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# THE HOWE DUALITY AND THE PROJECTIVE REPRESENTATIONS OF SYMMETRIC GROUPS

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ABSTRACT. The symmetric group $\mathfrak{S}_k$ possesses a nontrivial central extension, whose irreducible representations, different from the irreducible representations of $\mathfrak{S}_k$ itself, coincide with the irreducible representations of the algebra $\mathfrak{A}_k$ generated by indeterminates $\tau_{i,j}$ for $i \neq j$, $1 \leq i, j \leq n$ subject to the relations

$$\tau_{i,j} = -\tau_{j,i}, \quad \tau_{i,j}^2 = 1, \quad \tau_{i,j}\tau_{m,l} = -\tau_{m,l}\tau_{i,j} \text{ if } \{i,j\} \cap \{m,l\} = \emptyset;$$
$$\tau_{i,j}\tau_{j,m}\tau_{i,j} = \tau_{j,m}\tau_{i,j}\tau_{j,m} = -\tau_{i,m} \quad \text{for any } i, j, l, m.$$

Recently M. Nazarov realized irreducible representations of $\mathfrak{A}_k$ and Young symmetrizers by means of the Howe duality between the Lie superalgebra $\mathfrak{q}(n)$ and the Hecke algebra $H_k = \mathfrak{S}_k \circ Cl_k$, the semidirect product of $\mathfrak{S}_k$ with the Clifford algebra $Cl_k$ on $k$ indeterminates.

Here I construct one more analog of Young symmetrizers in $H_k$ as well as the analogs of Specht modules for $\mathfrak{A}_k$ and $H_k$.

## 1. SUMMARY

Lately, we have witnessed an increase of interest in the study of representations of symmetric groups. In particular, in their projective representations.

Recall that the symmetric group $\mathfrak{S}_k$ has a nontrivial central extension whose irreducible representations do not reduce to those of $\mathfrak{S}_k$ but coincide (may be identified) with the irreducible representations of the algebra $\mathfrak{A}_k$ determined by generators $\tau_{i,j}$ for $i \neq j$, $1 \leq i, j \leq n$ subject to the relations

$$\tau_{i,j} = -\tau_{j,i}, \quad \tau_{i,j}^2 = 1, \quad \tau_{i,j}\tau_{m,l} = -\tau_{m,l}\tau_{i,j} \text{ if } \{i,j\} \cap \{m,l\} = \emptyset;$$
$$\tau_{i,j}\tau_{j,m}\tau_{i,j} = \tau_{j,m}\tau_{i,j}\tau_{j,m} = -\tau_{i,m} \quad \text{for any } i, j, l, m. \tag{1.1}$$

In [N1] Nazarov realized irreducible representations of $\mathfrak{A}_k$ by means of an orthogonal basis constructed in each of the spaces of the representations and indicating the action of the generators $\tau_{i,i+1}$ on them (an analog of the Young orthogonal form). In [N2], with the help of an "odd" analog of the degenerate affine Hecke algebra Nazarov constructed elements of the algebra $H_k = \mathfrak{S}_k \circ Cl_k$, the semidirect product of $\mathfrak{S}_k$ with the Clifford algebra $Cl_k$ on $k$ indeterminates. Certain elements of $H_k$ serve as analogs of Young symmetrizers.

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Here I construct one more analog of Young symmetrizers in $H_k$ as well as analogs of Specht modules (cf. [Ja]) for the algebras $\mathfrak{A}_k$ and $H_k$. This construction is based on another form of expression of Young symmetrizers based on the notion of seminormal representation (an analog of Gelfand-Tsetlin basis) for the symmetric group (see [OV], [R], [Mu]) one can express Young symmetrizers in the form distinct from both the classical one (that of Weyl [W]) and that suggested in [Ju2]. Observe that the structure of Nazarov's symmetrizers is rather complicated; Jones simplified them [Jo1], [Jo2] but not completely. The symmetrizers constructed in this paper are simpler in structure. Our method is also applicable to the $q$-analogs of the Young symmetrizers of the usual and projective representations of $\mathfrak{S}_k$; cf. [JN].

This paper is an expanded and rewritten (thanks to a shrewd referee) version of my short note math.RT/9810148. The reader might also be interested in the results of [Ya1] and [Ya2], slightly related with this paper; namely, [Ya1] describes $\mathfrak{A}_k$ as a Howe dual of $\mathfrak{q}(n)$ and in [Ya2] a Howe dual of $\mathfrak{q}(m) \oplus \mathfrak{q}(n)$ is established to be the hyperoctahedral group.

1.1. For the general notion of a seminormal representation see [R]. In this section I will formulate it for $\mathfrak{S}_k$ in convenient terms (cf. [OV]).

Let $\mathfrak{S}_k$ be the symmetric group on $k$ symbols, $s_{ij} \in \mathfrak{S}_k$ the transposition. The Jucys–Murphy elements are defined as follows (cf. [Ju1], [Mu]):

$$x_1 = 0,\ x_2 = s_{12},\ x_3 = s_{13} + s_{23},\ \dots\ x_k = s_{1k} + s_{2k} + \cdots + s_{k-1,k}.$$

As in [Ma], we will identify the partition $\lambda$ of a positive integer $k$ with a Young diagram of $k$ cells. A *Young tableau of the form* $\lambda$ is the Young diagram corresponding to $\lambda$ filled in with the integers 1 to $k$. The tableau is called a *standard* one if the numbers that fill it do not decrease along the rows left to right and along the columns downwards.

For any partition $\lambda$ denote by $\mathcal{T}(\lambda)$ the set of standard tableaux of shape $\lambda$. For $\Lambda \in \mathcal{T}(\lambda)$ and $1 \le p \le k$, we set $c_\Lambda(p) = j - i$, where $(i, j)$ is the slot in $\Lambda$ occupied by $p$.

**1.2. Theorem** ([Mu], [OV], [R])**.** *Let $S^\lambda$ be the irreducible representation of $\mathfrak{S}_k$ corresponding to the partition $\lambda$. Then in $S^\lambda$ there exists a common eigenbasis for the Jucys–Murphy elements $x_1, \dots, x_k$; the basis can be indexed with the elements of $\mathcal{T}(\lambda)$ as follows:*

$$x_p v_\Lambda = c_\Lambda(p) v_\Lambda \quad \textit{for any } \Lambda \in \mathcal{T}(\lambda) \textit{ and } 1 \le p \le k.$$

Let us partially order the pertitions with respect to *dominance*, i.e., we set

$$\mu \ge \lambda \Longleftrightarrow \sum_{i=1}^{l} \mu_i \ge \sum_{i=1}^{l} \lambda_i \text{ for } l = 1, 2, \dots .$$

Let $\Lambda_c$ be a Young tableau of shape $\lambda$ consecutively filled in along columns (that is where the subscript comes from) from left to right. Set

$$\kappa_{\Lambda_c} = \prod_{i=1}^{k} (j - x_i), \tag{1.1.1}$$

where $j$ is the number of the column occupied by $i$. Since the elements $x_1, \dots, x_k$ pairwise commute, the order of factors in (1.1.1) is irrelevant.

**Example 1.** For

$$\Lambda_c = \begin{matrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & \end{matrix}$$

we have

$$\kappa_{\Lambda_c} = 1 \cdot (1 - x_2)(1 - x_3)(2 - x_4)(2 - x_5)(2 - x_6)(3 - x_7)(3 - x_8).$$

**1.3. Theorem.** *Let $S^\mu$ be the irreducible representation of $\mathfrak{S}_k$ corresponding to the partition $\mu$ of $k$. Then*

i) $\kappa_{\Lambda_c}(S^\mu) = 0$ *if* $\mu > \Lambda_c$*;*

ii) $\kappa_{\Lambda_c}(v_\Lambda) = 0$ *if* $\Lambda_c \neq \Lambda$ *for* $\Lambda \in \mathcal{T}(\lambda)$ *and the basis vectors* $v_{\Lambda_c}$ *as in Theorem 1.2;*

iii) $\kappa_{\Lambda_c}(v_{\Lambda_c}) = 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n} v_{\Lambda_c}$, *where* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ *are nonzero integers of the partition* $\lambda$.

*Proof.* Let $\mu \geq \lambda$ and $\Lambda$ be a standard tableau of shape $\mu$; let $v_\Lambda \in S^\mu$ be the corresponding vector for the seminormal basis (i.e., as in section 1.2). If $\Lambda \neq \Lambda_c$, then there exists a $j$ such that the $j$-th columns of the tableaux $\Lambda$ and $\Lambda_c$ are distinct whereas all the columns with lesser numbers are identical. Since the orders of the tableaux coincide and $\mu \geq \lambda$, it follows that $\mu'_j \leq \lambda'_j$ (these are the lengths of the $j$-th columns of the tableaux $\Lambda$ and $\Lambda_c$, respectively).

Let $p$ be the least of the numbers from the $j$-th column of $\Lambda$ (such a number exists because $\Lambda \neq \Lambda_c$ and $\mu'_j \leq \lambda'_j$). Since $\Lambda$ is standard, the slot that $p$ occupies is $(1, j+1)$. Hence,

$$x_p v_\Lambda = c_\Lambda(p) v_\Lambda = (j + 1 - 1) v_s = j v_\Lambda, \quad \text{i.e., } (j - x_p) v_\Lambda = 0.$$

Since $j - x_p$ enters $\kappa_{\Lambda_c}$ as a factor, $\kappa_{\Lambda_c} v_\Lambda = 0$. This proves i) and ii).

To prove iii), observe that

$$(j - x_p) v_{\Lambda_c} = (j - (j - i)) v_{\Lambda_c} = i v_{\Lambda_c},$$

where $i$ is the number of the row occupied by $x_p$. This proves iii). □

**1.4. Lemma.** *Let $p < k$ and let $\mathfrak{S}_p$ be considered as the subgroup of $\mathfrak{S}_k$ that preserves all but the first $p$ symbols. Let $U$ be a $\mathfrak{S}_k$-module, $\varepsilon$ the nontrivial one-dimensional representation and $a = \sum_{\sigma \in \mathfrak{S}_p} \varepsilon(\sigma)\sigma$. Let $v \in U$ be a nonzero vector fixed by a transposition $s_{lk}$ for $l \leq p$. Then*

$$a\Big(\sum_{i \in \mathfrak{S}_p} s_{ik}\Big) v = av.$$

*Proof.* Let $P = \{1, \dots, p\}$. Then

$$\begin{aligned} a\Big(\sum_{i \in P} s_{ik}\Big) v &= a\Big(\sum_{i \in P} s_{ik}\Big) s_{lk} v \\ &= a\Big(1 + \sum_{i \in P \setminus \{l\}} s_{ik} s_{kl}\Big) v = a\Big(1 + \sum_{i \in P \setminus \{l\}} s_{li} s_{ik}\Big) v \\ &= a\Big(1 - \sum_{i \in P \setminus \{l\}} s_{ik}\Big) v = a\Big(2 - \sum_{i \in P} s_{ik}\Big) v; \end{aligned}$$

hence, $a(\sum_{i \in P} s_{ik}) v = av$. □

**1.5. Corollary.** *Let $R_{\Lambda_c}$ and $C_{\Lambda_c}$ be the row and column stabilizers of the tableau $\Lambda_c$; let $U$ be a $\mathfrak{S}_k$-module and $v \in U$ a nonzero $R_{\Lambda_c}$-invariant vector. Set $a_{\Lambda_c} = \sum\limits_{\sigma \in C_{\Lambda_c}} \varepsilon(\sigma)\sigma$. Then $\kappa_{\Lambda_c} v = a_{\Lambda_c} v$.*

*Proof.* Induction on the number of columns in $\Lambda_c$. If $\Lambda_c$ consist of one column, then by induction on $k$ we easily verify the identity

$$(1 - s_{12})(1 - s_{13} - s_{23}) \dots (1 - s_{1k} - \dots - s_{k-1,k}) = \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma)\sigma.$$

So for one column the statement is true.

Let tableau $\Lambda_c$ consist of $n$ columns $\Lambda_1, \dots, \Lambda_n$; let $\Lambda_c^*$ be obtained from $\Lambda_c$ by deleting the last column. Then

$$\begin{aligned} \kappa_{\Lambda_c} v &= \kappa_{\Lambda_c^*} \prod_{i \in \Lambda_n} (n - x_i) v = \prod_{i \in \Lambda_n} (n - x_i) \cdot \kappa_{\Lambda_c^*} v \\ &= \text{(by induction)} \ = \prod_{i \in \Lambda_n} (n - x_i) \cdot a_{\Lambda_c^*} v = a_{\Lambda_c^*} \prod_{i \in \Lambda_n} (n - x_i) v. \end{aligned}$$

Let $i \in \Lambda_n$; then

$$a_{\Lambda_c^*}(n - x_i)v = a_{\Lambda_c^*}(n - \sum_{\alpha \in \Lambda_1} s_{\alpha i} - \sum_{\alpha \in \Lambda_2} s_{\alpha i} - \dots - \sum_{\alpha \in \Lambda_n,\ \alpha < i} s_{\alpha i})v.$$

By Lemma 1.4 $a_{\Lambda_c^*}(\sum\limits_{\alpha \in \Lambda_j} s_{\alpha i})v = a_{\Lambda_c^*} v$ for $j = 1, \dots, n-1$, so $a_{\Lambda_c^*}(n - x_i)v = a_{\Lambda_c^*}(1 - \sum\limits_{\alpha \in \Lambda_n,\ \alpha < i} s_{\alpha i})v$. Hence, if $\Lambda_n = \{i, i+1, \dots, i+l\}$, then

$$\begin{aligned} \kappa_{\Lambda_c} v &= a_{\Lambda_c^*}(1 - s_{i,i+1})(1 - s_{i,i+2} - s_{i+1,i+2}) \dots (1 - s_{i,i+l} - \dots - s_{i+l-1,i+l})v \\ &= a_{\Lambda_c^*}(\sum_{\sigma \in \Lambda_n} \varepsilon(\sigma)\sigma)v = a_{\Lambda_c} v. \end{aligned}$$

□

Thus, if $v = \sum\limits_{\sigma \in R_{\Lambda_c}} \sigma = b_{\Lambda_c}$, then

$$e_{\Lambda_c} = a_{\Lambda_c} b_{\Lambda_c} = \kappa_{\Lambda_c} b_{\Lambda_c},$$

i.e., we have expressed the Young symmetrizer $e_{\Lambda_c}$ as $\kappa_{\Lambda_c} b_{\Lambda_c}$. In §3 a similar construction will be used to construct projective analogs of Young symmetrizers.

## 2. Auxiliary data

Let $\mathfrak{S}_k$ be the symmetric group, $Cl_k$ the Clifford algebra generated by $k$ indeterminates $p_1, \dots, p_k$ subject to the relations

$$p_i^2 = -1,\ p_i p_j + p_j p_i = 0 \text{ for } i \neq j.$$

The symmetric group acts on $Cl_n$ permuting the generators, so we can form a semidirect product $H_k = \mathfrak{S}_k \circ Cl_k$, i.e., the linear span of the elements $f\sigma$, where $f \in Cl_k$ and $\sigma \in \mathfrak{S}_k$ with the multiplication satisfying

$$\sigma f = f^\sigma \cdot \sigma \text{ (here } f^\sigma \text{ is the result of the action of } \sigma \in \mathfrak{S}_k \text{ on } f \in Cl_k).$$

Set

$$\tau_{i,j} = \frac{1}{\sqrt{2}}(p_i - p_j)s_{i,j}.$$

As is not difficult to verify, the relations (1.1) hold; hence, the algebra generated by the $\tau_{i,j}$ is isomorphic to $\mathfrak{A}_n$. Besides, $\mathfrak{A}_n$ supercommutes with $Cl_n$; hence, $H_n = \mathfrak{A}_n \otimes Cl_n$, as *super*algebras if we define parity in $\mathfrak{A}_n$ by setting $p(\tau_{i,j}) = \bar{1}$.

Let $\mathbb{N}$ be the set of positive integers, $\bar{\mathbb{N}}$, another , "odd", copy of $\mathbb{N}$; its elements are barred. Let $V$ be a superspace of superdimension $(n, n)$ with the fixed basis $\{e_i\}_{i=1}^{n} \coprod \{e_{\bar{i}}\}_{\bar{i}=\bar{1}}^{\bar{n}}$. Define the odd operator $Q : V \longrightarrow V$ by setting

$$Q(e_i) = e_{\bar{i}}; \quad Q(e_{\bar{i}}) = -e_i.$$

The (super)centralizer of $Q$ in $\mathrm{Mat}(V)$ is denoted by $\mathrm{Q}(V)$; cf. [BL]. We denote the Lie superalgebras associated with the associative superalgebras $\mathrm{Mat}(V)$ and $\mathrm{Q}(V)$ by $\mathfrak{gl}(V)$ and $\mathfrak{q}(V)$, respectively. We set $\mathfrak{q}(n) = \mathfrak{q}(V)$ for $n = \dim V$.

Select a basis in $\mathfrak{q}(n)$: let the $e_i^*$ be the left dual basis to the $\{e_i : i = 1, \dots, n; \bar{1}, \dots, \bar{n}\}$; set

$$E_{i,j} = e_i \otimes e_j^* + e_{\bar{i}} \otimes e_{\bar{j}}^*; \quad F_{i,j} = e_i \otimes e_{\bar{j}}^* + e_{\bar{i}} \otimes e_j^*.$$

Then $\mathfrak{h} = Span(E_{i,i} \text{ and } F_{i,i} : i = 1, \dots, n)$ is a Cartan subalgebra in $\mathfrak{q}(n)$ and $\mathfrak{b} = Span(E_{i,j} \text{ and } F_{i,j} : i \leq j)$ is a Borel subalgebra; cf. [Pe]. Set further $\mathfrak{n}_- = Span(E_{i,j} \text{ and } F_{i,j} : i > j)$ and $\mathfrak{n}_+ = Span(E_{i,j} \text{ and } F_{i,j} : i < j)$. Then, clearly, $\mathfrak{q}(n) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$; any finite dimensional irreducible $\mathfrak{q}(n)$-module $U = U^\lambda$ is uniquely, up to an isomorphism, determined by its highest weight and the irreducible $\mathfrak{h}$-module $U^+$ of highest weight vectors annihilated by $\mathfrak{n}_+$, and for any $u \in U^+$ we have $hu = \lambda(h)u$ for any $h \in \mathfrak{h}_{\bar{0}}$; cf. [Pe]. Observe that since $\mathfrak{h}$ is, unlike the case of Lie algebras, not commutative, $U^+$ is seldom one-dimensional; cf. [S3]. This module $U$ will be denoted by $U^\lambda$.

We will stick to the following terminology; cf [BL], [L]. The representation of a superalgebra $A$ in the superspace $V$ is *irreducible* if it does not contain homogeneous (with respect to parity) subrepresentations distinct from 0 and $V$ itself.

Thus, there are two types of irreducible representations: those that do not contain any nontrivial subrepresentations (called of *general type* or of type $G$) and those that contain *in*homogeneous invariant subspaces (call them of *type* $Q$). If $V$ is of finite dimension, then in the first case its centralizer, as of $A$-module, is isomorphic to $\mathfrak{gl}(1)$, whereas in the second case to $\mathfrak{q}(1)$.

Let $V_1$ and $V_2$ be finite dimensional irreducible modules over $A_1$ and $A_2$, respectively. Then $V_1 \otimes V_2$ is an irreducible $A_1 \otimes A_2$-module except for the case where both $V_1$ and $V_2$ are of type $Q$. In the latter case, the centralizer of the $A_1 \otimes A_2$-module $V_1 \otimes V_2$ is isomorphic to $Cl_2$, the Clifford superalgebra with 2 generators. (For further interesting details see [Sch].)

If $e \in Cl_2$ is a minimal idempotent, then $e(V_1 \otimes V_2)$ is an irreducible $A_1 \otimes A_2$-module of type $G$ that we will denote by $2^{-1}(V_1 \otimes V_2)$.

Let $\mathbb{M} = \mathbb{N} \coprod \overline{\mathbb{N}}$ be ordered as follows:

$$\bar{1} < 1 < \bar{2} < 2 < \bar{3} < 3 \dots$$

We will call the elements from $\mathbb{N}$ "even" and those form $\overline{\mathbb{N}}$ "odd" ones; so we can encounter an "even" odd element, etc.

The partition $\lambda$ of $k$ is called a *strict* one if its parts are pairwise distinct. To a strict partition $\lambda$ we assign a *shifted diagram* (see [Ma]) obtained from the ordinary diagram by shifting its $i$-th row by $i-1$ cell to the right for $i = 1, 2, \dots$ . A shifted tableau of shape $\lambda$ is a filling of a shifted diagram of shape $\lambda$ with numbers from $\mathbb{M}$. The shifted tableau is called a *standard* one if:

i) the numbers that fill it do not decrease from left to right along rows and downwards along columns;

ii) the symbols from $\mathbb{N}$ strictly increase along columns (downwards);

iii) the symbols from $\overline{\mathbb{N}}$ strictly increase along rows (from left to right).

For the strict partition $\lambda$ denote by $ST(\lambda)$ the set of strict standard tableaux of shape $\lambda$ filled with the numbers $1, \dots, k$, where $k = \lambda_1 + \lambda_2 + \dots$. If $\Lambda \in ST(\lambda)$ and $1 \le p \le k$, then set $c_\Lambda(p) = \sqrt{(j-i)(j-i+1)}$, where $(i,j)$ is the slot in $\Lambda$ occupied by $p$.

Denote by $l(\lambda)$ the number of nonzero parts of the partition $\lambda$ and set

$$\delta(\lambda) = \begin{cases} 0 & \text{if the number of nonzero parts of } \lambda \text{ is even,} \\ 1 & \text{otherwise.} \end{cases}$$

As is shown in [S1], the algebras $\mathfrak{q}(n)$ and $H_k$ constitute a Howe-dual pair in the superspace $W_k = V^{\otimes k}$. The $\mathfrak{q}(n)$-action is uniquely determined by the identity action in $V$ extended as a superdifferentiation onto the tensor algebra $T(V) = \sum\limits_{k \ge 0} W_k$. The symmetric group $\mathfrak{S}_k$ acts on $W_k$ by the formula

$$\begin{aligned} s_{i,i+1}&(v_1 \otimes \dots \otimes v_i \otimes v_{i+1} \otimes \otimes \dots \otimes v_k \\ &= (-1)^{p(v_i)p(v_{i+1})} v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \otimes \dots \otimes v_k, \end{aligned}$$

where $p(v_i)$ and $p(v_{i+1})$ are parities of respective vectors. Recall that $\mathfrak{q}(n)$ is defined as the centralizer of an odd operator $Q$ whose square is $-\mathrm{id}$. The element $p_i \in H_k$ acts in $W_k$ as the operator

$$1 \otimes \dots \otimes 1 \otimes Q \otimes 1 \otimes \dots \otimes 1 \quad (Q \text{ occupies the } i\text{-th slot}).$$

**2.1. Lemma.** *Let $w \in W_{k-1}$, set $\pi_k = \sum\limits_{\alpha < k} \tau_{\alpha k}$, and also*

$$E^{(2)}_{im} = \sum_{j=1}^{n} (E_{ij}E_{jm} - F_{ij}F_{jm}), \quad F^{(2)}_{im} = \sum_{j=1}^{n} (F_{ij}E_{jm} - E_{ij}F_{jm}).$$

*Then*

i) $\pi_k(w \otimes e_i) = \frac{1}{\sqrt{2}} \sum\limits_{j=1}^{n} (F_{ij} - p_k E_{ij})(w \otimes e_j)$;

ii) $\pi_k^2(w \otimes e_i) = \frac{1}{2} \sum\limits_{m=1}^{n} \Big( (E^{(2)}_{im} w) \otimes e_m) + (F^{(2)}_{im} w) \otimes e_{\overline{m}}) \Big) + \sum\limits_{m=1}^{n} (E_{mm} w) \otimes e_i$.

*Proof.* i) It suffices to assume that $w = e_{i_1} \otimes \dots \otimes e_{i_{k-1}}$, where all the indices $i_1$, $\dots$, $i_{k-1}$ are "even", i.e., belong to $\mathbb{N}$, not $\overline{\mathbb{N}}$. Then

$$\begin{aligned} \pi_k(w \otimes e_i) = \sum_{\alpha < k} \tau_{\alpha k}(w \otimes e_i) &= \frac{1}{\sqrt{2}} \sum_{\alpha < k} (p_\alpha - p_k) s_{\alpha k}(w \otimes e_j) \\ &= \frac{1}{\sqrt{2}} \sum_{\alpha < k} (p_\alpha - p_k) w^i_\alpha \otimes e_{i_\alpha}, \end{aligned}$$

where $w^i_\alpha$ is obtained from $w = e_{i_1} \otimes \dots \otimes e_{i_{k-1}}$ by replacing $e_{i_\alpha}$ with $e_i$. Hence,

$$\pi_k(w \otimes e_i) = \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \sum_{i_\alpha = j} (p_\alpha - p_k) w^i_\alpha \otimes e_j.$$

But, on the other hand, $\sum_{i_\alpha=j} w^i_\alpha = E_{ij}w$ and $\sum_{i_\alpha=j} p_\alpha w^i_\alpha = F_{ij}w$ implying that

$$\begin{aligned}\pi_k(w\otimes e_i) &= \frac{1}{\sqrt{2}}\sum_{j=1}^{n}(F_{ij}w\otimes e_j - E_{ij}w\otimes e_{\bar{j}})\\ &= \frac{1}{\sqrt{2}}\sum_{j=1}^{n}\left(F_{ij}(w\otimes e_j) - w\otimes e_{\bar{i}} - p_kE_{ij}(w\otimes e_{\bar{i}}) + w\otimes e_{\bar{i}}\right)\\ &= \frac{1}{\sqrt{2}}\sum_{j=1}^{n}(F_{ij} - p_kE_{ij})(w\otimes e_j)\end{aligned}$$

and heading i) is proven.

ii)

$$\begin{aligned}\pi_k^2(w\otimes e_i) &= \pi_k\left(\frac{1}{\sqrt{2}}\sum_{j=1}^{n}(F_{ij}-p_kE_{ij})(w\otimes e_j)\right)\\ &= -\frac{1}{\sqrt{2}}\sum_{j=1}^{n}(F_{ij}-p_kE_{ij})\pi_k(w\otimes e_j)\\ &= -\frac{1}{2}\sum_{j=1}^{n}(F_{ij}-p_kE_{ij})\sum_{m=1}^{n}(F_{jm}-p_kE_{jm})(w\otimes e_m)\\ &= -\frac{1}{2}\sum_{j,m=1}^{n}(F_{ij}-p_kE_{ij})(F_{jm}-p_kE_{jm})(w\otimes e_m)\\ &= -\frac{1}{2}\sum_{j,m=1}^{n}\left((F_{ij}F_{jm}-E_{ij}E_{jm}) - p_k(E_{ij}F_{jm}-F_{ij}E_{jm})\right)(w\otimes e_m)\\ &= \frac{1}{2}\sum_{j,m=1}^{n}\left((E_{ij}E_{jm}-F_{ij}F_{jm}) + p_k(E_{ij}F_{jm}-F_{ij}E_{jm})\right)(w\otimes e_m).\end{aligned}$$

Let us calculate the sum by parts. We have

$$\begin{aligned}&(E_{ij}E_{jm}-F_{ij}F_{jm})(w\otimes e_m)\\ &\quad= E_{ij}(E_{jm}w\otimes e_m + w\otimes e_j) - F_{ij}(F_{jm}w\otimes e_m + w\otimes e_{\bar{j}})\\ &\quad= E_{ij}E_{jm}w\otimes e_m + E_{jm}w\otimes E_{ij}e_m + E_{ij}w\otimes e_j + w\otimes e_i\\ &\qquad - F_{ij}F_{jm}w\otimes e_m + F_{jm}w\otimes F_{ij}e_m - F_{ij}w\otimes e_{\bar{j}} - w\otimes e_i\\ &\quad= E_{ij}E_{jm}w\otimes e_m - F_{ij}F_{jm}w\otimes e_m + E_{mm}w\otimes e_i\\ &\qquad + F_{mm}w\otimes e_{\bar{i}} + E_{ij}w\otimes e_j - F_{ij}w\otimes e_{\bar{j}}.\end{aligned}$$

and

$$\begin{aligned}&p_k(E_{ij}F_{jm}-F_{ij}E_{jm})(w\otimes e_m)\\ &\quad= p_kE_{ij}(F_{jm}w\otimes e_m + w\otimes e_{\bar{j}}) - p_kF_{ij}(E_{jm}w\otimes e_m + w\otimes e_j)\\ &\quad= -E_{ij}E_{jm}w\otimes e_{\bar{m}} - F_{mm}w\otimes e_{\bar{i}} - E_{ij}w\otimes e_j - w\otimes e_i\\ &\qquad + F_{ij}E_{jm}w\otimes e_{\bar{m}} + E_{mm}w\otimes e_i + F_{ij}w\otimes e_{\bar{j}} + w\otimes e_i\\ &\quad= (F_{ij}E_{jm}-E_{ij}F_{jm})w\otimes e_{\bar{m}} + E_{mm}w\otimes e_i\\ &\qquad + F_{ij}w\otimes e_{\bar{j}} - E_{ij}w\otimes e_j - F_{mm}w\otimes e_{\bar{i}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2}\sum_{j,m=1}^{n}&\left((E_{ij}E_{jm}-F_{ij}F_{jm})+p_k(E_{ij}F_{jm}-F_{ij}E_{jm})\right)(w\otimes e_m)\\ &=\frac{1}{2}\sum_{j,m=1}^{n}\left((E_{ij}E_{jm}w\otimes e_m-F_{ij}F_{jm}w\otimes e_m)\right.\\ &\qquad\left.+(F_{ij}E_{jm}-E_{ij}F_{jm})w\otimes e_{\bar m}+2E_{mm}w\otimes e_i\right)\\ &=\frac{1}{2}\sum_{m=1}^{n}(E^{(2)}_{im}w\otimes e_m+F^{(2)}_{im}w\otimes e_{\overline{m}})+\sum_{m=1}^{n}E_{mm}w\otimes e_i.\end{aligned}$$

□

2.2. For the convenience of the reader we formulate the main result of [S1] in the following suitable form. Observe that if $\lambda$ is a strict partition of $k$ and $\lambda_{n+1}=0$, then $\lambda$ can be interpreted as a linear functional on $\mathfrak{h}_{\bar 0}$: set

(2.2.1) $$\lambda(E_{ii})=\lambda_i.$$

**Theorem.** *The superalgebras $H_k$ and $\mathfrak{q}(n)$ are mutual centralizers in the superspace $W_k$ and the following decomposition takes place:*

(2.2.2) $$W_k=\bigoplus_{\lambda}2^{-\delta(\lambda)}T^{\lambda}\otimes V^{\lambda},$$

*where $\lambda$ runs over strict partitions of $k$ such that $\lambda_{n+1}=0$; here $T^{\lambda}$ is an irreducible (in the super sence) $H_k$-module corresponding to $\lambda$, and $V^{\lambda}$ an irrreducible $\mathfrak{q}(n)$-module.*

*If $\delta(\lambda)=0$, then the corresponding representation $T^{\lambda}$ is of type $G$, it is of type $Q$ if $\delta(\lambda)=1$.*

*The dimension of the space (superstructure ignored) of the highest weight vectors $(V^{\lambda})^+$ in $V^{\lambda}$ is equal to $2^{(l(\lambda)+\delta(\lambda))/2}$ copies of $T^{\lambda}$.*

**2.3. Corollary.** *The $H_k$-module of highest weight vectors in $W_k$ is a free $Cl_k$-module.*

*Proof.* By Theorem $W_k^+=\bigoplus_{\lambda}2^{-\delta(\lambda)}T^{\lambda}\otimes(V^{\lambda})^+$, so the $H_k$-module of highest weight vectors in $W_k$ is equal to $2^{-\delta(\lambda)}T^{\lambda}\otimes(V^{\lambda})^+$. By considering the dimension of $(V^{\lambda})^+$ and taking into account the type ($G$ or $Q$) of its irreducibility we get the statement desired. □

2.4. Let us denote the $H_k$-module of highest weight vectors of weight $\lambda$ in $W_k$ by $R^{\lambda}$.

**Lemma.** *Let $\mathfrak{g}=\mathfrak{q}(n)$; let $\mathfrak{b}$ and $\mathfrak{h}$ be defined as above and $V$ a $\mathfrak{g}$-module on which $\mathfrak{h}_{\bar 0}$ acts diagonally. Let $(V^{\lambda})^+$ be the set of $\mathfrak{b}$-highest vectors of weight $\lambda$.*

*If $u\in U(\mathfrak{g})$ and $u((V^{\lambda})^+)\subset(V^{\lambda})^+$, then there exists $w\in U(\mathfrak{h})$ such that $u|_{(V^{\lambda})^+}=w|_{(V^{\lambda})^+}$.*

*Proof.* Let $u=\sum u_{\alpha}$ be the weight decomposition of $u\in U(\mathfrak{g})$ with respect to $\mathfrak{h}_{\bar 0}$. Observe that $(V^{\lambda})^+$ is the $\lambda$ weight space of $V$. Therefore, if $v\in V_{\lambda}$, then $u(v)=\sum_{\alpha}u_{\alpha}v$ and, if $u_{\alpha}v\neq 0$, then the weight of $u_{\alpha}v$ is equal to $\lambda+\alpha$. Thus, thanks to the condition $u(v)\in(V^{\lambda})^+$, we may assume that $u=u_0\in U(\mathfrak{g})^{\mathfrak{h}_{\bar 0}}$, where $U(\mathfrak{g})^{\mathfrak{h}_{\bar 0}}$ is the centralizer of $\mathfrak{h}_{\bar 0}$.

Further, thanks to [S2], we know that $U(\mathfrak{g})^{\mathfrak{h}_{\bar{0}}} \cong U(\mathfrak{h}) \oplus L$, where $L = U(\mathfrak{g})^{\mathfrak{h}_{\bar{0}}} \cap U(\mathfrak{g})\mathfrak{b}^+$ and where $\mathfrak{b}^+$ is the linear span of the positive roots in $\mathfrak{b}$, is a two-sided ideal in $U(\mathfrak{g})^{\mathfrak{h}_{\bar{0}}}$.

Hence, $u = w + u_1$, where $w \in U(\mathfrak{h})$ and $u_1 \in L$; this implies that $uv = wv$ for $v \in (V^\lambda)^+$. □

Let $T$ be a finite dimensional $H_k$-module. The $\mathfrak{q}(n)$-module $U$ corresponding to it via Howe duality is determined by the formula

$$U = \mathrm{Hom}_{H_k}(T, W_k) \text{ for } W_k = V^{\otimes k}.$$

It is not difficult to deduce from Theorem 2.2 that likewise $T$ is uniquely recoverable from $U$, namely,

$$T = \mathrm{Hom}_{\mathfrak{q}(n)}(U, W_k).$$

Further on, since $U$ is a $\mathfrak{q}(n)$-submodule of $W_k$, it follows that all its weights with respect to $\mathfrak{h}_{\bar{0}}$ are of the form $m_1\varepsilon_1 + \dots + m_n\varepsilon_n$, where the $\varepsilon_i$ are the weights of the standard $\mathfrak{q}(n)$-module $V$ and the $m_i$ are nonnegative integers. This implies that the formal character of the $\mathfrak{q}(n)$-module $U$ equal to $\sum m_\chi e^\chi$ (here $\chi$ runs over the set of weights of $U$ with respect to $\mathfrak{h}_{\bar{0}}$) is a polynomial in $x_1 = e^{\varepsilon_1}, \dots, x_n = e^{\varepsilon_n}$. This polynomial is called the *characteristic* of $T$.

**Example 2.** Consider the Clifford superalgebra $Cl_k$ as a module over $H_k$ ($\mathfrak{S}_k$ acts by permuting the generators of $Cl_k$ while $Cl_k$ acts on itself via the left regular representation). The corresponding characteristic $q_k$ is the coefficient of $t^k$ in the decomposition

$$\frac{\prod_{i=1}^{n}(1 + x_i t)}{\prod_{i=1}^{n}(1 - x_i t)} = \sum_{k \geq 0} q_k(x) t^k.$$

In [S1] I showed that the $x$-character of $T^\lambda$ for the strict partition $\lambda$ is equal to $2^{(l(\lambda)-\delta(\lambda))/2} Q_\lambda(x)$, where $Q_\lambda(x)$ is the projective Schur function (for its definition see [Ma]). Therefore, the characteristic of $R^\lambda$ is equal precisely to $Q_\lambda(x)$.

2.5. Set $P_\lambda(x) = 2^{-l(\lambda)} Q_\lambda(x)$. Denote by $SST(\lambda)$ the set of standard tableaux of arbitrary shape with elements from $\mathbb{M}$ and such that

a) scanning the tableau from left to right and downwards the last of the symbols encountered should be an "even" one;

b) for every $i = 1, 2, \dots$ the total number of occurrences of any of the symbols $i$ or $\bar{i}$ is equal to $\lambda_i$.

**Lemma.** *Let $\lambda$ be a strict partition of $k$ and $\lambda_1 > \dots > \lambda_n > 0$. Then*

$$P_{\lambda_1}(x) \dots P_{\lambda_n}(x) = \sum_{\mu} m_{\lambda\mu} P_\mu(x),$$

*where the sum runs over strict partitions $\mu$ such that $\mu \geq \lambda$ (with respect to dominance partial order) and $m_{\lambda\mu}$ is equal to the number of tableaux from $SST(\lambda)$ of shape $\mu$.*

Proof of this statement is an easy corollary of the multiplication table for projective Schur functions; cf. [P], §4.

2.6. **Even and odd Jucys-Murphy elements** (cf. [N2]). Set

$$x_1 = 0,\ x_2 = s_{12}(1 + p_1 p_2),\ \ldots,\ x_k = \sum_{i<k} s_{ik}(1 + p_i p_k).$$

The *odd analogs* of the Jucys-Murphy elements $\pi_i$ are defined as

$$\pi_1 = 0,\ \pi_2 = \tau_{12},\ \ldots,\ \pi_k = \sum_{i<k} \tau_{ik}.$$

Observe that $\pi_i = -\frac{1}{\sqrt{2}} p_i x_i$ and the distinction of $\pi_i$ from $x_i$ is that the former supercommute with the elements of the Clifford superalgebra $Cl_k$.

The following statement proven by M. Nazarov in [N2] (Th. 7.2) describes the action of $x_i$ in $R^\lambda$.

**Theorem.** *The $Cl_k$-module $R^\lambda$ is free with a basis $\{v_\Lambda : \Lambda \in ST(\lambda)\}$ and*

$$x_p v_\Lambda = c_\Lambda(p) v_\Lambda \textit{ for } 1 \le p \le k.$$

## 3. Specht modules over $H_k$

Let $\lambda$ be a strict partition of $k$ and $\lambda_{n+1} = 0$. Define the functional $\lambda$ on the Cartan subalgebra $\mathfrak{h}$ by setting

$$\lambda(E_{ii}) = \lambda_i; \quad \lambda(F_{ii}) = 0.$$

In $W_k = V^{\otimes k}$, where $V$ is the standard $\mathfrak{q}(n)$-module of dimension $(n, n)$, consider the $H_k$-submodule $M^\lambda$ consisting of the vectors of weight $\lambda$, i.e., $w \in W_k$ such that $E_{ii} w = \lambda_i w$. Let $\Lambda_r$ be a shifted tableau of shape $\lambda$ filled in consecutively along rows from left to right with the numbers 1 to $k$.

Let $\Lambda$ be an arbitrary shifted tableau of shape $\lambda$ and let $R_\Lambda$ be the row stabilizer of $\Lambda$. Let $I$ be the sequence obtained by reading the tableau $\Lambda$ from left to right and downwards. For $i \in I$ define

$$\pi_i = \sum_{\alpha \in I,\ \alpha \text{ preceeds } i} \tau_{\alpha, i}.$$

Set

$$\kappa_\Lambda = \prod_{i \in I} (\frac{1}{2} j(j+1) - \pi_i^2),$$

where $j$ is the number of the column occupied by $i$.

**Example 3.** For

$$\Lambda = \begin{matrix} 1 & 2 & 4 & 7 \\ & 3 & 5 & 8 \\ & & 6 & \end{matrix}$$

we have

$$\kappa_\Lambda = 1 \cdot (3 - \pi_2^2)(3 - \pi_3^2)(6 - \pi_4^2)(6 - \pi_5^2)(6 - \pi_6^2)(10 - \pi_7^2)(10 - \pi_8^2).$$

For every shifted tableau $\Lambda$ of shape $\lambda$ denote by $w_\Lambda$ the vector from $W_k$ equal to $v_1 \otimes \cdots \otimes v_k$, where $v_p = e_i$ if $p$ occupies the $i$-th row. Clearly, $w_\Lambda \in M^\lambda$.

**3.1. Theorem.** *$\kappa_\Lambda(M^\lambda) = Cl_k \kappa_\Lambda(w_\Lambda)$ and $\kappa_\Lambda(w_\Lambda) \in R^\lambda$ is nonzero.*

*Proof.* Since $M^\lambda$ is the module induced from the trivial representation of the row stabilizer of $\Lambda_r$, it follows that

$$\mathrm{Hom}_{H_k}(T, W_k) = S^{\lambda_1}(V) \otimes \cdots \otimes S^{\lambda_n}(V),$$

where $S^p$ is the operator of the $p$-th symmetric power and $V$ is the standard $\mathfrak{q}(n)$-module. Hence, the characteristic of $M^\lambda$ is equal to $q_{\lambda_1} \dots q_{\lambda_n}$ (see (2.4)). Lemma 2.5 implies that

$$q_{\lambda_1} \dots q_{\lambda_n} = \sum_{\mu \geq \lambda} 2^{l(\lambda) - l(\mu)} m_{\lambda\mu} Q_\mu.$$

Hence, $M^\lambda$ is equal to the direct sum of modules isomorphic to either $T^\mu$ or $\Pi(T^\mu)$. Further, let $\varphi_i : M^\lambda \longrightarrow M^\lambda$ be a homomorphism such that $\varphi_i(v_{\Lambda_r}) = p^i_{\Lambda_r}$ for $i = 1, \dots, l(\lambda)$, where $p^i_{\Lambda_r} = \sum_{j \text{ lies in the } i\text{-th row}} p_j$.

It is not difficult to verify that $\varphi_1, \dots, \varphi_{l(\lambda)}$ generate the Clifford algebra $Cl_{l(\lambda)}$. Therefore, $M^\lambda$ is equal to the direct sum of modules isomorphic to $R^\mu$ (or $\Pi(R^\mu)$) for $\mu \geq \lambda$ and the multiplicity of $R^\lambda$ in $M^\lambda$ is equal to 1.

It suffices to consider the case when $\Lambda = \Lambda_c$ is a shifted tableau of shape $\lambda$ filled from left to right and downwards, as usual. In each module $R^\mu$, select a basis $\{v_\Lambda\}$, where $\Lambda$ runs over standard shifted tableaux, as in Theorem 2.6. Let us demonstrate that $\kappa_{\Lambda_c}(R^\mu) = 0$ if $\mu > \lambda$ and $\kappa_{\Lambda_c}(R^\lambda) = Cl_k v_{\Lambda_c}$.

Consider a vector $v_\Lambda \in R^\mu$ for $\mu \geq \lambda$ and $\Lambda \neq \Lambda_c$. There exists a number $j$ such that the $j$-th columns of the tableaux $\Lambda$ and $\Lambda_c$ are distinct and all the respective columns with lesser numbers are identical. Since the tableaux are of the same order and $\mu \geq \lambda$, it follows that the length of the $j$-th column of the *shifted* tableau $\lambda$ is not less than the length of the $j$-th column of the shifted tableaux $\Lambda$. Let $p$ be the least of the numbers occupying the $j$-th column of $\Lambda_c$ which is not contained in the $j$-th column of $\Lambda$. Since $\Lambda$ is standard, we see theat $p$ occupies the $(j+1)$-st column and the first row of $\Lambda$.

By Theorem 2.6

$$\pi_p^2 v_\Lambda = \frac{1}{2} x_p^2 v_\Lambda = \frac{1}{2}(j + 1 - 1)(j + 1 - 1 + 1)v_\Lambda = \frac{1}{2} j(j+1) v_\Lambda.$$

Hence,

$$(\frac{1}{2} j(j+1) - \pi_p^2) v_\Lambda = 0.$$

Since $\frac{1}{2} j(j+1) - \pi_p^2$ enters $\kappa_{\Lambda_c}$ as a factor, $\kappa_{\Lambda_c}(v_\Lambda) = 0$. Moreover, since the $\pi_p$ for $p = 1, \dots, k$ supercommute with $Cl_k$, it follows that $\kappa_{\Lambda_c}(R^\mu) = 0$ if $\mu > \lambda$ and $\kappa_{\Lambda_c}(v_\Lambda) = 0$ if $v_\Lambda \in R^\lambda$ and $\Lambda \neq \Lambda_c$. Moreover,

$$(\frac{1}{2} j(j+1) - \pi_p^2) v_{\Lambda_c} = (\frac{1}{2} j(j+1) - \frac{1}{2}(j-i)(j-i+1)v_{\Lambda_c} = \frac{1}{2} i(2j + i - 1) v_{\Lambda_c},$$

where $(i, j)$ is the position of $p$. Hence, $\kappa_{\Lambda_c}(v_{\Lambda_c}) = \alpha v_{\Lambda_c}$ for a nonzero $\alpha$.

Thus, it is shown that $\kappa_{\Lambda_c}$ projects $M^\lambda$ onto the subspace $Cl_k v_{\Lambda_c}$. To prove the theorem, it suffices to establish that $\kappa_{\Lambda_c}(w_{\Lambda_c}) \neq 0$. We induct on $k$. For $k = 1$ the statement is obvious. Let $k > 1$, let $\Lambda_c^*$ be obtained from $\Lambda_c$ by deleting the last cell in the last column and let $(i, j)$ be the position of $k$. Then

$$\kappa_{\Lambda_c}(w_{\Lambda_c}) = (\frac{1}{2} j(j+1) - \pi_k^2) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*} \otimes e_i) = (\frac{1}{2} j(j+1) - \pi_k^2)(\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \otimes e_i).$$

By induction, $\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \neq 0$ while by Lemma 2.1 and in notations of Lemma 2.1 we have

$$\left(\frac{1}{2}j(j+1) - \pi_k^2)(\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \otimes e_i\right) = \sum_{m=1}^{n} (w_m \otimes e_m + \overline{w}_m \otimes e_{\overline{m}}),$$

where

$$w_m = \left(\frac{1}{2}j(j+1) - \frac{1}{2}E_{im}^{(2)} - \delta_{il}\sum_{l=1}^{n} E_{ll}\right)\kappa_{\Lambda_c^*}(w_{\Lambda_c^*})$$

and $\overline{w}_m = -\frac{1}{2}F_{im}^{(2)}\kappa_{\Lambda_c^*}(w_{\Lambda_c^*})$.

It remains to show that $w_i \neq 0$. Observe that since $\kappa_{\Lambda_c^*}(w_{\Lambda_c^*})$ is a highest weight vector with respect to $\mathfrak{q}(n)$, i.e.,

$$E_{ij}\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = F_{ij}\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = 0 \text{ for } i < j,$$

we have

$$\begin{aligned}
w_i &= \left(\frac{1}{2}j(j+1) - \frac{1}{2}\sum_{r=1}^{n}(E_{ir}E_{ri} - F_{ir}F_{ri}) - \sum_{r=1}^{n} E_{rr}\right)\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\
&= \left(\frac{1}{2}j(j+1) - \frac{1}{2}\sum_{r>i}^{n}(E_{ir}E_{ri} - F_{ir}F_{ri}) - \frac{1}{2}(E_{ii}^2 - F_{ii}^2) - \sum_{r=1}^{n} E_{rr}\right)\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\
&= \left(\frac{1}{2}j(j+1) - \frac{1}{2}\sum_{r>i}([E_{ir}, E_{ri}] - [F_{ir}, F_{ri}])\right. \\
&\qquad\qquad \left. - \frac{1}{2}(E_{ii}^2 - E_{ii}) - \sum_{r=1}^{n} E_{rr}\right)\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\
&= \left(\frac{1}{2}j(j+1) - \frac{1}{2}\sum_{r>i}(E_{ii} - E_{rr} - E_{ii} - E_{rr})\right. \\
&\qquad\qquad \left. - \frac{1}{2}(E_{ii}^2 - E_{ii}) - \sum_{r=1}^{n} E_{rr}\right)\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\
&= \left(\frac{1}{2}j(j+1) - \frac{1}{2}\sum_{r>i} E_{rr} - \frac{1}{2}(E_{ii}^2 - E_{ii}) - \sum_{r=1}^{n} E_{rr}\right)\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\
&= \left(\frac{1}{2}j(j+1) - \frac{1}{2}\sum_{r=1} iE_{rr} - \frac{1}{2}(E_{ii}^2 - E_{ii})\right)\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\
&= \left(\frac{1}{2}\lambda_1(\lambda_1+1) - (\lambda_1 + \cdots + \lambda_i - 1) - \frac{1}{2}(\lambda_i - 1)(\lambda_i - 2)\right)\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\
&= \frac{1}{2}\left(\lambda_1(\lambda_1+1) - 2(\lambda_1 + \cdots + \lambda_i) + 2 - (\lambda_i - 1)(\lambda_i - 2)\right)\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}).
\end{aligned}$$

If $i$ is the number of the row occupied by $k$, then

$$\lambda_1 = \lambda_i + i - 1,\ \lambda_2 = \lambda_i + i - 2,\ \ldots,\ \lambda_{i-1} = \lambda_i + 1,$$

and, therefore,

$$w_i = \frac{1}{2}\left((\lambda_i + i - 1)(\lambda_i + i) - 2(i\lambda_i + 1 + 2 + \cdots + i - 1)\right.$$
$$\left. +2 - (\lambda_i - 1)(\lambda_i - 2)\right)\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = \lambda_i \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \neq 0.$$

□

3.2. *Remark.* Observe that

$$\begin{aligned}\overline{w}_i &= -\frac{1}{2}F_{ii}^{(2)}\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = -\frac{1}{2}\sum_{r=1}^{n}(F_{ir}E_{ri} - E_{ir}F_{ri})\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \frac{1}{2}\sum_{i=1}^{n}(E_{ir}F_{ri} - F_{ir}E_{ri})\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = \frac{1}{2}\sum_{r>i}(E_{ir}F_{ri} - F_{ir}E_{ri})\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \frac{1}{2}\sum_{r>i}([E_{ir}, F_{ri}] - [F_{ir}, E_{ri}])\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = 0.\end{aligned}$$

Hence,

$$\kappa_{\Lambda_c}(w_{\Lambda_c}) = \lambda_i \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \otimes e_i + \sum_{r \neq i}(w_r + \overline{w}_r p_k) \otimes e_r$$

and by the inductive hypothesis applied to $\kappa_{\Lambda_c^*}(w_{\Lambda_c^*})$ we deduce that

$$\kappa_{\Lambda_c}(w_{\Lambda_c}) = \lambda_1! \ldots \lambda_n! w_{\Lambda_c} + \sum f_\Lambda w_\Lambda, \quad \text{where } f_\Lambda \in Cl_k \text{ and } w_\Lambda \neq w_{\Lambda_c}$$

and where the summation runs over the tableaux $\Lambda$ of order $k$ which contain $\lambda_1$ symbols 1 or $\bar{1}$, $\lambda_2$ symbols 2 or $\bar{2}$, etc., $\lambda_n$ symbols $n$ or $\bar{n}$.

3.3. **Specht modules.** Recall that for a strict partition $\lambda$ of $k$ the *Specht module* is the submodule in $M^\lambda$ generated by the vectors $\kappa_\Lambda w_\Lambda$ for all shifted tableaux $s$ of shape $\lambda$.

**3.3.1. Theorem.** *Specht module is equal to $R^\lambda$. It is isotypical and its $H_k$-centralizer is isomorphic to the Clifford algebra with $l(\lambda)$ generators.*

*Proof.* Recall that $R^\lambda$ is the set of highest weight vectors in $W_k = V^{\otimes k}$. By the Howe duality between $U(\mathfrak{q}(n))$ and $H_k$ the algebra of $H_k$-homomorphisms is generated by $U(\mathfrak{q}(n))$. But thanks to Lemma 2.4 we may assume that the algebra of $H_k$-homomorphisms is generated by $U(\mathfrak{h})$ for the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{q}(n)$. If $\lambda_i = 0$, then, in our notations for the basis of $\mathfrak{h}$, we have

$$F_{ii}^2 = E_{ii} = \lambda_i = 0 \text{ in } R^\lambda.$$

But $\ker F_{ii}$ is an $\mathfrak{h}$-submodule in the set $(V^\lambda)^+$ of highest weight vectors. But $(V^\lambda)^+$ is irreducible as an $\mathfrak{h}$-module by Theorem 2.2. Hence, $F_{ii}|_{(V^\lambda)^+} = 0$ which implies $F_{ii}|_{R^\lambda} = 0$. Thus, the algebra of $H_k$-homomorphisms is generated by the $F_{ii}$ for $i$ such that $\lambda_i \neq 0$.

By Theorem 3.1 the Specht module is contained in $R^\lambda$. Moreover, if $\Lambda$ is a shifted tableau of shape $\lambda$, then

$$F_{ii}\kappa_\Lambda w_\Lambda = \kappa_\Lambda F_{ii} w_\Lambda = \left(\sum_{\alpha \text{ lies in the } i\text{-th row of } \Lambda} p_\alpha\right)\kappa_\Lambda w_\Lambda.$$

Therefore, any homomorphism of $M^\lambda$ sends $R^\lambda$ into itself. Hence, the Specht module coincides with $R^\lambda$. □

**3.3.2. Corollary.** *Let $\Lambda$ be a shifted tableau. Set* $p^i_\Lambda = \sum\limits_{\alpha \text{ lies in the } i\text{-th row of } \Lambda} p_\alpha$. *Then for any $H_k$-module endomorphism $\varphi$ of $M^\lambda$ we have*

$$\varphi(\kappa_\Lambda w_\Lambda) = f \cdot \kappa_\Lambda w_\Lambda,$$

*where $f$ belongs to the subalgebra of $Cl_k$ generated by $p^1_\Lambda, \dots, p^{l(\lambda)}_\Lambda$.*

*Proof.* Recall that $M^\lambda$ is the subset of vectors of weight $\lambda$ in $W_k$. Therefore, any endomorphism of $M^\lambda$ may be identified with an element of $U(\mathfrak{q}(n))$; the restriction of this endomorphism onto $R^\lambda$ may be identified with an element of $U(\mathfrak{h})$.

But $U(\mathfrak{h})$ is generated by the $F_{ii}$ and to prove the corollary, it suffices to verify it for these elements. We have

$$F_{ii}(\kappa_\Lambda w_\Lambda) = \kappa_\Lambda(F_{ii} w_\Lambda) = \kappa_\Lambda(p^i_\Lambda w_\Lambda) = p^i_\Lambda \kappa_\Lambda(w_\Lambda).$$

□

**3.3.3. Corollary.** *Let $b_\Lambda$ be the sum of the elements of the row stabilizer of a shifted tableau $\Lambda$ and $\varphi : M^\lambda \longrightarrow M^\lambda$ be an $H_k$-module endomorphism given by the formula $\varphi(w_\Lambda) = b_\Lambda \kappa_\Lambda(w_\Lambda)$. Then $\varphi|_{R^\lambda} = c \in \mathbb{C}$.*

*Proof.* Let us show that $\varphi$ commutes with the endomorphisms $F_{ii}$. Indeed,

$$\begin{aligned}
\varphi \cdot F_{ii}(w_\Lambda) &= \varphi(p^i_\Lambda w_\Lambda) = p^i_\Lambda \varphi(w_\Lambda) = p^i_\Lambda b_\Lambda \kappa_\Lambda(w_\Lambda);\\
F_{ii} \cdot \varphi(w_\Lambda) &= F_{ii} b_\Lambda \kappa_\Lambda(w_\Lambda) = b_\Lambda \kappa_w(F_{ii} w_\Lambda)\\
&= b_\Lambda \kappa_\Lambda(p^i_\Lambda w_\Lambda) = b_\Lambda p^i_\Lambda \kappa_\Lambda(w_\Lambda) = p^i_\Lambda b_\Lambda \kappa_\Lambda(w_\Lambda).
\end{aligned}$$

The latter identity holds thanks to the fact that $p^i_\Lambda$ commutes with $b_\Lambda$.

Thus, $\varphi \cdot F_{ii}(w_\Lambda) = F_{ii} \cdot \varphi(w_\Lambda)$ and, since the elements $w_\Lambda$ generate the $H_k$-module $M^\lambda$, we have

$$\varphi \cdot F_{ii} = F_{ii} \cdot \varphi.$$

Further, $\varphi(R^\lambda) \subset R^\lambda$ for any endomorphism $\varphi$ of $M^\lambda$, so $\varphi|_{R^\lambda}$ is an element from the centralizer of $R^\lambda$. But by Theorem 3.3.1 the centralizer is the Clifford algebra $Cl_{l(\lambda)}$. But $\varphi$ is an even central element of $Cl_{l(\lambda)}$, hence, $\varphi$ is a constant. □

**3.3.4. Corollary.** *Set $e_\Lambda = \kappa_\Lambda b_\Lambda$. Then*

$$e^2_\Lambda = c \cdot e_\Lambda \text{ for } c \in \mathbb{C},\ c \neq 0;$$

*the algebra $e_\Lambda H_k e_\Lambda$ is isomorphic to $Cl_{l(\lambda)}$ and is generated by the $p^i_\Lambda$ for $1 \leq i \leq l(\lambda)$.*

*Proof.* Thanks to Corollary 3.3.3

$$\varphi(\kappa_\Lambda(w_\Lambda)) = \kappa_\Lambda b_\Lambda \kappa_\Lambda(w_\Lambda) = c\kappa_\Lambda(w_\Lambda)$$

or, equivalently,

$$e^2_\Lambda = c \cdot e_\Lambda.$$

If $c = 0$, then $\varphi(\kappa_\Lambda(w_\Lambda)) = 0$. But the $H_k$-submodule generated by $\kappa_\Lambda(w_\Lambda)$ coincides with the Specht module, which, thanks to Theorem 3.3.1 is equal to $R^\lambda$. So in this case $\varphi(R^\lambda) = 0$. On the other hand, Theorem 3.1 implies that $\kappa_\Lambda(w_\Lambda) = \alpha w_\Lambda + \sum \beta_T w_T$, where $\alpha \in \mathbb{C}$ and $\beta_T \in Cl_k$ and where $w_\Lambda \neq w_T$.

Therefore,

$$b_\Lambda \kappa_\Lambda(w_\Lambda) = \alpha\lambda_1! \dots \lambda_n! w_\Lambda + \sum \gamma_T w_T,$$

where $\gamma_T \in Cl_k$, the sum runs over the tableaux $T$ of order $k$ which contain $\lambda_1$ symbols 1 or $\bar{1}$, $\lambda_2$ symbols 2 or $\bar{2}$, etc., $\lambda_n$ symbols $n$ or $\bar{n}$, and where $w_\Lambda \neq w_T$ and $\varphi(w_\Lambda) = b_\Lambda \kappa_\Lambda(w_\Lambda) \neq 0$. This contradicts the requirement $\varphi(R^\lambda) = 0$. So $c \neq 0$.

The algebra $e_\Lambda H_k e_\Lambda$ is anti-isomorphic to the algebra $H_k$ of endomorphisms of $R^\lambda$ and, thanks to Theorem 3.3.1, it is the Clifford superalgebra $Cl_{l(\lambda)}$. □

3.3.5. Let $\lambda = (k)$ be the partition consisting of one part, $\Lambda = 1\,2\dots k$ the corresponding one-row tableau. Then $b_\Lambda = \sum\limits_{\sigma\in\mathfrak{S}_k} \sigma$ and

$$\kappa_\Lambda = 1 \cdot (3 - \pi_2^2)(6 - \pi_3^2)(10 - \pi_4^2) \dots (\frac{1}{2}k(k+1) - \pi_k^2).$$

Further, for any $i$, $j$ and $l$ we have $(\tau_{ij} + \tau_{jl} + \tau_{li})b_\Lambda = 0$, so

$$\begin{aligned}\pi_p^2 b_\Lambda &= \left(\sum_{i<p} \tau_{ip}\right)^2 b_\Lambda = \left(p - 1 - \sum_{i<j<p} (\tau_{ip}\tau_{jp} + \tau_{jp}\tau_{ip})\right) b_\Lambda \\ &= \left(p - 1 - \sum (\tau_{ij}(\tau_{ij} + \tau_{jp} + \tau_{pi}) - 1)\right) b_\Lambda = \frac{1}{2} p(p-1) b_\Lambda.\end{aligned}$$

Therefore, $\kappa_\Lambda b_\Lambda = k! b_\Lambda = e_\Lambda$ and, up to a factor, $e_\Lambda$ coincides with the respective Young symmetrizer.

Observe that the symmetrizer Nazarov constructed for the one-row tableau is equal to

$$\frac{1}{k!} \prod_{j=2}^{k} \left( \prod_{i=1}^{j-1} (1 + \frac{s_{ij}}{u_j - u_i} - \frac{p_i p_j s_{ij}}{u_j + u_i}) \right),$$

where $u_p = \sqrt{p(p-1)}$ and $s_{ij}$ is the transposition; so it does not coincide with $b_\Lambda$.

3.3.6. In the case $\lambda = (p+1, p)$ and

$$\Lambda = \begin{matrix} 1 & 2 & 4 & 6 & \dots & p \\ & 3 & 5 & 7 & \dots & 2p+1 \end{matrix}$$

we can simplify the expression for $\kappa_\Lambda b_\Lambda$. Namely, set

$$\begin{aligned} a_1 &= 1 + \tau_{12}\tau_{23} + \tau_{23}\tau_{12}, \\ a_2 &= 2 + (\tau_{14} + \tau_{24} + \tau_{34})\tau_{45} + \tau_{45}(\tau_{14} + \tau_{24} + \tau_{34}), \\ a_3 &= 3 + \pi_6\tau_{67} + \tau_{67}\pi_6, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ a_p &= p + \pi_{2p}\tau_{2p,2p+1} + \tau_{2p,2p+1}\pi_{2p}. \end{aligned}$$

Then it is possible to verify that

$$\kappa_\Lambda b_\Lambda = a_1 a_2 \dots a_p b_\Lambda.$$

The author does not know how to simplify the expression for $\kappa_\Lambda b_\Lambda$ in the general case similarly to the case of the symmetric group; cf. section 1.5.

## 4. The Specht modules over $\mathfrak{A}_k$

In this section, we construct for $\mathfrak{A}_k$ certain analogs of the modules $M^\lambda$ and the Specht modules $R^\lambda$. First of all, we need the following statement.

**4.1. Lemma.** *Let $\pi_p = \sum\limits_{\alpha<p} \tau_{\alpha p}$ for $p = 2, \dots, k$ be odd analogs of Jucys–Murphy's elements. Then for $k \geq 2$*

$$e_k = \prod_{p\geq 2} \frac{2}{p(p-1)}\pi_p^2$$

*is an idempotent and $e_k\mathfrak{A}_k e_k$ is isomorphic to the Clifford algebra $Cl_{k-1}$.*

*Proof.* Set $e_1 = e_2 = 1$. It is easy to verify by induction that for $k \geq 2$ we have

$$e_k = \frac{1}{k!}\sum_{p\geq 1}(-1)^p 2^{k-p-1}\Sigma_{2p+1},$$

where $\Sigma_{2p+1}$ is the sum of all elements from $\mathfrak{A}_k$ of the form $\tau_{i_1 i_2}\tau_{i_2 i_3}\dots\tau_{i_{2\alpha} i_{2\alpha+1}}$. This implies that $e_k$ is a central element that does not vary under automorphisms of $\mathfrak{A}_k$ that send $\tau_{ij}$ into $\tau_{\sigma(i)\sigma(j)}$ for any $\sigma \in \mathfrak{S}_k$. It is not difficult to verify that

$$(\tau_{12} + \tau_{23} + \tau_{31})\pi_2^2 = 0.$$

Hence, $(\tau_{12} + \tau_{23} + \tau_{31})e_k = 0$. Having applied the above described automorphisms to this relation we deduce that

$$(\tau_{ij} + \tau_{il} + \tau_{li})e_k = 0.$$

Further, arguments as in section 3.3.5 yield

$$\pi_p^2 e_k = \frac{1}{2}p(p-1)e_k,$$

hence,

$$e_k^2 = \left(\prod_{p\geq 2} \frac{2}{p(p-1)}\pi_i^2\right) e_k = e_k.$$

Furthermore, since $e_k$ is a central element, then $e_k\mathfrak{A}_k e_k = \mathfrak{A}_k e_k$. Let $I$ be the two-sided ideal in $\mathfrak{A}_k$ generated by the elements $\tau_{ij} + \tau_{jl} + \tau_{li}$. Then $e_k I = 0$. Set $\bar{\mathfrak{A}} = \mathfrak{A}_k/I$. In $\bar{\mathfrak{A}}$, then, the following relations hold:

$$\tau_{12} = \pi_2,\ \tau_{23} = \frac{1}{2}(\pi_3 - \pi_2),\ \tau_{34} = \frac{1}{3}(\pi_4 - \pi_3),\ \dots,\ \tau_{k-1,k} = \frac{1}{k-1}(\pi_k - \pi_{k-1}).$$

Hence, $\bar{\mathfrak{A}}$ is generated by the $\pi_p$ for $2 \leq p \leq k$. We showed above that condition $\tau_{ij}+\tau_{il}+\tau_{li} = 0$ in $\bar{\mathfrak{A}}$ implies that $\pi_i^2 = \frac{1}{2}(i-1)(i-2)$ in $\bar{\mathfrak{A}}$. Moreover, $\pi_i\pi_j+\pi_j\pi_i = 0$ for $i \neq j$. So $\bar{\mathfrak{A}}$ is the Clifford algebra generated by the images of the $\pi_i$ for $2 \leq i \leq k$.

Further on, $(1 - e_k)I = I$; so $\mathfrak{A}_k(1 - e_k) \supset I$, and since $\bar{\mathfrak{A}}$ is simple as a superalgebra, we deduce that $\mathfrak{A}_k(1 - e_k) = I$. Therefore, $\mathfrak{A}_k e_k \cong \mathfrak{A}_k/\mathfrak{A}_k(1 - e_k) \cong \mathfrak{A}_k/I \cong \bar{\mathfrak{A}}_k$. □

Let $\lambda$ be a strict partition of $k$ and $\Lambda$ a shifted $\lambda$-tableau. Let $\Lambda^{(i)} = \{\alpha_1, \dots \alpha_{\lambda_i}\}$ be the $i$-th row. Recall that

$$p_\Lambda^i = \sum_{\alpha\in\Lambda^{(i)}} p_\alpha.$$

Let further $\pi_p^{(i)} = \sum\limits_{j<p} \tau_{\alpha_j\alpha_p}$ and

$$e_\Lambda^i = \prod_{p=2}^{\lambda_i} \frac{2}{p(p-1)} \left(\pi_p^{(i)}\right)^2, \quad e_\Lambda = \prod_{i=1}^{l(\lambda)} e_\Lambda^i.$$

Let $V$ be an $(n,n)$-dimensional superspace with $n \geq l(\lambda)$ and $W_k = V^{\otimes k}$, where $k = \sum \lambda_i$. Let $M^\Lambda$ be the subspace of $W_k$ spanned by the vectors $w \in W_k$ such that

$$E_{ii}w = \lambda_i w; \quad F_{ii}w = p_\Lambda^i w \text{ for } 1 \leq i \leq l(\lambda).$$

Let $R^\Lambda = \{w \in M^\Lambda : E_{ij}w = F_{ij}w = 0\}$ be the set of highest weight vectors from $M^\Lambda$. Recall that the irreducible representations of *super*algebra $\mathfrak{A}_k$ are labelled by strict partitions $\lambda$ of $k$; moreover, the representation is of type $G$ if $k$ and $l(\lambda)$ are of the same parity, and of type $Q$, otherwise; see [N2].

4.2. **Specht modules.** Let $\lambda$ be a strict partition of $k$ and $\Lambda$ a shifted tableau of shape $\lambda$. The Specht module for tableau $\Lambda$ is the submodule in $M^\Lambda$ generated by the vectors $\kappa_T w_\Lambda$, where the tableau $T$ is of shape $\lambda$ and its rows have the same numbers as the respective rows of $\Lambda$.

**Theorem.** i) *As $\mathfrak{A}_k$-module, $M^\Lambda$ is isomorphic to $\mathfrak{A}_k e_\Lambda$.*

ii) *The Specht submodule of $M^\Lambda$ is equal to $R^\Lambda$ and its centralizer is isomorphic to the Clifford superalgebra $Cl_{k-l(\lambda)}$.*

*Proof.* i) Let $\mathfrak{A}_i$ be the subalgebra of $\mathfrak{A}_k$ generated by the $\tau_{\alpha\beta}$, where $\alpha$ and $\beta$ lie in the $i$-th row of $\Lambda$. Define the homomorphism $\varphi_i : \mathfrak{A}_i \longrightarrow Cl_{\lambda_i}$ into the Clifford superalgebra on $\lambda_i$ generators $p_{\alpha_1}, \dots, p_{\alpha_{\lambda_i}}$, where $\alpha_i$ runs numbers in the the $i$-th row of $\Lambda$ by setting

$$\varphi_i(\tau_{\alpha\beta}) = \frac{1}{\sqrt{2}}(p_\alpha - p_\beta).$$

From Lemma 4.1 we deduce that this homomorphism induces an isomorphism of $\mathfrak{A}_i e_\Lambda^i$ with the subalgebra of $Cl_{\lambda_i}$ generated by $p_\alpha - p_\beta$ for $\alpha, \beta \in \Lambda^{(i)}$ (recall that $\Lambda^{(i)}$ is the $i$-th row of $\Lambda$).

Therefore, we have an homomorphism

$$\varphi_1 \otimes \dots \otimes \varphi_{l(\lambda)} : \mathfrak{A}_1 \otimes \dots \otimes \mathfrak{A}_{l(\lambda)} \longrightarrow Cl_{\lambda_1} \otimes \dots \otimes Cl_{\lambda_{l(\lambda)}}$$

which induces an isomorphism of $\mathfrak{A}_1 \otimes \dots \otimes \mathfrak{A}_{l(\lambda)} e_\Lambda$ with a subalgebra $Cl_\Lambda$ in $Cl_{\lambda_1} \otimes \dots \otimes Cl_{\lambda_{l(\lambda)}} \cong Cl_k$. Moreover, we can see that this subsuperalgebra is generated by the $p_\alpha - p_\beta$, where $\alpha$ and $\beta$ lie in the same row of $\Lambda$. The dimension of this subalgebra is equal to $2^{\lambda_1-1}2^{\lambda_2-1}\dots 2^{\lambda_{l(\lambda)}-1} = 2^{k-l(\lambda)}$. This implies that the ideal $\mathfrak{A}_k e_\Lambda$ is isomorphic, as an $\mathfrak{A}_k$-module, to the module induced from the module $Cl_\Lambda$ over $\mathfrak{A}_1 \otimes \dots \otimes \mathfrak{A}_{l(\lambda)}$, is of dimension

$$\frac{k!}{\lambda_1!\lambda_2!\dots\lambda_{l(\lambda)}!} 2^{k-l(\lambda)}.$$

It is subject to direct verification that $\dim M^\Lambda$ is the same and the homomorphism $\varphi : \mathfrak{A}_k e_\Lambda \longrightarrow M^\Lambda$ such that $\varphi(e_\Lambda) = w_\Lambda$ is surjective. Hence, $\varphi$ is an isomorphism.

ii) By Theorem 3.1 $\kappa_T w_\Lambda = \kappa_T w_T \in R^\Lambda$, so the Specht module is contained in $R^\Lambda$.

Now, let us show that the centralizer of $R^t$ is isomorphic to the Clifford algebra $Cl_{k-l(\lambda)}$.

Obviously, $\mathfrak{A}_k$ and $Cl_k \otimes U(\mathfrak{q}(n))$ form a Howe-dual pair in $W_k$. By Theorem 3.3.1 the centralizer of the $H_k$-module $R^\lambda$ is the Clifford algebra generated by the $F_{ii}$ for $1 \leq i \leq l(\lambda)$. Therefore, the centralizer of the $\mathfrak{A}_k$-module $R^\lambda$ is the Clifford algebra $Cl_{k+l(\lambda)}$ generated by the $F_{ii}$ for $1 \leq i \leq l(\lambda)$ and $p_1, \ldots, p_k$.

The condition $F_{ii}w = p^i_\Lambda w$ is equivalent to the condition $ew = w$ for

$$e = \prod_{1 \leq i \leq l(\lambda)} \frac{1}{2}\left(1 - \frac{1}{\lambda_i} p^i_\Lambda F_{ii}\right) \text{ and } e^2 = e.$$

Therefore, the centralizer of $R^\Lambda$ is isomorphic to $eCl_{k+l(\lambda)}e$.

Let $Cl_\Lambda$ be the subalgebra of $Cl_k$ generated by the $p_i - p_j$ for $i$, $j$ that belong to the same row of $\Lambda$. Then it is not difficult to see that $eCl_{k+l(\lambda)}e \cong Cl_\Lambda$ and this proves that the centralizer of $R^\Lambda$ is isomorphic to $Cl_{k-l(\lambda)}$.

Let us prove now that the Specht module is actually equal to $R^\Lambda$. Indeed, $Cl_\Lambda$ naturally acts on $R^\Lambda$ by left multiplications and this action commutes with that of $\mathfrak{A}_k$. Hence, by the above, any endomorphism of the $\mathfrak{A}_k$-module $R^\Lambda$ is of the form $w \mapsto fw$, where $w \in R^\Lambda$ and $f \in Cl_\Lambda$, and the algebra of such endomorphisms is generated by multiplications by $p_\alpha - p_\beta$, where $\alpha$ and $\beta$ lie in the same row of $\Lambda$.

To prove that the Specht module is equal to $R^\Lambda$, it suffices to verify that every endomorphism of $R^\Lambda$ fixes the Specht module. By the above, it suffices to consider the homomorphism $\varphi$ given by multiplication by $p_\alpha - p_\beta$, where $\alpha$ and $\beta$ lie in the same row of $\Lambda$. We have

$$\varphi(p_\alpha - p_\beta)\kappa_T w_\Lambda = (p_\alpha - p_\beta)\kappa_T w_\Lambda = \sqrt{2}\tau_{\alpha\beta} s_{\alpha\beta}\kappa_T w_\Lambda = \sqrt{2}\tau_{\alpha\beta}\kappa_{s_{\alpha\beta}T} w_\Lambda.$$

But the rows of $s_{\alpha\beta}T$ consist of the same elements that constitute the respective rows of $\Lambda$. □

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