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# STABLE NILPOTENT ORBITAL INTEGRALS ON REAL REDUCTIVE LIE ALGEBRAS

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ABSTRACT. This paper proves a stable analog of Rossmann's formula for the number of $G(\mathbf{R})$-orbits in $\mathfrak{g} \cap \mathbf{O}$, where $\mathbf{O}$ is a nilpotent orbit in $\mathfrak{g}_{\mathbf{C}}$.

## 1. INTRODUCTION

1.1. **Notation.** Let $G$ be a connected reductive group over $\mathbf{R}$ and let $\mathfrak{g}$ denote its Lie algebra. For $g \in G(\mathbf{R})$ and $X \in \mathfrak{g}$ we denote by $g \cdot X$ the adjoint action of $g$ on $X$ (and the same for the adjoint action of $G(\mathbf{C})$ on the complexification $\mathfrak{g}_{\mathbf{C}}$ of $\mathfrak{g}$). For $X \in \mathfrak{g}$ we denote by $G_X$ the centralizer of $X$ in $G$.

1.2. **Stable conjugacy.** Let $X, Y$ be regular semisimple elements in $\mathfrak{g}$. Recall that $X, Y$ are said to be *stably conjugate* if there exists $g \in G(\mathbf{C})$ such that $g \cdot X = Y$. The inner automorphism $\mathrm{Int}(g)$ induces an isomorphism (over $\mathbf{R}$) from $G_X$ to $G_Y$, and this isomorphism is independent of the choice of $g$.

1.3. **Stable orbital integrals.** Let $X$ be a regular semisimple element of $\mathfrak{g}$. Choose Haar measures $dg$ and $dt$ on $G(\mathbf{R})$ and $G_X(\mathbf{R})$ respectively. For any Schwartz function $f$ on $\mathfrak{g}$ we write $O_X(f)$ for the orbital integral

$$O_X(f) := \int_{G(\mathbf{R})/G_X(\mathbf{R})} f(g \cdot X)\, dg/dt.$$

Similarly we write $SO_X(f)$ for the stable orbital integral

$$SO_X(f) := \sum_{X'} O_{X'}(f)$$

where the sum is taken over a set of representatives $X'$ for the conjugacy classes within the stable conjugacy class of $X$. We form the orbital integral $O_{X'}$ using $dg$ and the Haar measure $dt'$ on $G_{X'}(\mathbf{R})$ obtained from $dt$ via the isomorphism from $G_X$ to $G_{X'}$ defined in 1.2.

1.4. **Stably invariant tempered distributions.** Let $f$ be a Schwartz function on $\mathfrak{g}$. We write $f \sim 0$ if $SO_X(f) = 0$ for every regular semisimple element $X$ in $\mathfrak{g}$. We say that a tempered distribution $D$ on $\mathfrak{g}$ is *stably invariant* if $D(f) = 0$ for every Schwartz function $f$ on $\mathfrak{g}$ such that $f \sim 0$ (equivalently: $D$ is in the closure of the linear span of the tempered distributions $SO_X$, where closure is taken in the sense of the topology of pointwise convergence). It is obvious that any stably invariant distribution is invariant. Of course by an invariant distribution we mean

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one that is invariant under the adjoint action of $G(\mathbf{R})$. The definitions in 1.2–1.4 are Lie algebra analogs of ones made by Shelstad in [She79] (see also [Wal97] for the $p$-adic case).

1.5. **Nilpotent orbital integrals.** Now let $\mathbf{O}$ be a nilpotent $G(\mathbf{C})$-orbit in $\mathfrak{g}_{\mathbf{C}}$ and let $r_{\mathbf{O}}$ be the number of $G(\mathbf{R})$-orbits in $\mathfrak{g}\cap\mathbf{O}$. Each $G(\mathbf{R})$-orbit in $\mathfrak{g}\cap\mathbf{O}$ determines a tempered invariant distribution on $\mathfrak{g}$ (see [Rao72]), defined by integrating Schwartz functions $f$ on $\mathfrak{g}$ over the given orbit with respect to the (unique up to scalars) non-zero $G(\mathbf{R})$-invariant Radon measure on that orbit. The linear span of these distributions (one for each $G(\mathbf{R})$-orbit in $\mathfrak{g}\cap\mathbf{O}$) is an $r_{\mathbf{O}}$-dimensional space $\mathcal{D}_{\mathbf{O}}$ of tempered invariant distributions on $\mathfrak{g}$.

The main result of this paper is a formula for the dimension $s_{\mathbf{O}}$ of the subspace $\mathcal{D}_{\mathbf{O}}^{\mathrm{st}}$ of stably invariant elements in $\mathcal{D}_{\mathbf{O}}$. Our formula for $s_{\mathbf{O}}$ (see 1.8) is the stable analog of Rossmann's formula for $r_{\mathbf{O}}$, which we now recall.

1.6. $W^\sigma$**,** $W_{\mathbf{R}}$ **and** $\epsilon_I$**.** Let $T$ be a maximal torus in $G$ and let $W$ denote the Weyl group of $T(\mathbf{C})$ in $G(\mathbf{C})$. Then complex-conjugation, which we denote by $\sigma$, acts on $W$, and we can consider $W^\sigma$, the group of fixed points of $\sigma$ on $W$. Inside $W^\sigma$ we have the subgroup $W_{\mathbf{R}}$ consisting of elements in $W$ that can be realized by elements in $G(\mathbf{R})$ that normalize $T$.

Let $R_I$ denote the set of imaginary roots of $T$ (roots $\alpha$ of $T(\mathbf{C})$ in $\mathfrak{g}_{\mathbf{C}}$ such that $\sigma\alpha = -\alpha$). Then $W^\sigma$ permutes the elements of $R_I$, and we define a sign character $\epsilon_I$ on $W^\sigma$ in the usual way: $\epsilon_I(w) = (-1)^b$, where $b$ is the number of positive roots $\alpha \in R_I$ such that $w\alpha$ is negative. By restriction we also regard $\epsilon_I$ as a character on $W_{\mathbf{R}}$.

Similarly we use the set of real roots of $T$ (roots $\alpha$ such that $\sigma\alpha = \alpha$) to define another sign character $\epsilon_{\mathbf{R}}$ on $W^\sigma$: $\epsilon_{\mathbf{R}}(w) = (-1)^c$, where $c$ is the number of positive real roots such that $w\alpha$ is negative. It is well-known that $\epsilon_I\epsilon_{\mathbf{R}}$ coincides with the restriction to $W^\sigma$ of the usual sign character $\epsilon$ on the Weyl group $W$.

1.7. **Rossmann's formula.** Using Springer's correspondence [Spr76], we get from the nilpotent orbit $\mathbf{O}$ an irreducible character $\chi_{\mathbf{O}}$ of the abstract Weyl group $W_a$ of $G(\mathbf{C})$. We normalize the correspondence so that the trivial orbit $\mathbf{O} = \{0\}$ corresponds to the sign character $\epsilon$ of $W_a$. Of course we can also think of $\chi_{\mathbf{O}}$ as an irreducible character of the Weyl group $W$ of any maximal torus $T$ as above, so that we can consider the multiplicity $m_{W_{\mathbf{R}}}(\epsilon_I, \chi_{\mathbf{O}})$ with which $\epsilon_I$ occurs in the restriction of $\chi_{\mathbf{O}}$ to the subgroup $W_{\mathbf{R}}$. Rossmann [Ros90], 3.7 proved that

$$r_{\mathbf{O}} = \sum_T m_{W_{\mathbf{R}}}(\epsilon_I, \chi_{\mathbf{O}}), \tag{1.7.1}$$

where the sum is taken over a set of representatives for the $G(\mathbf{R})$-conjugacy classes of maximal tori $T$ in $G$.

1.8. **Main Theorem.** The main result of this paper is the following formula for the number $s_{\mathbf{O}}$:

$$s_{\mathbf{O}} = \sum_T m_{W^\sigma}(\epsilon_I, \chi_{\mathbf{O}}), \tag{1.8.1}$$

where the index set for the sum is the same as in (1.7.1) and where $m_{W^\sigma}(\epsilon_I, \chi_{\mathbf{O}})$ denotes the multiplicity with which $\epsilon_I$ appears in the restriction of $\chi_{\mathbf{O}}$ to $W^\sigma$.

To prove this formula for $s_{\mathbf{O}}$ we use the Fourier transform and must therefore check that the Fourier transform of a stably invariant tempered distribution on $\mathfrak{g}$ is stably invariant. This result (see 4.3, 4.4), which is due to Waldspurger [Wal97] in the $p$-adic case, follows easily from another theorem of Rossmann [Ros78].

The Fourier transforms of nilpotent orbital integrals are invariant eigendistributions on $\mathfrak{g}$ and hence by a theorem of Harish-Chandra are represented by locally integrable functions on $\mathfrak{g}$. It is easy to recognize when such a locally integrable function represents a stably invariant distribution. Moreover, a description of the image of $\mathcal{D}_{\mathbf{O}}$ under the Fourier transform is implicit in the work of Barbasch and Vogan [BV80], [BV82], [BV83], Ginzburg [Gin83], Hotta-Kashiwara [HK84] and Rossmann [Ros90]. (I am very much indebted to V. Ginzburg for this remark.) Our main theorem is a simple consequence of these observations, and all the methods used in this paper are straightforward adaptations of methods used in the papers just cited. For this reason much of the paper consists of a review of the relevant literature.

1.9. **Assem's conjecture.** Our main result is a formula for the dimension $s_{\mathbf{O}}$ of the subspace $\mathcal{D}^{\mathrm{st}}_{\mathbf{O}}$ of stably invariant elements in $\mathcal{D}_{\mathbf{O}}$. It would of course be better to find a basis for $\mathcal{D}^{\mathrm{st}}_{\mathbf{O}}$. Assem investigated this question in the $p$-adic case (for the group rather than its Lie algebra) and suggested (see the introduction to [Ass98]) a conjectural answer for split classical groups. Assem's conjecture has been proved by Waldspurger (see [Wal99], IX.16), who in fact treats all unramified classical groups.

Let us transpose Assem's conjecture to the real case and see what it says $s_{\mathbf{O}}$ ought to be. We assume that $G$ is quasi-split and classical, and we fix a nilpotent $G(\mathbf{C})$-orbit $\mathbf{O}$ in $\mathfrak{g}_{\mathbf{C}}$. First of all, Assem conjectures that $s_{\mathbf{O}} = 0$ unless $\mathbf{O}$ is special in Lusztig's sense. The truth of this conjecture follows from the main theorem of this paper in conjunction with the main result of [Kot98].

Next Assem defines a suitable notion of stable conjugacy for special nilpotent elements, intermediate between $G(\mathbf{R})$-conjugacy and $G(\mathbf{C})$-conjugacy. Assume that $\mathbf{O}$ is special. For $X \in \mathfrak{g} \cap \mathbf{O}$ we denote by $\mathcal{G}_X$ Lusztig's canonical (finite) quotient group of the centralizer $G_X$ (see [Lus84], 13.1, where the quotient group $\mathcal{G}_X$ is denoted by $\bar{A}(u)$). The quotient group and the canonical surjection $G_X \to \mathcal{G}_X$ are both defined over $\mathbf{R}$, and in fact the Galois group acts trivially on $\mathcal{G}_X$.

For $Y \in \mathfrak{g} \cap \mathbf{O}$ we choose $g \in G(\mathbf{C})$ such that $Y = \mathrm{Ad}(g)(X)$. Then $\sigma \mapsto g^{-1}\sigma(g)$ ($\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{R})$) is a 1-cocycle of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ in $G_X$, whose class $a_Y$ in $H^1(\mathbf{R}, G_X)$ is well-defined. Following Assem (except in terminology) we say that $Y$ is *stably conjugate* to $X$ if the image $\bar{a}_Y$ of $a_Y$ in $H^1(\mathbf{R}, \mathcal{G}_X)$ is trivial. Stable conjugacy is an equivalence relation, and the map $Y \mapsto \bar{a}_Y$ is an injection from the set of stable conjugacy classes in $\mathfrak{g} \cap \mathbf{O}$ to $H^1(\mathbf{R}, \mathcal{G}_X)$. In fact this injection is actually a bijection, not just over $\mathbf{R}$, but over any field of characteristic 0. (Only the orthogonal groups pose any difficulty; for them one needs to use the explicit description of Lusztig's quotient group.) Therefore the number of stable conjugacy classes in $\mathfrak{g} \cap \mathbf{O}$ is the cardinality of $H^1(\mathbf{R}, \mathcal{G}_X) = \mathcal{G}_X$. (The last equality follows from the fact that $\mathcal{G}_X$ is an elementary abelian 2-group on which the Galois group acts trivially.)

Let $X = X_1, \dots, X_r$ be representatives for the $G(\mathbf{R})$-conjugacy classes in the stable conjugacy class of $X$. Assem refers to $X_1, \dots, X_r$ as the *stability packet* of $X$. Consider the linear span of the orbital integrals $O_{X_1}, \dots, O_{X_r}$. Assem conjectures that the subspace of stably invariant elements in this linear span is 1-dimensional, and that the stable combinations so obtained, one for each stable conjugacy class, form a basis for $\mathcal{D}^{\mathrm{st}}_{\mathbf{O}}$. In case $G$ is split, Assem predicts which linear combination of $O_{X_1}, \dots, O_{X_r}$ is stable. Fix a non-zero $G(\mathbf{C})$-invariant differential form of top degree on $\mathbf{O}$ that is defined over $\mathbf{R}$. Associated to the differential form is a measure on $\mathfrak{g} \cap \mathbf{O}$. We use (the restriction of) this measure to define the orbital integrals $O_{X_1}, \dots, O_{X_r}$; then Assem predicts that

$$\sum_{i=1}^{r} O_{X_i}$$

is stable. In the quasi-split case he predicts only that one can obtain a stable combination by taking a linear combination of the form

$$\sum_{i=1}^{r} c_i \cdot O_{X_i}$$

with $c_i \in \{\pm 1\}$ for all $i$.

In any case Assem's conjecture implies that $s_{\mathbf{O}}$ should be equal to the number of stable conjugacy classes in $\mathfrak{g} \cap \mathbf{O}$, namely $|\mathcal{G}_X|$. This prediction is correct; it follows from the main theorem in this paper in conjunction with the main result of [Kot98]. Since Waldspurger has proved Assem's conjecture for unramified classical $p$-adic groups, and since Assem's conjecture in the real case predicts the correct value of $s_{\mathbf{O}}$, it seems likely that Assem's conjecture is valid in the real case.

## 2. Review of invariant eigendistributions

In this section we review Harish-Chandra's theory of invariant eigendistributions. The presentation in 2.6–2.9 has been strongly influenced by the paper [HK84] of Hotta and Kashiwara.

2.1. **Invariant eigendistributions of type** $\Lambda$**.** Let $\mathfrak{h}_a$ be the abstract Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$. It comes equipped with a root system $R_a \subset \mathfrak{h}_a^*$ and a positive system $R_a^+$ in $R_a$. The Weyl group $W_a$ of $R_a$ is called the abstract Weyl group of $\mathfrak{g}_{\mathbf{C}}$.

The algebra $S(\mathfrak{g}_{\mathbf{C}})^{G(\mathbf{C})}$ of $G$-invariant constant coefficient differential operators on $\mathfrak{g}$ can be identified with the algebra $\mathbf{C}[\mathfrak{h}_a^*]^{W_a}$ of $W_a$-invariant polynomial functions $P$ on $\mathfrak{h}_a^*$; we write $\partial(P)$ for the differential operator on $\mathfrak{g}$ corresponding to $P$.

Let $\Lambda$ be a $W_a$-orbit in $\mathfrak{h}_a^*$. Any polynomial $P \in \mathbf{C}[\mathfrak{h}_a^*]^{W_a}$ takes a constant value $P(\Lambda)$ on the orbit $\Lambda$. An *invariant eigendistribution of type* $\Lambda$ on $\mathfrak{g}$ is by definition a $G(\mathbf{R})$-invariant distribution $D$ on $\mathfrak{g}$ such that $\partial(P)(D) = P(\Lambda)D$ for all $P \in \mathbf{C}[\mathfrak{h}_a^*]^{W_a}$, and we denote by $\mathcal{D}^G(\Lambda)$ (or just $\mathcal{D}(\Lambda)$) the vector space of all invariant eigendistributions of type $\Lambda$.

2.2. **Fourier transforms of orbital integrals.** Choose a non-degenerate $G$-invariant symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$ and use it to identify $\mathfrak{g}$ with its dual. We define the Fourier transform $\hat{f}$ of a Schwartz function $f$ on $\mathfrak{g}$ by

$$\hat{f}(X) = \int_{\mathfrak{g}} f(Y) \exp(i(X,Y))\, dY,$$

where $dY$ is the self-dual Haar measure on $\mathfrak{g}$, and we extend the Fourier transform to tempered distributions in the usual way.

For any $X \in \mathfrak{g}$ we pick a non-zero $G(\mathbf{R})$-invariant measure on the $G(\mathbf{R})$-orbit of $X$ in $\mathfrak{g}$ and use it to define orbital integrals

$$O_X(f) = \int_{G(\mathbf{R})\cdot X} f$$

for Schwartz functions $f$ on $\mathfrak{g}$. The integral is convergent and $O_X$ is a tempered distribution on $\mathfrak{g}$. The Fourier transform $\hat{O}_X$ is an invariant eigendistribution, and if $X$ is nilpotent, then $\hat{O}_X$ lies in $\mathcal{D}(0)$.

2.3. **Lie algebra analogs of discrete series characters.** Now assume that $G(\mathbf{R})$ has a discrete series. In other words, assume that $G$ has an elliptic, maximal torus $T$. Let $W_{\mathbf{R}} \subset W^\sigma \subset W$ be the groups attached to $T$ in 1.6; since $T$ is elliptic, $W^\sigma$ equals $W$. The discrete series representations of $G(\mathbf{R})$ come in packets, each of size $[W : W_{\mathbf{R}}]$ (the packets consist of discrete series representations having the same infinitesimal and central characters).

Harish-Chandra constructed Lie algebra analogs of discrete series characters (see [HC65a], Thm. 2), in the following way. Start by taking invariant eigendistributions of the form $\hat{O}_X$ (as in 2.2) for regular elements $X \in \mathrm{Lie}(T)$. This provides Lie algebra analogs of discrete series characters with unitary central character. Let $A_G$ denote the maximal split torus in the center of $G$. As usual we regard $\mathrm{Lie}(A_G)$ as a direct summand of $\mathfrak{g}$. Now multiply $\hat{O}_X$ by functions on $\mathfrak{g}$ of the form $Y \mapsto \exp(\nu(Y))$, where $\nu \in \mathrm{Lie}(A_G)^*$; the resulting products are invariant eigendistributions, and they are the desired Lie algebra analogs of discrete series characters.

Let $H_a$ be the abstract maximal torus of $G_{\mathbf{C}}$; of course $\mathrm{Lie}(H_a) = \mathfrak{h}_a$ and $X^*(H_a) \otimes_{\mathbf{Z}} \mathbf{R}$ is a real structure on $\mathfrak{h}_a^*$. Let $\Lambda$ be a $W_a$-orbit in $\mathfrak{h}_a^*$. Suppose that $\Lambda$ is contained in the real subspace $X^*(H_a) \otimes_{\mathbf{Z}} \mathbf{R}$ (we then say that $\Lambda$ is *real*) and suppose further that each element in $\Lambda$ takes non-zero values on every (abstract) coroot $\alpha^\vee \in \mathfrak{h}_a$ (we then say that $\Lambda$ is *regular*). In this case the construction described above yields a $[W : W_{\mathbf{R}}]$-dimensional space of discrete series analogs in $\mathcal{D}(\Lambda)$.

2.4. **Parabolic induction.** Let $P = MN$ be a parabolic subgroup of $G$ with Levi component $M$ and unipotent radical $N$, and let $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ be the corresponding Lie algebras. Pick a maximal compact subgroup $K$ in $G(\mathbf{R})$ and define a continuous linear map $f \mapsto f^{(P)}$ from $C_c^\infty(\mathfrak{g})$ to $C_c^\infty(\mathfrak{m})$ by

$$f^{(P)}(X) = \int_{\mathfrak{n}} \int_K f(k \cdot (X+Y))\, dk\, dY \qquad (X \in \mathfrak{m}), \tag{2.4.1}$$

where $dk$ and $dY$ are Haar measures on $K$ and $\mathfrak{n}$ respectively. For an $M(\mathbf{R})$-invariant distribution $D$ on $\mathfrak{m}$ the complex number $D(f^{(P)})$ is independent of the choice of $K$ and the distribution $i_P^G(D)$ on $\mathfrak{g}$ defined by

$$i_P^G(D)(f) = D(f^{(P)})$$

is $G(\mathbf{R})$-invariant.

The linear map $i_P^G$ is Harish-Chandra's Lie algebra analog of parabolic induction from $M(\mathbf{R})$ to $G(\mathbf{R})$. Since the map $f \mapsto f^{(P)}$ extends continuously to the Schwartz spaces, parabolic induction takes tempered invariant distributions on $\mathfrak{m}$ to tempered invariant distributions on $\mathfrak{g}$. The linear map $i_P^G$ depends on the choice of Haar measures on $\mathfrak{g}$ and $\mathfrak{m}$; one can avoid this choice by regarding $i_P^G$ as a linear map from invariant generalized functions on $\mathfrak{m}$ to invariant generalized functions on $\mathfrak{g}$.

The map $f \mapsto f^{(P)}$ is compatible with the natural injective $\mathbf{C}$-algebra homomorphism

$$S(\mathfrak{g}_{\mathbf{C}})^{G(\mathbf{C})} \to S(\mathfrak{m}_{\mathbf{C}})^{M(\mathbf{C})},$$

and therefore $i_P^G(D)$ is an invariant eigendistribution on $\mathfrak{g}$ whenever $D$ is an invariant eigendistribution on $\mathfrak{m}$.

The map $f \mapsto f^{(P)}$ commutes with the Fourier transform. Therefore the map $i_P^G$ also commutes with the Fourier transform (of tempered invariant distributions).

2.5. **Invariant eigendistributions are functions.** Harish-Chandra proved (see [HC65b]) that any invariant eigendistribution $D$ on $\mathfrak{g}$ is represented by a locally integrable function $F$ that is real-analytic on the set $\mathfrak{g}'$ of regular semisimple elements in $\mathfrak{g}$ (choose a Haar measure on $\mathfrak{g}$ in order to do this, or else work with generalized functions rather than distributions). In particular, if $F$ is identically 0 on $\mathfrak{g}'$, then $D$ is 0.

Suppose that $D = i_P^G(D_M)$ for some invariant eigendistribution $D_M$ on $\mathfrak{m}$ (with notation as in 2.4), and let $F$ (respectively, $F_M$) be the locally integrable function on $\mathfrak{g}$ (respectively, $\mathfrak{m}$) representing $D$ (respectively, $D_M$). It is a well-known consequence of the Weyl integration formula that $F$ is obtained from $F_M$ in the following way. Let $X \in \mathfrak{g}'$. Then

$$F(X) = \sum_Y F_M(Y) \cdot |D_M^G(Y)|^{-1/2}, \tag{2.5.1}$$

where $Y$ runs over a set of representatives for the $M(\mathbf{R})$-conjugacy classes in the intersection of $\mathfrak{m}$ and the $G(\mathbf{R})$-conjugacy class of $X$, and where

$$D_M^G(Y) := \det(\mathrm{ad}(Y); \mathfrak{g}/\mathfrak{m}) \qquad (Y \in \mathfrak{m}).$$

Suppose that $M$ contains an elliptic maximal torus $T$. Then (2.5.1) becomes especially simple for regular elements $X \in \mathrm{Lie}(T)$:

$$F(X) = \sum_g F_M(g \cdot X) \cdot |D_M^G(g \cdot X)|^{-1/2} \tag{2.5.2}$$

where $g$ runs over the finite group $N_{G(\mathbf{R})}(M)/M(\mathbf{R})$.

2.6. **Properties of the locally integrable function $F$.** An *allowed embedding* $i : \mathfrak{h}_a \to \mathfrak{g}_{\mathbf{C}}$ is an injective $\mathbf{C}$-linear map whose image is the complexification of some Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ and which has the property that roots of $\mathfrak{h}_{\mathbf{C}}$ in $\mathfrak{g}_{\mathbf{C}}$ pull back under $i$ to elements in $R_a$. The Cartan subalgebra $\mathfrak{h}$ is uniquely determined by $i$, and we denote it by $\mathfrak{h}_i$. Moreover we denote by $\mathfrak{h}_a^i$ the subspace of $\mathfrak{h}_a$ obtained as the inverse image of $\mathfrak{h}$ under $i : \mathfrak{h}_a \to \mathfrak{h}_{\mathbf{C}}$. The subspace $\mathfrak{h}_a^i$ puts a real structure on $\mathfrak{h}_a$ and defines the notion of real and imaginary roots on $\mathfrak{h}_a^i$ (roots in $R_a$ that take real (respectively, imaginary) values on $\mathfrak{h}_a^i$). We define $(\mathfrak{h}_a^i)''$ to be the complement in $\mathfrak{h}_a^i$ of the real root hyperplanes.

Now fix a $W_a$-orbit $\Lambda$ in $\mathfrak{h}_a^*$ and consider an invariant eigendistribution $D \in \mathcal{D}(\Lambda)$. As in 2.5 we write $F$ for the corresponding real-analytic function on $\mathfrak{g}'$. For each allowed embedding $i$ we define a function $F_i$ on $\mathfrak{h}_a^i \cap \mathfrak{h}_a'$ (where $\mathfrak{h}_a'$ denotes the regular set in $\mathfrak{h}_a$) by

$$F_i(X) := \pi(X) F(i(X)) \qquad (X \in \mathfrak{h}_a^i \cap \mathfrak{h}_a'),$$

where $\pi$ is the polynomial function on $\mathfrak{h}_a$ defined as the product of all positive roots $\alpha \in R_a^+$.

Harish-Chandra [HC65b], Thm. 2 showed that each $F_i$ extends (uniquely, of course) to a real-analytic function on $(\mathfrak{h}_a^i)''$, which we still denote by $F_i$. The function $F_i$ satisfies the differential equations

$$\partial(P)(\varphi) = P(\Lambda)\varphi \qquad (P \in \mathbf{C}[\mathfrak{h}_a^*]^{W_a}). \tag{2.6.1}$$

Any solution $\varphi$ of the differential equations (2.6.1) on any connected component of $(\mathfrak{h}_a^i)''$ extends (again uniquely) to a holomorphic solution of the same differential equations on $\mathfrak{h}_a$.

Write $\mathcal{H}(\Lambda)$ for the space of holomorphic solutions of the differential equations (2.6.1) on $\mathfrak{h}_a$ (the advantage of this space is that it is independent of $i$). Then (see [Ste64]) $\mathcal{H}(\Lambda)$ is the space of functions on $\mathfrak{h}_a$ of the form

$$X \mapsto \sum_{\lambda \in \Lambda} \Phi_\lambda(X) \cdot \exp(\lambda(X)),$$

where $\Phi_\lambda$ is a $W_\lambda$-harmonic polynomial on $\mathfrak{h}_a$. Here $W_\lambda$ denotes the stabilizer of $\lambda$ in $W_a$, and by a $W_\lambda$-harmonic polynomial on $\mathfrak{h}_a$ we mean a polynomial on $\mathfrak{h}_a$ that is annihilated by all constant coefficient differential operators $\partial(P)$ on $\mathfrak{h}_a$ with $P \in \mathbf{C}[\mathfrak{h}_a^*]^{W_\lambda}$ and $P(0) = 0$. The group $W_a$ acts on $\mathcal{H}(\Lambda)$, and the resulting representation of $W_a$ is isomorphic to the regular representation (see [Ste64]).

It follows from this discussion that for each connected component $C$ of $(\mathfrak{h}_a^i)''$ there is a unique element $F_{i,C} \in \mathcal{H}(\Lambda)$ whose restriction to $C$ coincides with that of $F_i$. The group $G(\mathbf{R})$ acts on the left of the set $S$ of pairs $s = (i, C)$ consisting of an allowed embedding $i$ and a connected component $C$ of $(\mathfrak{h}_a^i)''$ ($g \in G(\mathbf{R})$ sends $(i, C)$ to $(\mathrm{Ad}(g) \circ i, C)$) and the group $W_a$ acts on the right of $S$ ($w \in W_a$ sends $(i, C)$ to $(i \circ w, w^{-1}(C))$. The two actions commute and the quotient set $G(\mathbf{R})\backslash S/W_a$ can be identified with the set of $G(\mathbf{R})$-conjugacy classes of Cartan subalgebras of $\mathfrak{g}$ (associate to $s = (i, C)$ the Cartan subalgebra $\mathfrak{h}_i$).

The $G(\mathbf{R})$-invariance of $D$ is equivalent to the following property of the family of functions $F_s$ ($s \in S$):

$$F_{gs} = F_s \qquad \text{for all } g \in G(\mathbf{R}). \tag{2.6.2}$$

Moreover it is obvious from the definition of $F_s$ that

$$w(F_s) = \epsilon(w) F_{sw^{-1}} \qquad \text{for all } w \in W_a. \tag{2.6.3}$$

(We are using the obvious left action of $W_a$ on $\mathcal{H}(\Lambda)$.)

Harish-Chandra proved that the family of functions $F_s$ satisfies a matching condition [HC65b], Lemma 18, which we now recall. Let $i$ be an allowed embedding and let $\alpha$ be a real root for $\mathfrak{h}_i$ (we use $i$ to regard $\alpha$ as a real root of $\mathfrak{h}_a^i$ as well). Then $\alpha$ determines a connected subgroup $M \subset G$ whose Lie algebra $\mathfrak{m}$ is the direct sum of $\mathfrak{h}_i$ and the root spaces for $\pm\alpha$ (thus the derived subalgebra of $\mathfrak{m}$ is isomorphic to $\mathfrak{sl}_2(\mathbf{R})$). Choose $m \in M(\mathbf{C})$ such that $j := \mathrm{Ad}(m) \circ i$ is an allowed embedding having the property that $\mathfrak{h}_j$ is an elliptic Cartan subalgebra of $\mathfrak{m}$. Note that the set of real roots for $\mathfrak{h}_a^j$ is a subset of the set of real roots for $\mathfrak{h}_a^i$. Moreover

$$\mathfrak{h}_a^i \cap \ker(\alpha) = \mathfrak{h}_a^i \cap \mathfrak{h}_a^j = \mathfrak{h}_a^j \cap \ker(\alpha).$$

Now let $C$ be a connected component of $(\mathfrak{h}_a^i)''$ such that $\ker(\alpha)$ is a wall of $C$. Let $\overline{C}$ be the closure of $C$ in $\mathfrak{h}_a^i$; then the interior of $\overline{C} \cap \ker(\alpha)$ is contained in a unique connected component $\tilde{C}$ of $\mathfrak{h}_a^j$. Let $w_\alpha \in W_a$ denote the reflection in $\alpha$. Then the functions $F_s$ ($s \in S$) satisfy the following condition. For all $i$, $\alpha$, $j$, $C$, $\tilde{C}$ as above, for all constant coefficient differential operators $\partial$ on $\mathfrak{h}_a$ such that $w_\alpha(\partial) = -\partial$

$$(\partial F_{i,C})(X) = (\partial F_{j,\tilde{C}})(X)$$

for all $X \in \mathfrak{h}_a$ such that $\alpha(X) = 0$. It is easy to see that a holomorphic function $f$ on $\mathfrak{h}_a$ satisfies

$$(\partial f)(X) = 0$$

for all $X \in \ker(\alpha)$ and all constant coefficient differential operators $\partial$ such that $w_\alpha(\partial) = -\partial$ if and only if it satisfies $w_\alpha(f) = f$. Therefore Harish-Chandra's matching condition on the family $F_s$ ($s \in S$) can be reformulated as follows: for all $i$, $\alpha$, $j$, $C$, $\tilde{C}$ as above

(2.6.4) $$F_{i,C} - F_{j,\tilde{C}} \text{ is fixed by } w_\alpha.$$

2.7. **The spaces $\mathcal{F}(\Lambda)$ and $\mathcal{F}(V)$.** We now define $\mathcal{F}(\Lambda)$ to be the vector space consisting of all families $(F_s)_{s\in S}$ of elements $F_s \in \mathcal{H}(\Lambda)$ satisfying the three conditions (2.6.2), (2.6.3), (2.6.4). The discussion in 2.6 then yields an injection

(2.7.1) $$\mathcal{D}(\Lambda) \to \mathcal{F}(\Lambda),$$

which turns out to be an isomorphism (see 2.9).

Note that the conditions (2.6.2), (2.6.3), (2.6.4) involve only the $W_a$-module structure on $\mathcal{H}(\Lambda)$. Now let $V$ be any finite dimensional $W_a$-module. We define $\mathcal{F}(V)$ to be the vector space consisting of all collections $(F_s)_{s\in S}$ of elements $F_s \in V$ satisfying the three conditions (2.6.2), (2.6.3), (2.6.4). It is obvious that

(2.7.2) $$\mathcal{F}(V_1 \oplus V_2) = \mathcal{F}(V_1) \oplus \mathcal{F}(V_2)$$

and that the dimension of $\mathcal{F}(V)$ depends only on the isomorphism class of $V$. In particular the dimension of $\mathcal{F}(\Lambda)$ is independent of $\Lambda$, since $\mathcal{H}(\Lambda)$ is always isomorphic to the regular representation of $W_a$.

2.8. **Filtration on $\mathcal{F}(V)$.** Let $V$ be a finite-dimensional $W_a$-module. We now define a decreasing filtration

$$\mathcal{F}(V) = \mathcal{F}(V)_0 \supset \mathcal{F}(V)_1 \supset \mathcal{F}(V)_2 \supset \cdots$$

on $\mathcal{F}(V)$. To do so we first define the *split rank* of a maximal torus $T$ in $G$ (or of the corresponding Cartan subalgebra Lie$(T)$) to be $\dim(A_T/A_G)$, where $A_T$ is the maximal split torus in $T$ and $A_G$ is the maximal split torus in the center of $G$. Thus $T$ is elliptic if and only if its split rank is 0. Now define $\mathcal{F}(V)_r$ to be the subspace of $\mathcal{F}(V)$ consisting of all families $(F_s)_{s\in S}$ such that $F_s = 0$ for all $s = (i, C)$ such that the split rank of $\mathfrak{h}_i$ is strictly less than $r$. It is obvious that

(2.8.1) $$\mathcal{F}(V_1 \oplus V_2)_r = \mathcal{F}(V_1)_r \oplus \mathcal{F}(V_2)_r.$$

Fix $r \geq 0$. Consider a family $(F_s)_{s\in S}$ lying in $\mathcal{F}(V)_r$, and let $s = (i, C)$ be an element of $S$ such that the split rank of $\mathfrak{h}_i$ is $r$. Then the condition (2.6.4) reduces to

(2.8.2) $$w_\alpha(F_s) = F_s$$

for every real root $\alpha$ of $\mathfrak{h}_i$ such that $\ker(\alpha)$ is a wall of $C$ (since the split rank of $\mathfrak{h}_j$ is strictly less than $r$). Let $\mathcal{T}_r$ be a set of representatives for the $G(\mathbf{R})$-conjugacy classes of maximal tori $T$ in $G$ with split rank equal to $r$. For each $T \in \mathcal{T}_r$ we choose an element $s_T = (i, C) \in S$ such that $\mathfrak{h}_i = \mathrm{Lie}(T)$, and we use $i$ to identify $W_{\mathbf{R}}$ with a subgroup of $W_a$ (recall from 1.6 the objects $W$, $W_{\mathbf{R}}$, $W^\sigma$, $\epsilon_{\mathbf{R}}$, $\epsilon_I$, $\epsilon$ associated to $T$). Then $W_{\mathbf{R}} = W_1 \ltimes W_2$ where $W_1$ is the stabilizer of $C$ in $W_{\mathbf{R}}$ and $W_2$ is the (normal) subgroup of $W_{\mathbf{R}}$ generated by reflections in real roots $\alpha$ of $T$. Using that $\epsilon = \epsilon_{\mathbf{R}}\epsilon_I$, one sees that $\epsilon_I$ is trivial on $W_2$ and equal to $\epsilon$ on $W_1$. We claim that $w(F_{s_T}) = \epsilon_I(w)F_{s_T}$ for all $w \in W_{\mathbf{R}}$; indeed, for $w \in W_1$ this follows from (2.6.2) and (2.6.3) and for $w \in W_2$ it follows from (2.8.2).

We conclude that the map $(F_s)_{s\in S} \mapsto (F_{s_T})_{T\in\mathcal{T}_r}$ induces an injection

$$\mathcal{F}(V)_r/\mathcal{F}(V)_{r+1} \to \bigoplus_{T\in\mathcal{T}_r} V^{W_{\mathbf{R}},\epsilon_I}, \tag{2.8.3}$$

where $V^{W_{\mathbf{R}},\epsilon_I}$ denotes the subspace of $V$ consisting of elements that transform by $\epsilon_I$ under $W_{\mathbf{R}}$.

We claim that the maps (2.8.3) are actually isomorphisms. Using (2.8.1), we see that it suffices to prove this in the special case that $V$ is the regular representation. So take $V$ to be $\mathcal{H}(\Lambda)$, for some orbit $\Lambda$ that is real and regular (in the sense of 2.3).

Suppose first that $r = 0$ and that $\mathcal{T}_0$ is non-empty (*i.e.* $G$ has elliptic maximal tori). Then Harish-Chandra's Lie algebra analogs of discrete series characters (see 2.3) provide enough elements in $\mathcal{F}(\Lambda)$ to show that (2.8.3) is surjective (for $r = 0$ and $V = \mathcal{H}(\Lambda)$) (use [HC65a], Thm. 2).

Now consider any $r \geq 0$. Let $T \in \mathcal{T}_r$ and let $M$ denote the centralizer of $A_T$ in $G$. Then, by inducing discrete series analogs on $\mathfrak{m} = \mathrm{Lie}(M)$, we get enough elements in $\mathcal{F}(\Lambda)_r$ to show that the image of (2.8.3) contains the direct summand $\mathcal{H}(\Lambda)^{W_{\mathbf{R}},\epsilon_I}$ indexed by $T \in \mathcal{T}_r$. Here we used (2.5.2) and the exact sequence

$$1 \to W_{\mathbf{R}}^M \to W_{\mathbf{R}} \to N_{G(\mathbf{R})}(M)/M(\mathbf{R}) \to 1,$$

where $W_{\mathbf{R}}^M$ is the analog of $W_{\mathbf{R}}$ for the group $M$. Therefore (2.8.3) is indeed surjective.

The fact that the maps (2.8.3) are isomorphisms implies that

$$\dim \mathcal{F}(V) = \sum_T \dim(V^{W_{\mathbf{R}},\epsilon_I}), \tag{2.8.4}$$

where $T$ runs over a set of representatives for the $G(\mathbf{R})$-conjugacy classes of maximal tori in $G$.

2.9. **The isomorphism $\mathcal{D}(\Lambda) \to \mathcal{F}(\Lambda)$.** In 2.7 we defined an injection (2.7.1) from $\mathcal{D}(\Lambda)$ to $\mathcal{F}(\Lambda)$. We will now check that (2.7.1) is an isomorphism. First suppose that $\Lambda$ is real and regular. Then in 2.8 we saw that parabolic induction of discrete series analogs provides enough elements in $\mathcal{D}(\Lambda)$ to get all elements in $\mathcal{F}(\Lambda)_r$ modulo $\mathcal{F}(\Lambda)_{r+1}$. Since $\mathcal{F}(\Lambda)_r$ is $\{0\}$ for large $r$, we see that $\mathcal{D}(\Lambda)$ does indeed map onto $\mathcal{F}(\Lambda)$.

Now consider arbitrary $\Lambda$. Let $G_{\mathrm{sc}}$ be the simply connected cover of the derived group of $G$. Then the spaces $\mathcal{D}(\Lambda)$, $\mathcal{F}(\Lambda)$ for $G$ are obtained by taking invariants under $G(\mathbf{R})/\operatorname{im} G_{\mathrm{sc}}(\mathbf{R})$ in the spaces $\mathcal{D}(\Lambda)$, $\mathcal{F}(\Lambda)$ for $G_{\mathrm{sc}}$. Thus we may as well assume that $G$ is simply connected, in which case $G(\mathbf{R})$ is connected. Then elements in $\mathcal{D}(\Lambda)$ can be identified with eigendistributions on $\mathfrak{g}$ that are annihilated by the adjoint action of $\mathfrak{g}$. Therefore it follows from [HK84], 6.7.3 that the dimension of $\mathcal{D}(\Lambda)$ is independent of $\Lambda$. As was noted in 2.7, the dimension of $\mathcal{F}(\Lambda)$ is also independent of $\Lambda$. Since we know that $\mathcal{D}(\Lambda) \to \mathcal{F}(\Lambda)$ is injective, it is enough to show that $\dim \mathcal{D}(\Lambda)$ and $\dim \mathcal{F}(\Lambda)$ are equal, which follows from the real, regular case that has already been treated.

## 3. Invariant eigendistributions of type 0

3.1. **The goal.** Let $\mathbf{O}$ be a nilpotent $G(\mathbf{C})$-orbit in $\mathfrak{g}_{\mathbf{C}}$. We let $\mathcal{D}_{\mathbf{O}}$ and $r_{\mathbf{O}}$ be as in 1.5. We consider the composed map

$$\mathcal{D}_{\mathbf{O}} \xrightarrow{\text{FT}} \mathcal{D}(0) \xrightarrow{(2.7.1)} \mathcal{F}(0), \tag{3.1.1}$$

where the first map is the Fourier transform (see 2.2). The map (3.1.1) is of course injective; we wish to describe its image, using Springer theory. As was mentioned in the introduction, such a description is implicit in the literature.

3.2. **The character $\chi_{\mathbf{O}}$ and integer $d_{\mathbf{O}}$.** The Springer correspondence [Spr76] associates to $\mathbf{O}$ an irreducible character $\chi_{\mathbf{O}}$ of the abstract Weyl group $W_a$. We normalize the correspondence so that the trivial orbit $\mathbf{O} = \{0\}$ corresponds to the sign character $\epsilon$ of $W_a$ and the regular orbit $\mathbf{O}$ corresponds to the trivial character 1 of $W_a$. We also associate to $\mathbf{O}$ a non-negative integer $d_{\mathbf{O}}$, defined by

$$d_{\mathbf{O}} = |R_a^+| - \tfrac{1}{2}\dim \mathbf{O}. \tag{3.2.1}$$

Recall that for any $X \in \mathbf{O}$ the dimension of the variety of Borel subalgebras of $\mathfrak{g}_{\mathbf{C}}$ containing $X$ is equal to $d_{\mathbf{O}}$.

3.3. **Gradings on $\mathcal{F}(0)$.** For $\Lambda = 0$ the space $\mathcal{H}(\Lambda)$ (see 2.6) is simply the space $\mathcal{H}$ of $W_a$-harmonic polynomials on $\mathfrak{h}_a$. The $W_a$-module $\mathcal{H}$ has several direct sum decompositions. For $d \geq 0$ let $\mathcal{H}_d$ be the space of harmonic polynomials of degree $d$; then $\mathcal{H} = \bigoplus_{d\geq 0} \mathcal{H}_d$. For $\chi \in W_a^\vee$ (where $W_a^\vee$ denotes the set of irreducible characters of $W_a$) let $\mathcal{H}_\chi$ denote the $\chi$-isotypic component of $\mathcal{H}$; then $\mathcal{H} = \bigoplus_{\chi\in W_a^\vee} \mathcal{H}_\chi$. For $d \geq 0$ and $\chi \in W_a^\vee$ let $\mathcal{H}_{d,\chi} = \mathcal{H}_d \cap \mathcal{H}_\chi$; then $\mathcal{H} = \bigoplus_{d\geq 0} \oplus_{\chi\in W_a^\vee} \mathcal{H}_{d,\chi}$. Each of these direct sum decompositions gives rise to a corresponding direct sum decomposition of $\mathcal{F}(0)$ (see 2.7.2):

$$\mathcal{F}(0) = \bigoplus_{d\geq 0} \mathcal{F}(0)_d \tag{3.3.1}$$

$$= \bigoplus_{\chi\in W_a^\vee} \mathcal{F}(0)_\chi \tag{3.3.2}$$

$$= \bigoplus_{d\geq 0} \bigoplus_{\chi\in W_a^\vee} \mathcal{F}(0)_{d,\chi}. \tag{3.3.3}$$

3.4. **Description of the image of $\mathcal{D}_{\mathbf{O}}$ in $\mathcal{F}(0)$.** We will now show that the image of $\mathcal{D}_{\mathbf{O}}$ in $\mathcal{F}(0)$ is equal to $\mathcal{F}(0)_{d_{\mathbf{O}},\chi_{\mathbf{O}}}$. As in 2.9 it is enough to consider the case in which $G$ is simply connected, so that $G(\mathbf{R})$ is connected. Then $\mathcal{D}(0)$ is the space of distributions on $\mathfrak{g}$ that are solutions of the cyclic $D$-module denoted by $\mathcal{M}_0^F$ in [HK84]. Here $D$ is the Weyl algebra of differential operators on $\mathfrak{g}_{\mathbf{C}}$ with polynomial coefficients.

All the distributions in $\mathcal{D}(0)$ are tempered. Let $\hat{\mathcal{D}}(0)$ denote the space of Fourier transforms of distributions in $\mathcal{D}(0)$. Of course $\hat{\mathcal{D}}(0)$ is the space of tempered distributions on $\mathfrak{g}$ that are solutions of the Fourier transform $\mathcal{M}_0$ of the $D$-module $\mathcal{M}_0^F$.

Hotta and Kashiwara decompose the $D$-module $\mathcal{M}_0$ in two ways. This comes about as follows. Both $\mathcal{M}_0$ and $\mathcal{M}_0^F$ are completely regular holonomic $D$-modules. Under the Riemann-Hilbert correspondence $\mathcal{M}_0^F$ goes over to the middle extension of the following local system on $\mathfrak{g}_{\mathrm{rs}}$, the set of regular semisimple elements in $\mathfrak{g}_{\mathbf{C}}$. Let $\tilde{\mathfrak{g}}_{\mathrm{rs}}$ denote the set of pairs $(X, \mathfrak{b})$, where $X \in \mathfrak{g}_{\mathrm{rs}}$ and $\mathfrak{b}$ is a Borel subalgebra in $\mathfrak{g}_{\mathbf{C}}$ containing $X$. Then $\tilde{\mathfrak{g}}_{\mathrm{rs}}$ is an unramified Galois covering of $\mathfrak{g}_{\mathrm{rs}}$ with Galois group $W_a$. Therefore any finite dimensional representation of $W_a$ gives rise to a local system on $\mathfrak{g}_{\mathrm{rs}}$. The local system coming from $\mathcal{M}_0^F$ corresponds to the representation $\mathcal{H} \otimes \epsilon$ of $W_a$ (as usual $\mathcal{H}$ is the space of harmonic polynomials on $\mathfrak{h}_a$, and $\epsilon$ is the sign character on $W_a$). Therefore any direct sum decomposition of the $W_a$-module $\mathcal{H}$ gives rise to direct sum decompositions of $\mathcal{M}_0^F$ and of its Fourier transform $\mathcal{M}_0$. In particular the decompositions of $\mathcal{H}$ considered in 3.3 give rise to decompositions

$$\mathcal{M}_0 = \bigoplus_{d \geq 0} \mathcal{M}_0(d), \tag{3.4.1}$$

$$\mathcal{M}_0 = \bigoplus_{\chi \in W_a^\vee} \mathcal{M}_0(\chi), \tag{3.4.2}$$

$$\mathcal{M}_0 = \bigoplus_{d \geq 0} \bigoplus_{\chi \in W_a^\vee} \mathcal{M}_0(d, \chi). \tag{3.4.3}$$

The $D$-module $\mathcal{M}_0$ is homogeneous (with respect to the obvious scaling action of the group of positive real numbers on $\mathfrak{g}$), and (3.4.1) is its homogeneous decomposition [HK84], §7 (with a different indexing). Moreover the $D$-module $\mathcal{M}_0$ is semisimple and (3.4.2) is its isotypic decomposition [HK84].

Thus our problem becomes the following. Let $\psi$ be a $D$-module map from $\mathcal{M}_0$ to the space of tempered distributions on $\mathfrak{g}$, and let $x \in \hat{\mathcal{D}}(0)$ be the image under $\psi$ of the canonical generator of $\mathcal{M}_0$. We must show that $x \in \mathcal{D}_{\mathbf{O}}$ if and only if $\psi$ factors through the projection map

$$\mathcal{M}_0 \to \mathcal{M}_0(d_{\mathbf{O}}, \chi_{\mathbf{O}}).$$

The image of $\psi$ is the cyclic $D$-module $D \cdot x$ generated by $x$. It follows from [HK84], Cor. 7.1.4 that $\mathcal{M}_0(d_{\mathbf{O}}, \chi_{\mathbf{O}})$ is a simple $D$-module supported on $\overline{\mathbf{O}}$ (the closure of $\mathbf{O}$) and that $\mathcal{H}_{d_{\mathbf{O}}, \chi_{\mathbf{O}}}$ is an irreducible $W_a$-module whose character is $\chi_{\mathbf{O}}$.

Now suppose that $x \in \mathcal{D}_{\mathbf{O}}$. We want to show that $\psi$ factors through $\mathcal{M}_0(d_{\mathbf{O}}, \chi_{\mathbf{O}})$. It follows from [HK84], Cor. 7.1.5 that $D \cdot x$ is either $\{0\}$ or isomorphic to $\mathcal{M}_0(d_{\mathbf{O}}, \chi_{\mathbf{O}})$. Therefore $\psi$ factors through projection on the $\mathcal{M}_0(d_{\mathbf{O}}, \chi_{\mathbf{O}})$-isotypic component of $\mathcal{M}_0$, namely $\mathcal{M}_0(\chi_{\mathbf{O}})$. Since $x$ is homogeneous of degree $\dim(\mathfrak{g}) - \frac{1}{2}\dim(\mathbf{O})$ (with respect to scaling by positive real numbers), $\psi$ also factors through projection on $\mathcal{M}_0(d_{\mathbf{O}})$. Therefore $\psi$ factors through $\mathcal{M}_0(d_{\mathbf{O}}, \chi_{\mathbf{O}})$, as desired.

Now suppose that $\psi$ factors through projection on $\mathcal{M}_0(d_{\mathbf{O}}, \chi_{\mathbf{O}})$. Then $x$ is homogeneous of degree $\dim(\mathfrak{g}) - \frac{1}{2}\dim(\mathbf{O})$ (since $\psi$ factors through $\mathcal{M}_0(d_{\mathbf{O}})$) and $x$ is supported on $\overline{\mathbf{O}} \cap \mathfrak{g}$ (since $\psi$ factors through $\mathcal{M}_0(d_{\mathbf{O}}, \chi_{\mathbf{O}})$ and $\mathcal{M}_0(d_{\mathbf{O}}, \chi_{\mathbf{O}})$ is supported on $\overline{\mathbf{O}}$). But any invariant distribution on $\mathfrak{g}$ with these two properties lies in $\mathcal{D}_{\mathbf{O}}$ by [BV80], Cor. 3.9.

3.5. **Rossmann's formula for $r_{\mathbf{O}}$.** It follows from 3.4 that the injection (3.1.1) identifies $\mathcal{D}_{\mathbf{O}}$ with $\mathcal{F}(\mathcal{H}_{d_{\mathbf{O}}, \chi_{\mathbf{O}}})$ (in the notation of 2.7). Using (2.8.4) to calculate the dimension of $\mathcal{F}(\mathcal{H}_{d_{\mathbf{O}},\chi_{\mathbf{O}}})$ and recalling that the character of $\mathcal{H}_{d_{\mathbf{O}},\chi_{\mathbf{O}}}$ is $\chi_{\mathbf{O}}$ (see 3.4), we see that

$$r_{\mathbf{O}} = \dim \mathcal{D}_{\mathbf{O}} = \sum_T m_{W_{\mathbf{R}}}(\epsilon_I, \chi_{\mathbf{O}}) \tag{3.5.1}$$

(notation as in the introduction). This is Rossmann's formula [Ros90] for $r_{\mathbf{O}}$.

## 4. Stably invariant tempered distributions

4.1. **Induction.** Let $P = MN$ be a parabolic subgroup in $G$ and write $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ for the corresponding Lie algebras. In 2.4 we reviewed the parabolic induction map $i_P^G$ from invariant distributions on $\mathfrak{m}$ to invariant distributions on $\mathfrak{g}$. Recall that for any $G$-regular semisimple element $Y \in \mathfrak{m}$

$$i_P^G(O_Y) = |D_M^G(X)|^{1/2} \cdot O_X \tag{4.1.1}$$

where $X$ is the image of $Y$ under the inclusion of $\mathfrak{m}$ in $\mathfrak{g}$, and where $D_M^G(X) := \det(\mathrm{ad}(X); \mathfrak{g}/\mathfrak{m})$. Using the well-known fact that

$$H^1(\mathbf{R}, M) \to H^1(\mathbf{R}, G) \tag{4.1.2}$$

is injective, one sees that if $Y_1, \dots, Y_r$ are a set of representatives for the $M(\mathbf{R})$-conjugacy classes in the stable conjugacy class of $Y$, then their images $X_1, \dots, X_r$ under the inclusion of $\mathfrak{m}$ in $\mathfrak{g}$ are representatives for the $G(\mathbf{R})$-conjugacy classes in the stable conjugacy class of $X$. It follows that

$$i_P^G(SO_Y) = |D_M^G(X)|^{1/2} \cdot SO_X. \tag{4.1.3}$$

We claim that if $D$ is any stably invariant tempered distribution on $\mathfrak{m}$, then $i_P^G(D)$ is stably invariant. Indeed, by (4.1.3) this is true if $D = SO_Y$ for $Y$ as above; therefore it is true in general, since the span of the distributions $SO_Y$ is dense in the space of stably invariant tempered distributions on $\mathfrak{m}$.

4.2. **Stably invariant eigendistributions.** Let $D$ be an invariant eigendistribution on $\mathfrak{g}$ and as before let $F$ be the corresponding real-analytic function on $\mathfrak{g}'$ (locally integrable on $\mathfrak{g}$). Then $D$ need not be tempered, so that our definition of stable invariance might not apply to $D$. However, if $D$ happens to be tempered, then $D$ is stably invariant if and only if the function $F$ is constant on every stable conjugacy class in $\mathfrak{g}'$. (The proof of this statement is essentially the same as Shelstad's proof of the analogous statement [She79], 5.1 on the group $G(\mathbf{R})$ and will therefore be omitted.) Thus it is reasonable to make the following definition: the (possibly non-tempered) invariant eigendistribution $D$ is said to be *stably invariant* if the function $F$ is constant on every stable conjugacy class in $\mathfrak{g}'$.

Let us make this definition more explicit. Let $T$ be a maximal torus in $G$, and let $W_{\mathbf{R}} \subset W^\sigma \subset W$ be as in 1.6. We need yet another subgroup of $W$, namely the imaginary Weyl group. Let $A$ be the maximal split subtorus in $T$, and let $M$ be the centralizer of $A$ in $G$. Then $M$ is a Levi subgroup of $G$, and $T$ is an elliptic maximal torus in $M$. The *imaginary Weyl group* of $T$ is the Weyl group $W_M$ of $T(\mathbf{C})$ in $M(\mathbf{C})$ (the roots of $T$ in $M$ are precisely the imaginary roots of $T$ in $G$); we view $W_M$ as a subgroup of $W$ and note that $W_M \subset W^\sigma$.

Let $X \in \mathrm{Lie}(T) \cap \mathfrak{g}'$. Then (see [She79], §2) the set $\mathcal{S}_M$ of $M(\mathbf{R})$-conjugacy classes within the stable conjugacy class of $X$ in $\mathfrak{m} = \mathrm{Lie}(M)$ corresponds bijectively to the set

$$(W_{\mathbf{R}} \cap W_M)\backslash W_M,$$

and the set $\mathcal{S}_G$ of $G(\mathbf{R})$-conjugacy classes within the stable conjugacy class of $X$ in $\mathfrak{g}$ corresponds bijectively to the set

$$W_{\mathbf{R}}\backslash W^\sigma;$$

since the natural map $\mathcal{S}_M \to \mathcal{S}_G$ is bijective (see 4.1) it follows that the natural map

$$(W_{\mathbf{R}} \cap W_M)\backslash W_M \to W_{\mathbf{R}}\backslash W^\sigma$$

is bijective (in other words, $W^\sigma = W_{\mathbf{R}} W_M$).

We conclude that the invariant eigendistribution $D$ is stable if and only if the function $F$ satisfies the following property for all maximal tori $T$ in $G$; the restriction of $F$ to $\mathrm{Lie}(T) \cap \mathfrak{g}'$ is invariant under $W^\sigma$, or, equivalently, invariant under the imaginary Weyl group $W_M$.

4.3. **Fourier transform.** Let $D$ be a stably invariant tempered distribution on $\mathfrak{g}$. We claim that the Fourier transform $\hat{D}$ of $D$ is also stably invariant. Indeed, unwinding the definition of stable invariance, we see that it is enough to show that the Fourier transform of $SO_X$ is stably invariant for all regular semisimple $X \in \mathfrak{g}$. Since $i_P^G$ preserves stability (see 4.1) and commutes with the Fourier transform (see 2.4), it is enough to show that the Fourier transform of $SO_X$ is stably invariant for all *elliptic* regular semisimple $X \in \mathfrak{g}$. We will check this in 4.4.

4.4. **Lie algebra analogs of stable discrete series characters.** We now resume the discussion in 2.3. We suppose that $G$ has an elliptic maximal torus $T_e$, and let $X$ be a regular element in $\mathrm{Lie}(T_e)$. We claim that the Fourier transform $\widehat{SO}_X$ of $SO_X$ is stably invariant.

Let $F$ be the real-analytic function on $\mathfrak{g}'$ that represents the tempered invariant distribution $\widehat{SO}_X$. To show that $\widehat{SO}_X$ is stable we must show that for every maximal torus $T$ in $G$, the restriction of $F$ to $\mathrm{Lie}(T) \cap \mathfrak{g}'$ is invariant under the imaginary Weyl group. In case $T$ is elliptic this restriction is given by a theorem of Rossmann (see the second corollary on p. 217 of [Ros78]) and the desired invariance (under all of $W$ in this case) is clear from Rossmann's result. The desired invariance for non-elliptic $T$ follows from the invariance for elliptic $T$, just as for stable discrete series characters on $G(\mathbf{R})$ (see the first part of the proof of Lemma 5.2 in [She79], which relies on Lemma 61 of [HC65a], and note that Harish-Chandra's proof of his Lemma 61 works equally well for Lie algebras).

As in 2.3 we can multiply $\widehat{SO}_X$ by functions on $\mathfrak{g}$ of the form $Y \mapsto \exp(\nu(Y))$, where $\nu \in \mathrm{Lie}(A_G)^*$; the resulting products are stably invariant eigendistributions (possibly non-tempered) and are the desired Lie algebra analogs of the stable discrete series characters on $G(\mathbf{R})$ (see [She79]).

Let $\Lambda$ be a $W_a$-orbit in $\mathfrak{h}_a^*$ and suppose that $\Lambda$ is real and regular (in the sense of 2.3). Then the construction above produces a 1-dimensional space of stably invariant eigendistributions of type $\Lambda$, and any non-zero element in this space has non-zero restriction to the elliptic Cartan subalgebra $\mathrm{Lie}(T_e)$ (again by Rossmann's theorem [Ros78]).

## 5. Stably invariant eigendistributions of type $\Lambda$

5.1. **The spaces $\mathcal{D}_{\mathrm{st}}(\Lambda)$ and $\mathcal{F}_{\mathrm{st}}(\Lambda)$.** Let $\Lambda$ be a $W_a$-orbit in $\mathfrak{h}_a^*$. Let $\mathcal{D}_{\mathrm{st}}(\Lambda)$ denote the space of stably invariant eigendistributions on $\mathfrak{g}$ of type $\Lambda$. Of course $\mathcal{D}_{\mathrm{st}}(\Lambda)$ is a subspace of $\mathcal{D}(\Lambda)$.

Let $\mathcal{F}_{\mathrm{st}}(\Lambda)$ denote the subspace of $\mathcal{F}(\Lambda)$ consisting of collections $(F_s)_{s\in S}$ satisfying the following additional condition: for every $s=(i,C)\in S$

$$F_{i,C} = F_{\mathrm{Ad}(g)\circ i,C} \tag{5.1.1}$$

for all $g\in G(\mathbf{C})$ such that the restriction of $\mathrm{Ad}(g)$ to $\mathfrak{h}_i$ is defined over $\mathbf{R}$ (recall that $\mathfrak{h}_i$ is the Cartan subalgebra of $\mathfrak{g}$ whose complexification is $i(\mathfrak{h}_a)$). It is clear from the discussion in 4.2 that the isomorphism (2.7.1) from $\mathcal{D}(\Lambda)$ to $\mathcal{F}(\Lambda)$ restricts to an isomorphism

$$\mathcal{D}_{\mathrm{st}}(\Lambda)\to\mathcal{F}_{\mathrm{st}}(\Lambda). \tag{5.1.2}$$

5.2. **The spaces $\mathcal{F}_{\mathrm{st}}(V)$.** Let $V$ be a finite dimensional representation of $W_a$. Just as in 2.7 we can define a variant $\mathcal{F}_{\mathrm{st}}(V)$ of $\mathcal{F}_{\mathrm{st}}(\Lambda)$; thus $\mathcal{F}_{\mathrm{st}}(V)$ consists of families $(F_s)_{s\in S}$ of elements $F_s\in V$ satisfying (2.6.2), (2.6.3), (2.6.4) and the additional condition (5.1.1) (of course (2.6.2) can be dropped since it follows from (5.1.1)). It is obvious that

$$\mathcal{F}_{\mathrm{st}}(V_1\oplus V_2)=\mathcal{F}_{\mathrm{st}}(V_1)\oplus\mathcal{F}_{\mathrm{st}}(V_2). \tag{5.2.1}$$

5.3. **Filtration on $\mathcal{F}_{\mathrm{st}}(V)$.** The filtration (see 2.8) on $\mathcal{F}(V)$ induces a filtration on the subspace $\mathcal{F}_{\mathrm{st}}(V)$:

$$\mathcal{F}_{\mathrm{st}}(V)=\mathcal{F}_{\mathrm{st}}(V)_0\supset\mathcal{F}_{\mathrm{st}}(V)_1\supset F_{\mathrm{st}}(V)_2\supset\cdots$$

where

$$\mathcal{F}_{\mathrm{st}}(V)_r := \mathcal{F}_{\mathrm{st}}(V)\cap\mathcal{F}(V)_r.$$

It is obvious that

$$\mathcal{F}_{\mathrm{st}}(V_1\oplus V_2)_r=\mathcal{F}_{\mathrm{st}}(V_1)_r\oplus\mathcal{F}_{\mathrm{st}}(V_2)_r. \tag{5.3.1}$$

Just as in 2.8 we see that there is an injection (analogous to (2.8.3))

$$\mathcal{F}_{\mathrm{st}}(V)_r/\mathcal{F}_{\mathrm{st}}(V)_{r+1}\to\bigoplus_{T\in\mathcal{T}_r}V^{W^\sigma,\epsilon_I} \tag{5.3.2}$$

where $V^{W^\sigma,\epsilon_I}$ denotes the subspace of $V$ consisting of vectors that transform by $\epsilon_I$ under $W^\sigma$.

We claim that the maps (5.3.2) are isomorphisms. Using (5.3.1) we see that it suffices to prove this in the special case that $V$ is the regular representation. So take $V$ to be $\mathcal{H}(\Lambda)$ for some orbit $\Lambda$ that is real and regular (in the sense of 2.3). Then, just as in 2.8, parabolic induction of stable discrete series analogs on suitable Levi subalgebras $\mathfrak{m}$ in $\mathfrak{g}$ provides enough elements in $\mathcal{F}_{\mathrm{st}}(\Lambda)_r$ to show that (5.3.2) is surjective.

The fact that the maps (5.3.2) are isomorphisms implies that

$$\dim\mathcal{F}_{\mathrm{st}}(V)=\sum_T\dim(V^{W^\sigma,\epsilon_I}), \tag{5.3.3}$$

where $T$ runs over a set of representatives for the $G(\mathbf{R})$-conjugacy classes of maximal tori in $G$.

## 6. Stably invariant eigendistributions of type 0

6.1. **Description of $\mathcal{D}^{\mathrm{st}}_{\mathbf{O}}$.** Let $\mathbf{O}$ be a nilpotent $G(\mathbf{C})$-orbit in $\mathfrak{g}_{\mathbf{C}}$. We let $\mathcal{D}^{\mathrm{st}}_{\mathbf{O}}$ and $s_{\mathbf{O}}$ be as in 1.5. We consider the composed map

$$\mathcal{D}^{\mathrm{st}}_{\mathbf{O}} \xrightarrow{\mathrm{FT}} \mathcal{D}_{\mathrm{st}}(0) \xrightarrow{(5.1.2)} \mathcal{F}_{\mathrm{st}}(0), \tag{6.1.1}$$

where the first map is the Fourier transform (see 2.2 and 4.3). The map (6.1.1) is of course injective, and its image is simply the intersection of $\mathcal{F}_{\mathrm{st}}(0)$ and $\mathcal{F}(0)_{d_{\mathbf{O}},\chi_{\mathbf{O}}}$ in $\mathcal{F}(0)$ (here we are using the notation and results of 3.4), and it follows from our definitions that this intersection is $\mathcal{F}_{\mathrm{st}}(\mathcal{H}_{d_{\mathbf{O}},\chi_{\mathbf{O}}})$.

6.2. **Main theorem.** We can now prove the formula (1.8.1) for $s_{\mathbf{O}}$. Indeed, from 6.1 we see that

$$s_{\mathbf{O}} = \dim \mathcal{F}_{\mathrm{st}}(\mathcal{H}_{d_{\mathbf{O}},\chi_{\mathbf{O}}}).$$

Then, recalling from 3.4 that the character of $\mathcal{H}_{d_{\mathbf{O}},\chi_{\mathbf{O}}}$ is $\chi_{\mathbf{O}}$, we see from (5.3.3) that

$$\dim \mathcal{F}_{\mathrm{st}}(\mathcal{H}_{d_{\mathbf{O}},\chi_{\mathbf{O}}}) = \sum_{T} m_{W^\sigma}(\epsilon_I, \chi_{\mathbf{O}}).$$

## References

[Ass98] M. Assem, *On stability and endoscopic transfer of unipotent orbital integrals on p-adic symplectic groups*, Mem. Amer. Math. Soc. **635** (1998). MR **98m:**22013

[BV80] D. Barbasch and D. Vogan, *The local structure of characters*, J. Funct. Anal. **37** (1980), 27–55. MR **82e:**22024

[BV82] D. Barbasch and D. Vogan, *Primitive ideals and orbital integrals in complex classical groups*, Math. Ann. **259** (1982), 153–199. MR **84h:**22038

[BV83] D. Barbasch and D. Vogan, *Primitive ideals and orbital integrals in complex exceptional groups*, J. Algebra **80** (1983), 350–382. MR **83m:**22026

[Gin83] V. Ginsburg, *Intégrales sur les orbites nilpotentes et représentations des groupes de Weyl*, C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), 249–252. MR **85b:**22019

[HC65a] Harish-Chandra, *Discrete series for semisimple Lie groups I*, Acta Math. **113** (1965), 241–318. MR **36:**2744

[HC65b] Harish-Chandra, *Invariant eigendistributions on a semisimple Lie algebra*, Inst. Hautes Études Sci. Publ. Math. **27** (1965), 5–54. MR **31:**4862c

[HK84] R. Hotta and M. Kashiwara, *The invariant holonomic system on a semisimple Lie algebra*, Invent. Math. **75** (1984), 327–358. MR **87i:**2204

[Kot98] R. Kottwitz, *Involutions in Weyl groups*, Represent. Theory **4** (2000), 1-16.

[Lus84] G. Lusztig, *Characters of reductive groups over a finite field*, Ann. of Math. Studies, 107, Princeton University Press, 1984. MR **86j:**20038

[Rao72] R. Rao, *Orbital integrals in reductive groups*, Ann. of Math. (2) **96** (1972), 505–510. MR **47:**8771

[Ros78] W. Rossmann, *Kirillov's character formula for reductive Lie groups*, Invent. Math. **48** (1978), 207–220. MR **81g:**22012

[Ros90] W. Rossmann, *Nilpotent orbital integrals in a real semisimple Lie algebra and representations of Weyl groups*, Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, Progr. Math., 92, Birkhäuser, Boston, 1990, pp. 263–287. MR **92c:**22022

[She79] D. Shelstad, *Characters and inner forms of a quasi-split group over* **R**, Compositio Math. **39** (1979), 11–45. MR **80m:**22023

[Spr76] T. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. **36** (1976), 173–207. MR **56:**491

[Ste64] R. Steinberg, *Differential equations invariant under finite reflection groups*, Trans. Amer. Math. Soc. **112** (1964), 392–400. MR **29:**4807

[Wal97] J.-L. Waldspurger, *Le lemme fondamental implique le transfert*, Compositio Math. **105** (1997), 153–236. MR **98h:**22023

[Wal99] J.-L. Waldspurger, *Intégrales orbitales nilpotentes et endoscopie pour les groupes classiques non ramifiés*, preprint, 1999.

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