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SYMMETRIC POLYNOMIALS AND $U_q(\widehat{sl}_2)$

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Abstract. We study the explicit formula of Lusztig's integral forms of the level one quantum affine algebra $U_q(\widehat{sl}_2)$ in the endomorphism ring of symmetric functions in infinitely many variables tensored with the group algebra of $\mathbb{Z}$. Schur functions are realized as certain orthonormal basis vectors in the vertex representation associated to the standard Heisenberg algebra. In this picture the Littlewood-Richardson rule is expressed by integral formulas, and is used to define the action of Lusztig's $\mathbb{Z}[q, q^{-1}]$-form of $U_q(\widehat{sl}_2)$ on Schur polynomials. As a result the $\mathbb{Z}[q, q^{-1}]$-lattice of Schur functions tensored with the group algebra contains Lusztig's integral lattice.

1. Introduction

The relation between vertex representations and symmetric functions is one of the interesting aspects of affine Kac-Moody algebras and the quantum affine algebras. In the late 1970's to the early 1980's the Kyoto school [DJKM] found that the polynomial solutions of KP hierarchies are obtained by Schur polynomials. This breakthrough was achieved in formulating the KP and KdV hierarchies in terms of affine Lie algebras. On the other hand, I. Frenkel [F1] identified the two constructions of the affine Lie algebras via vertex operators, which put the boson-fermion correspondence in a rigorous formulation. I. Frenkel [F2] further showed that the boson-fermion correspondence can give the Frobenius formula of the irreducible characters for the symmetric group $\mathfrak{S}_n$ (see also [J1]). Schur functions also played a key role in Lepowsky and Primc construction [LP] of certain bases for higher level representations of the affine Lie algebra $\widehat{sl}(2)$.

In [J1, J2] the vertex operator approach to classical symmetric functions was developed to study Schur's Q-functions and more generally Hall-Littlewood symmetric functions. These families of symmetric polynomials appear naturally as orthogonal bases in the vertex representation. The formulation of Hall-Littlewood polynomials in terms of the boson-fermion correspondence was found in [J4] afterwards. Since then the vertex operator approach to symmetric functions is used to study both old and new problems in symmetric functions (see [CT] and [Ga]).

In 1988 Macdonald introduced more general (orthogonal) symmetric polynomials, which are a two-parameter deformation of the Schur polynomials (cf. [M]). At the same time the vertex representation of the quantum affine algebra was

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constructed in [FJ]. The success of the vertex operator interpretation of Hall-Littlewood polynomials suggests a possible formulation for the Macdonald polynomial in the quantum vertex representation.

Recently J. Beck, I. Frenkel and the author [BFJ] have used the q-vertex operators to study the canonical bases for the level one irreducible modules for the quantum affine algebra $U_q(\widehat{sl}_2)$. The zonal Macdonald polynomials are shown as some "canonical" bases of the basic representation sitting between Kashiwara and Lusztig's canonical and dual canonical bases [L, K]. This essentially answered the question about the vertex realization of (zonal) Macdonald polynomials. The Macdonald basis constructed in [BFJ] also satisfy the characteristic properties of bar invariance and orthogonality under the Kashiwara form. The transition matrix from the canonical basis to the Macdonald basis is triangular, integral and bar-invariant and was conjectured to be positive. Since the transition matrix from Macdonald polynomials to (modified) Schur polynomials is also triangular, a natural problem is to determine the action of the quantum affine algebra on the Schur polynomials.

The main goal of this paper is to explicitly realize the quantum affine algebra $U_q(\widehat{sl}_2)$ by Schur functions with the help of the Littlewood-Richardson rule. We first modify the vertex operator realization of Schur functions and express all products of Schur and dual Schur vertex operators in terms of the Schur basis using the symmetry of Clifford algebras, which is an analog of the linkage symmetry for the weights of the Lie algebra $\mathfrak{sl}(n)$. We then use the idea [J3] of expressing Schur functions inside the basic representation of $U_q(\widehat{sl}_2)$ to realize the action of divided powers of Drinfeld generators of $U_q(\widehat{sl}_2)$.

Moreover, this enables us to extend the action to Lusztig's integral form $U_{\mathcal{A}}(\widehat{sl}_2)$, where $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. Let $\Lambda_F(x_1, x_2, \cdots)$ be the ring of symmetric functions in the x_n ($n \in \mathbb{N}$) over the ring F. We show that the lattice

$$V_{\mathcal{A}}(\Lambda_i) = \bigoplus_{m\in\mathbb{Z}} \Lambda_{\mathcal{A}}(x_1, x_2, \cdots) \otimes e^{m\alpha} e^{i\alpha/2}, \quad i = 0, 1$$

is invariant under Lusztig's $\mathcal{A}$-form $U_{\mathcal{A}}(\widehat{sl}_2)$ of divided powers, thus it contains the lattice $U_{\mathcal{A}}(\widehat{sl}_2)v_{\Lambda_i}$. From another direction in [CP, BCP] Beck, Chari and Pressley construct a PBW basis for the algebra $U_{\mathcal{A}}$, which partly generalizes Garland's work [G]. In the forthcoming paper with Chari [CJ], we will combine the two directions to study, among other things, the level one representations of Lusztig's integral form inside the basic representation.

We also find a vertex operator approach to the Littlewood-Richardson rule. In particular, an integral formula for the Littlewood-Richardson rule is found, and we use this to give a combinatorial description of divided powers of the Chevalley generators. The special case of only Chevalley generators corresponds to the deletion and insertion procedure on Young tableaux in the fermionic construction of $U_q(\widehat{sl}_2)$ used by Misra and Miwa in [MM] (see also [H]). Our explicit formulas of the divided powers of Chevalley generators suggest that there are corresponding formulas in the fermionic case. The fermionic picture together with our formulas will explain the meaning of the Littlewood-Richardson rule in the boson-fermion correspondence.

The method in this paper can be generalized to quantum affine algebras of ADE types, and this will provide more information about Schur functions as crystal bases

[BCP]. Our formulas will also be helpful in understanding the positivity conjecture of [BFJ].

The paper is organized as follows. In Section 2 we redevelop the vertex operator approach to Schur polynomials and express all mixed products of Schur and dual Schur vertex operators in terms of Schur functions, and we derive an integral formula for multiplication of Schur polynomials (Littlewood-Richardson rule). In Section 3 we first construct a Schur basis for the Frenkel-Jing vertex representation of $U_q(\widehat{sl}_2)$. We then use the Littlewood-Richardson rule to give explicit formulas for the action of the divided powers of the Drinfeld generators in terms of the Schur basis. In the last section (Sec. 4) we show that the $\mathcal{A}$-lattice of Schur functions tensored with the group algebra $\mathcal{A}[\mathbb{Z}\alpha]$ is a sublattice of a Lusztig's $\mathcal{A}$-lattice inside the vertex representation, which provides a simple combinatorial model for the homogeneous picture of the basic module for $U_{\mathcal{A}}(\widehat{sl}_2)$.

2. Schur functions and vertex operators

Let Λ_F be the ring of symmetric functions in infinitely many variables $x_1, x_2, \cdots$ over the ring F. In this section we take $F = \mathbb{Q}$, and later we will take $F = \mathbb{Q}(q)$ and $\mathbb{Z}[q, q^{-1}]$.

A partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ of n, denoted $\lambda \vdash n$, is a special decomposition of n: $n = \lambda_1 + \cdots + \lambda_l$ with $\lambda_1 \geq \cdots \geq \lambda_l \geq 1$. The number l is called the length of λ. We will identify $(\lambda_1, \cdots, \lambda_l)$ with $(\lambda_1, \cdots, \lambda_l, 0, \cdots, 0)$ if we want to view λ in $\mathbb{Z}^n$ when $n \geq l(\lambda)$. Sometime we prefer to use another notation for λ: $(1^{m_1}2^{m_2}\cdots)$ where m_i is the number of times that i appears among the parts of λ. The set of partitions will be denoted by $\mathcal{P}$.

There are several well-known bases in Λ_F parameterized by partitions: the power sum symmetric functions

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_l}$$

with $p_n = \sum x_i^n$ ($\mathbb{Q}$-basis); the monomial symmetric functions

$$m_\lambda(x_1, \cdots, x_n) = \sum_\sigma x^{\sigma(\lambda)} = \sum_\sigma x_1^{\sigma(\lambda_1)} \cdots x_n^{\sigma(\lambda_n)},$$

where σ runs through distinct permutations of λ as tuples; and the Schur functions s_λ form a basis over $\mathbb{Z}$. In terms of finitely many variables the Schur function is given by the Weyl character formula

$$s_\lambda(x_1, \cdots, x_n) = \frac{\sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma) x^{\sigma(\lambda+\delta)}}{\Pi_{i<j}(x_i - x_j)}, \tag{2.1}$$

where $\delta = (n-1, n-2, \cdots, 1, 0)$, $\lambda = (\lambda_1, \cdots, \lambda_n)$, and some λ_i may be zero.

Note that Eq. 2.1 gives a polynomial as long as $\lambda + \delta \in \mathbb{Z}_+^n$. In general for any n-tuple μ such that $\mu + \delta \in \mathbb{Z}_+^n$, $s_\mu = 0$ or $(-1)^{l(\sigma)} s_\lambda$, where $\lambda = \sigma(\mu + \delta) - \delta$ for some permutation σ and $l(\sigma)$ is the length of the permutation σ. This important property is still true for the Schur function in infinitely many variables, though there is a less satisfactory formula in algebraic combinatorics in that case. We will see that this symmetry property is manifested in our vertex operator approach.

We introduce an inner product on $\Lambda_{\mathbb{Z}}$ by setting

$$(s_\lambda, s_\mu) = \delta_{\lambda,\mu}. \tag{2.2}$$

It can be shown [M] that under this inner product

$$(p_\lambda, p_\mu) = z_\lambda \delta_{\lambda,\mu}, \tag{2.3}$$

where $z_\lambda = \prod_{i\geq 1} i^{m_i} m_i!$ for $\lambda = (1^{m_1} 2^{m_2} \cdots)$.

We recall the vertex operator approach to Schur functions [J1]. Let $\{b_n | n \neq 0\} \cup \{c\}$ be the set of generators of the Heisenberg algebra with defining relations

$$[b_m, b_n] = m\delta_{m,-n} c, \qquad [c, b_m] = 0. \tag{2.4}$$

The Heisenberg algebra has a canonical natural representation in the $\mathbb{Q}$-space $V = Sym(b_{-n}{'}s)$, the symmetric algebra generated by the b_{-n}, $n \in \mathbb{N}$. The action is given by

$$b_{-n}.v = b_{-n} v, \qquad b_n.v = n\frac{\partial v}{\partial b_{-n}}, \tag{2.5}$$

$$c.v = v\,. \tag{2.6}$$

It is clear that 1 is the highest weight vector in V.

Let us introduce two vertex operators (cf. $t = 0$ in [J2]):

$$\begin{aligned} S(z) &= exp(\sum_{n=1}^{\infty} \frac{b_{-n}}{n} z^n) exp(-\sum_{n=1}^{\infty} \frac{b_n}{n} z^{-n}) \\ &= \sum_{n\in\mathbb{Z}} S_n z^{-n}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} S^*(z) &= exp(-\sum_{n=1}^{\infty} \frac{b_{-n}}{n} z^n) exp(\sum_{n=1}^{\infty} \frac{b_n}{n} z^n) \\ &= \sum_{n\in\mathbb{Z}} S^*_n z^n. \end{aligned} \tag{2.8}$$

If we view z as a complex variable, then

$$S_n = \oint S(z) z^n \frac{dz}{z}, \quad S^*_n = \oint S^*(z) z^{-n} \frac{dz}{z},$$

where the contour integral is normalized so that $\oint \frac{dz}{z} = 1$.

It follows that for $n \geq 0$

$$S_n.1 = \delta_{n,0}, \quad S^*_{-n}.1 = \delta_{n,0}. \tag{2.9}$$

Lemma 2.1 ([J2])**.** *The components of $S(z)$ and $S^*(z)$ satisfy the following commutation relations.*

$$\begin{aligned} &S_m S_n + S_{n+1} S_{m-1} = 0, \qquad S^*_m S^*_n + S^*_{n-1} S^*_{m+1} = 0, \\ &S_m S^*_n + S^*_{n+1} S_{m+1} = \delta_{m,n}. \end{aligned}$$

The following result will be useful in our discussion.

Proposition 2.1. *Let $\delta = (n-1, n-2, \cdots, 1, 0)$. The operator products $S(z_1)S(z_2)\cdots S(z_n)z^\delta$ and $S^*(z_1)S^*(z_2)\cdots S^*(z_n)z^\delta$ are skew-symmetric under the action of $\mathfrak{S}_n$. For any $w \in \mathfrak{S}_n$ we have*

$$\begin{aligned} S(z_{w(1)})S(z_{w(2)})\cdots S(z_{w(n)})z^{w(\delta)} &= (-1)^{l(w)} S(z_1)S(z_2)\cdots S(z_n)z^\delta, \\ S^*(z_{w(1)})S^*(z_{w(2)})\cdots S^*(z_{w(n)})z^{w(\delta)} &= (-1)^{l(w)} S^*(z_1)S^*(z_2)\cdots S^*(z_n)z^\delta. \end{aligned}$$

Proof. The equations follow by repeatedly using the commutation relations satisfied by the Schur vertex and the dual vertex operators (see Lemma 2.1). ☐

The space V has a natural Hermitian product given by

$$b_n^* = b_{-n}. \tag{2.10}$$

Under the inner product, the elements $b_{-\lambda} = b_{-\lambda_1} \cdots b_{-\lambda_l} (\lambda \vdash n)$ span an orthogonal basis in V and

$$(b_{-\lambda}, b_{-\mu}) = \delta_{\lambda,\mu} z_\lambda. \tag{2.11}$$

Definition 2.2. The characteristic map ch from the vertex space V to the ring $\Lambda_{\mathbb{Q}}$ of symmetric functions is the $\mathbb{Q}$-linear map given by

$$b_{-\lambda} = b_{-\lambda_1} \cdots b_{-\lambda_l} \longrightarrow p_\lambda = p_{\lambda_1} \cdots p_{\lambda_l}.$$

It is clear that ch is an isomorphism of vector spaces. In particular, the vector $S_{-n}.1$ corresponds to the homogeneous symmetric polynomial s_n. For this reason we will simply write

$$s_n = S_{-n}.1, \qquad n \in \mathbb{Z}_+.$$

The generating function of s_n is given by

$$\sum_{n=0}^{\infty} s_n z^n = exp(\sum_{n=1}^{\infty} \frac{b_{-n}}{n} z^n). \tag{2.12}$$

The following result appeared in [J1, J4]. For completeness we include a modified proof.

Theorem 2.3. *The space V is isometrically isomorphic to $\Lambda_{\mathbb{Q}}$ under the map ch. The sets $\{h_{-\lambda} = s_{\lambda_1} s_{\lambda_2} \cdots s_{\lambda_l} : \lambda \vdash n, n \in \mathbb{Z}_+\}$ and $\{S_{-\lambda_1} S_{-\lambda_2} \cdots S_{-\lambda_l}.1 : \lambda \vdash n, n \in \mathbb{Z}_+\}$ are both $\mathbb{Q}$-basis. Moreover the basis $\{S_{-\lambda_1} S_{-\lambda_2} \cdots S_{-\lambda_l}.1\}$ is orthonormal and expressed explicitly by*

$$S_{-\lambda}.1 := S_{-\lambda_1} S_{-\lambda_2} \cdots S_{-\lambda_l}.1 = det(s_{\lambda_i - i + j}), \tag{2.13}$$

which corresponds to the Schur function s_λ in $\Lambda_{\mathbb{Q}}$.

Proof. To show the $\mathfrak{S}_n$-symmetry we consider the modified vertex operators associated with the root lattice $\mathbb{Z}\alpha$ with $(\alpha|\alpha) = 1$. Let $\tilde{V} = V \otimes \mathbb{Q}[\mathbb{Z}\alpha]$, where $\mathbb{Q}[\mathbb{Z}\alpha]$ is the group algebra generated by $e^{m\alpha}, m \in \mathbb{Z}$ over $\mathbb{Q}$. Define

$$\overline{S}(z) = S(z) e^{\alpha} z^{\partial} = \sum_{n \in \mathbb{Z}+1/2} \overline{S}_n z^{-n-1/2}, \tag{2.14}$$

$$\overline{S}^*(z) = S^*(z) e^{-\alpha} z^{-\partial} = \sum_{n \in \mathbb{Z}+1/2} \overline{S}_n^* z^{n-1/2}, \tag{2.15}$$

where the operators e^{α} and z^{∂} act on $\mathbb{Q}(q)[\mathbb{Z}\alpha]$ as follows:

$$e^{\alpha} e^{m\alpha} = e^{(m+1)\alpha}, \qquad z^{\partial} e^{m\alpha} = z^m e^{m\alpha},$$

The components satisfy the Clifford algebra relations

$$\{\overline{S}_m, \overline{S}_n\} = \{\overline{S}_m^*, \overline{S}_n^*\} = 0, \tag{2.16}$$

$$\{\overline{S}_m, \overline{S}_n^*\} = \delta_{m,n}, \tag{2.17}$$

where $m, n \in \mathbb{Z} + 1/2$.

For an l-tuple $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ satisfying $\lambda_i = \lambda_j - j + i$ for some i and j we have

$$\begin{aligned} S_{-\lambda_1} \cdots S_{-\lambda_l}.1 &= \overline{S}_{-\lambda_1+1/2}\overline{S}_{-\lambda_2-1/2} \cdots \overline{S}_{-\lambda_l-l+1/2}e^{-l\alpha} \\ &= \overline{S}_{-\lambda+\delta-(m-1/2)\mathbf{1}} \\ &= 0 \end{aligned}$$

due to $\lambda + \delta = (i,j)(\lambda+\delta)$ and (2.16–2.17). Here $\mathbf{1}$ denotes the integer vector $(1, 1, \cdots, 1)$, and similarly $m\mathbf{1} = (m, m, \cdots, m)$. In general for any other l-tuple μ we have

$$S_{-\mu_1} \cdots S_{-\mu_l}.1 = sgn(\mu) S_{-\lambda_1} \cdots S_{-\lambda_l}.1, \tag{2.18}$$

where λ is the partition related to μ: $\lambda = \sigma(\mu+\delta) - \delta$, and the sign $sgn(\mu) = (-1)^{l(\sigma)}$. See [J1] for details. □

We shall use vertex operators to give another proof that the Schur functions form a $\mathbb{Z}$-linear basis.

Definition 2.4. For an l-tuple μ we define the Schur function s_μ to be the symmetric function corresponding to $S_{-\mu_1} \cdots S_{-\mu_l}.1$ under the characteristic map. We will simply write

$$s_\mu = S_{-\mu}.1 = S_{-\mu_1} \cdots S_{-\mu_l}.1\,.$$

We say that two integral l-tuples μ and λ are related if there is a permutation σ such that $\mu + \delta = \sigma(\lambda + \delta)$. If there exists an odd permutation σ such that $\mu + \delta = \sigma(\mu+\delta)$, then we say that μ is degenerate. An integral tuple μ is said to be non-increasing if $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l$. In particular, a non-increasing positive integral tuple is a partition.

If there are no two parts μ_i, μ_j $(i<j)$ of a tuple μ such that $\mu_i = \mu_j - (j-i)$, then μ is non-degenerate and there exists a unique non-increasing tuple λ such that $\mu + \delta = \sigma(\lambda+\delta)$. We denote by $\pi(\mu)$ the associated non-increasing integral tuple λ of μ. For simplicity we let $sgn(\mu) = 0$ if the l-tuple μ is degenerate.

Remark 2.5. The above relation among integral tuples corresponds to the linkage symmetry in the weight theory of the Lie algebra $\mathfrak{sl}(n+1)$. The symmetric group $\mathfrak{S}_n$ is then viewed as the Weyl group.

Remark 2.6. In view of Definition 2.4 all symmetric functions in this paper are polynomials, though a Schur function of an arbitrary tuple can be a rational function as defined by (2.1). We do not attempt to distinguish the term "symmetric function" from that of "symmetric polynomial". It is apparent that the symmetry is only with respect to the variables $x_1, x_2, \cdots$ and not with respect to the power sum variable.

The following fact follows from Theorem 2.3 and Lemma 2.1.

$$s_\mu = \begin{cases} sgn(\sigma) s_{\pi(\mu)}, & \text{if } \pi(\mu) \in \mathcal{P}, \\ 0, & \text{if } \mu \text{ is degenerate or } \pi(\mu) \notin \mathcal{P}. \end{cases} \tag{2.19}$$

We also have similar results for the dual vertex operators $S^*(z)$, where the vector $S^*_n.1$ corresponds to the elementary symmetric function $(-1)^n e_n = (-1)^n s_{(1^n)}$.

For a partition $\lambda = (\lambda_1, \cdots, \lambda_l)$ we denote by $\lambda' = (\lambda'_1, \cdots, \lambda'_k)$ the dual partition, where $\lambda'_i = Card\{j : \lambda_j \geq i\}$. We also denote by (λ, μ) the juxtaposition of two partitions or tuples. Note that (λ, μ) is generally not a partition.

Theorem 2.7. *The vectors* $\{S^*_{\lambda_1}\cdots S^*_{\lambda_l}.1 \mid \lambda\vdash n, n\in\mathbb{Z}_+\}$ *are also orthonormal, and we have*

$$S^*_{\lambda_1}S^*_{\lambda_2}\cdots S^*_{\lambda_l}.1=(-1)^{|\lambda|}s_{\lambda'}=(-1)^{|\lambda|}det(s_{\lambda'_i-i+j}), \tag{2.20}$$

where $|\lambda|=\lambda_1+\lambda_2+\cdots+\lambda_l$. *Moreover, we have*

$$S_{-\mu_1}\cdots S_{-\mu_l}S^*_{\nu_1}\cdots S^*_{\nu_k}.1=(-1)^{|\nu|}sgn(\lambda,\mu)s_{\pi(\mu,\pi(\nu)')},$$
$$S^*_{\mu_1}\cdots S^*_{\mu_k}S_{-\nu_1}\cdots S_{-\nu_l}.1=(-1)^{|\mu|}sgn(\mu,\nu)s_{\pi(\mu,\pi(\nu)')'},$$

where (μ,ν) *is the juxtaposition of* μ *and* ν, *the associated partition* $\pi(\mu,\nu)$ *is obtained by* $\pi(\mu,\nu)=\sigma((\mu,\nu)+\delta)-\delta$ *for some* $\sigma\in\mathfrak{S}_{l+k}$, *and* $\pi(\mu,\pi(\nu)')'$ *denotes the dual partition of* $\pi(\mu,\pi(\nu)')$.

As a consequence of Theorem 2.7 it follows that

$$S_{-\lambda}.1=(-1)^{|\lambda|}S^*_\lambda.1.$$

Example 2.8.

$$S_{-1}S^*_2S^*_2.1=S_{-1}S_{-2}S_{-2}.1=0,$$
$$S_{-1}S^*_2S^*_1S^*_1S^*_1.1=-S_{-1}S_{-4}S_{-1}.1=S_{-3}S_{-2}S_{-1}.1=s_{(3,2,1)},$$

where $(1,4,1)+\delta=(3,5,1)\,\bar{\sim}\,(5,3,1)=(3,2,1)+\delta$.

To close this section we derive the Littlewood-Richardson rule in our picture, which will be used later to realize the action of the quantum affine algebra $U_q(\widehat{sl}_2)$ on $\Lambda_{\mathbb{Q}[q,q^{-1}]}$. In particular, we give a new proof of the integrality of Schur functions using vertex operators.

Proposition 2.2. *Let* λ *and* μ *be two partitions of lengths* m *and* n *respectively. Then* $s_\lambda s_\mu=\sum C_{MN}s_{(\lambda+M,\mu-N)}$, *where* C_{MN} *is the number of integral matrices* (k_{ij}) *such that*

$$(k_{11}+k_{12}+\cdots+k_{1n},\cdots,k_{m1}+\ldots+k_{mn})=M,$$
$$(k_{11}+k_{21}+\ldots+k_{m1},\cdots,k_{1n}+k_{2n}+\cdots+k_{mn})=N,$$

where $k_{ij}\geq 0$. *In particular, the Schur functions form a* $\mathbb{Z}$*-lattice in* $\Lambda_{\mathbb{Q}}$.

Proof. It follows from the definition that

$$s_\lambda s_\mu=\int S(z_1)\cdots S(z_m).1S(w_1)\cdots S(w_n).1z^{-\lambda}w^{-\mu}\frac{dz}{z}\frac{dw}{w}.$$

Observe that

$$S(z_1)\cdots S(z_m).1S(w_1)\cdots S(w_n).1$$
$$=\prod_{i,j}(1-\frac{w_j}{z_i})^{-1}S(z_1)\cdots S(z_m)S(w_1)\cdots S(w_n).1\,.$$

As an infinite series in $|w_j|<|z_i|$ we have

$$\begin{aligned}\prod_{i,j}(1-\frac{w_j}{z_i})^{-1}&=\prod_{i,j}(1+\frac{w_j}{z_i}+(\frac{w_j}{z_i})^2+\cdots)\\&=\prod_{i,j}(\sum_{k_{ij}}w_j^{k_{ij}}z_i^{-k_{ij}})\\&=\prod_{k=(k_{ij})}w_1^{k_{\cdot 1}}\cdots w_n^{k_{\cdot n}}z_1^{-k_{1\cdot}}\cdots z_m^{-k_{m\cdot}},\end{aligned}$$

with $k_{\cdot j}=k_{1j}+\cdots+k_{mj}$, $k_{i\cdot}=k_{i1}+\cdots+k_{in},k_{ij}\geq 0$.

Plugging the expansion into the integral and invoking Theorem (2.3), we prove the proposition. □

Remark 2.9. The number C_{MN} is equal to the index of $\mathfrak{S}_\lambda \pi \mathfrak{S}_\mu$ in $\mathfrak{S}_n$ [JK]. We can also write

$$s_\lambda s_\mu = \prod_{i,j}(1-R_{ij})^{-1} s_{(\lambda,\mu)},$$

where R_{ij} is the raising operator defined on the parts of the Schur functions $R_{ij} s_{(\cdots,\lambda_i,\cdots,\lambda_j,\cdots)} = s_{(\cdots,\lambda_i+1,\cdots,\lambda_j-1,\cdots)}$.

We remark that the characteristic map ch is actually defined over $\mathbb{Z}$ if we let $ch(S_{-\lambda}.1) = s_\lambda$. Then the space $V_\mathbb{Z}$ is isometrically isomorphic to $\Lambda_\mathbb{Z}$.

3. Vertex representations of $U_\mathcal{A}(\widehat{sl}_2)$

Let $\mathcal{A}$ be the ring $\mathbb{Z}[q,q^{-1}]$ of Laurent polynomials in q over $\mathbb{Z}$.

For $n \in \mathbb{Z}_+$ we define $[n] = \frac{q^n - q^{-n}}{q-q^{-1}}$. The q-factorial $[n]!$ is equal to $[n][n-1] \cdots [2][1]$ and then the q-Gaussian numbers are defined naturally by $\left[{n \atop m} \right] = \frac{[n]!}{[m]![n-m]!}$ for $n \geq m \geq 0$. By convention $[0] = [1] = 1$. For an element a in an algebra over $\mathbb{Q}(q)$ we use $a^{(n)}$ to denote the divided power $\frac{a^n}{[n]!}$.

The quantum affine algebra $U_q(\widehat{sl}_2)$ is the associative algebra over $\mathbb{Q}(q)$ generated by the Chevalley generators $e_i, f_i, K_i^{\pm 1}$ $(i = 0, 1)$ and $q^{\pm d}$ subject to the following defining relations:

$$\begin{aligned}
&K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad q^d q^{-d} = q^{-d} q^d = 1,\\
&K_i K_j = K_j K_i, \quad q^d K_i^{\pm 1} = K_i^{\pm 1} q^d,\\
&K_i e_j K_i^{-1} = q^{a_{ij}} e_j, \quad K_i f_j K_i^{-1} = q^{-a_{ij}} f_j,\\
&q^d e_i q^{-d} = q^{\delta_{i,0}} e_i, \quad q^d f_i q^{-d} = q^{-\delta_{i,0}} f_i,\\
&[e_i, f_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q-q^{-1}},\\
&\sum_{r=0}^{1-a_{ij}} (-1)^r e_i^{(r)} e_j e_i^{(1-a_{ij}-r)} = 0 \qquad \text{if } i \neq j,\\
&\sum_{r=0}^{1-a_{ij}} (-1)^r f_i^{(r)} f_j f_i^{(1-a_{ij}-r)} = 0 \qquad \text{if } i \neq j,
\end{aligned}$$

where $(a_{ij}) = \left({2 \atop -2} \; {-2 \atop 2} \right)$ is the extended Cartan matrix [Ka]. The element $K_0 K_1 = q^c$ is a central element of $U_q(\widehat{sl}_2)$.

Let $U_\mathcal{A}(\widehat{sl}_2)$ be the $\mathcal{A}$-subalgebra [L] of $U_q(\widehat{sl}_2)$ generated by $e_i^{(n)}$, $f_i^{(n)}$, $K_i^\pm$, $q^{\pm d}$ for $i = 0, 1$ and $n \in \mathbb{N}$. Then $U_\mathcal{A}(\widehat{sl}_2) \otimes_\mathcal{A} \mathbb{Q}(q) \simeq U_q(\widehat{sl}_2)$.

For $m \in \mathbb{Z}, r \in \mathbb{N}$ we define

$$\begin{bmatrix} K_i; m \\ r \end{bmatrix} = \prod_{s=1}^{r} \frac{K_i q^{m+1-s} - K_i^{-1} q^{s-1-m}}{q - q^{-1}},$$

which belong to the Cartan subalgebra of $U_\mathcal{A}(\widehat{sl}_2)$.

The module $V(\Lambda_i)$ $(i=0,1)$ is the simple highest weight module of $U_q(\widehat{sl}_2)$ generated by the highest weight vector v_{Λ_i} such that

$$e_i v_{\Lambda_i}=0, \quad K_j v_{\Lambda_i}=q^{\delta_{ij}}v_{\Lambda_i}, \quad q^d v_{\Lambda_i}=v_{\Lambda_i}, \qquad i=0,1.$$

An $\mathcal{A}$-subspace W of an $U_q(\widehat{sl}_2)$-module V is called an $\mathcal{A}$-lattice for $U_{\mathcal{A}}(\widehat{sl}_2)$ if W is invariant under $U_{\mathcal{A}}(\widehat{sl}_2)$ and $W\otimes_{\mathcal{A}}\mathbb{Q}(q)\simeq V$.

The (level one) modules $V(\Lambda_i)$ had been realized by vertex operators in [FJ] for $U_q(\widehat{sl}_2)$. Since we will construct the vertex representation of $U_{\mathcal{A}}(\widehat{sl}_2)$ we need to modify the original construction.

Let $U_q(\widehat{\mathfrak{h}})$ be the Heisenberg algebra with the generators $\{a_n | n\neq 0\}\cup\{C\}$ and the defining relations

$$[a_m,a_n]=\delta_{m,-n}\frac{m}{1+q^{2|m|}}C, \qquad [C,a_m]=0. \tag{3.1}$$

The level one irreducible representation $V(\Lambda_i)$ is realized on the vertex representation space

$$V(\Lambda_i)=Sym_{\mathbb{Q}(q)}(a_{-n}'s)\otimes\mathbb{Q}(q)[\mathbb{Z}\alpha]e^{i\alpha/2}, i=0,1,$$

where $Sym_{\mathbb{Q}(q)}(a_{-n}'s)$ denotes the $\mathbb{Q}(q)$-symmetric algebra generated by the Heisenberg generators $a_{-n}, n\in\mathbb{N}$. The element $e^{i\alpha/2}$ (the highest weight vector) is formally adjoined to $\mathbb{Q}(q)[\mathbb{Z}\alpha]=\mathbb{Q}(q)\langle e^{m\alpha}|m\in\mathbb{Z}\rangle$.

We define two kinds of operators on the vector space $\mathbb{Q}(q)[\mathbb{Z}\alpha]e^{i\alpha/2}$:

$$e^{n\alpha}.e^{m\alpha}e^{i\alpha/2}=e^{(m+n)\alpha}e^{i\alpha/2}, n\in\mathbb{Z}, \tag{3.2}$$

$$\partial.e^{m\alpha}e^{i\alpha/2}=(2m+i)e^{m\alpha}e^{i\alpha/2}. \tag{3.3}$$

In particular, $\mathbb{Q}(q)[\mathbb{Z}\alpha]e^{i\alpha/2}$ is a $\mathbb{Q}(q)[\mathbb{Z}\alpha]$-module.

The Heisenberg algebra $U_q(\widehat{\mathfrak{h}})$ acts on the $\mathbb{Q}(q)$-space $Sym_{\mathbb{Q}(q)}(a_{-n}'s)$ via

$$a_{-n}.v=a_{-n}v, \qquad a_n.v=\frac{n}{1+q^{2n}}\frac{\partial v}{\partial a_{-n}}, \tag{3.4}$$

$$C.v=v\,, \tag{3.5}$$

where $v\in U_q(\widehat{\mathfrak{h}})$.

We define the vertex operators associated to $U_q(\widehat{sl}_2)$ by

$$\begin{aligned} X^+(z)&=exp(\sum_{n=1}^{\infty}\frac{(1+q^{2n})q^{-n}}{n}a_{-n}z^n)exp(-\sum_{n=1}^{\infty}\frac{(1+q^{2n})q^{-n}}{n}a_nz^{-n})e^{\alpha}z^{\partial} \\ &=\sum_{n\in\mathbb{Z}}X_n^+z^{-n-1}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} X^-(z)&=exp(-\sum_{n=1}^{\infty}\frac{1+q^{2n}}{n}a_{-n}z^n)exp(\sum_{n=1}^{\infty}\frac{1+q^{2n}}{n}a_nz^{-n})e^{-\alpha}z^{-\partial} \\ &=\sum_{n\in\mathbb{Z}}X_n^-z^{-n-1}. \end{aligned} \tag{3.7}$$

The normal order of vertex operator products is defined by rearranging the exponential factors. For example,

$$
\begin{aligned}
&: X^+(z)X^+(w) : \\
&= exp(\sum_{n=1}^{\infty}\frac{(1+q^{2n})q^{-n}}{n}a_{-n}(z^n+w^n))exp(-\sum_{n=1}^{\infty}\frac{(1+q^{2n})q^{-n}}{n}a_n(z^{-n}+w^{-n})) \\
&\qquad \times e^{2\alpha}z^{\partial}w^{\partial}.
\end{aligned}
$$

The components of the Drinfeld generators satisfy some quadratic relations.

Lemma 3.1 ([FJ]). *The components of $X^{\pm}(z)$ satisfy the following commutation relations*

$$
\begin{aligned}
X_m^{\pm}X_n^{\pm} - q^{\pm 2}X_n^{\pm}X_m^{\pm} &= q^{\pm 2}X_{m-1}^{\pm}X_{n+1}^{\pm} - X_{n+1}^{\pm}X_{m-1}^{\pm}, \\
X_m^{+}X_n^{-} - X_n^{-}X_m^{+} &= \frac{1}{q-q^{-1}}\left(\psi_{m+n} - \phi_{m+n}\right),
\end{aligned}
$$

where the polynomials ψ_n and ϕ_{-n} in the a_n are defined by

$$
\Psi(z) = \sum_{n\geq 0}\psi_n z^{-n} = exp(\sum_{n\in\mathbb{N}}\frac{(q^{3n}-q^{-n})a_n}{n}z^{-n})q^{\partial}, \tag{3.8}
$$

$$
\Phi(z) = \sum_{n\geq 0}\phi_{-n} z^{n} = exp(\sum_{n\in\mathbb{N}}\frac{(q^{-n}-q^{3n})a_{-n}}{n}z^{n})q^{-\partial}. \tag{3.9}
$$

The following map defines the irreducible $U_q(\widehat{sl}_2)$-module structure for $V(\Lambda_i)$.

$$
\begin{aligned}
&e_1 \to X_0^+, \quad f_1 \to X_0^-, \quad K_1 \to q^{\partial}, \\
&e_0 \to X_1^- q^{-\partial}, f_0 \to q^{\partial}X_{-1}^+, K_0 \to q^{1-\partial}.
\end{aligned}
$$

The vertex space $V(\Lambda_i)$ is endowed with the standard inner product via

$$
\begin{aligned}
a_n^* = a_{-n}, \quad (e^{\alpha})^* = e^{-\alpha}, \\
(z^{\partial})^* = z^{-\partial}.
\end{aligned}
$$

It follows from the commutation relations (3.1) that

$$
(a_{-\lambda}e^{m\alpha}e^{i\alpha/2}, a_{-\mu}e^{n\alpha}e^{i\alpha/2}) = \delta_{mn}\delta_{\lambda\mu}z_{\lambda}\prod_{j\geq 1}\frac{1}{1+q^{2\lambda_j}}, \tag{3.10}
$$

where z_λ is as in Section 2.

By Section 2 there are two special bases in $V(\Lambda_i)$: the power sum basis $\{a_{-\lambda}e^{m\alpha}e^{i\alpha/2}\}$ and the Schur basis $\{s_\lambda e^{m\alpha}e^{i\alpha/2}\}$. However, the Schur basis is no longer orthogonal with respect to the inner product (3.10).

Let $b_{-n} = a_{-n}, b_n = (1+q^{2n})a_n$, $n \in \mathbb{N}$, then $\{b_{-n}\}$ generate a standard Heisenberg algebra as in Section 2. In terms of the new Heisenberg generators we have

$$
\begin{aligned}
S(z) &= exp(\sum_{n=1}^{\infty}\frac{1}{n}a_{-n}z^n)exp(-\sum_{n=1}^{\infty}\frac{1+q^{2n}}{n}a_n z^{-n}) = \sum_{n\in\mathbb{Z}}S_n z^{-n}, \\
S^*(z) &= exp(-\sum_{n=1}^{\infty}\frac{1}{n}a_{-n}z^n)exp(\sum_{n=1}^{\infty}\frac{1+q^{2n}}{n}a_n z^{-n}) = \sum_{n\in\mathbb{Z}}S_n^* z^{n},
\end{aligned}
$$

which generate the Schur function basis. We call $S(z)$ (or $S^*(z)$) the Schur (or dual Schur) vertex operator. For a partition λ and $m \in \mathbb{Z}$ we define the *Schur symmetric polynomial* in $V(\Lambda_i)$ (cf. Theorem 2.3):

$$s_\lambda e^{m\alpha} e^{i\alpha/2} := S_{-\lambda} e^{m\alpha} e^{i\alpha/2} = S_{-\lambda_1} \cdots S_{-\lambda_l} e^{m\alpha} e^{i\alpha/2}. \tag{3.11}$$

The element s_λ is a polynomial over $\mathbb{Q}$ in terms of the power sum a_μ, where $|\mu| = |\lambda|$. Note that $S(z)$ acts trivially on the lattice vector $e^{m\alpha} e^{i\alpha/2}$.

We need to recall some further terminology about partitions. Let λ and μ be two partitions; we write $\lambda \supset \mu$ if the Young diagram of λ contains that of μ. The set difference $\lambda \setminus \mu$ is called a skew diagram. The conjugate of a skew diagram $\theta = \lambda \setminus \mu$ is $\theta' = \lambda' \setminus \mu'$ and we define

$$|\theta| = \sum \theta_j = |\lambda| - |\mu|. \tag{3.12}$$

A skew diagram θ is a horizontal n-strip (resp. a vertical n-strip) if $|\theta| = n$ and $\theta'_j \leq 1$ (resp. $\theta_j \leq 1$) for each j. Thus a horizontal (resp. vertical) strip has at most one column (resp. rows) in its diagram.

For a partition μ and an integer m we let $\mathcal{V}_n = \mathcal{V}_n(m, \mu)$ be the set of the partitions of λ such that the skew diagram $\lambda \setminus \pi(m, \pi(\mu)')'$ is a vertical n-strip. We also let $\mathcal{H}_n = \mathcal{H}_n(m, \mu)$ be the set of partitions λ such that the skew diagram $\lambda \setminus \pi(m, \mu)$ is a horizontal n-strip. Note that $\mathcal{V}_n$ may be described as the set of partitions λ such that the skew diagram $\lambda \setminus \pi(1^m, \mu)$ is a vertical n-strip. The following is called the Pieri rule [M]:

$$s_n S_{-m} S_{-\mu}.1 = \sum_{\lambda \in \mathcal{H}_i(m,\mu)} sgn(m, \mu) s_\lambda, \tag{3.13}$$

$$s_{1^n} S_{-m} S_{-\mu}.1 = \sum_{\lambda \in \mathcal{V}_n(m,\mu)} (-1)^n sgn(m, \pi(\mu)')' s_\lambda. \tag{3.14}$$

We will use $\lambda - \mu$ to denote the difference of two integral tuples in $\mathbb{Z}^n$.

Theorem 3.2. *The quantum affine algebra $U_q(\widehat{sl}_2)$ is realized on the Fock space $\Lambda \otimes \mathbb{Q}(q)[\mathbb{Z}\alpha] e^{i\alpha/2}$ of symmetric functions by the following action:*

$$X_n^+ s_\mu e^{m\alpha} e^{i\alpha/2} = e^{(m+1)\alpha} e^{i\alpha/2}$$
$$\times \left(\sum_{j=0}^{l(\mu)-2m-n-1-i} q^{-2m-n-1-i-2j} sgn(-2m-n-1-i-j, \mu) \sum_{\lambda \in \mathcal{H}_j} s_\lambda \right),$$

where $\mathcal{H}_j = \mathcal{H}_j(-2m-n-1-i-j, \mu)$, the sign refers to $sgn(-2m-n-1-i-j, \mu) = (-1)^{l(\sigma)}$ such that $(-2m-n-1-i-j, \mu) + \delta = \sigma(\lambda + \delta)$ and λ is the partition of length at most $l(\mu) + 1$. Also we have

$$X_n^- s_\mu e^{m\alpha} e^{i\alpha/2} = (-1)^{n+1+i} e^{(m-1)\alpha} e^{i\alpha/2}$$
$$\times \left(\sum_{j=0}^{\mu_1+2m-n-1+i} q^{2j} sgn(2m-n-1-j+i, \mu') \sum_{\lambda \in \mathcal{V}_j} s_\lambda \right),$$

where $\mathcal{V}_j = \mathcal{V}_j(2m-n-1-j+i, \mu')$, the sign refers to $sgn(2m-n-1-i-j, \mu) = (-1)^{l(\sigma)}$ such that $(2m-n-1-j+i, \mu) + \delta = \sigma(\lambda + \delta)$ and λ is the partition of length at most $l(\mu) + 2m - n - 1 - j + i$.

This result will be proved in more generality later in Theorem 3.5.

We can reformulate the result in terms of the standard inner product in Section 2. Let $u_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ be the jth unit vector in $\mathbb{Z}^n$. Let $\mathbf{1}_{(l_1,\cdots,l_j)}$ be the sum of the unit vectors $u_{l_1}, \cdots, u_{l_j}$.

Proposition 3.1. *For $n \in \mathbb{Z}$ and a partition μ we have*

$$\begin{aligned} &X_n^- s_\mu e^{m\alpha} e^{i\alpha/2} \\ &= \sum_\lambda s_\lambda e^{(m-1)\alpha} e^{i\alpha/2} \sum_{j=0}^{l(\lambda)} (-q^2)^j \sum_{l_1<\cdots<l_j} (S^*_{2m+i-n-1-j} S_{-\mu}, S_{-(\lambda-\mathbf{1}_{(l_1,\cdots,l_j)})}), \end{aligned}$$

where λ runs through partitions of weight $|\mu|+2m-n-1+i$ such that $\lambda-\mathbf{1}_{(l_1,\cdots,l_j)}$ is the juxtaposition of $(1^{2m-n-j-1+i})$ and μ.

Later in Theorem 3.5 we will give another proof in terms of the dual vertex operator $S^*(z)$.

Example 3.3. Using Theorem 3.2 it is easy to compute the following:

$$\begin{aligned} &X_n^{\pm} e^{r\alpha} e^{i\alpha/2} = 0, \qquad \text{if } n > \mp 2r - 1 \mp i, \\ &X^+_{-2r+1-i} \cdots X^+_{-3-i} X^+_{-1-i} e^{i\alpha/2} = e^{r\alpha} e^{i\alpha/2}, \qquad r \geq 1, \\ &X^-_{2r-3+i} \cdots X^+_{1+i} X^+_{-1+i} e^{i\alpha/2} = e^{-r\alpha} e^{i\alpha/2}, \qquad r \geq 1. \end{aligned}$$

$$\begin{aligned} X^+_{-1} s_{(2,1)} e^{-\alpha} &= q^2 s_{(2,2,1)} - q^{-2}(s_5 + s_{(4,1)} + s_{(3,2)}) + q^{-6}(s_5 + s_{(4,1)}), \\ X_0^- s_1 e^{\alpha} &= -s_{(2)} + q^4 s_{(1^2)}. \end{aligned}$$

We can generalize the action to the divided powers of $X_n^{\pm(r)}$.

Lemma 3.4 ([BFJ]). *For $r \in \mathbb{N}$ we have*

$$\prod_{1\leq i<j\leq k} (z_i - q z_j) = \sum_{w \in \mathfrak{S}_k} (-q)^{\ell(w)} z^{w(\delta)} + \sum a_{\gamma_1,\ldots,\gamma_k} z_1^{\gamma_1} z_2^{\gamma_2} \ldots z_k^{\gamma_k},$$

where $\delta = (k-1, k-2, \ldots, 0)$ and the second sum consists of certain monomials such that some $\gamma_i = \gamma_j$, $i \neq j$ and $a_\gamma \in \mathbb{Z}[q]$, $a_\gamma(1) = 0$.

For $i = 0, 1$, we introduce the $\mathcal{A}$-linear subspace of $V(\Lambda_i)$

$$V_{\mathcal{A}}(\Lambda_i) = \bigoplus_{\lambda \in \mathcal{P}, m \in \mathbb{Z}} \mathbb{Z}[q, q^{-1}] s_\lambda e^{m\alpha} e^{i\alpha/2}. \tag{3.15}$$

It is clear that

$$V_{\mathcal{A}}(\Lambda_i) \otimes_{\mathcal{A}} \mathbb{Q}(q) \simeq V(\Lambda_i).$$

Theorem 3.5. *The $\mathcal{A}$-linear spaces $V_{\mathcal{A}}(\Lambda_i)$ are invariant under the action of the divided powers $X_n^{\pm(r)}$. More precisely, we have*

$$\begin{aligned} &X_n^{+(r)} s_\mu e^{m\alpha} e^{i\alpha/2} \\ &= q^{-3\binom{r}{2} - r(n+1+2m+i)} \sum_{l(\lambda)\leq r} q^{-2|\lambda|} s_\lambda \cdot s_{(-\lambda-2\delta-(n+1+2m+i)\mathbf{1},\mu)} e^{(m+r)\alpha} e^{i\alpha/2}, \\ &X_n^{-(r)} s_{\mu'} e^{m\alpha} e^{i\alpha/2} \\ &= (-1)^{r(n+1+i)} q^{\binom{r}{2}} \sum_{l(\lambda)\leq r} q^{2|\lambda|} s_{\lambda'} \cdot s_{\pi(-\lambda-2\delta-(n+1-2m-i)\mathbf{1},\mu)'} e^{(m-r)\alpha} e^{i\alpha/2}, \end{aligned}$$

where the summations run through all partitions λ of length $\leq r$, $\delta = (r-1, r-2, \cdots, 0)$, and $\mathbf{1} = (1, \cdots, 1) \in \mathbb{Z}^r$.

Proof. Let $z = (z_1, \cdots, z_r)$, $w = (w_1, \cdots, w_l)$, and $\mathbf{1} = (1, \cdots, 1) \in \mathbb{Z}^r$ in the following computation:

$$\begin{aligned}
&x_n^{+r} s_\mu e^{m\alpha} e^{i\alpha/2} \\
&= \oint X^+(z_1) \cdots X^+(z_r) S(w_1) \cdots S(w_l) z^{(n+1)\mathbf{1}} w^{-\mu} \, \frac{dzdw}{zw} \, e^{m\alpha} e^{i\alpha/2} \\
&= \oint \exp\big(\sum_{n=1}^{\infty} \frac{(q^n + q^{-n})}{n} a_{-n}(z_1^n + \cdots + z_r^n)\big) : S(w_1) \cdots S(w_l) : \\
&\quad \times \prod_{i<j} (z_i - z_j)(z_i - q^{-2} z_j)(1 - \frac{w_j}{w_i}) \prod_{i,j} (1 - q^{-1} \frac{w_j}{z_i}) \\
&\qquad \times z^{(2m+n+1+i)\mathbf{1}} e^{(m+r)\alpha} e^{i\alpha/2} \frac{dz}{z} \frac{dw}{w}.
\end{aligned}$$

Note that the integrand divided by $\prod_{j<k}(z_j - q^{-2} z_k)$ is an anti-symmetric function in $z_1, \ldots, z_r$. It follows from Lemma 3.4 that the terms $z_1^{\gamma_1} z_2^{\gamma_2} \ldots z_r^{\gamma_r}$ (for which some $\gamma_k = \gamma_j$) make no contribution to the integral. Therefore

$$\begin{aligned}
x_n^{+(r)} s_\mu e^{m\alpha} e^{i\alpha/2} &= \frac{1}{[r]!} \sum_{w \in \mathfrak{S}_r} \oint \exp\big(\sum_{n=1}^{\infty} \frac{(q^n + q^{-n})}{n} a_{-n}(z_1^n + \cdots + z_r^n)\big) \\
&\quad \times : S(w_1) \cdots S(w_l) : \prod_{i<j} (z_i - z_j) \prod_{i<j} (1 - \frac{w_j}{w_i}) \prod_{i,j} (1 - q^{-1} \frac{w_j}{z_i}) \\
&\quad (-q)^{-\ell(w)} z^{w(\delta) + (2m+n+1+i)\mathbf{1}} w^{-\mu} e^{(m+r)\alpha} e^{i\alpha/2} \frac{dzdw}{zw} \\
&\quad = q^{-\binom{r}{2}} \oint \exp\big(\sum_{n=1}^{\infty} \frac{q^{-2n}}{n} a_{-n}(z_1^n + \cdots + z_r^n)\big) S(qz_1) \cdots S(qz_r) \\
&\quad S(w_1) \cdots S(w_l) z^{2\delta + (2m+n+1+i)\mathbf{1}} w^{-\mu} e^{(m+r)\alpha} e^{i\alpha/2} \frac{dzdw}{zw},
\end{aligned}$$

where $\delta = (r-1, r-2, \ldots, 0)$ and we have used

$$\sum_{w \in \mathfrak{S}_r} q^{-2\ell(w)} = q^{-\binom{r}{2}} [r]!.$$

From the orthogonality of Schur functions [M] it follows that

$$\exp\big(\sum_{n=1}^{\infty} \frac{a_{-n}}{n} (z_1^n + \cdots + z_r^n)\big) = \sum_{l(\lambda) \leq r} s_\lambda(a_{-k}) s_\lambda(z_i),$$

where $s_\lambda(a_{-k})$ is the Schur function in terms of the power sum $a_{-\mu}$ and $s_\lambda(z_i)$ is the Schur polynomial in the variables $z_1, \cdots, z_r$. Replacing z_j by $q^{-1}z_j$ we get

$$\begin{aligned}
&x_n^{+(r)} s_\mu e^{m\alpha} e^{i\alpha/2}\\
&= q^{-3\binom{r}{2}-r(n+1+2m+i)} \sum_{l(\lambda)\le r} q^{-2|\lambda|} s_\lambda \oint s_\lambda(z) z^{2\delta+(n+1+2m+i)\mathbf{1}}\\
&\times S(z_1)\cdots S(z_r)S(w_1)\cdots S(w_l)e^{(m+r)\alpha}e^{i\alpha/2}w^{-\mu}\frac{dz}{z}\frac{dw}{w}\\
&= q^{-3\binom{r}{2}-r(n+1+2m+i)} \sum_{l(\lambda)\le r} s_\lambda \oint \sum_{w\in\mathfrak{S}_r} \frac{(-1)^{l(w)} z^{w(\lambda+\delta)}}{\prod_{j<k}(z_j - z_k)} z^{\delta+(n+1+2m+i)\mathbf{1}}\\
&\times z^\delta S(z_1)\cdots S(z_r)S(w_1)\cdots S(w_l)e^{(m+r)\alpha}e^{i\alpha/2}w^{-\mu}\frac{dz}{z}\frac{dw}{w}.
\end{aligned}$$

Applying the symmetry of the Schur vertex operators in Proposition 2.1 we see that the above expression becomes

$$\begin{aligned}
&q^{-3\binom{r}{2}-r(n+1+2m+i)} \sum_{l(\lambda)\le r} q^{-2|\lambda|} s_\lambda \sum_{w\in\mathfrak{S}_r} \oint \frac{(-1)^{l(w)} z^{\lambda+\delta}}{\prod_{j<k}(z_j - z_k)} z^{w(\delta)+(n+1+2m+i)\mathbf{1}}\\
&\times z^\delta S(z_1)\cdots S(z_r)S(w_1)\cdots S(w_l)e^{(m+r)\alpha}e^{i\alpha/2}w^{-\mu}\frac{dz}{z}\frac{dw}{w}\\
&= q^{-3\binom{r}{2}-r(n+1+2m+i)} \sum_{l(\lambda)\le r} q^{-2|\lambda|} s_\lambda \oint z^{\lambda+2\delta+(n+1+2m+i)\mathbf{1}}\\
&\times S(z_1)\cdots S(z_r)S(w_1)\cdots S(w_l)e^{(m+r)\alpha}e^{i\alpha/2}w^{-\mu}\frac{dz}{z}\frac{dw}{w},
\end{aligned}$$

where we have used the Weyl denominator formula (see $\lambda = 0$ in (2.1)) and the integral is taken along contours in z_i, w_i around the origin. The formula for $X_n^{+(r)}$ is then obtained by using Theorem 2.3.

The case of $X_n^{-(r)}$ is proved similarly with the help of the dual vertex operator $S^*(z)$.

$$\begin{aligned}
&X_n^{-(r)} s_{\mu'} e^{m\alpha} e^{i\alpha/2}\\
&= \frac{(-1)^{|\mu|}}{[r]!} \oint X^-(z_1)\cdots X^-(z_r)S^*(w_1)\cdots S^*(w_l)z^{n+1}w^{-\mu}e^{(m-r)\alpha}e^{i\alpha/2}\frac{dzdw}{zw}\\
&= \frac{(-1)^{|\mu|}}{[r]!} \oint :S^*(q^2z_1)\cdots S^*(q^2z_r): S^*(z_1)\cdots S^*(z_r)S^*(w_1)\cdots S^*(w_l)\\
&\qquad \times \prod_{i<j}(z_i - q^2 z_j) z^{\delta+(n+1-2m-i)\mathbf{1}} w^{-\mu}\ e^{(m-r)\alpha}e^{i\alpha/2}\frac{dzdw}{zw}\\
&= q^{\binom{r}{2}} \oint :S^*(q^2z_1)\cdots S^*(q^2z_r): S^*(z_1)\cdots S^*(z_r)S^*(w_1)\cdots S^*(w_l)\\
&\times z^{2\delta+(n+1-2m-i)\mathbf{1}} w^{-\mu}\ e^{(m-r)\alpha}e^{i\alpha/2}\frac{dzdw}{zw}(-1)^{|\nu|},
\end{aligned}$$

where we have used the skew-symmetry of the integrand and Lemma 3.4. Then the formula for $X_n^{-(r)}$ is obtained from the following identity and the Weyl denominator formula

$$\exp\big(-\sum_{n=1}^\infty \frac{a_{-n}}{n}(z_1^n + \cdots + z_r^n)\big) = \sum_{l(\lambda)\le r} s_{\lambda'}(a_{-k})s_\lambda(z_i). \quad \square$$

4. Combinatorial realization of $U_q(\widehat{sl}_2)$

We now describe the action of the generators of $U_{\mathcal{A}}(\widehat{sl}_2)$ on $V(\Lambda_i)$.

Recall that we have defined a special $\mathcal{A}$-linear subspace $V_{\mathcal{A}}(\Lambda_i)$ inside $V(\Lambda_i)$ in Eq. (3.15).

From Section 3 it follows that on $V_{\mathcal{A}}(\Lambda_i)$ we have

$$e_1^{(r)} s_\mu e^{m\alpha} e^{i\alpha/2} = q^{-3\binom{r}{2}-r(2m+i+1)} \times \left(\sum_{l(\lambda)\le r} q^{-2|\lambda|} s_\lambda s_{(-\lambda-2\delta-(2m+1+i)\mathbf{1},\mu)} \right) e^{(m+r)\alpha} e^{i\alpha/2}, \tag{4.1}$$

$$f_1^{(r)} s_{\mu'} e^{m\alpha} e^{i\alpha/2} = (-1)^{r(1+i)} q^{\binom{r}{2}} \times \left(\sum_{l(\lambda)\le r} q^{2|\lambda|} s_{\lambda'} s_{\pi(-\lambda-2\delta+(2m+i-1)\mathbf{1},\mu)'} \right) e^{(m-r)\alpha} e^{i\alpha/2}, \tag{4.2}$$

$$f_0^{(r)} s_\mu e^{m\alpha} e^{i\alpha/2} = q^{r(5-r)/2} \times \left(\sum_{l(\lambda)\le r} q^{-2|\lambda|} s_\lambda s_{(-\lambda-2\delta-(2m+i)\mathbf{1},\mu)} \right) e^{(m+r)\alpha} e^{i\alpha/2}, \tag{4.3}$$

$$e_0^{(r)} s_{\mu'} e^{m\alpha} e^{i\alpha/2} = (-1)^{ri} q^{-\binom{r}{2}-r(2m+i)} \times \left(\sum_{l(\lambda)\le r} q^{2|\lambda|} s_{\lambda'} s_{\pi(-\lambda-2\delta+(2m+i-2)\mathbf{1},\mu)'} \right) e^{(m-r)\alpha} e^{i\alpha/2}, \tag{4.4}$$

$$K_0 s_\mu e^{m\alpha} e^{i\alpha/2} = q^{1-2m-i} s_\mu e^{m\alpha} e^{i\alpha/2}, \tag{4.5}$$

$$K_1 s_\mu e^{m\alpha} e^{i\alpha/2} = q^{2m+i} s_\mu e^{m\alpha} e^{i\alpha/2}, \tag{4.6}$$

$$\begin{bmatrix} K_0; l \\ r \end{bmatrix} s_\mu e^{m\alpha} e^{i\alpha/2} = \begin{bmatrix} 1-2m-i+l \\ r \end{bmatrix} s_\mu e^{m\alpha} e^{i\alpha/2}, \tag{4.7}$$

$$\begin{bmatrix} K_1; l \\ r \end{bmatrix} s_\mu e^{m\alpha} e^{i\alpha/2} = \begin{bmatrix} 2m+i+l \\ r \end{bmatrix} s_\mu e^{m\alpha} e^{i\alpha/2}. \tag{4.8}$$

As a consequence of these formulas and the Littlewood-Richardson rule (2.2) we get the following theorem.

Theorem 4.1. *The $\mathcal{A}$-linear space $V_{\mathcal{A}}(\Lambda_i)$ is an $U_{\mathcal{A}}(\widehat{sl}_2)$-lattice in $V(\Lambda_i)$. In particular, $U_{\mathcal{A}}(\widehat{sl}_2)e^{i\alpha/2}$ is a sublattice of $V_{\mathcal{A}}(\Lambda_i)$.*

Proposition 4.1. *For $m \geq 0$ we have*

$$\begin{aligned} f_1^{(2m)} e^{m\alpha} &= (-1)^m q^{m(2m-1)} e^{-m\alpha}, \\ f_0^{(2m+1)} e^{-m\alpha} &= (-1)^m q^{-(2m+1)(m-2)} e^{-(m+1)\alpha}, \\ f_0^{(2m)} e^{-m\alpha} e^{\alpha/2} &= (-1)^m q^{-m(2m-5)} e^{m\alpha}, \\ f_1^{(2m+1)} e^{m\alpha} e^{\alpha/2} &= (-1)^m q^{m(2m+1)} e^{-(m+1)\alpha}. \end{aligned}$$

Proof. The four formulas are proved similarly. Take $f_1^{(2m)} e^{m\alpha}$. Observe that $-2\delta + (2m-1)\mathbf{1} = (-2m+1, -2m+3, \cdots, 2m-3, 2m-1)$ is of weight zero, thus only $\lambda = 0$ contributes to the summation in $f_1^{(2m)} e^{m\alpha}$ (see (4.2)). Since the longest element in $\mathfrak{S}_{2m}$ has inversion number $m(2m-1)$, the sign of $s_{-2\delta+(2m-1)\mathbf{1}}$ is $(-1)^{m(2m-1)} = (-1)^m$. ☐

Corollary 4.1. *For $m \geq 0$ we have*

$$\begin{aligned} f_1^{(2m)} f_0^{(2m-1)} \cdots f_1^{(2)} f_0.1 &= (-1)^m q^{3m^2} e^{-m\alpha}, \\ f_0^{(2m+1)} f_1^{(2m)} \cdots f_1^{(2)} f_0.1 &= q^{(m+1)(m+2)} e^{(m+1)\alpha}, \\ f_0^{(2m)} f_1^{(2m-1)} \cdots f_0^{(2)} f_1 e^{\alpha/2} &= (-1)^m q^{m(m+2)} e^{m\alpha} e^{\alpha/2}, \\ f_1^{(2m+1)} f_0^{(2m)} \cdots f_0^{(2)} f_1 e^{\alpha/2} &= q^{3m(m+1)} e^{-(m+1)\alpha} e^{\alpha/2}. \end{aligned}$$

Example 4.2. In the following we abbreviate $f_{i_1}^{(n_1)} \cdots f_{i_r}^{(n_r)}.1 = f_{i_1}^{(n_1)} \cdots f_{i_r}^{(n_r)}$ in the basic representation $V(\Lambda_0)$.

$$\begin{aligned} f_0 &= q^2 e^{\alpha}, \\ f_1 f_0 &= -q^2(1+q^2) s_1, \\ f_1^{(2)} f_0 &= -q^3 e^{-\alpha}, \\ f_0 f_1 f_0 &= q^2(q^2+1) s_1 e^{\alpha}, \\ f_1 f_0 f_1 f_0 &= -(q^4+q^2)(s_2 - q^4 s_{1^2}), \\ f_0 f_1^{(2)} f_0 &= -q^3(s_{1^2} + [3] s_2), \\ f_0 f_1 f_0 f_1 f_0 &= q^2(1+q^2)^2 (s_2 + s_{1^2}) e^{\alpha}, \\ f_0^{(2)} f_1^{(2)} f_0 &= q^4(s_2 + [3] s_{1^2}) e^{\alpha}, \\ f_0^{(3)} f_1^{(2)} f_0 &= q^6 e^{2\alpha}, \\ f_1^{(2)} f_0 f_1 f_0 &= q^5(1+q^2) s_1 e^{-\alpha}. \end{aligned}$$

Example 4.3. As in the last example we use $f_{i_1}^{(n_1)} \cdots f_{i_r}^{(n_r)}$ to denote $f_{i_1}^{(n_1)} \cdots f_{i_r}^{(n_r)} e^{\alpha/2}$ in the basic representation $V(\Lambda_1)$.

$$\begin{aligned}
f_1 &= e^{-\alpha}, \\
f_0 f_1 &= q^{-2}(1+q^{-2}) s_1, \\
f_0^{(2)} f_1 &= -q^3 e^{\alpha}, \\
f_1 f_0 f_1 &= -(q^{-2}+1) s_1 e^{-\alpha}, \\
f_0 f_1 f_0 f_1 &= (1+q^2)(s_2 - s_{1^2}), \\
f_1 f_0^{(2)} f_1 &= -q^5([3] s_{1^2} + s_2), \\
f_1 f_0 f_1 f_0 f_1 &= (1+q^2)^2 (s_2 + q^2 s_{1^2}) e^{-\alpha}, \\
f_1^{(2)} f_0^{(2)} f_1 &= q^5([3] s_2 + s_{1^2}) e^{-\alpha}, \\
f_1^{(3)} f_0^{(2)} f_1 &= q^6 e^{-2\alpha}, \\
f_0^{(2)} f_1 f_0 f_1 &= -[2] s_1 e^{\alpha}.
\end{aligned}$$

It would be interesting to see the relation between our formulas and the fermionic picture [LLT].

Finally we would like to remark on generalizing the result of this paper to quantum affine algebras of ADE type. Using the realization [FJ] it is clear that the action of divided powers of Drinfeld generators on the Schur symmetric functions in multi-sets of variables are definable, but not as explicit as in the sl_2-case due to the complexity of the interaction between adjacent simple roots. We do not know how to deal with the hypergeometric functions appearing in the interaction. Another reason for this difficulty is as follows. In [BFJ] we studied two lattices associated with the canonical and dual canonical basis of $U_q(\widehat{sl}_2)$. Both are equivalent under the Macdonald inner product but are not equivalent under the Schur inner product. The lattice associated with the canonical basis is easier to be formulated to the ADE cases as shown in [CJ], but the relation to the Littlewood-Richardson rule is sacrificed. We choose to study the lattice of dual canonical basis in order to use Littlewood-Richardson rule, and are also motivated by the transition relation with the Macdonald polynomial.

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