u}_{-2}$ belongs to the center of $\mathfrak{g}$ and the quotient $\mathfrak{g}/\mathfrak{u}_{-2}$ is simple. Moreover, let $D \in \mathfrak{u}_0$ determine an exterior derivation of $\mathfrak{g}$ without kernel on $\mathfrak{u}_{-2} \oplus \mathfrak{u}_+$, where $\mathfrak{u}_+ = \bigoplus_{i>0} \mathfrak{u}_i$.

Let us study the mk-prolong $\tilde{\mathfrak{g}} = (\mathfrak{g}_-, \mathfrak{g}_0 \oplus \mathbb{C}D)^{mk}_*$. The main result of this section: a description of two simple exceptional Lie superalgebras $\mathfrak{mb}(4|5) = (\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^m_*$ (Theorem 6.2) and $\mathfrak{ksle}(9|6) = (\mathfrak{hei}(8|6), \mathfrak{svect}_{3,4}(4))^k_*$ (Theorem 6.5).

6.1. Lemma. *Either* $\tilde{\mathfrak{g}}$ *is simple or* $\tilde{\mathfrak{g}} \cong \mathfrak{g} \ni\!\!\!\supset \mathbb{C}D$, *where* $\mathfrak{g} = (\mathfrak{u}_-, \mathfrak{g}_0)_*$.

Proof. First, let us prove that if $\tilde{\mathfrak{g}} \not\cong \mathfrak{g} \ni\!\!\!\supset \mathbb{C}D$, then $\mathfrak{u}_{-2}$ is not an ideal in $\tilde{\mathfrak{g}}$. Indeed, in this case there exist $g_{-1} \in \mathfrak{g}_{-1}$, $g_0 \in \mathfrak{g}_0$ and $g_1 \in \tilde{\mathfrak{g}}_1$ such that $[g_1, g_{-1}] = D + g_0$.

Let $\mathfrak{u}_{-2} = \mathbb{C}z$. Then $[g_{-1}, z] = [g_0, z] = 0$ and we have

$$\begin{aligned}[g_{-1}, [z, g_1]] &= [[g_{-1}, z], g_1] + (-1)^{p(g_{-1})p(z)}[z, [g_{-1}, g_1]]\\ &+(-1)^{p(g_{-1})p(z)}[z, D + g_0] = (-1)^{p(g_{-1})p(z)}[z, D] \neq 0.\end{aligned}$$

(We have taken into account that D has no kernel on $\mathfrak{u}_{-2}$.) Hence, $[z, g_1]$ is a nonzero element of $\mathfrak{g}_{-1}$. The rest of the proof mimics that of Lemma 4.1 □

6.2. Consider $\hat{\mathfrak{g}} = \mathfrak{b}_{1/2}(n; n)$. We have

$$\hat{\mathfrak{g}}_{-1} = \Pi(T(\overrightarrow{-1/2})); \quad \hat{\mathfrak{g}}_0 = \mathfrak{cvect}(0|n)$$

and the $\hat{\mathfrak{g}}_0$-action on $\hat{\mathfrak{g}}_{-1}$ preserves the nondegenerate superskew-symmetric form

$$B(\varphi\sqrt{vol}, \psi\sqrt{vol}) = \int \varphi\psi \cdot vol; \qquad p(B) \equiv n \pmod 2. \tag{6.1}$$

Now, let $\mathfrak{g} = \tilde{\mathfrak{c}}(\mathfrak{b}_{1/2}(3; 3))$ be the corresponding to (6.1) *nontrivial* central extension (we indicate this fact by tilde over $\mathfrak{c}$) of $\hat{\mathfrak{g}}$. The depth of $\tilde{\mathfrak{c}}(\mathfrak{b}_{1/2}(3; 3))$ is equal to 2. This central extension is naturally embedded into

$$\mathfrak{u} = \begin{cases} \mathfrak{m}(2^{n-1}) & \text{for } n \text{ odd},\\ \mathfrak{k}(1 + 2^{n-1}|2^{n-1}) & \text{for } n \text{ even}.\end{cases}$$

As the operator D described in sec. 6.1 we take the grading operator $d \in \mathfrak{u}_0$, i.e., $\mathfrak{g}_0 \oplus \mathbb{C}D \cong \mathfrak{c}(\mathfrak{g}_0)$.

Example. Let $n = 2$. Then $\mathfrak{g}_{-1} = \Pi(T(\overrightarrow{-1/2}))$ and $\mathfrak{g}_- = \mathfrak{hei}(2|2)$. We also have $\mathfrak{c}(\mathfrak{g}_0) = \mathfrak{cvect}(0|2) \cong \mathfrak{cosp}(2|2)$ and $(\mathfrak{hei}(2|2), \mathfrak{cvect}(0|2))^k_* \cong \mathfrak{k}(3|2)$; see sec. 0.6.

Theorem. 1) $(\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^m_*$ *is a simple Lie superalgebra.*

2) $(\mathfrak{ab}(2^{n-1}), \mathfrak{cvect}(0|n))^{mk}_* \cong \mathfrak{d}((\mathfrak{ab}(2^{n-1}), \mathfrak{vect}(0|n))^{mk}_*) \cong \mathfrak{d}(\mathfrak{b}(n; n))$ *for* $n > 3$.

Proof. Thanks to Lemma 6.1 heading 1) follows from the fact that the m-prolongation of $(\mathfrak{ab}(4), \mathfrak{cvect}(0|3))$ is bigger than $(\mathfrak{ab}(4), \mathfrak{vect}(0|3))^m_*$. We give explicit formulas in Appendix 3. Heading 2) is proved in Appendix 2. □

6.3. **The exceptional extension $\mathfrak{esle}^\circ(3)$.** Let us clarify the structure of the exceptional Lie superalgebra $(\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{mk}_*$ with the help of a construction similar to that from §4. To this end, we describe another remarkable property of $\mathfrak{sle}^\circ(3)$ that singles it out among the $\mathfrak{sle}^\circ(n)$.

The Lie superalgebra $\mathfrak{g} = \mathfrak{sle}^\circ(3)$ has a 2ε-dimensional nontrivial central extension $\mathfrak{esle}^\circ(3)$: the element M_1 of degree -2 with respect to the standard grading of $\mathfrak{sle}^\circ(3)$ extends $\mathfrak{sle}^\circ(3)$ to $\mathfrak{sb}^\circ(3)$ while z of degree -2 with respect to the grading of $\mathfrak{sle}^\circ(3; 3)$ is associated with the form B on the space $\mathfrak{g}_{-1}$ of half-densities with shifted parity (see (6.1)) in the realization $\mathfrak{g} = \mathfrak{sle}^\circ(3; 3)$.

The regrading R interchanges these central elements and establishes a nontrivial automorphism of $\mathfrak{esle}^\circ(3)$.

Now let $\mathfrak{g} = \tilde{\mathfrak{c}}(\mathfrak{b}_{1/2}(3;3))$ be the described in sec. 6.2 nontrivial central extension of depth 2 of $\mathfrak{b}_{1/2}(3;3)$; clearly, $\mathfrak{g}_{-2} = \mathbb{C}z$. The inverse regrading R^{-1} sends $\mathfrak{g}$ into the *nontrivial* central extension $\mathfrak{h} = \tilde{\mathfrak{c}}(\mathfrak{b}_{-3,1}(3))$ of $\mathfrak{b}_{-3,1}(3)$ (see sec. 0.7) and $\deg R^{-1}z = -1$.

From the very beginning we have an embedding $i_1 : \mathfrak{g} \longrightarrow \mathfrak{u} = \mathfrak{m}(4)$. Let $\tilde{\mathfrak{h}} = \mathfrak{esle}^\circ(3) \subset \mathfrak{h}$.

Then the map $i_2 = SIGN \circ i_1 R$ determines an embedding of $\tilde{\mathfrak{h}}$ to $\mathfrak{u}$ preserving the grading of $\tilde{\mathfrak{h}}$.

Observe that $\tilde{\mathfrak{h}}_0 = \mathfrak{spe}(3)$ and $\mathfrak{h}_0 = \mathfrak{pe}(3) \cong \tilde{\mathfrak{h}}_0 \oplus \mathbb{C} \cdot M_{\sum q_i\xi_i - 3\tau}$. On $\mathfrak{h}_{-1}$, $z = M_{\sum q_i\xi_i - 3\tau}$ acts by the formula:

$$z : M_q \mapsto 4M_q, \quad M_\xi \mapsto 2M_\xi.$$

This action of z coincides with the action of $-3d + \frac{1}{2}i_1(z)$.

Therefore, the embedding i_2 of $\mathfrak{h}_-$ can be extended to an embedding of the whole $\mathfrak{h}$. We have

$$i_1(\mathfrak{g}_- \oplus \mathfrak{g}_0) + i_2(\mathfrak{h}_- \oplus \mathfrak{h}_0) = \mathfrak{u}_- \oplus (\mathfrak{g}_0 \oplus \mathbb{C}d),$$

i.e., the nondirect sum of the images of i_1 and i_2 covers the whole nonpositive part of $(\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{mk}_*$.

Thus, we have two distinct embeddings of $\tilde{\mathfrak{c}}(\mathfrak{b}_{-3,1}(3))$, isomorphic to $\tilde{\mathfrak{c}}(\mathfrak{b}_{1/2}(3;3))$ as abstract, but not as graded, Lie superalgebras, into $(\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{mk}_*$:

$$i_1 : \tilde{\mathfrak{c}}(\mathfrak{b}_{1/2}(3;3)) \longrightarrow (\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{mk}_* \quad \text{with the grading of } \tilde{\mathfrak{c}}(\mathfrak{b}_{1/2}(3;3)) \text{ preserved}$$

and

$$i_2 : \tilde{\mathfrak{c}}(\mathfrak{b}_{-3,1}(3)) \longrightarrow (\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{mk}_* \quad \text{with the grading of } \tilde{\mathfrak{c}}(\mathfrak{b}_{-3,1}(3)) \text{ preserved}$$

such that

$$i_1(\tilde{\mathfrak{c}}(\mathfrak{b}_{1/2}(3;3))) + i_2(\tilde{\mathfrak{c}}(\mathfrak{b}_{-3,1}(3))) = (\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{mk}_*$$

(the sum here is *not* a direct one!). As a linear space, $(\mathfrak{cvect}(0|3))^{mk}_*$ is the quotient of $\tilde{\mathfrak{c}}(\mathfrak{b}_{1/2}(3;3)) \oplus \tilde{\mathfrak{c}}(\mathfrak{b}_{-3,1}(3))$ modulo the subspace $V = \{(SIGN \circ Rg \oplus -g) : g \in \mathfrak{esle}^\circ(3)\}$. The map φ defined by the formulas

$$\varphi|_{i_1(\tilde{\mathfrak{c}}(\mathfrak{b}_{1/2}(3;3)))} = SIGN \circ i_2 i_1^{-1}; \qquad \varphi|_{i_2(\tilde{\mathfrak{c}}(\mathfrak{b}_{-3,1}(3)))} = i_1 i_2^{-1}$$

determines a nontrivial automorphism of $(\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{mk}_*$.

6.4. **Description of $(\mathfrak{hei}(8|6), \mathfrak{svect}(0|4)_{3,4})^{mk}_*$.** Consider the nontrivial central extension $\mathfrak{g} = \tilde{\mathfrak{c}}(\mathfrak{sle}^\circ(n;n))$ of $\mathfrak{sle}^\circ(n;n)$ defined as follows. We have: $\mathfrak{g}_0 = \mathfrak{svect}(0|n)$; $\mathfrak{g}_{-1} = \Pi(T^0(\vec{0})/\mathbb{C}\cdot 1)$, where $T^0(\vec{0}) = \{f \in T(\vec{0}) : \int f vol(\xi) = 0\}$. Define the central extension with the help of the form ω on $\mathfrak{g}_{-1}$ given by the formula:

$$\omega(f, g) = \int fg \cdot vol(\xi).$$

The same arguments as in 6.2, show that $(\mathfrak{g}_{-1}, \mathfrak{g}_0)^{mk}_*$ can be embedded into $\mathfrak{k}(1 + 2^{n-1}|2^{n-1} - 2)$ for n even and into $\mathfrak{m}(2^{n-1} - 1)$ for n odd.

Let x be the operator determining the standard $\mathbb{Z}$-grading of $\mathfrak{svect}(0|n)$ and let z commute with $\mathfrak{svect}(0|n)$; let $a, b \in \mathbb{C}$. For any a, b the element $ax + bz$ determines an outer derivation of $\mathfrak{g}_0$. Set

$$\mathfrak{svect}_{a,b}(n) = \mathfrak{svect}(0|n) \supplus \mathbb{C}(ax + bz);$$

set also

$$(\mathfrak{g}_-, \mathfrak{svect}_{a,b}(n))^{mk}_* = (\mathfrak{g}_-, \mathfrak{g}_0 \supplus \mathbb{C}(ax + bz))^{mk}_*,$$

$$\text{where } \mathfrak{g}_- = \begin{cases} \mathfrak{ab}(2^{n-1} - 1) & \text{for } n \text{ odd,} \\ \mathfrak{hei}(2^{n-1}|2^{n-1} - 2) & \text{for } n \text{ even.} \end{cases}$$

Example. Let $n = 3$. Then $\mathfrak{g}_{-1} = \Pi(\xi_1, \xi_2, \xi_3, \xi_1\xi_2, \xi_1\xi_3, \xi_2\xi_3)$ and $\mathfrak{g}_- \cong (\mathfrak{b}_\lambda(3))_- = \mathfrak{ab}(3)$ for any a, b, while $\mathfrak{g}_0 = \mathfrak{svect}(0|3) \cong \mathfrak{spe}(3)$. The operator x becomes $\sum \xi_i \partial\xi_i$ and

$$\mathfrak{g}_0 \supplus \mathbb{C}(ax + bz) \cong \mathfrak{spe}(3) \supplus \mathbb{C}(a \sum \xi_i \partial\xi_i + bz) \cong (\mathfrak{b}_\lambda(3))_0 \text{ for } \lambda = -\frac{b}{3a}.$$

Therefore, $(\mathfrak{ab}(3), \mathfrak{svect}_{a,b}(3))^{mk}_* \cong \mathfrak{b}_\lambda(3)$ for $\lambda = -\frac{b}{3a}$. In particular,

$$(\mathfrak{ab}(3), \mathfrak{svect}_{1,3}(3))^{mk}_* \cong \mathfrak{sm}(3) \text{ and } (\mathfrak{ab}(3), \mathfrak{svect}_{1,0}(3))^{mk}_* \cong \mathfrak{b}(3).$$

6.5. Theorem. 1) $(\mathfrak{hei}(8|6), \mathfrak{svect}_{3,4}(4))^{mk}_*$ *is a simple Lie superalgebra.*

2) *Let* $\mathfrak{g}_- = \begin{cases} \mathfrak{ab}(2^{n-1} - 1) & \textit{for } n \textit{ odd} \\ \textit{Then} & \\ \mathfrak{hei}(2^{n-1}|2^{n-1} - 2) & \textit{for } n \textit{ even.} \end{cases}$

$$(\mathfrak{g}_-, \mathfrak{svect}_{a,b}(n))^{mk}_* \cong (\mathfrak{g}_-, \mathfrak{svect}(0|n))^{mk}_* \supplus \mathbb{C}(ax + bz) \textit{ if } n > 4 \textit{ or if } (a, b) \notin \mathbb{C}(3, 4) \textit{ and } n = 4.$$

Proof. As in Theorem 6.1, heading 1) follows from a direct calculation based on Lemma 6.1; for the explicit formulas see Appendix 3. Heading 2) is proved in Appendix 2. □

Let us clarify the structure of $(\mathfrak{hei}(8|6), \mathfrak{svect}_{3,4}(4))^k_*$. This Lie superalgebra is contained in $\mathfrak{u} = \mathfrak{k}(9|6)$. In sec. 6.3 we have already described the Lie superalgebra $\mathfrak{g} = \tilde{\mathfrak{c}}(\mathfrak{sle}^\circ(4; 4))$ and its embedding $i_1 : \mathfrak{g} \to \mathfrak{u}$.

Observe that $\mathfrak{g} \supset \mathfrak{as}$ and this embedding preserves the $\mathbb{Z}$-grading described in sec. 2.1:

$$\mathfrak{as}_{-2} = \mathfrak{g}_{-2}; \quad \mathfrak{as}_{-1} = \mathfrak{g}_{-1} = \Pi(\Lambda^2(\xi_1, \xi_2, \xi_3, \xi_4))$$
$$\mathfrak{as}_0 = \mathfrak{sl}(4) \subset \mathfrak{g}_0 = \mathfrak{svect}(0|4); \quad \mathfrak{as}_1 = \Pi(S^2(q_1, q_2, q_3, q_4)) \subset \mathfrak{g}_1.$$

For the role of $\mathfrak{h}$ (see 4.3 and 6.3) take $\mathfrak{kas}$. It follows from Theorem 3.2 that $\mathfrak{as} \subset \mathfrak{kas}$; set $\tilde{\mathfrak{h}} = \mathfrak{as}$. Let $R : \tilde{\mathfrak{h}} \longrightarrow \mathfrak{g}$ be the embedding that executes the isomorphism of two copies of $\mathfrak{as}$. (Notice that R preserves the $\mathbb{Z}$-grading (1.1) of $\mathfrak{as}$.) The map $i_2 = i_1 R$ determines an embedding of $\tilde{\mathfrak{h}}$ into $\mathfrak{k}(9|6)$.

But $\mathfrak{h}_0 = \tilde{\mathfrak{h}}_0 \oplus \mathbb{C}K_t$. It turns out that i_2 can be extended to an embedding $i_2 : \mathfrak{kas} \longrightarrow \mathfrak{u}$ and $i_1(\tilde{\mathfrak{c}}(\mathfrak{sle}^\circ(4; 4))) \cap i_2(\mathfrak{kas}) = \mathfrak{as}$.

As in the above examples, we have: $i_1(\mathfrak{g}_- \oplus \mathfrak{g}_0) + i_2(\mathfrak{h}_- \oplus \mathfrak{h}_0) \cong \mathfrak{u}_- \supplus \mathbb{C}(3x + 4z)$ (the sum in the left hand side here is *not* a direct one!). But, unlike the cases $\mathfrak{vle}(4|3)$ and $\mathfrak{mb}(4|5)$, the nondirect sum of $\mathfrak{g} = \tilde{\mathfrak{c}}(\mathfrak{sle}^\circ(4; 4))$ with $\mathfrak{h} = \mathfrak{kas}$ does not span the whole of $\mathfrak{ksle}(9|6)$. A description similar to the cases $\mathfrak{vle}(4|3)$ and $\mathfrak{mb}(4|5)$ will be given in §7.

7. The Weisfeiler regradings

7.1. **Weisfeiler regradings.** Let $\mathfrak{g} = \bigoplus_{j\geq -d}\mathfrak{g}_j$ be a $\mathbb{Z}$-graded vectorial superalgebra. By a *Weisfeiler regrading or W-regrading* of $\mathfrak{g}$ we understand a second structure of $\mathbb{Z}$-graded superalgebra $\mathfrak{g} = \bigoplus_{i\geq -D}\mathfrak{h}_i$ (with $D<\infty$) such that

1) the two gradings determine a bigrading: $\mathfrak{h}_i = \bigoplus_j \mathfrak{g}_j\cap\mathfrak{h}_i$;

2) $\mathfrak{h}^0 = \bigoplus_{j\geq 0}\mathfrak{h}_j$ is a maximal Lie subalgebra of finite codimension; and

3) the grading $\mathfrak{h}$ is transitive: for any non-zero $x\in\mathfrak{h}_k$ for $k\geq 0$ there is $y\in\mathfrak{h}_{-1}$ such that $[x,y]\neq 0$.

In what follows in this section $\mathfrak{h}$ denotes a Lie superalgebra obtained from $\mathfrak{g}$ by a regrading. We will describe all the W-regradings of the exceptional Lie superalgebras.

7.1.1. Lemma. *Let $\mathfrak{h} = \bigoplus_{i\geq -D}\mathfrak{h}_i$ be a W-regrading of the Lie superalgebra $\mathfrak{g} = \bigoplus_{j\geq -d}\mathfrak{g}_j$. It determines a bigrading:*

$$\mathfrak{h}_i = \bigoplus_{j\geq m(i)}^{M(i)} \mathfrak{h}_{i,j},\ \text{where } \mathfrak{h}_{i,j} = \mathfrak{h}_i\cap\mathfrak{g}_j. \tag{7.1.1}$$

Suppose $\dim(\mathfrak{h}_0\cap\mathfrak{g}_-) = 0|k$. *Then the number $M = M(-1)-m(-1)+1$ of the summands $\mathfrak{h}_{-1,j}$ in the decomposition (7.1.1) of $\mathfrak{h}_{-1}$ does not exceed the number of homogeneous components with respect to the $\mathbb{Z}$-grading of $\Lambda(\mathfrak{h}_0\cap\mathfrak{g}_-)$.*

In particular, if $\mathfrak{h}_0\cap\mathfrak{g}_-$ is homogeneous with respect to the $\mathbb{Z}$-grading in $\mathfrak{g}$, i.e., $\mathfrak{h}_0\cap\mathfrak{g}_-\subset\mathfrak{g}_j$ for some j, then $M\leq k+1$.

Proof. The component $V = \mathfrak{h}_{-1,M(-1)}$ of the maximal degree is invariant with respect to the subalgebra $\mathfrak{h}_{0,\geq 0} = \bigoplus_{j\geq 0}\mathfrak{h}_{0,j}$. Since the $\mathfrak{h}_0$-module $\mathfrak{h}_{-1}$ is irreducible, it must be a quotient of the induced module: as spaces, $\operatorname{ind}^{\mathfrak{h}_0}_{\mathfrak{h}_{0,\geq 0}} V\cong\Lambda(\mathfrak{h}_0\cap\mathfrak{g}_-)\otimes V$. □

7.1.2. Lemma. *If $\mathfrak{h}_0\cap\mathfrak{g}_- = 0$, then $\mathfrak{h}_0\subset\mathfrak{g}_0$ and $\mathfrak{h}_{-1}$ is homogeneous.*

Proof. Clearly, $\mathfrak{h}_0 = \bigoplus_{k\geq 0}\mathfrak{h}_{0,k}$ and $\mathfrak{h}_{-1} = \bigoplus_j\mathfrak{h}_{-1,j}$.

The Lie superalgebra $\mathfrak{h}_{0,0}$ transforms $\mathfrak{h}_{-1,j}$ into itself and the operators from $\mathfrak{h}_{0,k}$ send $\mathfrak{h}_{-1,j}$ into $\mathfrak{h}_{-1,j+k}$. Therefore, if the representation of $\mathfrak{h}_0$ on $\mathfrak{h}_{-1}$ is irreducible, then $\mathfrak{h}_{-1}$ is homogeneous with respect to the grading of $\mathfrak{g}$, i.e., $\mathfrak{h}_{-1} = \mathfrak{h}_{-1,j_0}$ for some j_0. But then $\mathfrak{h}_{0,k}$ for $k>0$ sends $\mathfrak{h}_{-1}$ to 0. Since $\mathfrak{h}$ is transitive, $\mathfrak{h}_{0,k} = 0$ for all $k>0$, i.e., $\mathfrak{h}_0 = \mathfrak{h}_{0,0}$. □

7.1.3. Lemma. *If $\mathfrak{h}_0\cap\mathfrak{g}_- = 0$ and there exists a nonzero $x\in\mathfrak{g}_{-1}\cap\mathfrak{h}_-$, then the gradings of $\mathfrak{h}$ and $\mathfrak{g}$ coincide.*

Proof. $\mathfrak{h}_{-1} = \mathfrak{h}_{-1,j_0}\subset\mathfrak{g}_{j_0}$ and, therefore, $\mathfrak{h}_{-k}\subset\mathfrak{g}_{k\cdot j_0}$. If $x\in\mathfrak{h}_{-k}$, then $-1 = k\cdot j_0$, implying $k=1$ and $j_0=-1$, i.e., $\mathfrak{h}_{-1}\subset\mathfrak{g}_{-1}$, and, therefore, $\mathfrak{h}_{-1} = \mathfrak{g}_{-1}$ and $\mathfrak{h}_i = \mathfrak{g}_i$ for all i. □

7.2. **How to describe all the W-regradings.** 1) Determine $\mathfrak{h}_0\cap\mathfrak{g}_{-1}$.

2) Construct a "minimal" (i.e., most tightly compressed) regrading with the given intersection, i.e., such that it preserves in $\mathfrak{h}_{-1}$ all the elements of $\mathfrak{g}_{-1}$ except for those that have to go away in view of the condition on the intersection.

3) If the "minimal" regrading is a Weisfeiler one, then with the help of Lemmas 7.1.2 and 7.1.3 we prove that any other W-regrading with the given intersection coincides with the minimal one.

4) If the "minimal" regrading is not a Weisfeiler one, then with the help of Lemma 7.1.1 we prove that there are no W-regradings with the given intersection.

7.3. **W-regradings of $\mathfrak{g} = \mathfrak{vle}(4|3)$.** We consider a realization of $\mathfrak{g}$ by a nondirect sum $i_1(\mathfrak{le}(3;3)) + i_2(\mathfrak{le}(3))$; cf. sec. 4.3.

Notations: $\xi_i' = i_1(\xi_i)$, $u_i' = i_1(u_i)$; $\xi_i = i_2(\xi_i)$, $u_i = i_2(u_i)$ for $i = 1, 2, 3$.

Then

$$\mathfrak{g}_{-1} = Span(\xi_i = \xi_i';\ u_k = \xi_i'\xi_j' \text{ for even permutations } (i, j, k);\ \xi_1'\xi_2'\xi_3').$$

For any regrading we have:

$$\begin{array}{ll} \deg \xi_i + \deg u_i = N & \text{for any } i = 1, 2, 3; \\ \deg \xi_i' + \deg u_i' = N' & \text{for any } i = 1, 2, 3; \\ \deg_{Lie}(f) = \deg(f) - N; & \deg_{Lie}(f') = \deg(f') - N'. \end{array} \tag{7.3.1}$$

The conditions $\xi_i = \xi_i'$ imply that

$$\deg u_i' = \deg u_i, \tag{7.3.2}$$

whereas the conditions $u_k = \xi_i'\xi_j'$ imply that

(7.3.3)
$$\deg u_k - N = \deg \xi_i' + \deg \xi_j' - N' \Longleftrightarrow \deg \xi_i' = u_j' - \deg \xi_k = \deg u_j' - \deg \xi_k'.$$

3a) $\mathfrak{h} = \mathfrak{vle}(4|3;1)$. Set

$$\begin{array}{l} \deg \xi_1 = 0,\ \deg u_1 = 2;\ \ N = 2; \\ \deg \xi_2 = \deg \xi_3 = \deg u_2 = \deg u_3 = 1. \end{array} \tag{7.3.4}$$

Formulas (7.3.2) and (7.3.3) imply that

$$\begin{array}{l} \deg \xi_1' = 0,\ \deg u_1' = 2;\ \ N' = 2; \\ \deg \xi_2' = \deg \xi_3' = \deg u_2' = \deg u_3' = 1. \end{array} \tag{7.3.4}$$

Hence, $\mathfrak{h} = \bigoplus_{i \geq -2} \mathfrak{h}_i$, where

(7.3.5)
$$\mathfrak{h}_{-2} = \mathbb{C} \cdot \xi_1 = \mathbb{C} \cdot \xi_1'$$
$$\mathfrak{h}_{-1} = Span(\xi_2, \xi_3, u_2, u_3) \otimes \Lambda(\xi_1) = Span(\xi_2', \xi_3', u_2', u_3') \otimes \Lambda(\xi_1')$$
$$\mathfrak{h}_0 = \left(\Lambda^2(\xi_2, \xi_3) \bigoplus \Lambda^1(\xi_2, \xi_3) \otimes S^1(u_2, u_3) \bigoplus S^2(u_2, u_3)\right) \otimes \Lambda(\xi_1) \ni \mathfrak{vect}(\xi_1)$$
$$+ \text{ (NOT } \oplus!) \text{ the same with } '$$

Observe that since $\mathrm{id}(\mathfrak{sl}(2)) \cong \mathrm{id}^*(\mathfrak{sl}(2))$, we have

$$\begin{array}{c} \Lambda^2(\xi_2, \xi_3) \bigoplus \Lambda^1(\xi_2, \xi_3) \otimes S^1(u_2, u_3) \bigoplus S^2(u_2, u_3) \\ \cong \mathfrak{pe}(2) \cong \mathfrak{sl}(2) \otimes \Lambda(\xi_1) \ni \mathfrak{vect}(\xi_1). \end{array} \tag{7.3.6}$$

Introduce a formal odd indeterminate η. Set

$$V = Span(\xi_2, \xi_3); \quad \eta V = Span(-u_3, u_2).$$

Then $\xi_1 V = Span(\xi_1\xi_2, \xi_1\xi_3)$ and $\xi_1\eta V = Span(-\xi_1 u_3, \xi_1 u_2)$; the action of $\xi_1\xi_2$ is identical with that of $\frac{\partial}{\partial\eta}$.

In these notations, $\mathfrak{g}_{-1} = V \otimes \Lambda(\xi_1, \eta) = V \otimes \Lambda(2)$ and the bracket on $\mathfrak{g}_{-1}$ is given by the formula:

$$[v_1 \otimes \varphi_1(\xi_1, \eta), v_2 \otimes \varphi_2(\xi_2, \eta)] = (-1)^{p(\varphi_1)p(v_2)} \xi_1 \omega_v(v_1, v_2) \int \varphi_1\varphi_2 vol, \tag{7.3.7}$$

where ω_v is a 2-form on V preserved by $\mathfrak{sl}(2) \cong \mathfrak{sp}(2)$.

The direct calculations show that $\mathfrak{h}_0 \cong \mathfrak{c}(\mathfrak{sl}(2) \otimes \Lambda(2) \supplus \mathfrak{vect}(0|2))$, where the subalgebra $\mathfrak{vect}(0|2) \subset \mathfrak{g}_0$ acts on $\mathfrak{h}_{-1} = V \otimes \Lambda(2)$ as $\mathrm{id} \otimes T^{1/2}$, where $T^{1/2}$ is the representation of $\mathfrak{vect}(0|2)$ in the space of half-densities, $T(\overrightarrow{-1/2})$.

Thus,

$$\mathfrak{vle}(4|3;1) = (\mathfrak{hei}(4|4), \mathfrak{c}(\mathfrak{sl}(2) \otimes \Lambda(2) \supplus \mathfrak{vect}(0|2)))^{mk} \subset \mathfrak{k}(5|4). \tag{7.3.8}$$

<u>3b) $\mathfrak{h} = \mathfrak{vle}(4|3;2)$</u>. Set

$$\begin{array}{l}\deg \xi_1 = \deg \xi_2 = 0, \deg u_1 = \deg u_2 = 2; N = 2;\\ \deg \xi_3 = \deg u_3 = 1\end{array} \tag{7.3.9}$$

Formulas (7.3.2) and (7.3.3) imply then that

$$\begin{array}{l}\deg u_1' = \deg u_2' = 2;\ N' = 3;\\ \deg \xi_1' = \deg \xi_2' = 1;\ \deg \xi_3' = 2;\ \deg u_3' = 1.\end{array} \tag{7.3.9$'$}$$

Hence, $\mathfrak{h} = \bigoplus_{i \geq -2} \mathfrak{h}_i$, where

$$\begin{aligned}
\mathfrak{h}_{-2} &= Span(\xi_1 = \xi_1', \xi_2 = \xi_2', \xi_1\xi_2 = u_3') \cong \Pi(\Lambda(2)/\mathbb{C} \cdot 1);\\
\mathfrak{h}_{-1} &= Span(\xi_3, u_3) \otimes \Lambda(\xi_1, \xi_2) = V \otimes \Lambda(2),\ \dim V = 1|1;\\
\mathfrak{h}_0 &= \mathfrak{c}\big(\mathfrak{pe}(1) \otimes \Lambda(2) \supplus \mathfrak{vect}(0|2)\big).
\end{aligned}$$

The action of $\mathfrak{h}_0$ in $\mathfrak{h}_{-1}$ is not irreducible; hence, $\mathfrak{vle}(4|3;2)$ is not a W-regrading.

<u>3c) $\mathfrak{h} = \mathfrak{vle}(4|3;3)$</u>. Set

$$\deg u_i = 2; \deg \xi_i = 0; \deg u_i' = \deg \xi_i' = 1;\quad N = N' = 2. \tag{7.3.10}$$

By sec. 4.3, $\mathfrak{h} = \mathfrak{vle}(4|3;3) \cong \mathfrak{vle}(4|3)$.

<u>3d) $\mathfrak{g} = \mathfrak{vle}(4|3;K)$</u>. Set

$$\deg u_i = \deg u_i' = 2;\ \deg \xi_i = \deg \xi_i' = 1;\quad N = N' = 3. \tag{7.3.11}$$

Hence, $\mathfrak{h} = \bigoplus_{i \geq -2} \mathfrak{h}_i$, where

(7.3.12)

$$\begin{aligned}
\mathfrak{h}_{-2} &= \Pi\left(Span(\xi_i = \xi_i' : i = 1, 2, 3)\right)\\
\mathfrak{h}_{-1} &= \Pi\left(Span(u_i = \xi_j'\xi_k', \xi_i\xi_j = u_k' \text{ for all even permutations } (i,j,k)\right)\\
\mathfrak{h}_0 &= \Pi\Big(Span(u_i\xi_j = u_i'\xi_j') \cap \mathfrak{sle}(3)\\
&\bigoplus Span(\xi_1\xi_2\xi_3, \xi_1'\xi_2'\xi_3', \sum u_i\xi_i - \sum u_i'\xi_i') \bigoplus \mathbb{C}(\sum u_i\xi_i + \sum u_i'\xi_i')\Big)\\
&\cong \mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathbb{C} \cdot z.
\end{aligned}$$

The $\mathfrak{h}_0$-action in $\mathfrak{h}_{-1}$ is $\mathrm{id}(\mathfrak{sl}(3)) \otimes \mathrm{id}(\mathfrak{sl}(2)) \otimes 1$.

7.3.1. Statement. *Any W-regrading of* $\mathfrak{g} = \mathfrak{vle}(4|3)$ *is either* $\mathfrak{vle}(4|3;1)$ *or* $\mathfrak{vle}(4|3;K)$.

Proof. Let $\mathfrak{h} = \bigoplus_{i \geq -d} \mathfrak{h}_i$ be a W-regrading of $\mathfrak{g}$. Since the elements $\xi_i = \xi_i'$ act in $\mathfrak{g}$ as $\frac{\partial}{\partial u_i}$, all of them must lie in $\mathfrak{h}_-$. Set $U = Span(u_1, u_2, u_3)$ and let $\dim U \cap \mathfrak{h}_0 = m$, where $0 \leq m \leq 3$.

<u>If $m = 0$</u>, then by Lemma 7.1.2 the gradings in $\mathfrak{g}$ and $\mathfrak{h}$ can differ only if $\xi_1'\xi_2'\xi_3' \in \mathfrak{h}_0$. But then, since $\mathfrak{h}_0 \not\ni u_k = \xi_i'\xi_j' = [u_k', \xi_1'\xi_2'\xi_3']$, it follows that $u_k' = \xi_i\xi_j \notin \mathfrak{h}_0$.

But then $\mathfrak{h}$ and $\mathfrak{g} = \mathfrak{vle}(4|3;K)$ satisfy the conditions of Lemma 7.1.3; hence, their gradings coincide.

If $m = 1$, then, up to linear automorphisms, we may assume that $U \cap \mathfrak{h}_0 = \mathbb{C} \cdot u_1$. Since ad u_1 acts as $\frac{\partial}{\partial \xi_1}$, we deduce that $\xi_1\xi_2, \xi_2\xi_3, \xi_1 u_2, \xi_1 u_3 \notin \mathfrak{h}_0$. So we can apply Lemma 7.1.2 to $\mathfrak{h}$ and $\mathfrak{g} = \mathfrak{vle}(4|3;1)$.

If $m = 2$, then, up to linear automorphisms, we may assume that $U \cap \mathfrak{h}_0 = \mathbb{C} \cdot Span(u_1, u_2)$. As $\xi_i \in \mathfrak{h}_-$ for all i, we have $\Lambda(\xi)/\mathbb{C} \cdot 1 \subset \mathfrak{h}_-$. Hence, $\xi_1'\xi_2'\xi_3' \notin \mathfrak{h}_-$, as an element dual to $\xi_1\xi_2\xi_3 \in \mathfrak{h}_-$.

If $\mathfrak{h} \neq \mathfrak{vle}(4|3;2)$, then $\mathfrak{h}_- \neq \mathfrak{vle}(4|3;2)_-$. Hence, either $\mathfrak{h} = \Lambda(\xi)/\mathbb{C} \cdot 1$, or $g \otimes \Lambda(\xi_1, \xi_2) \in \mathfrak{h}_-$ for some $g \in \mathfrak{g}_{\geq 0}$.

In the first case, the subspases $\Lambda(\xi_1, \xi_2)/\mathbb{C} \cdot 1$ and $\xi_3 \otimes \Lambda(\xi_1, \xi_2)$ cannot belong to the same component $\mathfrak{h}_{-k}$ because of $u_3 \notin \mathfrak{h}_0$. But none of these subspaces generates the other. Thus, $\mathfrak{h}$ is not a W-regrading.

In the second case, either $\mathfrak{h}_{-1}$ does not generate $\mathfrak{h}_-$, or $\mathfrak{h}_{-1}$ contradicts Lemma 7.1.1. In both cases, $\mathfrak{h}$ is not a W-regrading. □

If $m = 3$, then $\xi_i \notin \mathfrak{h}_0$. Hence, $\Lambda(\xi_1, \xi_2, \xi_3)/\mathbb{C} \cdot 1 \cap \mathfrak{h}_0 = 0$. So we can apply Lemma 7.1.3 to $\mathfrak{h}$ and $\mathfrak{g} = \mathfrak{vle}(4|3;3) = \mathfrak{vle}(4|3)$. □

7.4. **W-regradings of $\mathfrak{g} = \mathfrak{mb}(4|5)$.** We consider a realization of $\mathfrak{g}$ by a nondirect sum $(i_1(\mathfrak{b}_{1/2}(3;3)) + i_2(\mathfrak{b}_{-3,1}(3))$; cf. sec. 6.3.

Recall that if u_1, u_2, u_3, ξ_1, ξ_2, ξ_3 and τ are the standard indeterminates in $\mathfrak{m}(3)$, then each generating function of $\mathfrak{b}_{1/2}(3;3)$ is of the form

$$F = f + \frac{1}{\deg(\Delta(f)) + 1}\tau\Delta(f), \text{ where } f \in \mathbb{C}[u, \xi].$$

Therefore, we may use f instead of F and f' instead of F'. Thus, $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, where

$$\begin{aligned} \mathfrak{g}_{-2} &= \Pi(\mathbb{C} \cdot 1), \\ \mathfrak{g}_{-1} &= \Pi(Span(1',\ \xi_i = \xi_i,\ u_i = \xi_j'\xi_k',\ \xi_1'\xi_2'\xi_3')). \end{aligned} \tag{7.4.1}$$

4a) $\mathfrak{h} = \mathfrak{mb}(4|5;1)$. Then $\mathfrak{h}$ is described by (7.3.4)–(7.3.4). Moreover, the superspaces $\mathfrak{h}_{-1}$ and $\mathfrak{h}_0$ are the same as for $\mathfrak{vle}(4|3;1)$ whereas our component $\mathfrak{h}_{-2}$ is spanned by $\mathfrak{vle}(4|3;1)_{-2} = Span(\xi_1)$ together with 1 and $1'$.

To visualize the result better, redenote the elements of $\mathfrak{h}_{-2}$:

$$1 \mapsto \eta,\ 1' \mapsto \xi_1,\ \xi_1 \mapsto \xi_1\eta.$$

Then the bracket in $\mathfrak{h}_{-1}$ is of the form

$$[v_1 \otimes \varphi_1(\xi_1, \eta), v_2 \otimes \varphi_2(\xi_1, \eta)] = (-1)^{p(\varphi_1)p(\varphi_2)}\omega_v(v_1, v_2) \cdot \varphi_1\varphi_2.$$

Thus, with the natural action of vector fields on the functions, we have

$$\mathfrak{h}_- \cong (\mathfrak{hei}(2|0) \otimes \Lambda(2))/\mathbb{C} \cdot z \otimes 1 \text{ and } \mathfrak{h}_0 \cong \mathfrak{c}\left(\mathfrak{sl}(2) \otimes \Lambda(2) \ni \mathfrak{vect}(0|2)\right). \tag{7.4.2}$$

Observe that the only difference of nonpositive parts of $\mathfrak{mb}(4|5;1)$ and $\mathfrak{h}(2|4;2)$ is that the former has one element more — the center — in the component of degree 0.

4b) $\mathfrak{g} = \mathfrak{mb}(4|5;2)$. Relations (7.3.9)–(7.3.9′) mean that $1 \in \mathfrak{h}_{-2}$ while $1' \in \mathfrak{h}_{-3}$. Thus,

$$\mathfrak{h}_{-3} = \mathbb{C} \cdot 1';\ \ \mathfrak{h}_{-2} = \Pi(\Lambda(2));\ \ \mathfrak{h}_{-1} = V \otimes \Lambda(2).$$

Observe that $\dim V = 1|1$, so $V \cong \Lambda(1)$; hence, $V \otimes \Lambda(2) \cong \Lambda(1) \otimes \Lambda(2) \cong \Lambda(3)$ with the action of $\mathfrak{pe}(1) \cong T^{1/2}(\mathfrak{vect}(0|1))$ on V.

The bracket $[\mathfrak{h}_{-1}, \mathfrak{h}_{-2}]$ is given by the formula

$$[\varphi_1(\xi_3; \xi_1, \xi_2), \varphi_2(\xi_1, \xi_2)] = c \cdot \int \varphi_1 \varphi_2 vol(\xi_1, \xi_2, \xi_3).$$

Thus,

$$\mathfrak{h}_0 \cong \mathfrak{c}\left(T^{1/2}(\mathfrak{vect}(0|1)) \otimes \Lambda(2) \ni \mathfrak{vect}(0|2)\right).$$

Since the $\mathfrak{h}_0$-action on $\mathfrak{h}_{-1}$ is reducible, this is not a W-filtration.

4c) $\mathfrak{mb}(4|5;3)$. The regrading is determined by (7.3.10). By sec. 6.3 $\mathfrak{mb}(4|5;3) \cong \mathfrak{mb}(4|5)$.

4d) $\mathfrak{mb}(4|5;K)$. Relations (7.3.11) mean that $\mathfrak{h}_{-3} = Span(1, 1')$. The other nonpositive components are the same as for $\mathfrak{vle}(4|3;K)$.

Statement. *Any W-regrading of* $\mathfrak{mb}(4|5)$ *is isomorphic to* $\mathfrak{mb}(4|5;1)$ *or to* $\mathfrak{mb}(4|5;K)$.

Proof is similar to that of Statement 7.3.1. □

7.5. Statement. $\mathfrak{g} = \mathfrak{vas}(4|4)$ *has no W-regradings.*

Proof. As in Theorem 2.3, realize the elements of $\mathfrak{g} = \mathfrak{vas}(4|4)$ in the form

$$D_f + c \cdot Z, \text{ where } f \in \mathfrak{sle}^{\circ}(4).$$

Suppose $\mathfrak{h} = \bigoplus_{i \geq -d} \mathfrak{h}_i$ is a W-regrading of $\mathfrak{g}$ distinct from the initial one. By Lemmas 7.1.2 and 7.1.3 we see that $\mathfrak{h}_0 \cap \mathfrak{g}_{-1} \neq 0$. Observe that if $\deg_\xi f \leq 1$, then $D_f = Le_f$. In particular, $[D_{f(u)}, D_{\xi_i}] = \frac{\partial f}{\partial u_i}$. Hence, $D_{\xi_i} \in \mathfrak{h}_-$ for all $i = 1, \dots, 4$; hence, $V = Span(D_{q_i} : i = 1, \dots, 4) \cap \mathfrak{h}_0 \neq 0$.

Lemma. $\dim V \leq 1$.

Proof. If $\dim V > 1$, then, up to linear transformations, we may assume that at least $D_{u_1}, D_{u_2} \in \mathfrak{h}_0$. But $[D_{u_i}, D_{\xi_i \xi_j}] = D_{u_j}$ and, therefore, $D_{\xi_1 \xi_i}, D_{\xi_2 \xi_j} \in \mathfrak{h}_-$ for any $i = 2, 3, 4$ and $j = 1, 3, 4$. In particular, $D_{\xi_1 \xi_4}, D_{\xi_2 \xi_3} \in \mathfrak{h}_-$. But $[D_{\xi_1 \xi_4}, D_{\xi_2 \xi_3}] = \lambda Z \in \mathfrak{h}_-$.

Since ad$Z$ determines the $\mathbb{Z}$-grading of $\mathfrak{h}$, the element Z belongs to $\mathfrak{h}_0$. The contradiction obtained completes the proof of Lemma. □

Let us continue with the proof of Statement. It remains to consider the case $\dim V = 1$. Let, for definiteness, $D_{u_1} \in \mathfrak{h}_0$, whereas $D_{u_i} \notin \mathfrak{h}_0$ for $i \neq 1$.

In principle, such regradings are possible: for example, for $\mathfrak{vas}(4|4;1)$ we set $\deg \xi_1 = 0$, $\deg u_1 = 3$; $\deg \xi_i = 2$, $\deg u_i = 1$ for $i > 1$. We see that $\mathfrak{h}_{-3} = \mathbb{C}D_{\xi_1}$; $\mathfrak{h}_{-2} = Span(D_{\xi_1 q_i}, D_{u_i};\ i > 1)$; and $\mathfrak{h}_{-1} = \mathfrak{vas}(4|4;1)_{-1}$ contains three components of the bidegree:

$$\mathfrak{h}_{-1,-1} = Span(D_{\xi_i};\ i > 1),\ \mathfrak{h}_{-1,0} = Span(D_{\xi_1 \xi_i};\ i > 1),\ \mathfrak{h}_{-1,1} = Span(D_{S^2(u_i;\ i>1) \otimes \Lambda(\xi_1)});$$

by Lemma 7.1.1 $\mathfrak{vas}(4|4;1)$ is not a W-grading.

Let $\mathfrak{h} = \bigoplus_{i \geq -d} \mathfrak{h}_i$ be a W-regrading of $\mathfrak{g} = \mathfrak{vas}(4|4)$ distinct from $\mathfrak{vas}(4|4;1)$ but such that $D_{u_1} \in \mathfrak{h}_0$. Let $D_{\xi_i} \in \mathfrak{h}_{-n_i}$ for $n_i \geq 1$. Then $D_{\xi_1 \xi_i} \in \mathfrak{h}_{-n_i}$ but

$$D_{\xi_1 \xi_i} = \xi_1 \frac{\partial}{\partial u_i} - \xi_i \frac{\partial}{\partial u_1} + \lambda(u_j \frac{\partial}{\partial \xi_k} - u_k \frac{\partial}{\partial \xi_j})$$

implying $D_{u_j} \in \mathfrak{h}_{-n_i-n_k}$. Similarly, $D_{u_iu_j} \in \mathfrak{h}_{-n_k}$. Thus, all the elements of the form D_{ξ_1}, D_{ξ_i}, and D_{u_i}, $D_{\xi_1\xi_i}$, as well as $D_{u_iu_j}$ and $D_{\xi_1u_iu_j}$ with $i > 1$ belong to $\mathfrak{h}_-$.

Observe that neither D_{ξ_1} nor D_{u_i} can belong to $\mathfrak{h}_{-1}$. But $\mathfrak{h}_{-1}$ generates the whole of $\mathfrak{h}_-$. Hence,

1) $D_{\xi_i} \in \mathfrak{g}_{-1}$ for at least one i (and then $D_{\xi_1\xi_i} \in \mathfrak{h}_{-1}$ and $D_{\xi_1\xi_i} \in \mathfrak{g}_0$); and

2) $\mathfrak{h}_{-1}$ contains at least one element from $\mathfrak{g}_{>0} = \bigoplus_{i>0} \mathfrak{g}_i$: otherwise we cannot obtain $D_{\xi_1u_iu_j} \in \mathfrak{g}_1 \cap \mathfrak{h}_-$.

By Lemma 7.1.1 $\mathfrak{h}$ is not a W-regrading. □

7.6. **W-regradings of $\mathfrak{g} = \mathfrak{kas} \subset \mathfrak{k}(1|6)$.** Clearly, the regradings of $\mathfrak{k}(1|6)$ induce the regradings of $\mathfrak{kas}$. However, due to nonsymmetry of $\mathfrak{kas}$ inside of $\mathfrak{k}(1|6)$ isomorphic regradings of $\mathfrak{k}(1|6)$ obtained by the replacement $\xi \longleftrightarrow \eta$ may produce distinct regradins of $\mathfrak{kas}$. For definiteness, fix a realization: $\mathfrak{kas} = \mathfrak{kas}^\xi$.

6a) $\mathfrak{k}(1|6;1)$ induces two regradings of $\mathfrak{kas}$ that we will shorthand as 1ξ and 1η: with the degree of one of the ξ's (resp. η's) set equal to zero, e.g.:

$$\mathfrak{kas}(;1\xi): \quad \begin{aligned} &\deg \xi_1 = 0, \deg \eta_1 = \deg t = 2,\\ &\deg \xi_i = \deg \eta_i = 1 \text{ for } i > 1.\end{aligned} \tag{7.6.1}$$

Set $\deg K_f = \deg f - 2$. We see that $\mathfrak{k}(1|6;1)_- \subset \mathfrak{kas}(;1\xi)_-$ which means that $\mathfrak{k}(1|6;1)_- = \mathfrak{kas}(;1\xi)_-$. Moreover,

$$\begin{aligned} &\mathfrak{k}(1|6;1)_0 \subset \mathfrak{k}(1|6)_{-1} \oplus \mathfrak{k}(1|6)_0 \oplus \mathfrak{k}(1|6)_1;\\ &\mathfrak{k}(1|6;1)_0 \cap \mathfrak{k}(1|6)_1 = \big(\mathbb{C}t \oplus \Lambda^2(\xi_2, \eta_2, \xi_3, \eta_3)\big) \otimes \xi_1.\end{aligned} \tag{7.6.2}$$

From the explicit description of $\mathfrak{kas}_1$ (see (3.3.1)) we derive that $K_{t\xi_1} \in \mathfrak{kas}$ and of the 6 elements of $\Lambda^2(\xi_2, \eta_2, \xi_3, \eta_3) \otimes \xi_1$ only 3 belong to $\mathfrak{kas}$; on the space they span, $\mathfrak{o}(4) \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ acts as $\mathrm{ad} \otimes \chi_0$, where χ_0 is the trivial character. Finally,

$$\begin{aligned} &\mathfrak{kas}(;1\xi)_{-1} = \mathrm{id}(\mathfrak{sl}(2)) \otimes \mathrm{id}(\mathfrak{gl}(2)) \otimes \Lambda(1);\\ &\mathfrak{kas}(;1\xi)_0 = \mathfrak{sl}(2) \oplus \mathfrak{gl}(2) \ni \mathfrak{vect}(0|1).\end{aligned} \tag{7.6.3}$$

Similarly, we prove that

$$\mathfrak{kas}(;1\xi) \cong \mathfrak{kas}(;1\eta). \tag{7.6.4}$$

Under this isomorphism the two copies of $\mathfrak{sl}(2)$ constituting $\mathfrak{o}(4)$ change places. We denote these Lie superalgebras uniformly: $\mathfrak{kas}(;1)$.

6b) $\mathfrak{g} = \mathfrak{k}(1|6;2)$ induces three regradings of $\mathfrak{kas}$ that we will shorthand as 2ξ, $1\xi + 1\eta$ and 2η. To describe them, consider, first, the nonpositive terms of $\mathfrak{k}(1|6;2)$:

-2	-1	0
$\Lambda(2)$	$V_2 \otimes \Lambda(2)$	$\mathfrak{co}(2) \otimes \Lambda(2) \ni \mathfrak{vect}(0\vert 2)$

Here $\dim V_2 = 0|2$ and the $\mathfrak{g}_0$-action in $\mathfrak{g}_{-1}$ is reducible.

In each of the cases $\mathfrak{h} = \mathfrak{kas}(;2\xi)$ and $\mathfrak{kas}(;2\eta)$ and $\mathfrak{kas}(;1\xi + 1\eta)$ their $\mathfrak{h}_{-2}$ are identical with $\mathfrak{g}_{-2}$ and $\mathfrak{h}_0$ is the subalgebra of $\mathfrak{g}_0$ preserving $\mathfrak{h}_{-1}$. So let us describe $\mathfrak{h}_{-1}$. It is obtained from $\mathfrak{g}_{-1}$ by deleting one element:

Case 2ξ. $\mathfrak{h}_{-2} = \Lambda(\xi_1, \xi_2)$,

$$\mathfrak{h}_{-1} = \xi_3 \otimes \Lambda(\xi_1, \xi_2) \bigoplus \eta_3 \otimes (1 \otimes \Lambda(\xi_1, \xi_2)) \cong \Pi(\Lambda(\xi_1, \xi_2)) \bigoplus \mathbb{C}^{2|1};$$

$$\begin{aligned} \mathfrak{h}_0 &= (t + \Phi) \otimes \Lambda(\xi_1, \xi_2) \ni Span(\eta_1, \eta_2, \xi_i\eta_j, t\xi_1, t\xi_2)\\ &\cong (t + \Phi) \otimes \Lambda(\xi_1, \xi_2) \ni \mathfrak{gl}(2|1),\end{aligned}$$

where $\Phi = \sum \xi_i \eta_i$. Recall that $\mathfrak{gl}(2|1) \cong \mathfrak{c}(\mathfrak{vect}(0|2))$. So the $\mathfrak{h}_0$-action in $\mathfrak{h}_{-1}$ is as follows: $(t+\Phi)\otimes\Lambda(\xi_1,\xi_2)$ commutes with $\mathbb{C}^{2|1}$ and acts by multiplication on $\Pi(\Lambda(\xi_1,\xi_2))$ while $\mathfrak{gl}(2|1)\cong\mathfrak{c}(\mathfrak{vect}(0|2))$ acts as $T^{1/2}\otimes\chi_0$ in the vectorial realization in $\Pi(\Lambda(\xi_1,\xi_2))$ and as $\mathrm{id}(\mathfrak{gl}(2|1))$ in $\mathbb{C}^{2|1}$. So $\mathfrak{h}_{-1}$, as $\mathfrak{h}_0$-module, is reducible and the regrading is not a W- regrading.

Cases $1\xi + 1\eta$ and 2η are similar to case 2ξ with the same outcome; they are not W- regradings.

6c) $\mathfrak{g} = \mathfrak{k}(1|6;3)$ induces four regradings of $\mathfrak{kas}$: 3ξ, $2\xi+1\eta$, $1\xi+2\eta$ and 3η. To describe them, consider the nonpositive terms of $\mathfrak{k}(1|6;3)$:

-1	0
$\Lambda(3)$	$\Lambda(3) \supplus \mathfrak{vect}(0|3)$

Consider two cases:

(i) $\mathfrak{h} = \mathfrak{kas}(;2\xi+1\eta)$ or $\mathfrak{kas}(;3\xi)$. Then $\mathfrak{h}_{-1} = \Lambda(3)$, $\mathfrak{h}_0 = \Lambda(3)\supplus\mathfrak{sl}(1|3)$; here $\mathfrak{sl}(1|3)$ is considered as the subalgebra of $\mathfrak{vect}(0|3)$.

(ii) $\mathfrak{h} = \mathfrak{kas}(;1\xi+2\eta)$ or $\mathfrak{kas}(;3\eta)$. Then $\mathfrak{h}_{-1} = Vol_0(0|3)$, $\mathfrak{h}_0 = \mathfrak{c}(\mathfrak{vect}(0|3))$.

Statement. *All the W-regradings of* $\mathfrak{kas}$ *are isomorphic to one of the following:* $\mathfrak{kas}(;1)$*,* $\mathfrak{kas}(;3\xi)$ *or* $\mathfrak{kas}(;3\eta)$*.*

Proof is similar to that of Statement 7.3.1. □

7.7. **W-regradings of $\mathfrak{g} = \mathfrak{ksle}(9|6)$.** Let ξ_i, q_i be the standard indeterminates of $\mathfrak{sle}^\circ(4)$; $\mathfrak{g}_{-2} = \mathbb{C}\cdot c$.

7a) $\mathfrak{g} = \mathfrak{ksle}(9|6;1)$ is determined by the formulas

$$\deg\xi_1 = 0;\ \ \deg q_1 = 2;\ \ \deg\xi_i = \deg q_i = 1 \text{ for } i>1.$$

Then $\mathfrak{h}_{-2} = \mathbb{C}\cdot\xi_1$ and $\mathfrak{h}_{-1} = Span(\xi_i, \xi_1 q_i, q_i, \xi_1\xi_i \text{ for } i>1; c, (\xi_2\xi_3\xi_4)^*)$, where $(\xi_2\xi_3\xi_4)^*$ is the dual to $\xi_2\xi_3\xi_4 \in \mathfrak{g}_1$.

It is not difficult to see that the bracket determines a nondegenerate even skew 2-form on $\mathfrak{h}_{-1}$; clearly, $\dim\mathfrak{h}_{-1} = 8|6$. This means that $\mathfrak{ksle}(9|6;1)\cong\mathfrak{ksle}(9|6)$ as $\mathbb{Z}$-graded Lie superalgebras.

Denote this isomorphism by RG. Let $\mathfrak{h} = \tilde{c}(\mathfrak{sle}^\circ(4;4)) \subset \mathfrak{ksle}(9|6)$; sec. 6.5. Then $RG(\mathfrak{h}) \subset \mathfrak{ksle}(9|6)$ is isomorphic to $\mathfrak{h}$ as an abstract superalgebra, but not as $\mathbb{Z}$-graded one. We have

$$\begin{aligned} \mathfrak{h}\cap RG(\mathfrak{h}) &= \mathbb{C}\cdot c + (\mathfrak{sb}^\circ(\xi_2,\xi_3,\xi_4)\otimes\Lambda(\xi_1))/\mathbb{C}\cdot 1 \supplus \mathbb{C}\cdot\partial_{\xi_1} \\ &\cong \tilde{\mathfrak{c}}(\mathfrak{sb}^\circ(3)\otimes\Lambda(1))/\mathbb{C}\cdot 1)\supplus\mathfrak{svect}(1). \end{aligned} \tag{7.7.1}$$

The regrading RG determines a nontrivial automorphism of $\mathfrak{h}\cap RG(\mathfrak{h})$, generated by the automorphism R of $\mathfrak{sle}^\circ(3)$ (sec. 4.3) and permuting the central elements c and ξ_1.

The nondirect sum of the linear spaces $\mathfrak{h}$ and $RG(\mathfrak{h})$ spans the whole of $\mathfrak{ksle}(9|6)$:

$$\mathfrak{h} + RG(\mathfrak{h}) = \mathfrak{ksle}(9|6).$$

7b) $\mathfrak{h} = \mathfrak{ksle}(9|6;2)$ is determined by the formulas

$$\deg\xi_1 = \deg\xi_2 = \deg q_1 = \deg q_2 = 1;\ \ \deg\xi_3 = \deg\xi_4 = 0;\ \ \deg q_3 = \deg q_4 = 2.$$

Then $\mathfrak{h}_{-2} = Span(c,\xi_3,\xi_4,\xi_3\xi_4)$ and $\dim\mathfrak{h}_{-2} = 3|1$.

$$\begin{aligned} &\mathfrak{h}_{-1} = Span(\xi_1,\xi_2,q_1,q_2)\otimes\Lambda(\xi_3,\xi_4)\cong V_2\otimes\Lambda(3); \\ &\mathfrak{h}_0 \cong (\mathfrak{sl}(2)\otimes\Lambda(3))\supplus\mathfrak{sl}(3|1), \end{aligned} \tag{7.7.2}$$

where $\mathfrak{sl}(3|1) \subset \mathfrak{vect}(0|3)$. The $\mathfrak{h}_0$-action in $\mathfrak{h}_{-1}$ is the natural one.

7c) $\mathfrak{g} = \mathfrak{ksle}(9|6; K)$. Set

$$\deg q_i = 2; \quad \deg \xi_i = 1 \text{ for all } i.$$

Then we obtain the compatible grading

$$\mathfrak{h}_{-2} = \mathrm{id}, \ \ \mathfrak{h}_{-2} = \Lambda^2(\mathrm{id}), \ \mathfrak{h}_0 = \mathfrak{sl}(5). \tag{7.7.3}$$

Statement. *Any W-regrading of* $\mathfrak{ksle}(9|6)$ *is isomorphic to* $\mathfrak{ksle}(9|6; 2)$ *or to* $\mathfrak{ksle}(9|6; K)$

Proof is similar to that of Statement 7.3.1. □

Appendix 1. The solution of the system (2.1)–(2.4)

Let $D = \sum_{1\leq i\leq 4}(P_i\frac{\partial}{\partial\xi_i} + Q_i\frac{\partial}{\partial u_i}) \in \mathfrak{g}^\lambda$ be an homogeneous (with respect to parity) vector field. Then by Lemma 2.1 it satisfies the system (2.1)–(2.4).

Equations (2.3) imply that there exists a function $f = f(u, \xi)$ such that $Q_i = -(-1)^{p(D)}\frac{\partial f}{\partial \xi_i}$. Equations (2.1) imply further that

$$P_i = \frac{\partial f}{\partial u_i} + \varphi_i(u; \xi_i) = \frac{\partial f}{\partial u_i} + \varphi_i{}^\circ(u) + \varphi_i^1(u)\cdot\xi_i$$

or, in other words, $D = Le_f + \sum(\varphi_i{}^\circ(u) + \varphi_i^1(u)\xi_i)\frac{\partial}{\partial\xi_i}$. Equations (2.2) now imply that

$$\varphi_i^1(u) = \frac{\partial\varphi_i}{\partial\xi_i} = \frac{1}{2}(-1)^{p(D)}\sum\frac{\partial Q_j}{\partial u_j} = -\frac{1}{2}\Delta(f), \ \text{ where } \Delta = \sum\frac{\partial^2}{\partial u_i\partial\xi_i}; \tag{2.2$'$}$$

whereas equations (2.4) imply that

$$\frac{\partial\varphi_i}{\partial u_j} - \frac{\partial\varphi_j}{\partial u_i} = -2\lambda\frac{\partial^2 f}{\partial\xi_l\partial\xi_k}. \tag{2.4$'$}$$

Remark. Let $\psi_i = \psi_i(u)$, where $i = 1, \dots, 4$, be a set of functions such that $\frac{\partial\psi_i}{\partial u_j} - \frac{\partial\psi_j}{\partial u_i} = 0$. Then there exists a function $\psi(u)$ for which $\psi_i = \frac{\partial\psi}{\partial u_i}$ and $\sum\psi_i(u)\frac{\partial}{\partial\xi_i} = Le_\psi = D_\psi$ (see heading 2) from the corollary in Theorem 2.3). Thus, for any function f it suffices to find any collection of functions $\varphi_i{}^\circ$.

With the help of the differential forms

$$\alpha = \sum_{i\leq 4}\varphi_i du_i \text{ and } \omega_0(f) = \sum_{(i,j,k,l)\in A_4}\frac{\partial^2 f}{\partial\xi_l\partial\xi_k}du_j\wedge du_i$$

equations (2.2$'$) and (2.4$'$) can be expressed in the form

$$d\alpha = \frac{1}{2}\Delta(f)\cdot\omega - 2\lambda\cdot\omega_0(f). \tag{A.1}$$

Equation (A.1) is solvable if and only if the form in the right hand side is exact or, since our considerations are local, equivalently, if and only if it is closed.

Direct calculations show that the condition that the form in the right hand side is closed is equivalent to the system

$$\begin{cases}\frac{\partial f}{\partial\xi_i} = 0 \quad \text{for all } i = 1,2,3,4 & \text{(A.1.1)}\\ \frac{1}{2}\frac{\partial\Delta f}{\partial u_i} - 2\lambda\frac{\partial^3 f}{\partial\xi_j\partial\xi_k\partial\xi_l} = 0 \quad \text{for all } i = 1,2,3,4 \text{ and } (i,j,k,l)\in A_4 & \text{(A.1.2)}\end{cases}$$

Equations (A.1.1) imply that $\Delta(f)$ only depends on u; equations (A.1.2) imply that $\deg_\xi f \leq 3$.

First, suppose that $\deg_\xi f \leq 2$. Then equations (A.1.2) imply that

$$\Delta(f) = const. \tag{A.2}$$

Denote $-\frac{1}{4}\Delta(f)$ by c. Thus, $f = -c\sum_{i\leq 4} u_i\xi_i + f_0$, where $\Delta(f_0) = 0$. By (2.2′) then, $\varphi_i^1(u) = 2c$ and $D = Le_{-c\sum u_i\xi_i + f_0} + \sum \varphi_i{}^\circ \frac{\partial}{\partial\xi_i} + 2c\sum \xi_i \frac{\partial}{\partial\xi_i} = Le_{f_0} + \sum \varphi_i{}^\circ \frac{\partial}{\partial\xi_i} + cZ$. Replace f with f_0. Then $\Delta(f) = 0$ and we have

$$\frac{\partial\varphi_i{}^\circ}{\partial u_j} - \frac{\partial\varphi_j{}^\circ}{\partial u_i} = -2\lambda\frac{\partial^2 f}{\partial\xi_l\partial\xi_k}. \tag{2.4″}$$

If $\deg_\xi f < 2$, then the right hand side of (2.4″) is equal to 0. So due to Remark we can take $\varphi_i{}^\circ = 0$. In this case

$$D = Le_f + cZ = D_f + cZ. \tag{A.3}$$

Let $\deg_\xi f = 2$. As $\Delta(f) = 0$, we can set $g = \Delta^{-1}(f)$. Then $\deg_\xi g = 3$ and $f = \Delta(g)$. For the role of functions $\varphi_i{}^\circ$ satisfying equation (2.4″) we can take

$$\varphi_i{}^\circ = 2\lambda\frac{\partial^3 g}{\partial\xi_j\partial\xi_k\partial\xi_l}, \text{ where } (i,j,k,l) \in A_4.$$

We get

$$D_f = \mathrm{Le}_f + 2\lambda \sum_{(i,j,k,l)\in A_4;\ 1\leq i\leq 4} \frac{\partial^3 g}{\partial\xi_j\partial\xi_k\partial\xi_l}\cdot\frac{\partial}{\partial\xi_i} + cZ = D_f + cZ. \tag{A.4}$$

Let $\deg_\xi f = 3$. Let us represent f in the form $f = f_3 + f_{<3}$, where f_3 is a homogeneous (with respect to the degree in ξ) polynomial of degree 3, while $f_{<3}$ is a polynomial of lesser degree.

Since $\Delta(f)$ only depends on u, we see that $\Delta(f_3) = 0$ and, therefore, we can introduce $H = \Delta^{-1}(f_3) = F(u)\xi_1\xi_2\xi_3\xi_4$ for some function $F(u)$.

From (2.4″) we deduce that

$$\frac{\partial\varphi_i^1}{\partial u_j} = 2\lambda\frac{\partial F}{\partial u_j} \text{ or, with (2.2′), } \varphi_i^1 = 2\lambda F = -\frac{1}{2}\Delta(f).$$

Therefore, $\Delta(f) = \Delta(f_{<3}) = -4\lambda F$. Set

$$\hat{f} = 4\Delta^{-1}(F).$$

We see that $f = \Delta(H) - \lambda\hat{f} + g$ for some function g such that $\Delta(g) = 0$ and $\deg_\xi g < 3$. But we have already described the solutions for all such g. So now we assume $g = 0$. In this case we can take $\varphi_i{}^\circ = 0$ (due to Remark). We get

$$\begin{aligned} D &= \mathrm{Le}_f + \lambda(-\mathrm{Le}_{\hat{f}} + 2F\sum \xi_i\frac{\partial}{\partial\xi_i}) \\ &= \mathrm{Le}_f + \lambda\left(-\mathrm{Le}_{\hat{f}} + 2\sum_{(i,j,k,l)\in A_4;\ 1\leq i\leq 4}\frac{\partial^3 H}{\partial\xi_j\partial\xi_k\partial\xi_l}\cdot\frac{\partial}{\partial\xi_i}\right) = D_f. \end{aligned} \tag{A.5}$$

Formulas (A.3)–(A.5) prove Theorem 2.3.

Appendix 2. Proof of headings 2 of Theorems 4.2, 6.2 and 6.5

A.2.1. Lemma. *Let* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ *be simple; let the* $\mathfrak{g}_0$*-module* $\mathfrak{g}_{-1}$ *be irreducible. If* $\mathfrak{h} = (\mathfrak{g}_{-1}, \mathfrak{c}\mathfrak{g}_0)_*$ *is also simple, then for every* $v \in \mathfrak{g}_{-1}$ *there exists an* $F \in \mathfrak{g}_1$ *such that* $[v, F] \notin \mathfrak{g}_0$.

The same applies to $(\mathfrak{g}_-, \mathfrak{g}_0)_*^{mk}$ *and* $\mathfrak{h} = (\mathfrak{g}_-, \mathfrak{c}\mathfrak{g}_0)_*^{mk}$.

Proof. By simplicity of $(\mathfrak{c}\mathfrak{g}_0)_*$ (due to Lemma 4.1) we have $[\mathfrak{h}_{-1}, \mathfrak{h}_1] = c\mathfrak{g}_0$, i.e., there exists $v_0 \in \mathfrak{g}_{-1}, F_0 \in \mathfrak{g}_1$ such that $[v_0, F_0] \notin \mathfrak{g}_0$.

Let

$$V_1 = \{v_1 \in \mathfrak{g}_{-1} : [g, v_1] = v_0 \text{ for some } g \in \mathfrak{g}_0\}.$$

Then for any $v_1 \in V_1$ we have

$$\mathfrak{g}_0 \not\ni [[g, v_1], F_0] = \pm[[g, F_0], v_1] \pm [g, [v_1, F_0]],$$

where the signs are governed by Sign Rule. Therefore, one of the two cases holds:

1) $[v_1, F_0] \notin \mathfrak{g}_0$, hence, $F = F_0$;

2) $[v_1, F_0] \in \mathfrak{g}_0$.

In case 2) we have $[g, [v_1, F_0]] \in \mathfrak{g}_0$; hence,

$$[F, v_1] \notin \mathfrak{g}_0 \text{ for } F = [g, F_0].$$

Similarly, introduce the sequence of spaces

$$V_2 = \{v_2 \in \mathfrak{g}_{-1} : [g, v_2] \in V_1 + < Span(v_0) \text{ for some } g \in \mathfrak{g}_0\}, \text{ etc.}$$

By irreducibility of $\mathfrak{g}_0$-action on $\mathfrak{g}_{-1}$ for every $v \in \mathfrak{g}_{-1}$ there exists an n such that $v \in V_n$ and, therefore, $F = F_n$, where $[v, F_n] \notin \mathfrak{g}_0$.

The arguments for depth 2 are literally the same. □

A.2.2. Corollary. *Let* $\mathfrak{g}_{-1} = \Pi(\Lambda(n)/\mathbb{C}1), \mathfrak{g}_0 = \mathfrak{vect}(0|n)$*, i.e.,* $\mathfrak{g}_* = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{le}(n; n)$*. Then* $(\mathfrak{g}_{-1}, \mathfrak{c}(\mathfrak{g}_0))_*$ *is not simple for* $n > 3$.

Hereafter in this appendix we often abuse the notations and denote the elements by their generating functions.

Proof. By Lemma A.2.1 the simplicity of $\mathfrak{g}_*$ implies that for any $v \in \mathfrak{g}_{-1}$ there exists $F \in \mathfrak{g}_{-1}$ such that $[v, F] \notin \mathfrak{g}_0$.

Take $v = \xi_1 \dots \xi_n$; let d be the central element of $\mathfrak{c}(\mathfrak{g}_0)$ normalized so that $\operatorname{ad} d|_{\mathfrak{g}_{-1}} = -\operatorname{id}$. Let $F \in \mathfrak{g}_1$ be such that

$$[v, F] = d + g, \quad \text{for } g \in \mathfrak{g}_0.$$

Then

$$\pm[v, [F, v_1]] \overset{\text{by Jacobi id.}}{=} [[v, F], v_1] = (d + g)v_1 = -v_1 + gv_1. \tag{A.2.0}$$

In other words, $g_1 = [F, v_1]$ maps v to $-v_1 + gv_1$ up to a sign. But in the $\mathfrak{g}_0$-module considered, the image of v can only be a function of degree $\geq n - 1$.

Hence, $gv_1 = v_1 + \varphi(v_1)$, where $\deg \varphi \geq n - 1$, for any v_1 of degree $< n - 1$. Consequently, the projection g_0 of g on the zeroth component of $\mathfrak{vect}(0|n)$ with respect to the standard $\mathbb{Z}$-grading, i.e., on $\mathfrak{gl}(n)$, satisfies the condition

$$g_0|_{Span(v_1 \colon \deg v_1 < n-1)} = \text{id}.$$

But in $\mathfrak{vect}(0|n)$ the dimension of the maximal torus $Span(\varepsilon_i \partial_i : 1 \leq i \leq n)$ is equal to n and there is no operator whose restrictions to the spaces of homogenuous functions in ξ of at least two distinct degrees are scalar operators.

Since $n-2\geq 2$ for $n>3$, the Lie superalgebra $(\mathfrak{g}_{-1}, \mathfrak{cg}_0)_*$ is not simple. □

A.2.3. Corollary. *Let*

$$\mathfrak{g}_- = \begin{cases} \mathfrak{ab}(2^{n-1}) & \text{for } n \text{ odd,}\\ \mathfrak{hei}(2^{n-1}|2^{n-1}) & \text{for } n \text{ even.}\end{cases}$$

The Lie superalgebra $(\mathfrak{g}_-, \mathfrak{cvect}(0|n))_*^{mk}$ *is not simple for* $n>3$.

Proof follows the lines of the proof of Corollary A.2.1 with the correction that (A.2.0) is now true not for all $v_1\in\mathfrak{g}_{-1}$ but only for those which satisfy $[v_1, v]=0$. Such elements v_1 are represented by functions $f\in\Lambda(n)$ such that $0<\deg f\leq n-1$. There are ≥ 2 distinct degrees which satisfy this inequality for $n>3$. □

A.2.4. Corollary. *Let*

$$\mathfrak{g}_- = \begin{cases} \mathfrak{ab}(2^{n-1}-1) & \text{for } n \text{ odd,}\\ \mathfrak{hei}(2^{n-1}|2^{n-1}-2) & \text{for } n \text{ even.}\end{cases}$$

Then $\mathfrak{g} = (\mathfrak{g}_-, \mathfrak{svect}_{a,b}(0|n))_*^{mk}$ *is not a simple Lie superalgebra if either* $n>4$ *or* $n=4$ *and* $(a,b)\notin\mathbb{C}(3,4)$.

Proof is obtained by a slight modification of the proof of Corollary A.2.2. As v we now take $\xi_1\dots\xi_{n-1}\in\mathfrak{g}_{-1}$; let F be such that $[v,F]=ax+bd+g$, where $g\in\mathfrak{svect}(0|n)$. Then

$$[[F, v_1], v] = \pm[[v, F], v_1] = (ak-b)v_1 + gv_1 \tag{A.2.1}$$

for any monomial $v_1\in\mathfrak{g}_{-1}$ of degree k and distinct from ξ_n. Since every element from $\mathfrak{g}_0$ lowers the degree of any monomial not more than by 1, we see that the projection g_0 of g on $\mathfrak{svect}(0|n)_0$ satisfies the relation

$$g_0v_1 = (b-ak)v_1 \tag{A.2.2}$$

for any monomial $v_1\in\mathfrak{g}_{-1}$ of degree $k<n-2$ and distinct from ξ_n. In particular, for $n>4$ this means that g_0 acts on $Span(\xi_1,\dots,\xi_{n-1})$ by multiplication by $b-a$ and on $\Lambda^2(\xi)$ by multiplication by $b-2a$. Hence, $g_0=0$, i.e., $a=b=0$.

In A.2.2 $k<n-2$. So if $n=4$, then $k=1$. The component g_0 is defined by its action on ξ_1, ... , ξ_n. But A.2.2 gives the action of g_0 only on ξ_1, ... , ξ_{n-1}. Its action on ξ_n can be arbitrary with only one condition: $g_0\in\mathfrak{svect}(0|4)$; this is what A.2.3 means:

$$g_0(\xi_1\xi_4) = -2(b-a)\xi_1\xi_4 + c_1\xi_1\xi_2 + c_2\xi_1\xi_3. \tag{A.2.3}$$

Look at formula (A.2.1) with $v=\xi_1\xi_2\xi_3$ and $v_1=\xi_1\xi_4$. It means that $ad[F,v_1]$ (which is an element from $\mathfrak{svect}(0|4)\ni\mathbb{C}(ax+bd)$) sends $\xi_1\xi_2\xi_3$ to $(2a-b)\xi_1\xi_4+g(\xi_1\xi_4)$. Since no vector field can send v to v_1, we deduce that $g_0(v_1)$ must compensate $(2a-b)\xi_1\xi_4$. But from formula (A.2.3) we derive that $b-2a=-2b+2a$, implying $3b=4a$. □

Due to Lemmas 4.1, 6.1, Corollaries A.2.2–A.2.4 are equivalent to the headings 2) of Theorems 4.2, 6.2 and 6.5, respectively.

Appendix 3. Proof of simplicity of the Lie superalgebras $\mathfrak{mb}(4|5) = (\mathfrak{ab}(4), \mathfrak{cvect}(0|3))_*^m$ and $\mathfrak{ksle}(9|6) = (\mathfrak{hei}(8|6), \mathfrak{svect}(4)_{3,4})_*^k$

In this appendix $\mathfrak{g}$ is either $\mathfrak{mb}(4|5)$ or $\mathfrak{ksle}(9|6)$. Due to Lemma 6.1, to prove the simplicity of $\mathfrak{g}$ it suffices to exhibit an element $\hat F\in\mathfrak{g}_1$ such that

(A.3.1) $[\mathfrak{g}_{-1}, \hat F]$ is not entirely contained in $\mathfrak{vect}(0|3)$ and $\mathfrak{svect}(0|3)$, respectively.

A.3.1. *Simplicity of* $(\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^m_*$. First, let us show how to embed $\mathfrak{g} = \mathfrak{cvect}(0|3)^m_*$ into $\mathfrak{m}(4)$. We consider $\mathfrak{m}(4)$ as preserving the Pfaff equation given by the form $\alpha_0 = d\tau + \sum_{i=0}^{3}(\eta_i du_i + u_i d\eta_i)$. Denote the basis elements of $\mathfrak{g}$ as follows:

$\mathfrak{g}_{-2}$	A basis of $\mathfrak{g}_{-1} = \Pi(Vol^{\frac{1}{2}})$	notations of the corres. functions that generate $\mathfrak{g}_{-1} \subset \mathfrak{m}(4)$
	$\xi_1\xi_2\xi_3$	η_0
M_1	$\xi_2\xi_3,\quad \xi_3\xi_1,\quad \xi_1\xi_2$	$u_1,\quad u_2,\quad u_3$
	$\xi_1,\quad \xi_2,\quad \xi_3$	$\eta_1,\quad \eta_2,\quad \eta_3$
	1	u_0

The following is an explicit realization of the embedding $i : \mathfrak{g}_0 = \mathfrak{vect}(0|3) \longrightarrow \mathfrak{le}(3)$. We only indicate the generating functions of the image:

$\deg D$	$D \in \mathfrak{vect}(0|3)$
-1	$\partial_1,\quad \partial_2,\quad \partial_3$
0	$\xi_i\partial_j$ for $i \neq j$
0	$\xi_1\partial_1,\quad \xi_2\partial_2,\quad \xi_3\partial_3$
1	$\xi_2\xi_3\partial_1,\quad \xi_3\xi_1\partial_2,\quad \xi_1\xi_2\partial_3$
1	$\xi_1(\xi_2\partial_2 - \xi_3\partial_3),\quad \xi_2(\xi_3\partial_3 - \xi_1\partial_1),\quad \xi_3(\xi_1\partial_1 - \xi_2\partial_2)$
1	$\xi_1(\xi_2\partial_2 + \xi_3\partial_3),\quad \xi_2(\xi_3\partial_3 + \xi_1\partial_1),\quad \xi_3(\xi_1\partial_1 + \xi_2\partial_2)$
2	$\xi_1\xi_2\xi_3\partial_i$

The respective images $i(D)$ are as follows:

$-u_0u_1 + \eta_2\eta_3,\quad -u_0u_2 + \eta_3\eta_1,\quad -u_0u_3 + \eta_1\eta_2$
$-u_i\eta_j$ for $i \neq j$; $i, j > 0$
$\frac{1}{2}(-u_0\eta_0 - u_1\eta_1 + u_2\eta_2 + u_3\eta_3)$
$\frac{1}{2}(-u_0\eta_0 + u_1\eta_1 - u_2\eta_2 + u_3\eta_3)$
$\frac{1}{2}(-u_0\eta_0 + u_1\eta_1 + u_2\eta_2 - u_3\eta_3)$
$-\frac{1}{2}u_1^2,\quad -\frac{1}{2}u_2^2,\quad -\frac{1}{2}u_3^2$
$-u_2u_3,\quad -u_1u_3,\quad -u_1u_2$
$\eta_0\eta_1,\quad \eta_0\eta_2,\quad \eta_0\eta_3$
$-\frac{1}{2}u_i\eta_0$

To check condition (A.3.1), take

$$\hat{F} = M_F, \text{ where } F = 2\tau u_1 - 2\eta_0\eta_2\eta_3 + u_0^2\eta_1.$$

Then the brackets with $\mathfrak{g}_{-1}$ are

$$\begin{array}{lll} & \{F, u_0\}_{m.b.} = -2u_0u_1 + 2\eta_2\eta_3; & \{F, \eta_1\}_{m.b.} = 2\tau; \\ \text{(A.3.2)} & \{F, u_1\}_{m.b.} = -3u_1^2; & \{F, \eta_i\}_{m.b.} = -2u_1\eta_i \ (i = 0, 2, 3); \\ & \{F, u_2\}_{m.b.} = -2u_1u_2 - 2\eta_0\eta_3; & \{F, u_3\}_{m.b.} = -2u_1u_3 + 2\eta_0\eta_2. \end{array}$$

We get M_τ, while the remaining elements in the right hand sides of (A.3.2) lie in $\mathfrak{vect}(0|3)$. $\square$

A.3.2. *Simplicity of* $(\mathfrak{hei}(8|6), \mathfrak{svect}(0|4)_{3,4})^k_*$. We first embed $(\mathfrak{hei}(8|6), \mathfrak{svect}(0|4))^k_*$ into $\mathfrak{k}(9|6)$. We realize $\mathfrak{k}(9|6)$ as preserving the Pfaff equation given by the form

$$\alpha_1 = dt - \sum_{i\leq 4}(p_i dq_i - dq_i p_i) - \sum_{j\leq 3}(\eta_j d\xi_j + \xi_j d\eta_j).$$

Let us redenote the basis elements of $\mathfrak{g}_{-1}$:

A basis of $\mathfrak{g}_{-1}$ $= \Pi(T^0_0(\vec{0}))$	notations of the corresp. functions that generate $\mathfrak{g}_{-1} \subset \mathfrak{k}(9\|6)$
$\xi_1, \quad \xi_2, \quad \xi_3, \quad \xi_4$	$p_1, \quad p_2, \quad p_3, \quad p_4$
$\xi_1\xi_2, \quad \xi_1\xi_3, \quad \xi_1\xi_4$	$\eta_1, \quad \eta_2, \quad \eta_3$
$-\xi_3\xi_4, \quad \xi_2\xi_4, \quad -\xi_2\xi_3$	$\zeta_1, \quad \zeta_2, \quad \zeta_3$
$\xi_2\xi_3\xi_4, \quad -\xi_1\xi_3\xi_4, \quad \xi_1\xi_2\xi_4, \quad -\xi_1\xi_2\xi_3$	$q_1, \quad q_2, \quad q_3, \quad q_4$

The following is an explicit realization of the embedding $i : \mathfrak{g}_0 = \mathfrak{svect}(0|4) \longrightarrow \mathfrak{h}(8|6)_0$. We only indicate the generating functions of the image. For $D \in \mathfrak{svect}(0|4)$ we have

<u>$\deg D = -1:$</u>

$$\begin{array}{lcl}
\partial_{\xi_1} & \mapsto & \zeta_1 p_2 + \zeta_2 p_3 + \zeta_3 p_4 \\
\partial_{\xi_2} & \mapsto & -\zeta_1 p_1 + \eta_2 p_4 - \eta_3 p_3 \\
\partial_{\xi_3} & \mapsto & -\zeta_2 p_1 + \eta_3 p_2 - \eta_1 p_4 \\
\partial_{\xi_4} & \mapsto & -\zeta_3 p_1 - \eta_2 p_2 + \eta_1 p_3
\end{array}$$

<u>$\deg D = 0:$</u>

$$\begin{array}{llll}
\xi_1\partial_2 & \mapsto -p_1q_2 - \eta_2\eta_3 & \xi_2\partial_3 & \mapsto -p_2q_3 + \eta_1\zeta_2 \\
\xi_2\partial_1 & \mapsto -p_2q_1 + \zeta_2\zeta_3 & \xi_3\partial_2 & \mapsto -p_3q_2 + \eta_2\zeta_1 \\
\xi_1\partial_3 & \mapsto -p_1q_3 + \eta_1\eta_3 & \xi_2\partial_4 & \mapsto -p_2q_4 + \eta_1\zeta_3 \\
\xi_3\partial_1 & \mapsto -p_3q_1 - \zeta_1\zeta_3 & \xi_4\partial_2 & \mapsto -p_4q_2 + \eta_3\zeta_1 \\
\xi_1\partial_4 & \mapsto -p_1q_4 - \eta_1\eta_2 & \xi_3\partial_4 & \mapsto -p_3q_4 + \eta_2\zeta_3 \\
\xi_4\partial_1 & \mapsto -p_4q_1 + \zeta_1\zeta_2 & \xi_4\partial_3 & \mapsto -p_4q_3 + \eta_3\zeta_2
\end{array}$$

$$\begin{array}{lcl}
\xi_1\partial_1 - \xi_2\partial_2 & \mapsto & -p_1q_1 + p_2q_2 + \eta_2\zeta_2 + \eta_3\zeta_3 \\
\xi_2\partial_2 - \xi_3\partial_3 & \mapsto & -p_2q_2 + p_3q_3 + \eta_1\zeta_1 - \eta_2\zeta_2 \\
\xi_3\partial_3 - \xi_4\partial_4 & \mapsto & -p_3q_3 + p_4q_4 + \eta_2\zeta_2 - \eta_3\zeta_3 \\
\sum \xi_i\partial_i & \mapsto & -\sum p_iq_i - 2t
\end{array} \tag{A.3.3}$$

<u>$\deg D = 1:$</u>

$$\xi_1\xi_2\partial_3 \mapsto -\eta_1 q_3, \ \xi_2\xi_3\partial_1 \mapsto -\eta_3 q_1, \ \xi_3\xi_1\partial_2 \mapsto -\eta_2 q_2$$

$$\begin{array}{llll}
\xi_1\xi_2\partial_1 + \xi_2\xi_3\partial_3 & \mapsto -q_1\eta_1 + q_3\zeta_3 & \xi_1\xi_4\partial_1 + \xi_4\xi_2\partial_2 & \mapsto -q_1\eta_3 + q_2\zeta_2 \\
\xi_1\xi_2\partial_1 + \xi_2\xi_4\partial_4 & \mapsto -q_1\eta_1 - q_4\zeta_2 & \xi_1\xi_4\partial_1 + \xi_4\xi_3\partial_3 & \mapsto -q_1\eta_3 - q_3\zeta_1 \\
\xi_1\xi_3\partial_1 + \xi_3\xi_2\partial_2 & \mapsto -q_1\eta_2 - q_2\zeta_3 & \xi_1\xi_2\partial_2 - \xi_1\xi_3\partial_3 & \mapsto -q_2\eta_1 + q_3\eta_2 \\
\xi_1\xi_3\partial_1 + \xi_3\xi_4\partial_4 & \mapsto -q_1\eta_2 + q_4\zeta_1 & \xi_1\xi_2\partial_2 - \xi_1\xi_4\partial_4 & \mapsto -q_2\eta_1 + q_4\eta_3
\end{array}$$

<u>$\deg D = 2:$</u> The image under i is generated by q_iq_j for any $1 \leq i, j \leq 4$; it is inessential to us since $\mathfrak{svect}(0|4)_2$ is generated by $\mathfrak{svect}(0|4)_1$.

Now, set

$$x_0 = K_{-\sum p_i q_i - 2t}, \qquad x_1 = K_{-p_1q_1+p_2q_2+\eta_2\zeta_2+\eta_3\zeta_3},$$
$$x_2 = K_{-p_2q_2+p_3q_3+\eta_1\zeta_1-\eta_2\zeta_2}, \quad x_3 = K_{-p_3q_3+p_4q_4+\eta_2\zeta_2-\eta_3\zeta_3};$$

see (A.3.3). Set

$$f = t + \sum_{i\leq 3} p_iq_i + 3p_4q_4 + \eta_1\zeta_1 + \eta_2\zeta_2 - \eta_3\zeta_3.$$

Then

$$K_f = \frac{1}{2}x_1 + x_2 + \frac{3}{2}x_3 - \frac{3}{2}x_0 - 2K_t \in \mathfrak{svect}(0|4) \ni \mathbb{C}(3x_0 + 4K_t).$$

To check the condition (A.3), take $\hat{F} = K_F$, where

$$F = tp_4 + p_4(\sum_{i\leq 4} p_iq_i + \eta_1\zeta_1 + \eta_2\zeta_2 - \eta_3\zeta_3) - 2\zeta_1\zeta_2p_1 + 2\zeta_1\eta_3p_2 + 2\zeta_2\eta_3p_3.$$

The commutators of F with $\mathfrak{k}_{-1}(9|6)$ are of the form:

$$\{q_i, F\}_{k.b.} = q_i\tfrac{\partial F}{\partial t} + \tfrac{\partial F}{\partial p_i}; \qquad \{\eta_i, F\}_{k.b.} = \eta_i\tfrac{\partial F}{\partial t} - \tfrac{\partial F}{\partial \zeta_i};$$
$$\{p_i, F\}_{k.b.} = p_i\tfrac{\partial F}{\partial t} - \tfrac{\partial F}{\partial q_i}; \qquad \{\zeta_i, F\}_{k.b.} = \zeta_i\tfrac{\partial F}{\partial t} - \tfrac{\partial F}{\partial \eta_i}.$$

Hence,

$$\begin{aligned}
&\{q_4, F\}_{k.b.} = f;\\
&\{\eta_1, F\}_{k.b.} = 2(\eta_1p_4 + \zeta_2p_1 - \eta_3p_2) \mapsto -2\partial_3;\\
&\{\eta_2, F\}_{k.b.} = 2(\eta_2p_4 - \zeta_1p_1 - \eta_3p_3) \mapsto 2\partial_2;\\
&\{\eta_3, F\}_{k.b.} = \{\zeta_1, F\}_{k.b.} = \{\zeta_2, F\}_{k.b.} = 0;\\
&\{\zeta_3, F\}_{k.b.} = 2(\zeta_3p_4 + \zeta_1p_2 + \zeta_2p_3) \mapsto 2\partial_1;\\
&\{q_1, F\}_{k.b.} = 2(q_1p_4 - \zeta_1\zeta_2) \mapsto -2\xi_4\partial_1;\\
&\{q_2, F\}_{k.b.} = 2(q_2p_4 + \zeta_1\eta_3) \mapsto -2\xi_4\partial_2;\\
&\{q_3, F\}_{k.b.} = 2(q_3p_4 + \zeta_2\eta_3) \mapsto -2\xi_4\partial_3;\\
&\{p_i, F\}_{k.b.} = 0 \quad \text{for } i = 1, 2, 3, 4.
\end{aligned}$$

So we get K_f, while the remaining brackets lie in $\mathfrak{svect}(0|3)$. □

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