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VERIFYING KOTTWITZ' CONJECTURE BY COMPUTER

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Abstract. In these notes I will discuss the computations that were used to verify the main conjecture of Kottwitz (1997) for the groups E_6, E_7, E_8, and the subsidiary one for F_4 and E_6. At the end I will include tables of the relevant computer output. I begin by recalling briefly what is to be computed.

The main conjecture

Suppose W to be a finite Weyl group. An **involution** in W is any element of order at most two. If σ is an involution, let W_σ be the centralizer of σ. A root λ is called **imaginary** if $\sigma\lambda = -\lambda$ (as opposed to **real** if $\sigma\lambda = \lambda$). Let I_σ be the set of all imaginary roots of σ. Any element commuting with σ permutes I_σ. Therefore, if

$$P_\sigma = \prod_{\lambda>0, \lambda\in I_\sigma} \lambda,$$

then for any w in W_σ

$$wP_\sigma = \mathrm{sgn}_\sigma(w) P_\sigma$$

where $\mathrm{sgn}_\sigma = \pm 1$ is a multiplicative homomorphism from W_σ to $\{\pm 1\}$. It can be calculated explicitly as $(-1)^{\ell_\sigma(w)}$ where

$$\ell_\sigma(w) = \#\{\lambda \in I_\sigma \mid \lambda > 0, w^{-1}\lambda < 0\} = I_\sigma \cap \Lambda_w$$

if

$$\Lambda_w = \{\lambda > 0 \mid w^{-1}\lambda < 0\}\ .$$

Kottwitz' conjecture concerns the multiplicity of sgn_σ in the restriction to W_σ of irreducible representations of W. In other words, we must compute

$$m(\sigma, E) = \langle \mathrm{sgn}_\sigma, E \mid W_\sigma, =\rangle \frac{1}{\# W_\sigma} \sum_{W_\sigma} \mathrm{sgn}_\sigma(w)\chi_E(w)$$

for the irreducible representations E of W.

Some cases are simple. If $\sigma = 1$, then it has no imaginary roots and sgn_σ is the trivial character of W. In this case $m(\sigma, E) = 0$ unless E is equal to the trivial character. If the longest element w_ℓ of W happens to be -1 and $\sigma = w_\ell$, then all roots are imaginary and sgn_σ is then the usual sign-character sgn_W of W. Again in this case $m(\sigma, E) = 0$ unless $E = \mathrm{sgn}_W$ itself. Along these lines, it can be seen more generally that if -1 lies in W, then $m(\sigma, E) = m(\sigma_*, E_*)$ whenever $\sigma_* = -\sigma$, $E_* = E \cdot \mathrm{sgn}_W$.

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Kottwitz' conjecture asserts that the sum

$$m(E) = \sum_{\sigma} m(\sigma, E)$$

(where the sum is over representatives of all conjugacy classes of involutions) is equal to the Lusztig-Fourier transform of a function φ_0 which I do not define here. (But I shall recall later exactly what the computation has to agree with.) For classical groups as well as G_2 and F_4, Kottwitz was able to verify his conjecture by hand. This leaves the exceptional groups E_6 ($51,840$ elements), E_7 ($2,903,040$ elements), and E_8 ($696,729,600$ elements), for which it was apparently necessary to do the computations by machine.

At first it looked as though it would be a great deal of work just getting the known character and conjugacy class information into computer-readable form, but luckily — and just in time — Meinolf Geck made available to us some recently developed programs for use with the well known algebra package `GAP`, which were able to produce exactly the information we needed. (These files are part of a larger project called `CHEVIE` involving Geck and several of his colleagues, and are available for public use as extensions to `GAP`. For information, see the reference to `CHEVIE` at the end of this note.)

Once we had the character and conjugacy class data that we needed in a form which a program could read, it was not too difficult to write a program that could calculate the multiplicities.

The original program searched through the whole Weyl group to pick out those in the centralizer W_σ, since I was not aware at this time how simple the centralizer was. Several people pointed out to me later that my exhaustive calculations could be short-circuited. It was also pointed out to me that Dean Alvis had undoubtedly made similar calculations a long time ago, as evidenced by his work with Lusztig. Most valuable (and embarassing) was a note from John Stembridge, who has developed a Maple package for finite Weyl groups which could do this particular calculation very quickly. I shall say a few words about the early version, however, because someone else might face a similar problem without a shortcut.

Because of the crude technique used, it was something of a challenge to construct an efficient program, since the groups E_6, E_7, and E_8 are so large. The basic technique in all cases was the same — for each one of the involution classes $\{\sigma\}$ in W the whole group was scanned to find elements in the centralizer W_σ (the order of the centralizer is known, and is one of the data produced by `GAP`). For each w in W_σ the parity of $I_\sigma \cap \Lambda_w$ was calculated as well as its conjugacy class in W, from which the terms $m(\sigma, E)$ could be calculated, again from the `GAP` data.

The details of the calculation were important, if the program were to be fast. The fastest known way to scan the group, as far as I can see, is that described in [7], and involves building and then traversing the automaton describing the `ShortLex` language of strings of simple generators for elements in W which are of minimal length and lexicographically least. The fastest way to perform multiplication in the group is to use the ideas of du Cloux [8], based in turn on ideas of Deodhar, representing an element of W as a sequence of elements of Weyl group cosets (with respect to smaller Weyl groups), using canonical representatives of these cosets. Here the basic calculation is to multiply an element of the group, represented as a sequence of coset representatives, by a simple generator s_i. The group E_6 is in fact small enough so that in fact the whole table of ws_i can be stored in an array. It would have been possible to use tables of representatives already constructed by du Cloux, but in fact the program built these tables on the fly, using the multiplication algorithm described in [7], which in turn is based on ideas of Brink and Howlett [4].

There seems to be no uniform efficient way to handle the conjugacy class problem for these groups. Here again, the group E_6 is small enough that the most efficient and dependable thing to do was simply to build a list, in the obvious way, of classes for every element in the group. For E_8, Geck suggested using the fact that the map from conjugacy classes of E_8 to those in the group of permutations of its 240 roots is an embedding. In other words, a conjugacy class in E_8 is distinguished by its representation in terms of cycles in $\mathfrak{S}_{240}$. There are simple and efficient algorithms for finding the cycle representation of a permutation, but nonetheless to do this several million times for permutations of 240 items is necessarily a slow business.

For E_7 there are 126 roots. The map from conjugacy classes into conjugacy classes of $\mathfrak{S}_{126}$ fails to be an injection, but the map into conjugacy classes in E_8 is injective. Those classes in E_7 which coincide in $\mathfrak{S}_{126}$ were distinguished in this way.

Thus in traversing the group there were essentially three things to do: (1) tell whether the element w commutes with the current involution σ; (2) calculate $\mathrm{sgn}_\sigma(w)$; (3) find the conjugacy class of w. The data needed for these calculations is easily updated in going from w to ws with $ws > w$, which is the way in which the automaton is traversed.

The amount of memory needed to run the program was negligible, but the amount of time required was substantial. I used several machines to develop the program with the small A_n and F_4, to compare with Kottwitz' calculations, and then to deal with the cases E_6 and E_7, which are still relatively small. They took a few seconds and a few minutes, respectively. The group E_8 is large by almost any standard, however. It contains almost a billion elements, and for each one of these a large number of calculations had to be made. The final run for E_8 took about 36 hours on a SPARC 20.

The tables of $m(\sigma, E)$ will be exhibited at the end of this note. In order to understand exactly how they imply Kottwitz' conjecture, I present here some relevant data mentioned in Kottwitz' article, but in a tabular form so that immediate comparison is straightforward. According to Lusztig [11], the irreducible representations E of W are partitioned into families $\mathcal{F}$. To each family is associated a finite group $\mathcal{G}$, and to $\mathcal{G}$ is associated a finite set $\mathcal{M}(\mathcal{G})$ of conjugacy classes of pairs (g, ρ), where g is an element of $\mathcal{G}$ and ρ an irreducible representation of the centralizer $\mathcal{G}_g$. For this theory, refer to [11], Chapter 13 of [6], and [12]. Each family maps injectively into a subset of $\mathcal{M}(\mathcal{G})$, but this map is not surjective. In the tables below, the image (g, ρ) of each E is indicated. Kottwitz' conjecture concerning the $m(E)$ is that they agree with the Lusztig-Fourier transform of a certain function φ_0 on $\mathcal{M}(\mathcal{G})$ defined in §2.10 of Kottwitz' article. Kottwitz has listed the relevant values of $\widehat{\varphi}_0$ in his article; it will be simpler for the reader to verify his conjecture from my calculations if he has these in a tabular form.

Throughout the tables, Carter's name conventions for both conjugacy classes and representations of W are followed (see Carter (1972) and Carter (1985)). Carter's naming scheme for representations refers to a representation $\varphi_{n,d}$ where n is its dimension and d is the lowest degree it appears in the canonical representation of W on $S(V)$, the symmetric algebra of the root space V. That for conjugacy classes takes advantage of the fact that most conjugacy classes in a Weyl group are

Coxeter elements in some Weyl subgroup. The roots, hence the numbering of the elementary reflections indexed in the reduced expressions, are numbered as in [3]. The characters and other conjugacy class data were provided by `GAP`.

Table of $\widehat{\varphi}_0$

$\mathcal{G}$	(g,ρ)	$\widehat{\varphi}_0(g,\rho)$
$\mathfrak{S}_1$	$(1,1)$	1
$\mathfrak{S}_2$	$(1,1)$	2
	$(g_2,1)$	0
	$(1,\varepsilon)$	0
$\mathfrak{S}_3$	$(1,1)$	2
	$(g_2,1)$	0
	$(1,r)$	1
	$(g_3,1)$	1
	$(1,\varepsilon)$	0
$\mathfrak{S}_4$	$(1,1)$	3
	$(1,\lambda^1)$	1
	$(1,\lambda^2)$	0
	$(1,\sigma)$	2
	$(g_2,1)$	0
	(g_2,ε'')	0
	$(g_2',1)$	1
	(g_2',ε'')	0
	(g_2',ε')	0
	$(g_3,1)$	1
	$(g_4,1)$	0
$\mathfrak{S}_5$	$(1,1)$	3
	$(g_3,1)$	2
	$(g_2',1)$	1
	$(1,\nu)$	2
	$(1,\lambda^1)$	2
	$(g_5,1)$	1
	(g_3,ε)	0
	$(1,\nu')$	1
	(g_2',ε'')	0
	$(1,\lambda^2)$	0
	(g_2',ε')	0
	$(1,\lambda^3)$	0
	$(g_2,1)$	0
	$(g_4,1)$	0
	$(g_6,1)$	0
	(g_2,r)	0
	(g_2,ε)	0

As Kottwitz mentions, it was the computer results for E_7 (which appeared before those for E_8) which forced him to deal with the six exceptional representations (two of E_7, four of E_8) specially. As far as I know, it is only in [13] that any theoretical explanation of some of the phenomena attached to these occurs in the literature.

I include here also the tables for the exceptional groups G_2 and F_4, which will perhaps allow the reader to orient himself in reading these tables.

The subsidiary conjecture

Kottwitz' second conjecture is made in the introduction to his paper. It asserts that if σ is an involution, then the number of involutions in any right cell $[\Gamma]$ and conjugate to σ is equal to $m(\sigma, \pi_\Gamma)$, the multiplicity with which the character sgn_σ of W_σ occurs in the representation π_Γ determined by Γ. As he points out, this is true for type A_n because of known facts about the Robinson-Schensted correspondence. If $\pi_\Gamma = \sum n_i E_i$ is its decomposition into irreducibles, then the claim is that

$$\#\{x \in \Gamma \mid x \sim \sigma\} = \sum n_i m(\sigma, E_i)$$

so that we can use previous calculations to verify this if we can count conjugacy classes of involutions in the cells. I have done this for the groups F_4, E_6 as well as a selection of smaller classical groups. This program was more interesting than the one used previously, since it involved computing explicitly all the W-graphs associated to the right cells of W. (It was also more interesting because I implemented it more than one year after the other, in the relatively new programming language `Java`, which is inefficient but extremely flexible.)

A few people have suggested that these calculations might have been carried out by hand, based on known facts about cells, but my feeling about this is that a well constructed program that deals uniformly with a wide range of Coxeter groups is more valuable than an *ad hoc* collection of techniques tailored to particular cases.

At any rate, we have the following table of data (see **Cells for** F_4 below), which in combination with the table of the $m(\sigma, E)$ for E an irreducible representation of F_4 implies the result. Each row in this table concerns an equivalence class of cells. The first column records the number of cells in the class $\{\Gamma\}$. The last column records the decomposition of the representation π_Γ into irreducible components. The middle column records the conjugacy classes of involutions occurring in the cell, with its multiplicity in brackets [ ].

The case E_6 was decided long after the case of F_4. The group E_6 is not from a mathematical point of view more complicated than F_4, but its much larger size creates serious computational difficulties. The program first calculated and stored in a file the W-graph of E_6, and then later read this file to verify the conjecture. (The plain text file took up about 5 megabytes, which gives you some idea of how difficult the group E_7 will be.) A number of tricks were required in order not to exceed memory or time limitations. The principal one was an efficient way to the Bruhat order, suggested by du Cloux, following Deodhar. I should add that du Cloux himself has calculated all the Kazhdan-Lusztig polynomials of E_6, and that I could have used his data in the second stage of my calculations, but for technical reasons it was just as easy to calculate the W-graph directly. I have intended for a long time now to make available on the Internet data files storing the W-graphs of a wide selection of Coxeter groups up to about the size of E_6, as well as partial graphs of a number of infinite Coxeter groups. What has deterred me is that the size of the files required is huge, and recently I have been exploring interesting ways to navigate them.

Cells for F_4

Number of equivalent cells	Involution classes in cell	Decomposition
1	1	$\varphi_{1,0}$
2	A_1, $\widetilde{A}_1$	$\varphi''_{2,4} + \varphi_{4,1}$
2	A_1, $\widetilde{A}_1$	$\varphi'_{2,4} + \varphi_{4,1}$
9	$A_1 \times \widetilde{A}_1$	$\varphi_{9,2}$
8	$\widetilde{A}_1$	$\varphi''_{8,3}$
8	A_1	$\varphi'_{8,3}$
8	A_1^3	$\varphi''_{8,9}$
8	$A_1^2 \times \widetilde{A}_1$	$\varphi'_{8,9}$
9	$A_1 \times \widetilde{A}_1$	$\varphi_{9,10}$
1	[5] $A_1 \times \widetilde{A}_1$, [2] A_1^2	$\varphi''_{1,12} + 2\varphi''_{9,6} + \varphi''_{6,6} + \varphi_{12,4} + \varphi''_{4,7} + \varphi_{16,5}$
3	[4] $A_1 \times \widetilde{A}_1$, A_1^2	$\varphi''_{9,6} + \varphi'_{6,6} + \varphi_{12,4} + \varphi''_{4,7} + \varphi_{16,5}$
4	[2] A_1^2, [5] $A_1 \times \widetilde{A}_1$	$\varphi_{4,8} + \varphi''_{9,6} + \varphi'_{9,6} + \varphi''_{6,6} + \varphi_{12,4} + 2\varphi_{16,5}$
3	[4] $A_1 \times \widetilde{A}_1$, A_1^2	$\varphi'_{9,6} + \varphi'_{6,6} + \varphi_{12,4} + \varphi'_{4,7} + \varphi_{16,5}$
2	A_1^3, $A_1^2 \times \widetilde{A}_1$	$\varphi''_{2,16} + \varphi_{4,13}$
1	[2] A_1^2, [5] $A_1 \times \widetilde{A}_1$	$\varphi'_{1,12} + 2\varphi'_{9,6} + \varphi''_{6,6} + \varphi_{12,4} + \varphi'_{4,7} + \varphi_{16,5}$
2	A_1^3, $A_1^2 \times \widetilde{A}_1$	$\varphi'_{2,16} + \varphi_{4,13}$
1	A_1^4	$\varphi_{1,24}$

I should also remark here that my calculations undoubtedly reproduce some that were made much earlier by the redoubtable Dean Alvis.

The tables

The group G_2. It has 12 elements.

Conjugacy class data:

Carter's name	Representative reduced word expression	Conjugacy class size
1	$\emptyset$	1
$\widetilde{A}_1$	[1]	3
A_1	[2]	3
$A_1 \times \widetilde{A}_1$	[121212]	1

Multiplicities:

$\mathcal{G}$	E	(g,ρ)	$m(\sigma,E)$				
			1	$\widetilde{A}_1$	A_1	$A_1 \times \widetilde{A}_1$	$m(E)$
$\mathfrak{S}_1$	$\varphi_{1,0}$		1	0	0	0	1
	$\varphi_{1,6}$		0	0	0	1	1
$\mathfrak{S}_3$	$\varphi_{2,1}$	$(1,1)$	0	1	1	0	2
	$\varphi'_{1,3}$	$(1,r)$	0	0	1	0	1
	$\varphi''_{1,3}$	$(g_3,1)$	0	1	0	0	1
	$\varphi_{2,2}$	$(g_2,1)$	0	0	0	0	0

The group F_4. It has 1152 elements.

Conjugacy class data:

Carter's name	Representative reduced word expression	Conjugacy class size
1	$\emptyset$	1
A_1^4	[121321323432132343213234]	1
A_1^2	[2323]	18
A_1	[1]	12
A_1^3	[232343234]	12
$\widetilde{A}_1$	[3]	12
$A_1^2 \times \widetilde{A}_1$	[121321323]	12
$A_1 \times \widetilde{A}_1$	[13]	72

Multiplicities: The group F_4 is unusual, in that Kondo's names are still commonly used, and in particular in [10]. They are therefore given here, just after those of Carter.

$\mathcal{G}$	E	$(Kondo)$	(g,ρ)	$m(\sigma,E)$								
				1	A_1^4	A_1^2	A_1	A_1^3	$\widetilde{A}_1$	$A_1^2\times\widetilde{A}_1$	$A_1\times\widetilde{A}_1$	$m(E)$
$\mathfrak{S}_1$	$\varphi_{1,0}$	1_1		1	0	0	0	0	0	0	0	1
	$\varphi_{1,24}$	1_4		0	1	0	0	0	0	0	0	1
	$\varphi_{9,10}$	9_4		0	0	0	0	0	0	0	1	1
	$\varphi''_{8,3}$	8_1		0	0	0	0	0	1	0	0	1
	$\varphi'_{8,3}$	8_3		0	0	0	1	0	0	0	0	1
	$\varphi''_{8,9}$	8_4		0	0	0	0	1	0	0	0	1
	$\varphi'_{8,9}$	8_2		0	0	0	0	0	0	1	0	1
	$\varphi_{9,2}$	9_1		0	0	0	0	0	0	0	1	1
$\mathfrak{S}_2$	$\varphi_{4,1}$	4_2	$(1,1)$	0	0	0	1	0	1	0	0	2
	$\varphi''_{2,4}$	2_1	$(g_2,1)$	0	0	0	0	0	0	0	0	0
	$\varphi'_{2,4}$	2_3	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{4,13}$	4_5	$(1,1)$	0	0	0	0	1	0	1	0	2
	$\varphi''_{2,16}$	2_4	$(g_2,1)$	0	0	0	0	0	0	0	0	0
	$\varphi'_{2,16}$	2_2	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_4$	$\varphi_{12,4}$	12_1	$(1,1)$	0	0	1	0	0	0	0	2	3
	$\varphi''_{9,6}$	9_2	$(g'_2,1)$	0	0	0	0	0	0	0	1	1
	$\varphi'_{9,6}$	9_3	$(1,\lambda^1)$	0	0	0	0	0	0	0	1	1
	$\varphi''_{1,12}$	1_2	(g'_2,ε')	0	0	0	0	0	0	0	0	0
	$\varphi'_{1,12}$	1_3	$(1,\lambda^2)$	0	0	0	0	0	0	0	0	0
	$\varphi_{4,8}$	4_1	(g'_2,ε'')	0	0	0	0	0	0	0	0	0
	$\varphi''_{4,7}$	4_3	$(g_4,1)$	0	0	0	0	0	0	0	0	0
	$\varphi'_{4,7}$	4_4	(g_2,ε'')	0	0	0	0	0	0	0	0	0
	$\varphi'_{6,6}$	6_1	$(g_3,1)$	0	0	0	0	0	0	0	1	1
	$\varphi''_{6,6}$	6_2	$(1,\sigma)$	0	0	1	0	0	0	0	1	2
	$\varphi_{16,5}$	16_1	$(g_2,1)$	0	0	0	0	0	0	0	0	0

The group E_6. It has $51,840$ elements.

Conjugacy class data:

Carter's name	Representative reduced word expression	Conjugacy class size
1	$\emptyset$	1
A_1^4	[343243543245]	45
A_1^2	[14]	270
A_1	[1]	36
A_1^3	[146]	540

Multiplicities:

$\mathcal{G}$	E	(g,ρ)	$m(\sigma,E)$					
			1	A_1^4	A_1^2	A_1	A_1^3	$m(E)$
$\mathfrak{S}_1$	$\varphi_{1,0}$		1	0	0	0	0	1
	$\varphi_{1,36}$		0	1	0	0	0	1
	$\varphi_{6,1}$		0	0	0	1	0	1
	$\varphi_{6,25}$		0	0	0	0	1	1
	$\varphi_{20,10}$		0	0	0	0	0	0
	$\varphi_{20,2}$		0	0	1	0	0	1
	$\varphi_{20,20}$		0	1	0	0	0	1
	$\varphi_{24,6}$		0	0	1	0	0	1
	$\varphi_{24,12}$		0	1	0	0	0	1
	$\varphi_{60,5}$		0	0	0	0	1	1
	$\varphi_{60,11}$		0	0	0	0	1	1
	$\varphi_{64,4}$		0	0	1	0	0	1
	$\varphi_{64,13}$		0	0	0	0	1	1
	$\varphi_{81,6}$		0	0	1	0	0	1
	$\varphi_{81,10}$		0	0	1	0	0	1
$\mathfrak{S}_2$	$\varphi_{30,15}$	$(1,1)$	0	0	0	0	2	2
	$\varphi_{15,17}$	$(1,\varepsilon)$	0	0	0	0	0	0
	$\varphi_{15,16}$	$(g_2,1)$	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{30,3}$	$(1,1)$	0	0	0	1	1	2
	$\varphi_{15,5}$	$(1,\varepsilon)$	0	0	0	0	0	0
	$\varphi_{15,4}$	$(g_2,1)$	0	0	0	0	0	0
$\mathfrak{S}_3$	$\varphi_{80,7}$	$(1,1)$	0	0	0	0	2	2
	$\varphi_{10,9}$	$(g_3,1)$	0	0	0	0	1	1
	$\varphi_{20,10}$	$(1,\varepsilon)$	0	0	0	0	0	0
	$\varphi_{60,8}$	$(g_2,1)$	0	0	0	0	0	0
	$\varphi_{90,8}$	$(1,r)$	0	0	0	0	1	1

The group E_7. It has $2,903,040$ elements.

Conjugacy class data:

Carter's name	Representative reduced word expression	Conjugacy class size
1	$\emptyset$	1
A_1^6	[767567456724567345672456345243]	63
$(A_1'')^4$	[545245345243]	315
A_1^2	[75]	945
$(A_1')^4$	[7523]	3780
A_1^7	[767567456724567345672456345243134567245634524313456724563452431]	1
A_1	[7]	63
$(A_1')^3$	[752]	315
A_1^5	[7545245345243]	945
$(A_1'')^3$	[753]	3780

Multiplicities: The exceptional classes are marked *Exc.*

$\mathcal{G}$	E	(g,ρ)	$m(\sigma,E)$										
			1	A_1^6	$(A_1'')^4$	A_1^2	$(A_1')^4$	A_1^7	A_1	$(A_1')^3$	A_1^5	$(A_1'')^3$	$m(E)$
$\mathfrak{S}_1$	$\varphi_{1,0}$		1	0	0	0	0	0	0	0	0	0	1
	$\varphi_{1,63}$		0	0	0	0	0	1	0	0	0	0	1
	$\varphi_{7,46}$		0	1	0	0	0	0	0	0	0	0	1
	$\varphi_{7,1}$		0	0	0	0	0	0	1	0	0	0	1
	$\varphi_{21,36}$		0	0	1	0	0	0	0	0	0	0	1
	$\varphi_{21,3}$		0	0	0	0	0	0	0	1	0	0	1
	$\varphi_{27,2}$		0	0	0	1	0	0	0	0	0	0	1
	$\varphi_{27,37}$		0	0	0	0	0	0	0	0	1	0	1
	$\varphi_{105,6}$		0	0	0	0	1	0	0	0	0	0	1
	$\varphi_{105,21}$		0	0	0	0	0	0	0	0	0	1	1
	$\varphi_{105,12}$		0	0	1	0	0	0	0	0	0	0	1
	$\varphi_{105,15}$		0	0	0	0	0	0	0	1	0	0	1
	$\varphi_{168,6}$		0	0	0	1	0	0	0	0	0	0	1
	$\varphi_{168,21}$		0	0	0	0	0	0	0	0	1	0	1
	$\varphi_{189,22}$		0	0	0	0	1	0	0	0	0	0	1
	$\varphi_{189,5}$		0	0	0	0	0	0	0	0	0	1	1
	$\varphi_{189,20}$		0	0	1	0	0	0	0	0	0	0	1
	$\varphi_{189,7}$		0	0	0	0	0	0	0	1	0	0	1
	$\varphi_{210,6}$		0	0	0	1	0	0	0	0	0	0	1
	$\varphi_{210,21}$		0	0	0	0	0	0	0	0	1	0	1
	$\varphi_{210,10}$		0	0	0	0	1	0	0	0	0	0	1
	$\varphi_{210,13}$		0	0	0	0	0	0	0	0	0	1	1
	$\varphi_{378,14}$		0	0	0	0	1	0	0	0	0	0	1
	$\varphi_{378,9}$		0	0	0	0	0	0	0	0	0	1	1
$\mathfrak{S}_2$	$\varphi_{56,3}$	$(1,1)$	0	0	0	0	0	0	1	0	0	1	2
	$\varphi_{35,4}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{21,6}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{56,30}$	$(1,1)$	0	1	0	0	1	0	0	0	0	0	2
	$\varphi_{35,31}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{21,33}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{120,4}$	$(1,1)$	0	0	0	1	1	0	0	0	0	0	2
	$\varphi_{15,7}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{105,5}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{120,25}$	$(1,1)$	0	0	0	0	0	0	0	0	1	1	2
	$\varphi_{15,28}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{105,26}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{405,8}$	$(1,1)$	0	0	0	0	2	0	0	0	0	0	2
	$\varphi_{216,9}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{189,10}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{405,15}$	$(1,1)$	0	0	0	0	0	0	0	0	0	2	2
	$\varphi_{216,16}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{189,17}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{420,10}$	$(1,1)$	0	0	0	1	1	0	0	0	0	0	2
	$\varphi_{84,12}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{336,11}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{420,13}$	$(1,1)$	0	0	0	0	0	0	0	0	1	1	2
	$\varphi_{84,15}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{336,14}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
Exc	$\varphi_{512,11}$	$(1,1)$	0	0	0	0	0	0	0	0	0	1	1
	$\varphi_{512,12}$	$(g_2,1)$	0	0	0	0	1	0	0	0	0	0	1
	$\varphi_{512,12}$	$(1,\varepsilon)$	0	0	0	0	1	0	0	0	0	0	1
$\mathfrak{S}_3$	$\varphi_{315,7}$	$(1,1)$	0	0	0	0	0	0	0	0	0	2	2
	$\varphi_{280,8}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{280,9}$	$(1,r)$	0	0	0	0	0	0	0	0	0	1	1
	$\varphi_{70,9}$	$(g_3,1)$	0	0	0	0	0	0	0	0	0	1	1
	$\varphi_{35,13}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_3$	$\varphi_{315,16}$	$(1,1)$	0	0	0	0	2	0	0	0	0	0	2
	$\varphi_{280,17}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{280,18}$	$(1,r)$	0	0	0	0	1	0	0	0	0	0	1
	$\varphi_{70,18}$	$(g_3,1)$	0	0	0	0	1	0	0	0	0	0	1
	$\varphi_{35,22}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0

The group E_8. It has $696,729,600$ elements.

Conjugacy class data:

Carter's name	Representative reduced word expression	Conjugacy class size
1	$\emptyset$	1
A_1^8	[87867856784567814567834567814567345614534113456781456734561453411345678145634567814563451431345671456345143134567145634514 31]	1
$(A_1')^4$	[545245345243]	3150
A_1^2	[61]	3780
A_1^6	[767567456724567345672456345243]	3780
$(A_1'')^4$	[7523]	113400
A_1	[3]	120
A_1^7	[767567456724567345672456345243134567245634524313456724563452431]	120
A_1^3	[861]	37800
A_1^5	[7545245345243]	37800

Multiplicities. The exceptional classes are marked *Exc.*

$\mathcal{G}$	E	(g,ρ)	$m(\sigma,E)$										
			1	A_1^8	$(A_1')^4$	A_1^2	A_1^6	$(A_1'')^4$	A_1	A_1^7	A_1^3	A_1^5	$m(E)$
$\mathfrak{S}_1$	$\varphi_{1,0}$		1	0	0	0	0	0	0	0	0	0	1
	$\varphi_{1,120}$		0	1	0	0	0	0	0	0	0	0	1
	$\varphi_{35,2}$		0	0	0	1	0	0	0	0	0	0	1
	$\varphi_{35,74}$		0	0	0	0	1	0	0	0	0	0	1
	$\varphi_{525,12}$		0	0	1	0	0	0	0	0	0	0	1
	$\varphi_{525,36}$		0	0	1	0	0	0	0	0	0	0	1
	$\varphi_{567,6}$		0	0	0	1	0	0	0	0	0	0	1
	$\varphi_{567,46}$		0	0	0	0	1	0	0	0	0	0	1
	$\varphi_{2100,20}$		0	0	1	0	0	0	0	0	0	0	1
	$\varphi_{2835,14}$		0	0	0	0	0	1	0	0	0	0	1
	$\varphi_{2835,22}$		0	0	0	0	0	1	0	0	0	0	1
	$\varphi_{6075,14}$		0	0	0	0	0	1	0	0	0	0	1
	$\varphi_{6075,22}$		0	0	0	0	0	1	0	0	0	0	1
	$\varphi_{8,1}$		0	0	0	0	0	0	1	0	0	0	1
	$\varphi_{8,91}$		0	0	0	0	0	0	0	1	0	0	1
	$\varphi_{560,5}$		0	0	0	0	0	0	0	0	1	0	1
	$\varphi_{560,47}$		0	0	0	0	0	0	0	0	0	1	1
	$\varphi_{3240,9}$		0	0	0	0	0	0	0	0	1	0	1
	$\varphi_{3240,31}$		0	0	0	0	0	0	0	0	0	1	1
	$\varphi_{4200,15}$		0	0	0	0	0	0	0	0	0	1	1
	$\varphi_{4200,21}$		0	0	0	0	0	0	0	0	1	0	1
	$\varphi_{4536,13}$		0	0	0	0	0	0	0	0	1	0	1
	$\varphi_{4536,23}$		0	0	0	0	0	0	0	0	0	1	1

			1	A_1^8	$(A_1')^4$	A_1^2	A_1^6	$(A_1'')^4$	A_1	A_1^7	A_1^3	A_1^5	$m(E)$
$\mathfrak{S}_5$	$\varphi_{4480,16}$	$(1,1)$	0	0	0	0	0	3	0	0	0	0	3
	$\varphi_{3150,18}$	$(g_3,1)$	0	0	0	0	0	2	0	0	0	0	2
	$\varphi_{4200,18}$	$(g_2',1)$	0	0	0	0	0	1	0	0	0	0	1
	$\varphi_{4536,18}$	$(1,\nu)$	0	0	0	0	0	2	0	0	0	0	2
	$\varphi_{5670,18}$	$(1,\lambda^1)$	0	0	0	0	0	2	0	0	0	0	2
	$\varphi_{420,20}$	$(g_5,1)$	0	0	0	0	0	1	0	0	0	0	1
	$\varphi_{1134,20}$	(g_3,ε)	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{1400,20}$	$(1,\nu')$	0	0	0	0	0	1	0	0	0	0	1
	$\varphi_{2688,20}$	(g_2',ε'')	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{1680,22}$	$(1,\lambda^2)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{168,24}$	(g_2',ε')	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{70,32}$	$(1,\lambda^3)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{7168,17}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{1344,19}$	$(g_4,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{2016,19}$	$(g_6,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{5600,19}$	(g_2,r)	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{448,25}$	(g_2,ε)	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{112,3}$	$(1,1)$	0	0	0	0	0	0	1	0	1	0	2
	$\varphi_{84,4}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{28,8}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{112,63}$	$(1,1)$	0	0	0	0	0	0	0	1	0	1	2
	$\varphi_{84,64}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{28,68}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{210,4}$	$(1,1)$	0	0	0	1	0	1	0	0	0	0	2
	$\varphi_{50,8}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{160,7}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{210,52}$	$(1,1)$	0	0	0	0	1	1	0	0	0	0	2
	$\varphi_{50,56}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{160,55}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{700,6}$	$(1,1)$	0	0	0	1	0	1	0	0	0	0	2
	$\varphi_{400,7}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{300,8}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{700,42}$	$(1,1)$	0	0	0	0	1	1	0	0	0	0	2
	$\varphi_{400,43}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{300,44}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{2268,30}$	$(1,1)$	0	0	0	0	1	1	0	0	0	0	2
	$\varphi_{972,32}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{1296,33}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{2268,10}$	$(1,1)$	0	0	0	1	0	1	0	0	0	0	2
	$\varphi_{972,12}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{1296,13}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{2240,28}$	$(1,1)$	0	0	0	0	0	2	0	0	0	0	2
	$\varphi_{1400,29}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{840,31}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{2240,10}$	$(1,1)$	0	0	0	0	0	2	0	0	0	0	2
	$\varphi_{1400,11}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{840,13}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0

			1	A_1^8	$(A_1')^4$	A_1^2	A_1^6	$(A_1'')^4$	A_1	A_1^7	A_1^3	A_1^5	$m(E)$
Exc	$\varphi_{4096,11}$	$(1,1)$	0	0	0	0	0	0	0	0	1	0	1
	$\varphi_{4096,12}$	$(1,\varepsilon)$	0	0	0	0	0	1	0	0	0	0	1
Exc	$\varphi_{4096,26}$	$(1,1)$	0	0	0	0	0	1	0	0	0	0	1
	$\varphi_{4096,27}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	1	1
$\mathfrak{S}_2$	$\varphi_{4200,12}$	$(1,1)$	0	0	0	0	0	2	0	0	0	0	2
	$\varphi_{840,14}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{3360,13}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{4200,24}$	$(1,1)$	0	0	0	0	0	2	0	0	0	0	2
	$\varphi_{840,26}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{3360,25}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{2800,13}$	$(1,1)$	0	0	0	0	0	0	0	0	1	1	2
	$\varphi_{700,16}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{2100,16}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{2800,25}$	$(1,1)$	0	0	0	0	0	0	0	0	1	1	2
	$\varphi_{700,28}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{2100,28}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{5600,15}$	$(1,1)$	0	0	0	0	0	0	0	0	2	0	2
	$\varphi_{3200,16}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{2400,17}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{5600,21}$	$(1,1)$	0	0	0	0	0	0	0	0	0	2	2
	$\varphi_{3200,22}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{2400,23}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_3$	$\varphi_{1400,7}$	$(1,1)$	0	0	0	0	0	0	0	0	2	0	2
	$\varphi_{1344,8}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{1008,9}$	$(1,r)$	0	0	0	0	0	0	0	0	1	0	1
	$\varphi_{448,9}$	$(g_3,1)$	0	0	0	0	0	0	0	0	1	0	1
	$\varphi_{56,19}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_3$	$\varphi_{1400,37}$	$(1,1)$	0	0	0	0	0	0	0	0	0	2	2
	$\varphi_{1344,38}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{1008,39}$	$(1,r)$	0	0	0	0	0	0	0	0	0	1	1
	$\varphi_{448,39}$	$(g_3,1)$	0	0	0	0	0	0	0	0	0	1	1
	$\varphi_{56,49}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_3$	$\varphi_{1400,8}$	$(1,1)$	0	0	0	0	0	2	0	0	0	0	2
	$\varphi_{1050,10}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{1575,10}$	$(1,r)$	0	0	0	0	0	1	0	0	0	0	1
	$\varphi_{175,12}$	$(g_3,1)$	0	0	0	0	0	1	0	0	0	0	1
	$\varphi_{350,14}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_3$	$\varphi_{1400,32}$	$(1,1)$	0	0	0	0	0	2	0	0	0	0	2
	$\varphi_{1050,34}$	$(g_2,1)$	0	0	0	0	0	0	0	0	0	0	0
	$\varphi_{1575,34}$	$(1,r)$	0	0	0	0	0	1	0	0	0	0	1
	$\varphi_{175,36}$	$(g_3,1)$	0	0	0	0	0	1	0	0	0	0	1
	$\varphi_{350,38}$	$(1,\varepsilon)$	0	0	0	0	0	0	0	0	0	0	0

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