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ON SQUARE-INTEGRABLE REPRESENTATIONS OF CLASSICAL p-ADIC GROUPS II

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Abstract. In this paper, we continue our study of non-supercuspidal discrete series for the classical groups $Sp(2n, F)$, $SO(2n+1, F)$, where F is p-adic.

1. Introduction

This paper is a continuation of [Jan4].

Suppose ρ is an irreducible, unitary supercuspidal representation of $GL_n(F)$ and σ an irreducible, supercuspidal representation of $S_r(F) = Sp_{2r}(F)$ or $SO_{2r+1}(F)$. Then, if $\rho \not\cong \tilde{\rho}$ (self-contragredient), we have $\text{Ind}(|det_n|^x \rho \otimes \sigma)$ is irreducible for all $x \in \mathbb{R}$. Otherwise, there is a unique $\alpha \geq 0$ such that $\text{Ind}(|det_n|^x \rho \otimes \sigma)$ is reducible for $x = \pm\alpha$ and irreducible for all $x \in \mathbb{R} \setminus \{\pm\alpha\}$ (cf. [Sil2], [Tad5]). Assuming certain conjectures, Mœglin [Mœ2] and Zhang [Zh] have shown that $\alpha \in \frac{1}{2}\mathbb{Z}$ (also, cf. [M-R], [Re], [Sha1], [Sha2]). In [Jan4], we showed that the problem of classifying non-supercuspidal discrete series could be reduced to classifying discrete series with supercuspidal support in

$$\mathcal{S}((\rho, \alpha); \sigma) = \{\nu^x \rho, \nu^{-x} \rho\}_{x \in \alpha + \mathbb{Z}} \cup \{\sigma\}$$

(though most of this result is from [Jan2] and [Tad4]).

In [Jan4], to an irreducible representation π supported on $\mathcal{S}((\rho, \alpha); \sigma)$, we associated an element $\chi_0(\pi)$ in the minimal Jacquet module for π (minimal meaning with respect to the smallest parabolic subgroup admitting a nonzero Jacquet module for π). We note that there is an order $\succ$ on the components of the (semisimplified) minimal Jacquet module of π such that $\chi_0(\pi)$ is minimal with respect to $\succ$. From $\chi_0(\pi)$, we can read off a representation $\delta_0(\pi)$ having the form

$$\delta_0(\pi) = \delta([\nu^{-b_1} \rho, \nu^{a_1} \rho]) \otimes \cdots \otimes \delta([\nu^{-b_k} \rho, \nu^{a_k} \rho]) \otimes \sigma,$$

where $\delta([\nu^{-b_i} \rho, \nu^{a_i} \rho])$ denotes the generalized Steinberg (for a general linear group) which has $\nu^{a_i} \rho \otimes \nu^{a_i - 1} \rho \otimes \cdots \otimes \nu^{-b_i} \rho$ as its (minimal) Jacquet module. We have $\pi \hookrightarrow \text{Ind}(\delta_0(\pi))$. Further, one can determine whether π is square-integrable just from $\delta_0(\pi)$. Our goal is then to constrain the candidates for δ_0 for square-integrable representations, thereby constraining where one needs to look for square-integrable representations. Of course, the ultimate goal is exhaustion: if the constraints are strong enough, every possible δ_0 listed will actually occur as $\delta_0(\pi)$ for some discrete series π. That is the case for the theorem below when $\rho = 1_{F^\times}$, $\sigma = 1_{S_0(F)}$ (cf. [Mœ1]); we expect it holds in general.

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In a sense, our point-of-view may be regarded as an extension of that of [B-Z] and [Zel]. Their descriptions of representations of general linear groups are given in combinatorial terms, with the combinatorial descriptions essentially independent of which particular supercuspidal representations appear in the support. Here, we work with discrete series for symplectic and odd-orthogonal groups in an analogous setting; while our results are not independent of the ρ, σ which appear, they depend on ρ, σ only insofar as ρ, σ determine α. (Roughly speaking, $\alpha = 1$ for the GL counterpart, so this issue does not arise.) Thus, from our point-of-view, α is essentially treated as input data, with discrete series characterizations as the output.

The main result in this paper is the following:

Theorem 1.1. *Suppose ρ is an irreducible, unitary, supercuspidal representation of $GL_n(F)$ and σ an irreducible, supercuspidal representation of $S_r(F)$. Also, suppose $Ind(|det_n|^x\rho \otimes \sigma)$ is reducible for $x = \pm\alpha$ and irreducible for $x \in \mathbb{R} \setminus \{\pm\alpha\}$, where $\alpha \geq 0$ has $\alpha \in \frac{1}{2}\mathbb{Z}$. If π is an irreducible, square-integrable representation supported on $\mathcal{S}((\rho,\alpha);\sigma)$ with*

$$\delta_0(\pi) = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma,$$

the following conditions must be satisfied:

1. *basic conditions:*
   (a) $a_k \geq a_{k-1} \geq \cdots \geq a_1 > 0$.
   (b) $a_i > b_i$ *for all* i.
   (c) *There is a β with $\alpha + 1 \geq \beta > 0$ such that each of $\{-\beta, -\beta - 1, \ldots, -\alpha\}$ appears exactly once among $b_1, b_2, \ldots, b_k$ and there are no other negative b_i's.*
2. $a_i \geq \beta$ *for all* i*;* $b_i \geq \beta - 1$ *for all* $b_i \geq 0$.
3. $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$ *are all distinct.*
4. *We do not have* $a_i > a_j > b_i > b_j$ *for any* $i > j$.

Remarks 1.2.
1. Coupled with Theorem 3.2.1 of [Jan4], Theorem 1.1 above applies to any irreducible, square-integrable representation if we assume the half-integrality hypothesis in Theorem 1.1 holds in general. That is, for any such (ρ, σ), the value of $x \in \mathbb{R}$, $x \geq 0$ which makes $\mathrm{Ind}(|det|^x\rho \otimes \sigma)$ reducible (if any) lies in $\frac{1}{2}\mathbb{Z}$. (The half-integrality hypothesis follows from [Mœ2] or [Zh] if certain conjectures are assumed.)
2. We make two remarks concerning converse directions.
   (a) If π is an irreducible representation and $\delta_0(\pi) = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma$ has $a_1, \ldots, a_k, b_1, \ldots, b_k$ satisfying 1–4 of Theorem 1.1, then π is square-integrable. (In fact, condition 1 of Theorem 1.1 is sufficient; cf. Theorem 4.2.1, [Jan4].)
   (b) If $a_1, \ldots, a_k, b_1, \ldots, b_k$ satisfy 1–4 of Theorem 1.1, one can ask whether there exists an irreducible representation π having $\delta_0(\pi) = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma$. We expect that this is the case, but it is not proved. (Note that in the case where $a_k > \cdots > a_1 > b_1 > \cdots > b_k$, such a π is constructed in section 7; in the case where $a_k > b_k > a_{k-1} > b_{k-1} > \cdots > a_1 > b_1$, the existence of such a π follows from the results in [Tad5].)
3. We note that we can have irreducible, square-integrable representations π_1, π_2 which have $\pi_1 \not\cong \pi_2$ but $\delta_0(\pi_1) = \delta_0(\pi_2)$ (cf. Theorem 7.7; examples may also be easily obtained from [Tad5]).

The basic proof is by induction on the parabolic rank (cf. 0.3.4 [B-W]) of the supercuspidal support; the whole paper is essentially one large inductive proof. Thus, we establish Theorem 1.1 and complementary results (Theorem 7.7 and Corollaries 7.8 and 7.9) for a fixed parabolic rank *p.r.* assuming they all hold when the parabolic rank of the supercuspidal support is less than *p.r.* We note that if $\delta_0(\pi) = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma$, then the supercuspidal support of π has parabolic rank $(a_1 + b_1 + 1) + \cdots + (a_k + b_k + 1)$. To avoid repeating the hypotheses of Theorem 1.1 in the rest of the theorems in this paper, let us simply let **(H)** denote the hypotheses of Theorem 1.1.

We now discuss the contents section by section. The next section reviews notation and some background results. We also record Theorem 2.4, which says that the first condition in Theorem 1.1 holds (it is a combination of results from [Jan4]).

In the third section, we verify condition 2 of the theorem. The basic idea in proving this is to show that should condition 2 fail, there must be something in the Jacquet module of π lower than the $\chi_0(\pi)$ we started with, a contradiction. For the most part, this can be done with a simple string of embeddings and equivalences; there is one subtler case which occupies most of this section.

The fourth and fifth sections verify that conditions 3 and 4 hold. Conditions 3 and 4 are essentially conditions on pairs of segments, and are treated as such. In the fourth section, we assume that $b_i > a_{i-1}$ for some i. Roughly speaking, such segments can be removed from consideration. What remains then has lower parabolic rank and the inductive hypothesis allows us to finish this case. The fifth section deals with the case where $b_i \leq a_{i-1}$ for all i. Here we need to show conditions 3 and 4 directly. The basic idea is to show that if 3 or 4 fails, then π may be embedded into an induced representation where the inducing representation is lower than $\delta_0(\pi)$, a contradiction. To do this, we use Jacquet module arguments to compare certain induced representations. We note that in the case where σ is generic, condition 3 may be deduced from [Mu].

In order to do the comparisons for section 5, we need to know the existence of square-integrable representations with certain properties. The last two sections are geared toward this. In the sixth section, we prove a technical result which is needed in the seventh section. In the seventh section, we show the existence of square-integrable representations with certain specific δ_0's, in particular, δ_0 having $b_i < a_{i-1}$ for all i. This is done by analyzing the induced representation

$$\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t,$$

where $\delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$ is a square-integrable representation (by the inductive hypothesis, already constructed) which has $\delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma$ as its δ_0.

2. Notation and preliminaries

We retain the notation introduced in section 1.2, [Jan4] and will forgo reviewing it here. We will also forgo reviewing certain standard results already discussed in [Jan4]. The Langlands classification and Casselman criteria for $S_r(F)$ ([B-W], [Sil1], [Cas] and [Tad1]) are discussed in section 3.1, [Jan4]. The Langlands classification for $GL_n(F)$ is discussed in section 2.4, [Jan4]. For clarity, we use $\mathcal{L}$ (resp. L) when the Langlands subrepresentation is for a general linear group (resp., classical group). We briefly review some of the notation and results from the remainder of [Jan4], and we introduce a few additional items needed for this paper.

Let ρ be an irreducible unitary supercuspidal representation of $GL_n(F)$, σ an irreducible supercuspidal representation of $S_r(F)$. Also, suppose $\nu^x\rho\rtimes\sigma$ is reducible for $x=\pm\alpha$ and irreducible for $x\in\mathbb{R}\setminus\{\pm\alpha\}$, where $\alpha\geq 0$ has $\alpha\in\frac{1}{2}\mathbb{Z}$.

Definition 2.1. Let π be an irreducible representation supported on $\mathcal{S}((\rho,\alpha);\sigma)$. Set

$$X(\pi)=\left\{\chi\leq s_{min}(\pi)\;\middle|\;\begin{array}{l}\chi=\nu^{x_1}\rho\otimes\cdots\otimes\nu^{x_m}\rho\otimes\sigma \text{ has } x_1+\cdots+x_m\\ \text{minimal for } s_{min}(\pi)\end{array}\right\}.$$

Then, let $\chi_0(\pi)\in X(\pi)$, which is minimal in the lexicographic ordering.

The following is Lemma 4.1.2, [Jan4] with a minor change of notation. (As in section 5, [Jan4], it is more convenient to incorporate a negative into the b_i's.)

Lemma 2.2. *$\chi_0(\pi)$ has the form*

$$\chi_0(\pi)=(\nu^{a_1}\rho\otimes\nu^{a_1-1}\rho\otimes\cdots\otimes\nu^{-b_1}\rho)\otimes\cdots\otimes(\nu^{a_k}\rho\otimes\nu^{a_k-1}\rho\otimes\cdots\otimes\nu^{-b_k}\rho)\otimes\sigma,$$

with $a_1\leq a_2\leq\cdots\leq a_k$.

Definition 2.3. With notation as above, if

$$\chi_0(\pi)=(\nu^{a_1}\rho\otimes\nu^{a_1-1}\rho\otimes\cdots\otimes\nu^{-b_1}\rho)\otimes\cdots\otimes(\nu^{a_k}\rho\otimes\nu^{a_k-1}\rho\otimes\cdots\otimes\nu^{-b_k}\rho)\otimes\sigma,$$

set

$$\delta_0(\pi)=\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho])\otimes\cdots\otimes\delta([\nu^{-b_k}\rho,\nu^{a_k}\rho])\otimes\sigma.$$

We also recall that if $\chi_0(\pi),\delta_0(\pi)$ as above, $M=GL_{(a_1-b_1+1)n}(F)\times\cdots\times GL_{(a_k-b_k+1)n}(F)\times S_r(F)$, then

$$\pi\hookrightarrow i_{GM}(\delta_0(\pi))$$

(cf. Lemma 4.1.4, [Jan4]).

We also note the following, a consequence of Theorem 4.2.1 of [Jan4], Lemma 4.4.1 of [Jan4], and Lemma 4.4.2 of [Jan4].

Theorem 2.4. *Suppose* **(H)**. *Then, condition 1 in Theorem 1.1 holds.*

We now introduce a bit of notation which will be used in the rest of this paper. In Definition 2.1 above, let $t.e.(\chi_0(\pi))$ denote the minimal value of $x_1+\cdots+x_m$ which arises. More generally, we let $t.e.(\pi)=t.e.(\delta_0(\pi))$ denote the same value. Implicit in Definition 2.1 is an ordering: $\chi_1\succ\chi_2$ if $t.e.(\chi_1)>t.e.(\chi_2)$ or $t.e.(\chi_1)=t.e.(\chi_2)$ and χ_1 is lexicographically higher than χ_2. We extend this ordering as follows: $\pi_1\succ\pi_2$ if $\chi_0(\pi_1)\succ\chi_0(\pi_2)$. (Typically, π_1 and π_2 will be representations of Levi factors of standard parabolic subgroups.)

We introduce one other piece of notation. At times, it will be convenient to write s_{app} when the Jacquet module of S_m is taken with respect to the appropriate standard parabolic subgroup. This will only be used when what constitutes the appropriate parabolic subgroup is clear from context.

We now recall Definition 5.2.1 of [Jan4]. It uses the Jacquet module structures developed in [Tad2]. Since a summary of the notation and results needed from [Tad2] was given in section 3.1 of [Jan4], we will forgo a discussion here.

Definition 2.5. Suppose τ is an irreducible representation of $GL_m(F)$ and π a representation of $S_n(F)$. Write

$$\mu^*(\pi)=\sum_i m_i\xi_i\otimes\theta_i,$$

where $\xi_i \otimes \theta_i$ is irreducible and m_i is its multiplicity. Let $I_\tau = \{i|\xi_i = \tau\}$. We set

$$\mu^*_\tau(\pi) = \sum_{i\in I_\tau} m_i\xi_i \otimes \theta_i = \sum_{i\in I_\tau} m_i\tau \otimes \theta_i.$$

Similarly, if ξ is a representation of $GL_r(F)$ and

$$M^*(\xi) = \sum_j n_j\xi_j^{(1)} \otimes \xi_j^{(2)},$$

let $J_\tau = \{j|\xi_j^{(1)} = \tau\}$. We set

$$M^*_\tau(\xi) = \sum_{j\in J_\tau} n_j\xi_j^{(1)} \otimes \xi_j^{(2)} = \sum_{j\in J_\tau} n_j\tau \otimes \xi_j^{(2)}.$$

It will be convenient to extend this definition to the case where $\tau = \tau_1 \otimes \cdots \otimes \tau_j$. In this case, we write $s_{\tau_1\otimes\cdots\otimes\tau_j}$ rather than $\mu^*_{\tau_1\otimes\cdots\otimes\tau_j}$ (since $s_{\tau_1\otimes\cdots\otimes\tau_j} \leq s_{app}$ not μ^*).

We close with a lemma which will expedite certain calculations.

Lemma 2.6. 1. *Suppose $\eta \leq \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma$. Assume τ is an irreducible representation of a general linear group such that $\{\nu^{b_1}\rho, \ldots, \nu^{b_k}\rho, \nu^{a_1}\rho, \ldots, \nu^{a_k}\rho\} \cap supp(\tau) = \emptyset$. Then for any representation ξ of a general linear group, we have*

$$\mu^*_\tau(\xi \rtimes \eta) = M^*_\tau(\xi) \rtimes (1 \otimes \eta).$$

2. *Suppose $\{x|M^*_{\nu^x\rho}(\xi) \neq 0\} \cap support(\tau) = \emptyset$. Then,*

$$\mu^*_\tau(\xi \rtimes \eta) = (1 \otimes \xi) \rtimes \mu^*_\tau(\eta).$$

Further, we note that $M^*_{\nu^x\rho}(\tau) \neq 0$ if and only if the following holds: either $r_{min}(\tau)$ contains a term of the form $\nu^x\rho \otimes \ldots$ or $r_{min}(\tau)$ contains a term of the form $\cdots \otimes \nu^{-x}\rho$.

Proof. Write $M^*(\xi) = \sum_i \tau_i^{(1)} \otimes \tau_i^{(2)}$ and $\mu^*(\eta) = \sum_j \tau_j \otimes \theta_j$. Then,

$$\mu^*(\xi \rtimes \eta) = \sum_{i,j} \tau_i^{(1)} \times \tau_j \otimes \tau_i^{(2)} \rtimes \theta_j.$$

Observe that if $\tau_j \neq 1$, we must have $r_{min}(\tau_j) = \sum_h \nu^{\alpha_h}\rho \otimes \ldots$ with $\alpha_h \in \{b_1, \ldots, b_k, a_1, \ldots, a_k\}$ for all h. But, for τ_j to contribute to μ^*_τ, we must have $\nu^{\alpha_h}\rho \in \text{supp}(\tau)$. Since this is not the case, we must have $\tau_j = 1$. Part 1 of the lemma follows.

The proof of 2 is straightforward and essentially the same as that of Lemma 5.2.2, [Jan4]; we omit it. □

3. Condition 2

The main result in this section is Theorem 3.7, which says that condition 2 in Theorem 1.1 must be satisfied. This condition is interesting only when $\beta \geq 2$. We note that when $\beta > 2$, the proof is fairly straightforward. The case $\beta = 2$ occupies most of this section.

In this section, we begin the inductive proof. Note that by the inductive hypothesis, we may assume Theorem 1.1, Theorem 7.7 (and Corollaries 7.8 and 7.9) hold when the parabolic rank of the supercuspidal support is less than $p.r.$ By Theorem 2.4, we actually know that condition 1 of Theorem 1.1 holds in general.

Suppose π is an irreducible subquotient of $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho])\times\cdots\times\delta([\nu^{-b_k}\rho,\nu^{a_k}\rho]) \rtimes \sigma$. For $c \geq 0$, let $m(c)$ denote the number of copies of $\nu^{\pm c}\rho$ which appear in an element of $s_{min}(\pi)$. We note that $m(c)$ is well-defined and independent of the particular subquotient π (it is simply a matter of supercuspidal support). Set

$$n_+(c) = \text{number of } a_i \text{ or } b_i \text{ which equal } c,$$

$$n_-(c) = \text{number of } a_i \text{ or } b_i \text{ which equal } -c.$$

For $c = 0$, we simply write $n(0)$. If

$$\delta_0(\pi) = \delta([\nu^{-b'_1}\rho,\nu^{a'_1}\rho])\otimes\cdots\otimes\delta([\nu^{-b'_k}\rho,\nu^{a'_k}\rho])\otimes\sigma,$$

we similarly define

$$n'_+(c) = \text{number of } a'_i \text{ or } b'_i \text{ which equal } c,$$

$$n'_-(c) = \text{number of } a'_i \text{ or } b'_i \text{ which equal } -c.$$

Again, for $c = 0$, we simply write $n'(0)$. The lemma below then follows from supercuspidal support considerations (it is essentially an extension of Proposition 5.3.2, [Jan4]):

Lemma 3.1. *For $c \geq 1$, we have the following:*

1. *For $c > 1$,*
$$n'_+(c-1) - n'_-(c) = m(c-1) - m(c) = n_+(c-1) - n_-(c).$$
2. *For $c = 1$,*
$$n'(0) - n'_-(1) = 2m(0) - m(1) = n(0) - n_-(1).$$

(At times, it will be convenient to let $n_+(0) = n(0)$, $n'_+(0) = n'(0)$ and just write $n'_+(c-1) - n'_-(c) = n_+(c-1) - n_-(c)$ for $c \geq 1$.)

Proof. Consider the contribution of $\delta([\nu^{-b_i}\rho,\nu^{a_i}\rho])$. Suppose $a_i, b_i \geq 0$. Then $\delta([\nu^{-b_i}\rho,\nu^{a_i}\rho])$ contributes 1 to both $n_+(a_i) - n_-(a_i+1)$ and $n_+(b_i) - n_-(b_i+1)$. Also, $\delta([\nu^{-b_i}\rho,\nu^{a_i}\rho])$ contributes 1 to both $m(a_i) - m(a_i+1)$ (resp., $2m(0) - m(1)$ if $a_i = 0$) and $m(b_i) - m(b_i+1)$ (resp., $2m(0) - m(1)$ if $b_i = 0$). The cases $a_i \geq 0, b_i < 0$ and $a_i < 0, b_i \geq 0$ may be done similarly ($a_i, b_i < 0$ is not possible). Combining these and the corresponding observations for $\delta([\nu^{-b'_i}\rho,\nu^{a'_i}\rho])$ gives the lemma. □

We now note the following observation:

Lemma 3.2. *Suppose π is an irreducible representation with*

$$\delta_0(\pi) = \delta([\nu^{-b_1}\rho,\nu^{a_1}\rho])\otimes\cdots\otimes\delta([\nu^{-b_k}\rho,\nu^{a_k}\rho])\otimes\sigma.$$

Then, there is an irreducible representation θ having

$$\delta_0(\theta) = \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])\otimes\cdots\otimes\delta([\nu^{-b_k}\rho,\nu^{a_k}\rho])\otimes\sigma.$$

Proof. Choose θ irreducible such that

1. $\mu^*(\pi) \geq \delta([\nu^{-b_1}\rho,\nu^{a_1}\rho])\otimes\theta$,
2. $s_{app}(\theta) \geq \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])\otimes\cdots\otimes\delta([\nu^{-b_k}\rho,\nu^{a_k}\rho])\otimes\sigma$.

Since $s_{app}(\pi) \geq \delta([\nu^{-b_1}\rho,\nu^{a_1}\rho])\otimes\delta_0(\theta)$, we see that

$$\delta_0(\theta) \succeq \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])\otimes\cdots\otimes\delta([\nu^{-b_k}\rho,\nu^{a_k}\rho])\otimes\sigma.$$

The lemma then follows immediately. □

Lemma 3.3. *If $\beta > 2$, then condition 2 of Theorem 1.1 holds.*

Proof. By the preceding lemma, Theorem 4.2.1, [Jan4], and the inductive hypothesis, $[\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]$ satisfy the conditions of Theorem 1.1. If $b_1 < 0$, we must have $b_1 = -\beta$ by Remark 4.4.3, [Jan4], and therefore $a_1 \geq \beta$ as well. Thus, the only way condition 2 could fail for $\delta_0(\pi)$ is to have $\beta - 2 \geq b_1 \geq 0$ (noting that if $a_1 \leq \beta - 1$, then $b_1 \leq \beta - 2$). In particular, this also means $\beta \in \{-b_2, \ldots, -b_k, \alpha+1\}$, and therefore $b_i \geq \beta - 1$ for $i \geq 2$.

First, we check that $a_1 \geq \beta - 1$. If not, we would have $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_i}\rho, \nu^{a_i}\rho])$ irreducible for all $i \geq 2$ (if $b_i < 0$, then $a_i \geq -b_i \geq \beta > a_1 + 1$; if $b_i \geq 0$, then $a_i > b_i \geq \beta - 1 \geq a_1 > b_1$). Then, we could "commute" $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho])$ to the right to get

$$\begin{aligned}
\pi \hookrightarrow \; & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \\
& \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\
\cong \; & \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \\
& \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\
& \vdots \\
\cong \; & \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \\
& \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \sigma \\
\cong \; & \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \\
& \times \delta([\nu^{-a_1}\rho, \nu^{b_1}\rho]) \rtimes \sigma,
\end{aligned}$$

by the irreducibility of $\delta([\nu^{-a_1}\rho, \nu^{b_1}\rho]) \rtimes \sigma$ (cf. Theorem 13.2, [Tad3]). However, by Frobenius reciprocity, this contradicts the minimality of $\delta_0(\pi)$ (just by *t.e.* considerations). Thus, $a_1 \geq \beta - 1$.

For the moment, let us assume $\alpha \in \mathbb{Z}$. Then,

$$\begin{aligned}
\pi & \hookrightarrow \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\
& \hookrightarrow \delta([\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{-1}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma.
\end{aligned}$$

Now, $\delta([\nu^{-b_1}\rho, \nu^{-1}\rho]) \times \delta([\nu^{-b_i}\rho, \nu^{a_i}\rho])$ is irreducible for all $i \geq 2$ (if $b_i < 0$, then $a_i \geq -b_i \geq \beta > 1$; if $b_i \geq 0$, then $a_i > b_i \geq \beta - 1 > b_1$). Thus, commuting $\delta([\nu^{-b_1}\rho, \nu^{-1}\rho])$ to the right, we get

$$\begin{aligned}
\pi & \hookrightarrow \delta([\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{-1}\rho]) \rtimes \sigma \\
& \cong \delta([\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \times \delta([\nu\rho, \nu^{b_1}\rho]) \rtimes \sigma
\end{aligned}$$

by the irreducibility of $\delta([\nu^{-b_1}\rho, \nu^{-1}\rho]) \rtimes \sigma$ (since $b_1 < \alpha$). Since $\delta([\nu^{-b_i}\rho, \nu^{a_i}\rho]) \times \delta([\nu\rho, \nu^{b_1}\rho])$ is irreducible for all $i \geq 2$ (if $b_i < 0$, then $a_i \geq -b_i \geq \beta > b_1 + 1$; if $b_i \geq 0$, then $a_i > b_1$ and $b_i \geq \beta - 1 > -1$), we have

$$\begin{aligned}
\pi \hookrightarrow \; & \delta([\rho, \nu^{a_1}\rho]) \times \delta([\nu\rho, \nu^{b_1}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\
\cong \; & \delta([\nu\rho, \nu^{b_1}\rho]) \times \delta([\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\
\hookrightarrow \; & \delta([\nu\rho, \nu^{b_1}\rho]) \times \delta([\nu^{\beta-1}\rho, \nu^{a_1}\rho]) \times \delta([\rho, \nu^{\beta-2}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \\
& \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma
\end{aligned}$$

(noting that $a_1 > \beta - 2 \geq b_1$). Now, the same sort of argument as above, applied to $\delta([\rho, \nu^{\beta-2}\rho])$ this time, gives

$$\begin{aligned}\pi \hookrightarrow\ & \delta([\nu\rho, \nu^{b_1}\rho]) \times \delta([\nu^{\beta-1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \\ & \times \delta([\rho, \nu^{\beta-2}\rho]) \rtimes \sigma \\ \cong\ & \delta([\nu\rho, \nu^{b_1}\rho]) \times \delta([\nu^{\beta-1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \\ & \times \delta([\nu^{-\beta+2}\rho, \rho]) \rtimes \sigma \\ \cong\ & \delta([\nu\rho, \nu^{b_1}\rho]) \times \delta([\nu^{-\beta+2}\rho, \rho]) \times \delta([\nu^{\beta-1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \\ & \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma,\end{aligned}$$

noting that $\beta > 2$ is required for the irreducibility of $\delta([\nu^{\beta-1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-\beta+2}\rho, \rho])$. By Frobenius reciprocity and the Casselman criteria, this contradicts the square-integrability of π.

The case $\alpha \in \frac{1}{2} + \mathbb{Z}$ is essentially the same. □

Lemma 3.4. *Suppose*

$$\delta_0(\pi) = \delta([\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^2\rho, \nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{\alpha}\rho, \nu^{a_\alpha}\rho]) \otimes \sigma.$$

Then we must have $a_1 < a_2$.

Proof. Of course, we automatically have $a_\alpha > \cdots > a_2 \geq a_1$. Thus we only need to show that $a_1 \neq a_2$. Suppose $a_1 = a_2$. Then,

$$\begin{aligned}\pi \hookrightarrow\ & \delta([\rho, \nu^{a_1}\rho]) \times \delta([\nu^2\rho, \nu^{a_1}\rho]) \times \delta([\nu^3\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{a_\alpha}\rho]) \rtimes \sigma \\ \cong\ & \delta([\nu^2\rho, \nu^{a_1}\rho]) \times \delta([\rho, \nu^{a_1}\rho]) \times \delta([\nu^3\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{a_\alpha}\rho]) \rtimes \sigma \\ \hookrightarrow\ & \delta([\nu^2\rho, \nu^{a_1}\rho]) \times \delta([\nu^2\rho, \nu^{a_1}\rho]) \times \delta([\rho, \nu\rho]) \times \delta([\nu^3\rho, \nu^{a_3}\rho]) \\ & \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{a_\alpha}\rho]) \rtimes \sigma \\ \cong\ & \delta([\nu^2\rho, \nu^{a_1}\rho]) \times \delta([\nu^2\rho, \nu^{a_1}\rho]) \times \delta([\nu^3\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{a_\alpha}\rho]) \\ & \times \delta([\rho, \nu\rho]) \rtimes \sigma \\ \cong\ & \delta([\nu^2\rho, \nu^{a_1}\rho]) \times \delta([\nu^2\rho, \nu^{a_1}\rho]) \times \delta([\nu^3\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{a_\alpha}\rho]) \\ & \times \delta([\nu^{-1}\rho, \rho]) \rtimes \sigma,\end{aligned}$$

contradicting the minimality of $\delta_0(\pi)$ (*t.e.* considerations are enough). □

Lemma 3.5. *Suppose $k = \alpha$ and $a_\alpha > \cdots > a_1 > 0$. Suppose*

$$\pi \hookrightarrow \delta([\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^2\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_\alpha}\rho]; \sigma)$$

(noting that the inducing representation exists by Theorem 7.7 and the inductive hypothesis). Then,

$$\pi \hookrightarrow \rho \rtimes \delta([\nu\rho, \nu^{a_1}\rho], [\nu^2\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_\alpha}\rho]; \sigma).$$

Proof. Let

$$\pi' = \delta([\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^2\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_\alpha}\rho]; \sigma)$$

$$\pi'' = \rho \rtimes \delta([\nu\rho, \nu^{a_1}\rho], [\nu^2\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_\alpha}\rho]; \sigma)$$

and

$$\pi^* = \rho \times \delta([\nu\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^2\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_\alpha}\rho]; \sigma).$$

Then, $\pi', \pi'' \leq \pi^*$. Next, we claim

$$\begin{aligned}\mu^*_{\delta([\rho,\nu^{a_1}\rho])}(\pi') = \mu^*_{\delta([\rho,\nu^{a_1}\rho])}(\pi'') = \mu^*_{\delta([\rho,\nu^{a_1}\rho])}(\pi^*) \\ = 2 \cdot \delta([\rho,\nu^{a_1}\rho]) \otimes \delta([\nu^2\rho,\nu^{a_2}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_\alpha}\rho];\sigma).\end{aligned}$$

For π', π^*, this follows from Lemma 2.6. For π'', it is enough to show $\mu^*(\pi'') \geq 2 \cdot \delta([\rho,\nu^{a_1}\rho]) \otimes \delta([\nu^2\rho,\nu^{a_2}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_\alpha}\rho];\sigma)$ (since $\pi'' \leq \pi^*$). This follows immediately from the observation that $M^*(\rho) \geq 2\rho \otimes 1$ and $\mu^*(\delta([\nu\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^2\rho,\nu^{a_2}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_\alpha}\rho];\sigma)) \geq \delta([\nu\rho,\nu^{a_1}\rho]) \otimes \delta([\nu^2\rho,\nu^{a_2}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_\alpha}\rho];\sigma)$. By Frobenius reciprocity, one then has that π must be a subquotient, and therefore (by unitarity) a subrepresentation, of π'', as needed. □

Corollary 3.6. *Suppose*

$$\pi \leq \delta([\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^2\rho,\nu^{a_2}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_\alpha}\rho];\sigma)$$

with π irreducible and $a_\alpha > \dots > a_2 > a_1 > 0$. Then, either

$$\delta_0(\pi) = \rho \otimes \delta([\nu\rho,\nu^{a_1}\rho]) \otimes \delta([\nu^2\rho,\nu^{a_2}\rho]) \otimes \dots \otimes \delta([\nu^{\alpha}\rho,\nu^{a_\alpha}\rho]) \otimes \sigma$$

or there are terms of the form $\nu^x\rho$ with $x < 0$ which appear in $\chi_0(\pi)$.

Proof. We consider three cases based upon Lemma 3.1.

<u>Case 1:</u> $n'_-(1) = 1$.

We note the following:

$$n'_-(x) = 0 \text{ for all } x > \alpha.$$

The proof of this is fairly simple: for $x > \alpha$, $\nu^x\rho$ does not appear as a lower segment end by Lemma 4.4.1, [Jan4]. Also, $\nu^{-x}\rho$ does not appear as an upper segment end since $s_{min}(\pi) \geq \nu^y\rho \otimes \dots$ has $y \in \{b_1,\dots,b_k,a_1,\dots,a_k\}$, which implies $c_1 \geq -\alpha$, and therefore $c_i \geq -\alpha$ for all i. (Alternatively, the assumption that no negative exponents occur in $\chi_0(\pi)$ rules out the possibility that $\nu^{-x}\rho$ appears as an upper segment end.) Now, from

$$n'(0) - n'_-(1) = n(0) - n_-(1),$$

we see that since $n(0) = 1$, $n_-(1) = 0$, and $n'_-(1) = 1$, we must have $n'(0) = 2$. For $c > 1$, we have $n'_-(c) = n_-(c)$, which implies $n'_+(c-1) = n_+(c-1)$. Thus,

$$\delta_0(\pi) = \delta([\nu^{-d_1}\rho,\nu^{c_1}\rho]) \otimes \dots \otimes \delta([\nu^{-d_{\alpha+1}}\rho,\nu^{c_{\alpha+1}}\rho]) \otimes \sigma$$

with $d_1,\dots,d_{\alpha+1},c_1,\dots,c_{\alpha+1}$ equal to $-\alpha,\dots,-1,0,0,a_1,\dots,a_\alpha$ up to permutation. In order to have no negative exponents in $\chi_0(\pi)$, we must have $-\alpha,\dots,-1$ as d_i's and $a_1,\dots,a_\alpha$ as c_i's. This accounts for everything but the two zeros, one of which must be the remaining d_i, the other the remaining c_i. Thus,

$$\delta_0(\pi) = \delta([\nu^{-d_1}\rho,\rho]) \otimes \delta([\nu^{-d_2}\rho,\nu^{a_1}\rho]) \otimes \dots \otimes \delta([\nu^{-d_{\alpha+1}}\rho,\nu^{a_\alpha}\rho]) \otimes \sigma.$$

Next, observe that we must have $d_1 = 0$; if $d_1 = -1$, we do not have $n'_-(1) = 1$. By Lemma 3.2, there is an irreducible θ having

$$\delta_0(\theta) = \delta([\nu^{-d_2}\rho,\nu^{a_1}\rho]) \otimes \dots \otimes \delta([\nu^{-d_{\alpha+1}}\rho,\nu^{a_\alpha}\rho]) \otimes \sigma.$$

By Theorem 4.2.1, [Jan4], θ is square-integrable. By the inductive hypothesis, the conditions of Theorem 1.1 are satisfied. The only way this can happen (cf. conditions 3 and 4 of that theorem) is if $d_2 > \dots > d_{\alpha+1}$. Thus $d_2 = -1,\dots,d_{\alpha+1} = -\alpha$. Thus

$$\delta_0(\pi) = \rho \otimes \delta([\nu\rho,\nu^{a_1}\rho]) \otimes \dots \otimes \delta([\nu^{\alpha}\rho,\nu^{a_\alpha}\rho]) \otimes \sigma$$

is the only possibility remaining in Case 1.

Case 2: $n'_-(1) = 0$ and $n'_-(2) = 1$.

By Lemma 3.1, one easily gets

$$\delta_0(\pi) = \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \otimes \delta([\nu^{-d_2}\rho, \nu^{c_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_\alpha}\rho, \nu^{c_\alpha}\rho]) \otimes \sigma$$

with $d_1, \ldots, d_\alpha, c_1, \ldots, c_\alpha$ equal to $-\alpha, \ldots, -2, 0, a_1, \ldots, a_\alpha$ up to permutation. Again, to avoid picking up any negative exponents, we must have $a_1, \ldots, a_\alpha$ as c_i's, and therefore $-\alpha, \ldots, -2, 0$ as d_i's. Thus,

$$\delta_0(\pi) = \delta([\nu^{-d_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-d_2}\rho, \nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_\alpha}\rho, \nu^{a_\alpha}\rho]) \otimes \sigma.$$

As in Case 1, there is an irreducible θ with

$$\delta_0(\theta) = \delta([\nu^{-d_2}\rho, \nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_\alpha}\rho, \nu^{a_\alpha}\rho]) \otimes \sigma.$$

As in Case 1, θ is square-integrable and the inductive assumption and Theorem 1.1 tell us we must have $d_2 > \cdots > d_\alpha$. Further, condition 2 of Theorem 1.1 also tells us we cannot have one of $d_2, \ldots, d_\alpha$ equal to 0. Thus the only possibility remaining in Case 2 is

$$\delta_0(\pi) = \delta([\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^2\rho, \nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{\alpha}\rho, \nu^{a_\alpha}\rho]) \otimes \sigma.$$

However, by Lemma 3.5, this cannot occur. Thus Case 2 can be eliminated.

Case 3: $n'_-(1) = n'_-(2) = 0$.

In this case, let β' be the minimal value having $n'_-(\beta') = 1$. If $n'_-(x) = 0$ for all x, we take $\beta' = \alpha + 1$. Then, $n'_-(1) = \cdots = n'_-(\beta'-1) = 0$, $n'_-(\beta') = \cdots = n'_-(\alpha) = 1$, and $n'_-(x) = 0$ for all $x > \alpha$. Since $n'_-(1) = n_-(1) = 0$, $n'(0) - n'_-(1) = n(0) - n_-(1)$ tells us $n'(0) = n(0) = 1$.

Next, we check that $a_x = x$ for $1 \leq x \leq \beta' - 2$. First, $n'_+(1) - n'_-(2) = n_+(1) - n_-(2)$ has $n_-(2) = 0$ and $n_-(2) = 1$. Since $n_+(1) \leq 1$ and $n'_+(1) \geq 0$, we see that $n_+(1) = 1$ and $n'_+(1) = 0$. To have $n_+(1) = 1$, we must have $a_1 = 1$. We iterate this argument until we get to the following: $n'_+(\beta'-2) - n'_-(\beta'-1) = n_+(\beta'-2) - n_-(\beta'-1)$ has $n'_-(\beta'-1) = 0$ and $n_-(\beta'-1) = 1$. Therefore, $n_+(\beta'-2) = 1$ and $n'_+(\beta'-2) = 0$. For $n_+(\beta'-2) = 1$, we must have $a_{\beta'-2} = \beta'-2$.

Finally, observe that for $x \geq \beta'$, we have $n'_-(x) = n_-(x)$. Therefore, since $n'_+(x-1) - n'_-(x) = n_+(x-1) - n_-(x)$, we have $n'_+(y) = n_+(y)$ for $y \geq \beta' - 1$. Thus, if

$$\delta_0(\pi) = \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \otimes \delta([\nu^2\rho, \nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_j}\rho, \nu^{c_j}\rho]) \otimes \sigma,$$

we must have $d_1, \ldots, d_j, c_1, \ldots, c_j$ equal to $-\alpha, \ldots, -\beta', 0, a_{\beta'-1}, \ldots, a_\alpha$ up to permutation. In order that no negative exponents appear, $a_{\beta'-1}, \ldots, a_\alpha$ must be c_i's and therefore $-\alpha, \ldots, -\beta', 0$ must be d_i's. An argument like that used in the previous case then allows us to conclude that

$$\delta_0(\pi) = \delta([\rho, \nu^{a_{\beta'-1}}\rho]) \otimes \delta([\nu^{\beta'}\rho, \nu^{a_{\beta'}}\rho]) \otimes \cdots \otimes \delta([\nu^{\alpha}\rho, \nu^{a_\alpha}\rho]) \otimes \sigma.$$

However, since $\beta' > 2$, this is in contradiction to Theorem 2.4. Thus we can also eliminate Case 3. The corollary now follows. □

Theorem 3.7. *Suppose* **(H)**. *Suppose that Theorems 1.1 and 7.7 (and Corollaries 7.8 and 7.9) hold when the parabolic rank of the supercuspidal support is less than p.r. Then, condition 2 in Theorem 1.1 holds.*

Proof. By Lemma 3.3, we may assume $\beta = 2$ (which means α must be an integer). As in the proof of Lemma 3.3 (first paragraph), if $b_1 < 0$, we have $b_1 = -\beta$ and can easily deduce that condition 2 of Theorem 1.1 holds. Also, as in the proof of Lemma 3.3 (first paragraph), if $b_1 \geq 0$, the only way condition 2 of Theorem 1.1 could fail is if $\beta - 2 \geq b_1 \geq 0$, i.e., $b_1 = 0$. Therefore, all we need to do is show $b_1 \neq 0$. So, let us suppose $b_1 = 0$. Then,

$$\delta_0(\pi) = \delta([\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma.$$

Next, let $[\nu^{-d_1}\rho, \nu^{c_1}\rho], \dots, [\nu^{-d_k}\rho, \nu^{c_k}\rho]$ denote a permutation of $[\nu^{-b_1}\rho, \nu^{a_1}\rho], \dots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]$ satisfying $d_1 \geq \cdots \geq d_k$. To make it unique, note that only $d_i = 0$ can occur for more than one value of i, and then only for two. If we should have $b_i = 0$ for some $i > 1$, then $[\rho, \nu^{a_i}\rho]$ should appear before $[\rho, \nu^{a_1}\rho]$ in the above list. We claim that

$$\pi \hookrightarrow \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_k}\rho, \nu^{c_k}\rho]) \rtimes \sigma$$

which we may write as

$$\begin{aligned}\pi \hookrightarrow \quad & \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_j}\rho, \nu^{c_j}\rho]) \times \delta([\rho, \nu^{a_1}\rho]) \times \delta([\nu^{2}\rho, \nu^{c_{j+2}}\rho]) \\ & \times \delta([\nu^{\alpha}\rho, \nu^{c_k}\rho]) \rtimes \sigma.\end{aligned}$$

Suppose $[\nu^{-d_1}\rho, \nu^{c_1}\rho] = [\nu^{-b_j}\rho, \nu^{a_j}\rho]$. Observe that for $i < j$,

$$[\nu^{-b_i}\rho, \nu^{a_i}\rho] \subset [\nu^{-b_j}\rho, \nu^{a_j}\rho]$$

so

$$\begin{aligned}& \delta([\nu^{-b_i}\rho, \nu^{a_i}\rho]) \times \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \\ & \qquad \cong \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \delta([\nu^{-b_i}\rho, \nu^{a_i}\rho]) \quad \text{(irreducible)}.\end{aligned}$$

Thus, we may commute $\delta([\nu^{-d_1}\rho, \nu^{c_1}\rho])$ to the left:

$$\begin{aligned}\pi \hookrightarrow \quad & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_{j-2}}\rho, \nu^{a_{j-2}}\rho]) \times \delta([\nu^{-b_{j-1}}\rho, \nu^{a_{j-1}}\rho]) \\ & \times \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \delta([\nu^{-b_{j+1}}\rho, \nu^{a_{j+1}}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\ \cong \quad & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_{j-2}}\rho, \nu^{a_{j-2}}\rho]) \times \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \\ & \times \delta([\nu^{-b_{j-1}}\rho, \nu^{a_{j-1}}\rho]) \times \delta([\nu^{-b_{j+1}}\rho, \nu^{a_{j+1}}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\ & \vdots \\ \cong \quad & \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_{j-2}}\rho, \nu^{a_{j-2}}\rho]) \\ & \times \delta([\nu^{-b_{j-1}}\rho, \nu^{a_{j-1}}\rho]) \times \delta([\nu^{-b_{j+1}}\rho, \nu^{a_{j+1}}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma.\end{aligned}$$

Iterating this argument, we next commute $\delta([\nu^{-d_2}\rho, \nu^{c_2}\rho])$ to the left, etc., and we get the result claimed.

Now, by Lemma 5.5, [Jan2],

$$\begin{aligned}\pi \hookrightarrow \quad & \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_j}\rho, \nu^{c_j}\rho]) \times \delta([\rho, \nu^{a_1}\rho]) \times \delta([\nu^{2}\rho, \nu^{c_{j+2}}\rho]) \\ & \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{c_k}\rho]) \rtimes \sigma \\ & \Downarrow \\ \pi \hookrightarrow \quad & \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_j}\rho, \nu^{c_j}\rho]) \rtimes \theta\end{aligned}$$

for some irreducible $\theta \leq \delta([\rho, \nu^{a_1}\rho]) \times \delta([\nu^{2}\rho, \nu^{c_{j+2}}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{c_k}\rho]) \rtimes \sigma$. Since

$$s_{app}(\pi) \geq \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_j}\rho, \nu^{c_j}\rho]) \otimes \delta_0(\theta)$$

we must have

$$t.e.(\delta_0(\theta)) \geq t.e.(\delta([\rho,\nu^{a_1}\rho]) \otimes \delta([\nu^2\rho,\nu^{c_{j+2}}\rho]) \otimes \cdots \otimes \delta([\nu^{\alpha}\rho,\nu^{c_k}\rho]) \otimes \sigma).$$

By Corollary 3.6,

$$\delta_0(\theta) = \rho \otimes \delta([\nu\rho,\nu^{a_1}\rho]) \otimes \delta([\nu^2\rho,\nu^{c_{j+2}}\rho]) \otimes \delta([\nu^{\alpha}\rho,\nu^{c_k}\rho]) \otimes \sigma.$$

Then,

$$\begin{aligned} \pi \hookrightarrow\ & \delta([\nu^{-d_1}\rho,\nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_j}\rho,\nu^{c_j}\rho]) \rtimes \theta \\ \hookrightarrow\ & \delta([\nu^{-d_1}\rho,\nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_j}\rho,\nu^{c_j}\rho]) \times \rho \times \delta([\nu\rho,\nu^{a_1}\rho]) \\ & \times \delta([\nu^2\rho,\nu^{c_{j+2}}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho,\nu^{c_k}\rho]) \rtimes \sigma \\ \cong\ & \rho \times \delta([\nu^{-d_1}\rho,\nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_j}\rho,\nu^{c_j}\rho]) \times \delta([\nu\rho,\nu^{a_1}\rho]) \\ & \times \delta([\nu^2\rho,\nu^{c_{j+2}}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho,\nu^{c_k}\rho]) \rtimes \sigma, \end{aligned}$$

contradicting the square-integrability of π. The theorem follows. □

4. Conditions 3 and 4—the first part of proof

In this section, we begin the proof that conditions 3 and 4 in Theorem 1.1 hold. We will view conditions 3 and 4 of Theorem 1.1 as conditions on pairs of segments. By Lemma 3.2, Theorem 4.2.1, [Jan4], and the inductive hypothesis, we know that $[\nu^{-b_i}\rho,\nu^{a_i}\rho]$, $[\nu^{-b_j}\rho,\nu^{a_j}\rho]$ satisfy these conditions when $i,j>1$ with $i \neq j$. Thus, our goal is to show $[\nu^{-b_1}\rho,\nu^{a_1}\rho]$, $[\nu^{-b_i}\rho,\nu^{a_i}\rho]$ satisfy these conditions for $i>1$. In this section, we address the case where $b_i > a_1$ for some $i \geq 2$. For any such i, we have $a_i > b_i > a_1 > b_1$, so conditions 3 and 4 of Theorem 1.1. certainly hold for $[\nu^{-b_1}\rho,\nu^{a_1}\rho]$, $[\nu^{-b_i}\rho,\nu^{a_i}\rho]$. Roughly speaking, we show that there is an irreducible representation θ, where $\delta_0(\theta)$ is essentially $\delta_0(\pi)$ with such $\delta([\nu^{-b_i}\rho,\nu^{a_i}\rho])$ removed. We can then use the inductive hypothesis to show $\delta_0(\theta)$ satisfies conditions 3. and 4. of Theorem 1.1, which is enough to finish this case.

Let us take a moment to review where we stand with respect to the inductive hypothesis. Of course, we may assume Theorems 1.1 and 7.7 (and Corollaries 7.8 and 7.9) hold when the parabolic rank of the supercuspidal support is less than $p.r.$ In addition, we may assume that condition 1 of Theorem 1.1 holds in general (Theorem 2.4) and that condition 2 of Theorem 1.1 holds when the parabolic rank of the supercuspidal support is equal to $p.r.$ (Theorem 3.7 and the inductive hypothesis).

The proof of the following proposition is essentially the same as that used in the proof of Theorem 3.7; we will not go through the details.

Proposition 4.1. *Suppose* $\delta_0(\pi) = \delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho,\nu^{a_k}\rho]) \otimes \sigma$. *Then there exist (unique)* $\ell, m, \ell+m+1=k$ *and permutation*

$$[\nu^{-b'_1}\rho,\nu^{a'_1}\rho],\ldots,[\nu^{-b'_\ell}\rho,\nu^{a'_\ell}\rho],[\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b''_1}\rho,\nu^{a''_1}\rho],\ldots,[\nu^{-b''_m}\rho,\nu^{a''_m}\rho]$$

of $[\nu^{-b_1}\rho,\nu^{a_1}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho]$ *satisfying the following conditions:*

1. $b'_1 > b'_2 > \cdots > b'_\ell > a_1$,
2. $a''_1 < a''_2 < \cdots < a''_m$ *with* $b''_i \leq a_1$ *for all* i.

Furthermore,

$$\begin{aligned} \pi \hookrightarrow\ & \delta([\nu^{-b'_1}\rho,\nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_\ell}\rho,\nu^{a'_\ell}\rho]) \times \delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \\ & \times\delta([\nu^{-b''_1}\rho,\nu^{a''_1}\rho]) \times \cdots \times \delta([\nu^{-b''_m}\rho,\nu^{a''_m}\rho]) \rtimes \sigma. \end{aligned}$$

The proof for conditions 3 and 4 of Theorem 1.1 will be broken into two parts—we address the case $\ell \geq 1$ here; the case $\ell = 0$ will be addressed in the next section. So, for the remainder of this section, we assume $\ell \geq 1$.

Observe that by Lemma 3.2, Theorem 4.2.1, [Jan4], and the inductive hypothesis, conditions 3 and 4 of Theorem 1.1 hold for $[\nu^{-b''_1}\rho, \nu^{a''_1}\rho], \dots, [\nu^{-b''_m}\rho, \nu^{a''_m}\rho]$. By construction, we cannot have $a''_i > b''_i > a''_j > b''_j$ for $i > j$. Therefore, we have $a''_m > a''_{m-1} > \cdots > a''_1 > b''_1 > b''_2 > \cdots > b''_m$. Then, by Lemma 5.5, [Jan2],

$$\begin{gathered}\pi \hookrightarrow \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \\ \times \delta([\nu^{-b''_1}\rho, \nu^{a''_1}\rho]) \times \cdots \times \delta([\nu^{-b''_{m-1}}\rho, \nu^{a''_{m-1}}\rho]) \rtimes (\delta([\nu^{-b''_m}\rho, \nu^{a''_m}\rho]) \rtimes \sigma) \\ \Downarrow \\ \pi \hookrightarrow \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \\ \times \delta([\nu^{-b''_1}\rho, \nu^{a''_1}\rho]) \times \cdots \times \delta([\nu^{-b''_{m-1}}\rho, \nu^{a''_{m-1}}\rho]) \rtimes \xi\end{gathered}$$

for some irreducible $\xi \leq \delta([\nu^{-b''_m}\rho, \nu^{a''_m}\rho]) \rtimes \sigma$. By the inductive hypothesis, ξ is one of the representations from Theorem 7.7; by *t.e.* (and Frobenius reciprocity), to avoid contradicting the minimality of $\delta_0(\pi)$, we must have $\xi = \delta([\nu^{-b''_m}\rho, \nu^{a''_m}\rho]; \sigma)_t$. Then,

$$\begin{gathered}\pi \hookrightarrow \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b''_1}\rho, \nu^{a''_1}\rho]) \\ \times \cdots \times \delta([\nu^{-b''_{m-2}}\rho, \nu^{a''_{m-2}}\rho]) \rtimes (\delta([\nu^{-b''_{m-1}}\rho, \nu^{a''_{m-1}}\rho]) \rtimes \delta([\nu^{-b''_m}\rho, \nu^{a''_m}\rho]; \sigma)_t) \\ \Downarrow \\ \pi \hookrightarrow \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \\ \times \delta([\nu^{-b''_1}\rho, \nu^{a''_1}\rho]) \times \cdots \times \delta([\nu^{-b''_{m-1}}\rho, \nu^{a''_{m-1}}\rho]) \rtimes \xi\end{gathered}$$

for some irreducible $\xi \leq \delta([\nu^{-b''_{m-2}}\rho, \nu^{a''_{m-2}}\rho]) \rtimes \delta([\nu^{-b''_m}\rho, \nu^{a''_m}\rho]; \sigma)_t$. Again, by the inductive hypothesis, ξ is one of the representations from Theorem 7.7; again, by *t.e.* considerations, we get $\xi = \delta([\nu^{-b''_{m-1}}\rho, \nu^{a''_{m-1}}\rho], [\nu^{-b''_m}\rho, \nu^{a''_m}\rho]; \sigma)_t$. Iterating this argument, we get

$$\pi \hookrightarrow \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]) \rtimes \theta$$

for some irreducible $\theta \leq \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b''_1}\rho, \nu^{a''_1}\rho], \dots, [\nu^{-b''_m}\rho, \nu^{a''_m}\rho]; \sigma)_t$.

Lemma 4.2. *With notation as above,*

$$s_{\delta([\nu^{-b'_1}\rho,\nu^{a'_1}\rho])\otimes\cdots\otimes\delta([\nu^{-b'_\ell}\rho,\nu^{a'_\ell}\rho])}(\pi) = c \cdot \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]) \otimes \theta$$

for some nonzero integer c.

Proof. First, by Lemma 5.5, [Jan2], there is an irreducible $\theta'_\ell \leq \delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]) \rtimes \theta$ such that

$$\pi \hookrightarrow \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_{\ell-1}}\rho, \nu^{a'_{\ell-1}}\rho]) \rtimes \theta'_\ell.$$

Iterating this argument, we obtain $\theta'_{\ell-1}, \theta'_{\ell-2}, \dots, \theta'_2$ such that each is irreducible, $\theta'_i \leq \delta([\nu^{-b'_i}\rho, \nu^{a'_i}\rho]) \rtimes \theta'_{i+1}$, and $\pi \hookrightarrow \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_{i-1}}\rho, \nu^{a'_{i-1}}\rho]) \rtimes \theta'_i$.

First, consider $\delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \rtimes \theta'_2$. We claim

$$\mu^*_{\delta([\nu^{-b'_1}\rho,\nu^{a'_1}\rho])}(\delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \rtimes \theta'_2) = 2 \cdot \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \otimes \theta'_2.$$

Write $\mu^*(\theta_2') = \sum_h \tau_h \otimes \xi_h$. Then,

$$\mu^*(\delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho]) \rtimes \theta_2') = \sum_h \sum_{i=-b_1'}^{a_1'+1} \sum_{j=i}^{a_1'+1} \delta([\nu^{-i+1}\rho, \nu^{b_1'}\rho]) \times \delta([\nu^{j}\rho, \nu^{a_1'}\rho]) \times \tau_h \otimes \delta([\nu^{i}\rho, \nu^{j-1}\rho]) \rtimes \xi_h.$$

We now need an observation (the importance of which was underscored for the author by [Mœ3]). Suppose $\delta([\nu^{-b_s}\rho, \nu^{a_s}\rho])$ appears to the left of $\delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho])$ in the original $\delta_0(\pi)$. For $s = 1$, we have $a_1' > b_1' > a_1 > b_1$ by construction. For $s > 1$, we have that $[\nu^{-b_1'}\rho, \nu^{a_1'}\rho], [\nu^{-b_s}\rho, \nu^{a_s}\rho]$ satisfy the conditions of Theorem 1.1 (by Lemma 3.2, Theorem 4.2.1, [Jan4], and the inductive hypothesis). Therefore, either $a_1' > b_1' > a_s > b_s$ or $a_1' > a_s > b_s > b_1'$; we can rule out the latter since b_1' is known to be larger than b_s by construction. Now, suppose $\delta([\nu^{-b_s}\rho, \nu^{a_s}\rho])$ appears to the right of $\delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho])$ in the original $\delta_0(\pi)$. Again, since $[\nu^{-b_1'}\rho, \nu^{a_1'}\rho], [\nu^{-b_s}\rho, \nu^{a_s}\rho]$ satisfy the conditions of Theorem 1.1, either $a_s > b_s > a_1' > b_1'$ or $a_s > a_1' > b_1' > b_s$; we can rule out the former since b_1' is known to be larger than b_s by construction. In particular, the observation we need is the following: there is no element of $\{a_1, \dots, a_k, b_1, \dots, b_k\} \setminus \{b_1', a_1'\}$ in the interval $[b_1', a_1']$. Therefore, to contribute to $\mu^*_{\delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho])}$, we must have $j \leq b_1'+1$ in the displayed equation above.

Suppose $j = b_1' + 1$. Then, $\tau_h = \delta([\nu^{-b_1'}\rho, \nu^{-i}\rho])$. If $i \neq b_1' + 1$, then by central character considerations,

$$\begin{aligned} &\theta_2' \hookrightarrow \nu^{-i}\rho \times \nu^{-i-1}\rho \times \cdots \times \nu^{-b_1'}\rho \rtimes \xi_h \\ &\qquad \Downarrow \\ &\pi \hookrightarrow \delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho]) \times \nu^{-i}\rho \times \nu^{-i-1}\rho \times \cdots \times \nu^{-b_1}\rho \rtimes \xi_h \\ &\quad \cong \nu^{-i}\rho \times \nu^{-i-1}\rho \times \cdots \times \nu^{-b_1'}\rho \times \delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho]) \rtimes \xi_h, \end{aligned}$$

contradicting the square-integrability of π. Thus, $i = b_1'+1$, so $\tau_h = 1$ and $\xi_h = \theta_2'$. This contributes one copy of $\delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho]) \otimes \theta_2'$ (and nothing else) to

$$\mu^*_{\delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho])}(\delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho]) \rtimes \theta_2').$$

If $j \neq b_1' + 1$, then we must have $i = -b_1'$ (since $\nu^{b_1'}\rho$ appear only once in $r_{min}(\delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho])))$. A similar argument tells us we must again have $\tau_h = 1$, $\xi_h = \theta_2'$. So, $j = -b_1'$ contributes a second copy of $\delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho]) \otimes \theta_2'$ to $\mu^*_{\delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho])}(\delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho]) \rtimes \theta_2')$. Thus,

$$\mu^*_{\delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho])}(\delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho]) \rtimes \theta_2') = 2 \cdot \delta([\nu^{-b_1'}\rho, \nu^{a_1'}\rho]) \otimes \theta_2',$$

as claimed.

Let us briefly discuss the first iteration of this argument. Our claim is that

$$\mu^*_{\delta([\nu^{-b_2'}\rho, \nu^{a_2'}\rho])}(\delta([\nu^{-b_2'}\rho, \nu^{a_2'}\rho]) \rtimes \theta_3') = 2 \cdot \delta([\nu^{-b_2'}\rho, \nu^{a_2'}\rho]) \otimes \theta_3'.$$

The analogue to the observation above is that there is nothing in $\{a_1, \dots, a_k, b_1, \dots, b_k\} \setminus \{a_1', b_1'\}$ which is strictly between $b_2' + 1$ and a_2'. Again, let us write

$\mu^*(\theta'_3) = \sum_h \tau_h \otimes \xi_h$. Then,

$$\mu^*(\delta([\nu^{-b'_2}\rho, \nu^{a'_2}\rho]) \rtimes \theta'_3) = \sum_h \sum_{i=-b'_2}^{a'_2+1} \sum_{j=i}^{a'_2+1} \delta([\nu^{-i+1}\rho, \nu^{b'_2}\rho]) \times \delta([\nu^{j}\rho, \nu^{a'_2}\rho]) \times \tau_h \otimes \delta([\nu^{i}\rho, \nu^{j-1}\rho]) \rtimes \xi_h.$$

Again, to contribute to $\mu^*_{\delta([\nu^{-b'_2}\rho,\nu^{a'_2}\rho])}$, we must have $j \leq b'_2+1$. If $j = b'_2+1$, we get $i = b'_2+1$ as before: if $i \neq b'_2+1$, then central character considerations tell us

$$\begin{array}{rcl} \theta'_3 & \hookrightarrow & \nu^{-i}\rho \times \nu^{-i-1}\rho \times \cdots \times \nu^{-b'_1}\rho \rtimes \xi_h \\ & \Downarrow & \\ \pi & \hookrightarrow & \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \delta([\nu^{-b'_2}\rho, \nu^{a'_2}\rho]) \times \nu^{-i}\rho \times \nu^{-i-1}\rho \times \cdots \times \nu^{-b'_2}\rho \rtimes \xi_h \\ & \cong & \nu^{-i}\rho \times \nu^{-i-1}\rho \times \cdots \times \nu^{-b'_2}\rho \times \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \delta([\nu^{-b'_2}\rho, \nu^{a'_2}\rho]) \rtimes \xi_h, \end{array}$$

which contradicts the square-integrability of π. When $i = b'_2+1$, we get a contribution of one copy of $\delta([\nu^{-b'_2}\rho, \nu^{a'_2}\rho]) \otimes \theta'_3$ to $\mu^*_{\delta([\nu^{-b'_2}\rho,\nu^{a'_2}\rho])}(\delta([\nu^{-b'_2}\rho, \nu^{a'_2}\rho]) \rtimes \theta'_3)$. If $j < b'_2+1$, then we must have $i = -b'_2$ and $\tau_h = \delta([\nu^{-b'_2}\rho, \nu^{j-1}\rho])$. The same argument as before tells us we must have $j = -b'_2$. Therefore, $\tau_h = 1$ and $\xi_h = \theta'_3$, giving a contribution of one more copy of $\delta([\nu^{-b'_2}\rho, \nu^{a'_2}\rho]) \otimes \theta'_3$ to $\mu^*_{\delta([\nu^{-b'_2}\rho,\nu^{a'_2}\rho])}(\delta([\nu^{-b'_2}\rho, \nu^{a'_2}\rho]) \rtimes \theta'_3)$, as claimed.

The lemma follows by iterating. □

It follows immediately from properties of Jacquet functors that there is an irreducible representation θ'' satisfying

1. $s_{app}(\pi) \geq \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]) \otimes \theta''$,
2. $s_{app}(\theta'') \geq \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b''_1}\rho, \nu^{a''_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b''_m}\rho, \nu^{a''_m}\rho]) \otimes \sigma$.

The corollary below then follows immediately from Lemma 4.2.

Corollary 4.3. *With notation as above, $\theta = \theta''$.*

Theorem 4.4. *Suppose* **(H)**. *Suppose that Theorems 1.1 and 7.7 (and Corollaries 7.8 and 7.9) are proved when the parabolic rank of the supercuspidal support is less than p.r. and that condition 2 of Theorem 1.1 is also proved when the parabolic rank of the supercuspidal support is equal to p.r. Then, if $\ell \geq 1$, conditions 3 and 4 of Theorem 1.1 hold (ℓ as in Proposition 4.1).*

Proof. By the inductive hypothesis, Lemma 3.2, and Theorem of 4.2.1 [Jan4], we have that $[\nu^{-b_i}\rho, \nu^{a_i}\rho], [\nu^{-b_k}\rho, \nu^{a_j}\rho]$ satisfy conditions 3 and 4 of Theorem 1.1 when $i > j \geq 2$. Therefore, it is enough to show that $[\nu^{-b_1}\rho, \nu^{a_1}\rho], [\nu^{-b_i}\rho, \nu^{a_i}\rho]$, $i \geq 2$ satisfy conditions 3 and 4. Further, by construction $[\nu^{-b'_i}\rho, \nu^{a'_i}\rho], [\nu^{-b_1}\rho, \nu^{a_1}\rho]$ satisfy conditions 3 and 4 Thus, it remains to show that $[\nu^{-b_1}\rho, \nu^{a_1}\rho], [\nu^{-b''_i}\rho, \nu^{a''_i}\rho]$ satisfy conditions 3 and 4.

To show $[\nu^{-b_1}\rho, \nu^{a_1}\rho], [\nu^{-b''_i}\rho, \nu^{a''_i}\rho]$ satisfy conditions 3 and 4, it is enough (by the inductive hypothesis) to show the following:

$$\delta_0(\theta) = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b''_1}\rho, \nu^{a''_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b''_m}\rho, \nu^{a''_m}\rho]) \otimes \sigma$$

(θ as above). The theorem then follows immediately.

Consider the permutation of $[\nu^{-b'_1}\rho, \nu^{a'_1}\rho], \ldots, [\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]$ where $a'_1, \ldots, a'_\ell$ appear in increasing order of size. Since it is this permutation we need for the remainder of this proof, let us abuse notation somewhat and simply denote it by $[\nu^{-b'_1}\rho, \nu^{a'_1}\rho], \dots, [\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]$, where $a'_1 < \cdots < a'_\ell$. Then,

$$\begin{aligned}\pi \hookrightarrow \quad & \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \\ & \times \delta([\nu^{-b''_1}\rho, \nu^{a''_1}\rho]) \times \cdots \times \delta([\nu^{-b''_m}\rho, \nu^{a''_m}\rho]) \rtimes \sigma\end{aligned}$$

(as in the proof of Theorem 3.7, any $\delta([\nu^{-b'_i}\rho, \nu^{a'_i}\rho])$, $\delta([\nu^{-b'_{i+1}}\rho, \nu^{a'_{i+1}}\rho])$ which has $a'_{i+1} < a'_i$ can be transposed by an irreducibility argument; a sequence of such transpositions suffices to rearrange terms in the order described).

First, observe that by construction,

$$\delta_0(\theta'') \preceq \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b''_1}\rho, \nu^{a''_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b''_m}\rho, \nu^{a''_m}\rho]) \otimes \sigma.$$

Also, since

$$s_{app}(\pi) \geq \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]) \otimes \delta_0(\theta''),$$

we see that

$$t.e.(\delta_0(\theta'')) \geq t.e.(\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b''_1}\rho, \nu^{a''_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b''_m}\rho, \nu^{a''_m}\rho]) \otimes \sigma).$$

Therefore,

$$t.e.(\delta_0(\theta'')) = t.e.(\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b''_1}\rho, \nu^{a''_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b''_m}\rho, \nu^{a''_m}\rho]) \otimes \sigma).$$

Now, suppose

$$\delta_0(\theta'') = \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-c_s}\rho, \nu^{c_s}\rho]) \otimes \sigma.$$

If we assume $\delta_0(\theta'') \succ \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b''_1}\rho, \nu^{a''_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b''_m}\rho, \nu^{a''_m}\rho]) \otimes \sigma$, we must have $\delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-c_s}\rho, \nu^{c_s}\rho]) \otimes \sigma$ lower than $\delta_0(\theta'') \succeq \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b''_1}\rho, \nu^{a''_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b''_m}\rho, \nu^{a''_m}\rho]) \otimes \sigma$ lexicographically.

Suppose $\delta([\nu^{-d_1}\rho, \nu^{c_1}\rho])$ is less than $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho])$ lexicographically. Then, $a'_j > b'_j > a_1 \geq c_1$. If $d_1 \leq b'_\ell$, then $\delta([\nu^{-b'_j}\rho, \nu^{a'_j}\rho]) \times \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho])$ is irreducible for all j. By Corollary 4.3,

$$\begin{aligned}\pi \hookrightarrow \quad & \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]) \times \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \\ & \times \cdots \times \delta([\nu^{-d_s}\rho, \nu^{c_s}\rho]) \rtimes \sigma \\ \cong \quad & \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]) \\ & \times \cdots \times \delta([\nu^{-d_s}\rho, \nu^{c_s}\rho]) \rtimes \sigma,\end{aligned}$$

contradicting the minimality of $\delta_0(\pi)$.

If $d_1 > b'_\ell$, then

$$\begin{aligned}\delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]) \times \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) = {} & \delta([\nu^{-b'_\ell}\rho, \nu^{c_1}\rho]) \times \delta([\nu^{-d_1}\rho, \nu^{a'_\ell}\rho]) \\ & + \mathcal{L}(\delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]), \delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]))\end{aligned}$$

(noting that $c_1 \in \{b_1, b''_1, \dots, b''_m, a_1, a''_1, \dots, a''_m\}$ forces $c_1 > -b'_\ell$). Therefore, by Lemma 5.5, [Jan4],

$$\begin{aligned}\pi \hookrightarrow {} & \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_{\ell-1}}\rho, \nu^{a'_{\ell-1}}\rho]) \times \delta([\nu^{-b'_\ell}\rho, \nu^{c_1}\rho]) \\ & \times \delta([\nu^{-d_1}\rho, \nu^{a'_\ell}\rho]) \times \delta([\nu^{-d_2}\rho, \nu^{c_2}\rho]) \times \cdots \times \delta([\nu^{-d_s}\rho, \nu^{c_s}\rho]) \rtimes \sigma\end{aligned}$$

or

$$\pi \hookrightarrow \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_{\ell-1}}\rho, \nu^{a'_{\ell-1}}\rho]) \times \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \\ \times \delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]) \times \delta([\nu^{-d_2}\rho, \nu^{c_2}\rho]) \times \cdots \times \delta([\nu^{-d_s}\rho, \nu^{c_s}\rho]) \rtimes \sigma.$$

In the first case, we now argue as we did when $d_1 \leq b'_\ell$ to obtain

$$\pi \hookrightarrow \delta([\nu^{-b'_\ell}\rho, \nu^{c_1}\rho]) \times \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_{\ell-1}}\rho, \nu^{a'_{\ell-1}}\rho]) \\ \times \delta([\nu^{-d_1}\rho, \nu^{a'_\ell}\rho]) \times \delta([\nu^{-d_2}\rho, \nu^{c_2}\rho]) \times \cdots \times \delta([\nu^{-d_s}\rho, \nu^{c_s}\rho]) \rtimes \sigma,$$

again contradicting the minimality of $\delta_0(\pi)$. In the second case, we can iterate the argument above: if $d_1 \leq b'_{\ell-1}$, we can again commute $\delta([\nu^{-d_1}\rho, \nu^{c_1}\rho])$ forward to get a contradiction. If $d_1 > b'_{\ell-1}$, we can again look at $\delta([\nu^{-b_{\ell-1}}\rho, \nu^{b'_{\ell-1}}\rho]) \times \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho])$ and repeat the argument above. In either case, we eventually get

$$\pi \hookrightarrow \delta([\nu^{-x}\rho, \nu^{c_1}\rho]) \times \dots,$$

with $x = d_1$ or b'_j for some j. In any case, we contradict the minimality of $\delta_0(\pi)$. Therefore, we could not have had $\delta([\nu^{-d_1}\rho, \nu^{c_1}\rho])$ less than $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho])$ lexicographically. Thus $c_1 = a_1$ and $d_1 = b_1$.

Since $\delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho])$, let us now suppose $\delta([\nu^{-d_2}\rho, \nu^{c_2}\rho])$ is lexicographically lower than $\delta([\nu^{-b''_1}\rho, \nu^{a''_1}\rho])$. We can easily see that

$$\pi \hookrightarrow \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_\ell}\rho, \nu^{a'_\ell}\rho]) \\ \times \delta([\nu^{-d_2}\rho, \nu^{c_2}\rho]) \times \cdots \times \delta([\nu^{-d_s}\rho, \nu^{c_s}\rho]) \rtimes \sigma.$$

Now, take i such that $a'_i < a''_1 < a'_{i+1}$ (subject to the obvious convention if $a''_1 < a'_1$ or $a''_1 > a'_\ell$). Then, $[\nu^{-b'_j}\rho, \nu^{a'_j}\rho] = [\nu^{-b_{j+1}}\rho, \nu^{a_{j+1}}\rho]$ for $1 \leq j \leq i$. Further, $[\nu^{-b''_1}\rho, \nu^{a''_1}\rho] = [\nu^{-b_{i+2}}\rho, \nu^{a_{i+2}}\rho]$. Repeating the argument above, we get

$$\begin{aligned} \pi &\hookrightarrow \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{-b'_i}\rho, \nu^{a'_i}\rho]) \\ &\quad \times \delta([\nu^{-x}\rho, \nu^{c_2}\rho]) \times \dots \\ &= \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_{i+1}}\rho, \nu^{a_{i+1}}\rho]) \\ &\quad \times \delta([\nu^{-x}\rho, \nu^{c_2}\rho]) \times \dots \end{aligned}$$

for $x = d_2$ or b'_j some $j > i$. Again, this contradicts the minimality of $\delta_0(\pi)$. Thus we must have $d_2 = b''_1$ and $c_2 = a''_1$.

We can now iterate this argument to see that

$$\delta_0(\theta'') = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b''_1}\rho, \nu^{a''_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b''_m}\rho, \nu^{a''_m}\rho]) \otimes \sigma,$$

as claimed. This finishes the proof. □

5. Conditions 3 and 4—the second part of proof

In this section, we address the second part of the proof of conditions 3 and 4, the case where $\ell = 0$. The basic idea here is this: we assume $\delta_0(\pi)$ fails to satisfy these conditions, then show that π can be embedded into another induced representation, one which corresponds to a δ_0 lower than assumed.

Let us review where we stand with respect to the inductive hypothesis. Of course, we may assume Theorems 1.1 and 7.7 (and Corollaries 7.8 and 7.9) hold when the parabolic rank of the supercuspidal support is less than $p.r.$ In addition, we may assume that condition 1 of Theorem 1.1 holds in general (Theorem 2.4) and that condition 2 of Theorem 1.1 holds when the parabolic rank of the supercuspidal support is equal to *p.r.* (Theorem 3.7 and the inductive hypothesis). We are also free to assume that conditions 3 and 4 of Theorem 1.1 hold when the parabolic rank of the supercuspidal support is equal to *p.r.* and $\ell \geq 1$, but we do not actually use this in dealing with the case $\ell = 0$.

Again, by Lemma 3.2, Theorem 4.2.1 of [Jan4], and the inductive hypothesis, $[\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]$ satisfy the conditions of Theorem 1.1. Therefore, for $i > j \geq 2$, we must have either $a_i > b_i > a_j > b_j$ or $a_i > a_j > b_j > b_i$. However, to have $\ell = 0$, the former cannot occur. Thus, $a_i > a_j > b_j > b_i$ for $i, j \geq 2$. Therefore, $a_k > \cdots > a_2 > b_2 > \cdots > b_k$. Further, we must have $b_2 \leq a_1$.

By construction, we have $a_1 \leq a_2$. We now show that the inequality is strict.

Lemma 5.1. *$a_1 \neq a_2$.*

Proof. Suppose not. Let us use a_1 for both a_1 and a_2. Since

$$\begin{aligned}\pi \hookrightarrow\ & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \\ & \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\ \cong\ & \delta([\nu^{-b_2}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \\ & \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma,\end{aligned}$$

we see that in order for $\delta_0(\pi)$ to be minimal lexicographically, we must have $b_1 \geq b_2$.

First, suppose $b_1 < 0$. By Theorem 2.4, we have $b_1 \neq b_2$. Then,

$$\begin{aligned}\pi \hookrightarrow\ & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \\ & \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\ \cong\ & \delta([\nu^{-b_2}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \\ & \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\ \hookrightarrow\ & \delta([\nu^{-b_2}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_1+1}\rho, \nu^{a_1}\rho]) \times \nu^{-b_1}\rho \times \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \\ & \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\ \cong\ & \delta([\nu^{-b_2}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_1+1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \\ & \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \times \nu^{-b_1}\rho \rtimes \sigma \\ \cong\ & \delta([\nu^{-b_2}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_1+1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \\ & \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \times \nu^{b_1}\rho \rtimes \sigma,\end{aligned}$$

where $\nu^{-b_1}\rho \times \delta([\nu^{-b_i}\rho, \nu^{a_i}\rho])$ is irreducible for $i \geq 3$ since $-b_1 < -b_i - 1$. This contradicts the minimality of $\delta_0(\pi)$ (by Frobenius reciprocity and total exponent considerations). Thus, we are now free to assume $b_1 \geq 0$.

Next, we note that if $b_2 < 0$, then by Theorem 3.7 we have $b_1 \geq |b_2| - 1 = \beta - 1$. Thus (whether $b_2 < 0$ or $b_2 \geq 0$), Theorem 7.7 and the inductive hypothesis allow us to define

$$\pi_t = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t,$$

$$\pi'_t = \delta([\nu^{-a_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t,$$

$$\begin{aligned}\pi^*_t = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{b_1+1}\rho, \nu^{a_1}\rho]) \\ \rtimes \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t\end{aligned}$$

(noting that $[\nu^{-b_2}\rho, \nu^{b_1}\rho] = \emptyset$ is possible). By Corollary 7.8, we see that $\pi_t \leq \pi_t^*$. By Lemma 5.4, [BDK],

$$\pi_t^* = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-a_1}\rho, \nu^{-b_1-1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t,$$

so $\pi_t' \leq \pi_t^*$ as well. It follows fairly easily from Lemma 2.6 that

$$\begin{aligned}\mu^*_{\delta([\nu^{b_1+1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{b_1+1}\rho, \nu^{a_1}\rho])}(\pi_t^*) &= \mu^*_{\delta([\nu^{b_1+1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{b_1+1}\rho, \nu^{a_1}\rho])}(\pi_t') \\ &= \delta([\nu^{b_1+1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{b_1+1}\rho, \nu^{a_1}\rho]) \\ &\quad \otimes \delta([\nu^{-b_1}\rho, \nu^{b_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t.\end{aligned}$$

Since $\pi \hookrightarrow \pi_t$, we have $\mu^*_{\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_1}\rho])}(\pi) \neq 0$. Now,

$$\begin{gathered} m^*_{\delta([\nu^{b_1+1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{b_1+1}\rho, \nu^{a_1}\rho])}(\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_1}\rho])) \neq 0 \\ \Downarrow \\ \mu^*_{\delta([\nu^{b_1+1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{b_1+1}\rho, \nu^{a_1}\rho])}(\pi) \neq 0. \end{gathered}$$

From above, this implies $\pi \leq \pi_t'$. By unitarity, $\pi \hookrightarrow \pi_t'$, contradicting the square-integrability of π. Thus we could not have had $a_1 = a_2$, as claimed. □

If $b_1 > b_2$, we have $a_k > \cdots > a_2 > a_1 > b_1 > b_2 > \cdots > b_k$, which certainly satisfies conditions 3 and 4 of Theorem 1.1. So, let us assume $b_1 \leq b_2$. Then, $a_2 > a_1 \geq b_2 \geq b_1$. Consider the possibility that $b_2 = a_1$; write a_1 for both. Observe that

$$\begin{aligned} \pi \hookrightarrow\ & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \rtimes \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t \\ \cong\ & \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t \\ & \Downarrow \\ \pi \hookrightarrow\ & \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \rtimes \theta, \end{aligned}$$

for some irreducible $\theta \leq \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$ (Lemma 5.5, [Jan2]).

Now, consider $\mu^*_{\delta([\nu^{-a_1}\rho, \nu^{a_2}\rho])}(\pi)$. Write $\mu^*(\theta) = \sum_h \tau_h \otimes \theta_h$. Then,

$$\begin{aligned}\mu^*(\delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \rtimes \theta) = \sum_h \sum_{i=-a_1}^{a_2+1} \sum_{j=i}^{a_2+1} [&\delta([\nu^{-i+1}\rho, \nu^{a_1}\rho]) \\ &\times \delta([\nu^j\rho, \nu^{a_2}\rho]) \times \tau_h] \otimes \delta([\nu^i\rho, \nu^{j-1}\rho]) \rtimes \theta_h.\end{aligned}$$

To contribute to $\mu^*_{\delta([\nu^{-a_1}\rho, \nu^{a_2}\rho])}$, we must pick up a $\nu^{-a_1}\rho$. There are three possible sources. If the $\nu^{-a_1}\rho$ comes from $\delta([\nu^j\rho, \nu^{a_2}\rho])$, we must have $j = -a_1$. Then $i = -a_1$ and $\tau_h = 1$, giving a contribution of one copy of $\delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \otimes \theta$. Suppose $\delta([\nu^{-i+1}\rho, \nu^{a_1}\rho])$ contributes the $\nu^{-a_1}\rho$. Then, $i = a_1 + 1$. By Lemma 2.6, we then have $\tau_h = 1$. Therefore, $j = a_1 + 1$. Thus we get a contribution of one more copy of $\delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \otimes \theta$. Finally, suppose τ_h contributes the $\nu^{-a_1}\rho$. Since anything in $r_{min}(\tau_h)$ has the form $\nu^x\rho \otimes \ldots$ with $x \in \{b_1, b_3, \ldots, b_k, a_1, a_3, \ldots, a_k\}$, we see that $r_{min}(\tau_h)$ must contain something of the form $(\nu^x\rho \otimes \nu^{x-1}\rho \otimes \cdots \otimes \nu^{-a_1}\rho) \otimes \ldots$ for such an x. Further, since $a_3, \ldots, a_k > a_2$, we must have $x \leq a_1$. Now, we have that $s_{min}(\theta)$ contains something of the form $(\nu^x\rho \otimes \nu^{x-1}\rho \otimes \cdots \otimes \nu^{-a_1}\rho) \otimes \ldots$. Therefore, by central character considerations,

$$\theta \hookrightarrow \nu^x\rho \times \cdots \times \nu^{x-1}\rho \times \cdots \times \nu^{-a_1}\rho \rtimes \theta'$$

for some irreducible θ'. Then,

$$\begin{aligned} \pi \hookrightarrow &\quad \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \rtimes \theta \\ \hookrightarrow &\quad \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \nu^x\rho \times \nu^{x-1}\rho \times \cdots \times \nu^{-a_1}\rho \rtimes \theta' \\ \cong &\quad \nu^x\rho \times \nu^{x-1}\rho \times \cdots \times \nu^{-a_1}\rho \times \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \rtimes \theta'. \end{aligned}$$

However, by Frobenius reciprocity, this contradicts the square-integrability of π. Thus we cannot have τ_h contributing the $\nu^{-a_1}\rho$. It follows immediately that we must have

$$\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{a_2}\rho])}(\pi) = c \cdot \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \otimes \theta$$

for some $c \leq 2$.

On the other hand, from properties of the Jacquet functors, we know that there is an irreducible θ'' such that

1. $\mu^*(\pi) \geq \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \otimes \theta''$,
2. $s_{app}(\theta'') \geq \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma$.

From the above calculations, it follows that $\theta = \theta''$. We now claim $\delta_0(\theta) = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma$. As the argument is similar to the proof of Theorem 4.4 (but easier), we will be somewhat brief. First, it follows easily from $\theta = \theta''$ that $t.e.(\delta_0(\theta)) = t.e.(\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma)$. Suppose

$$\delta_0(\theta) = \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \otimes \delta([\nu^{-d_2}\rho, \nu^{c_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_h}\rho, \nu^{c_h}\rho]) \otimes \sigma$$

is lower than $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma$ lexicographically. Observe that $\delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho])$ is either irreducible or is equal to $\delta([\nu^{-a_1}\rho, \nu^{c_1}\rho]) \times \delta([\nu^{-d_1}\rho, \nu^{a_2}\rho]) + \mathcal{L}(\delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]), \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]))$. In any case, we get either

$$\begin{gathered} \pi \hookrightarrow \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \delta([\nu^{-d_2}\rho, \nu^{c_2}\rho]) \times \dots \\ \Downarrow \text{ (Lemma 5.5, [Jan2])} \\ \pi \hookrightarrow \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \delta([\nu^{-d_2}\rho, \nu^{c_2}\rho]) \times \dots \end{gathered}$$

or

$$\delta([\nu^{-a_1}\rho, \nu^{c_1}\rho]) \times \delta([\nu^{-d_1}\rho, \nu^{a_2}\rho]) \times \delta([\nu^{-d_2}\rho, \nu^{c_2}\rho]) \times \dots.$$

Whichever holds, if $[\nu^{-d_1}\rho, \nu^{c_1}\rho]$ is lexicographically lower than $[\nu^{-b_1}\rho, \nu^{a_1}\rho]$ we get a contradiction to the minimality of $\delta_0(\pi)$. Therefore, $d_1 = b_1$ and $c_1 = a_1$. But, in that case,

$$\begin{aligned} \pi \hookrightarrow &\quad \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-d_2}\rho, \nu^{c_2}\rho]) \times \dots \\ \cong &\quad \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \delta([\nu^{-d_2}\rho, \nu^{c_2}\rho]) \times \dots, \end{aligned}$$

so $\delta([\nu^{-d_2}\rho, \nu^{c_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_h}\rho, \nu^{c_h}\rho]) \otimes \sigma$ cannot be lexicographically lower than $\delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma$. The claim follows. By Theorem 4.2.1, [Jan4], θ is square-integrable; by the inductive hypothesis, the conditions of Theorem 1.1 must be satisfied. Since $b_3 < b_2 \leq a_1$ (from the assumption $\ell = 0$), we can easily conclude that $a_k > \cdots > a_3 > a_1 > b_1 > b_3 > \cdots > b_k$. We are now ready to prove the following:

Lemma 5.2. $a_1 > b_2$.

Proof. Suppose $b_2 = a_1$. Observe that

$$\begin{aligned}
\pi &\hookrightarrow \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \rtimes \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t \\
&\cong \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t \\
&\qquad \Downarrow \ \text{(Lemma 5.5, [Jan2])} \\
\pi &\hookrightarrow \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \rtimes \theta
\end{aligned}$$

for some irreducible $\theta \leq \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$. By Theorem 7.7 (and the inductive hypothesis), we know the possibilities for $\delta_0(\theta)$. Since $s_{app}(\pi) \geq \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \otimes \delta_0(\theta)$, we see that only $\theta = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_{t'}$ does not contradict the minimality of $\delta_0(\pi)$. Note that if $k > 2$, we have $t' = t$ automatically. (If $k = 2$, we could argue that $t' = t$, but it is not needed.)

Since $b_1 > b_3$, we may now define

$$\begin{aligned}
\pi'_t &= \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \rtimes \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_{t'}, \\
\pi''_t &= \delta([\nu^{-a_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_1}\rho, \nu^{a_2}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_{t'}, \\
\pi^*_t &= \delta([\nu^{-a_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{a_1+1}\rho, \nu^{a_2}\rho]) \\
&\qquad \rtimes \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_{t'}.
\end{aligned}$$

By Corollary 7.8 (and the inductive hypothesis), $\pi'_{t'}, \pi''_{t'} \leq \pi^*_{t'}$. By Lemma 2.6 (and Corollary 7.8)

$$\begin{aligned}
&\mu^*_{\delta([\nu^{a_1+1}\rho, \nu^{a_2}\rho])}(\pi'_{t'}) = \mu^*_{\delta([\nu^{a_1+1}\rho, \nu^{a_2}\rho])}(\pi''_{t'}) = \mu^*_{\delta([\nu^{a_1+1}\rho, \nu^{a_2}\rho])}(\pi^*_{t'}) \\
&\quad = \delta([\nu^{a_1+1}\rho, \nu^{a_2}\rho]) \otimes \delta([\nu^{-a_1}\rho, \nu^{a_1}\rho]) \\
&\qquad \rtimes \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_{t'}.
\end{aligned}$$

Now, $\mu^*_{\delta([\nu^{-a_1}\rho, \nu^{a_2}\rho])}(\pi) \neq 0$ implies $\mu^*_{\delta([\nu^{a_1+1}\rho, \nu^{a_2}\rho])}(\pi) \neq 0$. Therefore, $\pi \leq \pi''_{t'}$; by unitarity, $\pi \hookrightarrow \pi''_{t'}$. However, this contradicts the square-integrability of π (or minimality of $\delta_0(\pi)$). Thus, we could not have had $b_2 = a_1$. □

We now show the following:

Lemma 5.3. *Assuming $a_2 > a_1 > b_2 \geq b_1$, we have $b_1 \neq b_2$.*

Proof. Suppose $b_1 = b_2$. By Theorem 2.4, we must have $b_1 = b_2 \geq 0$.

As in the beginning of section 4 (the argument immediately preceding the statement of Lemma 4.2), if $\delta_0(\pi)$ is as assumed, we have

$$\pi \hookrightarrow \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_1}\rho, \nu^{a_2}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t,$$

for some t. By Theorem 7.7 and the inductive hypothesis, we may define

$$\begin{aligned}
\pi_t &= \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_1}\rho, \nu^{a_2}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t, \\
\pi'_t &= \delta([\nu^{-b_1}\rho, \nu^{b_1}\rho]) \rtimes \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t, \\
\pi^*_t &= \delta([\nu^{b_1+1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{b_1}\rho]) \\
&\qquad \rtimes \delta([\nu^{-b_1}\rho, \nu^{a_2}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t.
\end{aligned}$$

It follows easily from Corollary 7.8 that $\pi_t, \pi'_t \leq \pi^*_t$. Further, a straightforward calculation like that in Lemma 5.2.5, [Jan4] gives

$$\begin{aligned}\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}(\pi_t) &= \mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}(\pi'_t) = \mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}(\pi^*_t)\\ &= \delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \otimes \delta([\nu^{-b_1}\rho,\nu^{b_1}\rho])\\ &\quad \rtimes \delta([\nu^{-b_1}\rho,\nu^{a_2}\rho],[\nu^{-b_3}\rho,\nu^{a_3}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t.\end{aligned}$$

Now, $\pi \hookrightarrow \pi_t$ implies $\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}(\pi) \neq 0$. It now follows from the equalities on $\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}$ above that $\pi \leq \pi'_t$. By unitarity, $\pi \hookrightarrow \pi'_t$. However, by Frobenius reciprocity, this contradicts the minimality of $\delta_0(\pi)$. Thus we could not have had $b_1 = b_2$. □

Theorem 5.4. *Suppose* **(H)**. *Suppose that Theorems 1.1 and 7.7 (and Corollaries 7.8 and 7.9) are proved when the parabolic rank of the supercuspidal support is less than p.r. and that condition 2 of Theorem 1.1 is also proved when the parabolic rank of the supercuspidal support is equal to p.r. Then, if $\ell = 0$ (ℓ as in Proposition 4.1), $[\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_i}\rho,\nu^{a_i}\rho]$ satisfy conditions 3 and 4 of Theorem 1.1 for $i \geq 2$.*

Proof. It suffices to show $b_1 > b_2$; then Lemma 5.1 tells us $a_k > \cdots > a_2 > a_1 > b_1 > b_2 > \cdots > b_k$, which certainly satisfies conditions 3 and 4. Note that if $b_2 < 0$, one already has $b_1 > b_2$: if $b_1 \geq 0$ it is automatic; if $b_1 < 0$ it follows from Remark 4.4.3, [Jan4]. Thus, we assume $b_2 \geq 0$.

Suppose $b_2 \geq b_1$. By Lemma 5.3, $b_2 > b_1$. We have

$$\begin{aligned}\pi \hookrightarrow\ & \delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \times \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho]) \times \delta([\nu^{-b_3}\rho,\nu^{a_3}\rho])\\ & \quad \times \cdots \times \delta([\nu^{-b_k}\rho,\nu^{a_k}\rho]) \rtimes \sigma\\ \hookrightarrow\ & \delta([\nu^{b_2+1}\rho,\nu^{a_1}\rho]) \times \delta([\nu^{-b_1}\rho,\nu^{b_2}\rho]) \times \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho]) \times \delta([\nu^{-b_3}\rho,\nu^{a_3}\rho])\\ & \quad \times \cdots \times \delta([\nu^{-b_k}\rho,\nu^{a_k}\rho]) \rtimes \sigma.\end{aligned}$$

The same argument used to prove Lemma 3.2 tells us there is an irreducible π_0 with the following properties:

1. $\mu^*(\pi) \geq \delta([\nu^{b_2+1}\rho,\nu^{a_1}\rho]) \otimes \pi_0$,
2. $\delta_0(\pi_0) = \delta([\nu^{-b_1}\rho,\nu^{b_2}\rho]) \otimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho,\nu^{a_k}\rho]) \otimes \sigma$.

By Theorem 4.2.1, [Jan4], we see that π_0 is square-integrable. However, having $\delta_0(\pi_0)$ as above is then a contradiction to Lemma 5.2. Thus, we could not have had $b_1 \leq b_2$. □

6. A technical result

The purpose of this section is to prove a technical result (Proposition 6.3). It is essentially a generalization of Proposition 5.3.2, [Jan4] (cf. Remark 6.2 below).

Let us pause to recall where we are with respect to the inductive argument. By the inductive hypothesis, we may assume Theorem 7.7 (and Corollaries 7.8 and 7.9) hold when the parabolic rank of the supercuspidal support is less than *p.r.* The inductive hypothesis coupled with Theorems 2.4, 3.7, 4.4, and 5.4 allow us to assume Theorem 1.1 holds when the parabolic rank is less than or equal to *p.r.* (again noting condition 1 of Theorem 1.1 holds in general).

An easy calculation gives the following:

Lemma 6.1.

$$t.e.(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho])\otimes\cdots\otimes\delta([\nu^{-b_k}\rho,\nu^{a_k}\rho]))=\sum_{i=1}^{k}\frac{a_i(a_i+1)}{2}-\sum_{i=1}^{k}\frac{b_i(b_i+1)}{2}.$$

Remark 6.2. In the case where $\alpha\leq\frac{1}{2}$, the following proposition is an immediate consequence of Theorem 2.4 and Lemma 3.1. (In fact, in the notation of the proposition, one can take weaker conditions on the segment ends and still get the stronger result that $h=k$.)

Proposition 6.3. *Suppose $a_k>\cdots>a_1>b_1>\cdots>b_k$ satisfy conditions 1–4 of Theorem 1.1. Suppose*

$$\pi\leq\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho])\rtimes\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$

is irreducible. Write

$$\delta_0(\pi)=\delta([\nu^{-d_1}\rho,\nu^{c_1}\rho])\otimes\cdots\otimes\delta([\nu^{-d_h}\rho,\nu^{c_h}\rho])\otimes\sigma.$$

Then, $h\leq k$.

Proof. First, suppose $b_1<0$. We start by showing that $n'_-(c)\leq n_-(c)$ for all $c>0$. Observe that

$$n_-(c)=\begin{cases}0 & \text{if } c<\beta,\\ 1 & \text{if } \beta\leq c\leq\alpha,\\ 0 & \text{if } c>\alpha.\end{cases}$$

If $c<\beta$, then $n'_-(c)=0$ since $\nu^{\pm c}$ does not appear in the supercuspidal support. If $c=\beta=\frac{1}{2}$, the fact that $\nu^{\pm\frac{1}{2}}\rho$ appears only once in the supercuspidal support implies $n'_-(\frac{1}{2})\leq 1$. If $c=\beta\geq 1$, Lemma 3.1 gives

$$n'_+(\beta-1)-n'_-(\beta)=n_+(\beta-1)-n_-(\beta).$$

Since $n'_+(\beta-1)=n_+(\beta-1)=0$ (again, $\nu^{\pm(\beta-1)}\rho$ does not appear in the supercuspidal support), we have $n'_-(\beta)=n_-(\beta)=1$. Thus $n'_-(c)\leq n_-(c)$ for $c=\beta$. For $\beta+1\leq c\leq\alpha$, Lemma 4.4.2, [Jan4] and the fact that there is nothing of the form $\nu^{-c}\rho\otimes\ldots$ in $s_{min}(\delta([\nu^{-\beta+1}\rho,\nu^{a_2}\rho],\ldots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma))$ tells us $n'_-(c)\leq 1$. When $c>\alpha$, Lemma 4.4.1, [Jan4] implies $n'_-(c)=0$. Combining these observations, we see that $n'_-(c)\leq n_-(c)$ for $c\geq\frac{1}{2}$.

It now follows from Lemma 3.1 that $n'_+(c)\leq n_+(c)$ for $c\geq 0$. Since $n'_\pm(c)\leq n_\pm(c)$ for all c, we see that $h\leq k$. This finishes the case $b_1<0$.

We may now assume $b_1\geq 0$. Suppose $h>k$. First, since there are no terms of the form $\nu^x\rho\otimes\ldots$ in $s_{min}(\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)$ with $x\leq 0$ (by the Casselman criteria), we can conclude that if $\nu^x\rho\otimes\ldots$ appears in $s_{min}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho])\rtimes\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)$, then $x\in\{b_1,\ldots,b_k,a_1,\ldots,a_k\}\setminus\{-\beta,-\beta-1,\ldots,-\alpha\}$. Therefore, $c_1\geq 0$. Further, since the non-negative values of b_i are all greater than or equal to $\beta-1$ (cf. Theorem 3.7), we also have $c_1\geq\beta-1$. Since the c_i's are all non-negative, Lemma 4.4.2, [Jan4] tells us the negative values in $\{d_1,\ldots,d_k,c_1,\ldots,c_k\}$ (all d_i's) may be written as $\{-\beta',-\beta'-1,\ldots,-\alpha\}$. If $\beta'\geq\beta$, then $n'_-(c)\leq n_-(c)$ for all $c>0$. By Lemma 3.1, $n'_+(c)\leq n_+(c)$ for all $c\geq 0$. This forces $h\leq k$. Thus, we must have $\beta'<\beta$. In this case, for $\beta'\leq c<\beta$, we have $n'_-(c)=1$ and $n_-(c)=0$. Therefore,

$n'_+(c) = n_+(c) + 1$ for $\beta' - 1 \leq c < \beta - 1$ with $c \geq 0$; it follows easily from Lemma 3.1 that $n'_+(c) = n_+(c)$ for all other values of c. Therefore, $h - k = \beta' - \beta$, so

$$\begin{aligned}\{c_1, \ldots, c_h, d_1, \ldots, d_h\} = \{b_1, \ldots, b_k, a_1, \ldots, a_k\} \\ \cup \{-\beta + 1, -\beta + 2, \ldots, -\beta + (h - k)\} \\ \cup \{\beta - 2, \beta - 3, \ldots, \beta - (h - k) - 1\}.\end{aligned}$$

Observe that x in the last set has $0 \leq x \leq \beta - 2$, b_i for any $b_i > 0$. So $d_1, \ldots, d_h$, $c_1, \ldots, c_h$ are distinct. Note that if β is not large enough, this automatically cannot occur. We also remark that the arguments above imply

$$|\{c_i\}| \leq |\{b_1, \ldots, b_k, a_1, \ldots, a_k\} \setminus \{-\beta, -\beta - 1, \ldots, -\alpha\}|,$$

or $h - k \leq |\{i | b_i \geq 0\}|$.

Claim 1: $c_h = a_k$ when $k \geq 2$.

Of course, if $a_k = c_i$ for some i, we must have $i = h$. So, suppose $a_k = d_m$. Then, since $[\nu^{-a_k}\rho, \nu^{c_m}\rho] \supset [\nu^{-d_i}\rho, \nu^{c_i}\rho]$ for all $i < m$, we can commute $\delta([\nu^{-a_k}\rho, \nu^{c_m}\rho])$ to the left:

$$\begin{aligned}\pi \hookrightarrow\ & \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{m-1}}\rho, \nu^{c_{m-1}}\rho]) \times \delta([\nu^{-a_k}\rho, \nu^{c_m}\rho]) \\ & \times \delta([\nu^{-d_{m+1}}\rho, \nu^{c_{m+1}}\rho]) \times \cdots \times \delta([\nu^{-d_h}\rho, \nu^{c_h}\rho]) \rtimes \sigma \\ \cong\ & \delta([\nu^{-a_k}\rho, \nu^{c_m}\rho]) \times \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{m-1}}\rho, \nu^{c_{m-1}}\rho]) \\ & \times \delta([\nu^{-d_{m+1}}\rho, \nu^{c_{m+1}}\rho]) \times \cdots \times \delta([\nu^{-d_h}\rho, \nu^{c_h}\rho]) \rtimes \sigma \\ & \Downarrow \\ & \mu^*_{\delta([\nu^{-a_k}\rho, \nu^{c_m}\rho])}(\pi) \neq 0.\end{aligned}$$

Now,

$$\begin{aligned}\mu^*(\pi) &\leq \mu^*(\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_{k-1}}\rho, \nu^{a_{k-1}}\rho]) \rtimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) \\ &= M^*(\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_{k-1}}\rho, \nu^{a_{k-1}}\rho])) \rtimes \mu^*(\delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t).\end{aligned}$$

Now, $\nu^{-a_k}\rho$ does not appear in $M^*(\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_{k-1}}\rho, \nu^{a_{k-1}}\rho]))$. Therefore, to have $\mu^*_{\delta([\nu^{-a_k}\rho, \nu^{c_m}\rho])}(\pi) \neq 0$, we must have

$$\mu^*_{\delta([\nu^{-a_k}\rho, \nu^{x}\rho])}(\delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) \neq 0$$

for some $x \geq -a_k$. However, since $x \leq b_k$, this contradicts the Casselman criteria for the square-integrability of $\delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$. Thus we could not have had $a_k = d_m$, and the claim follows.

Claim 2: $c_{m+h-k} = a_m$ for $m \geq 2$.

Of course, the first claim takes care of the case $m = k$.

Suppose Claim 2 does not hold. Let m be the largest value for which it fails. Then, $c_{m+h-k+1} = a_{m+1}, \ldots, c_h = a_k$. As above, if $a_m = c_i$ for some i, we would need to have $i = m$. So, we have $a_m = d_r$ for some r.

As above,

$$\begin{aligned}\pi \hookrightarrow\ & \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{r-1}}\rho, \nu^{c_{r-1}}\rho]) \times \delta([\nu^{-a_m}\rho, \nu^{c_r}\rho]) \\ & \times \delta([\nu^{-d_{r+1}}\rho, \nu^{c_{r+1}}\rho]) \times \cdots \times \delta([\nu^{-d_h}\rho, \nu^{c_h}\rho]) \rtimes \sigma \\ \cong\ & \delta([\nu^{-a_m}\rho, \nu^{c_r}\rho]) \times \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{r-1}}\rho, \nu^{c_{r-1}}\rho]) \\ & \times \delta([\nu^{-d_{r+1}}\rho, \nu^{c_{r+1}}\rho]) \times \cdots \times \delta([\nu^{-d_h}\rho, \nu^{c_h}\rho]) \rtimes \sigma \\ & \Downarrow \\ & \mu^*_{\delta([\nu^{-a_m}\rho, \nu^{c_r}\rho])}(\pi) \neq 0.\end{aligned}$$

Now, noting that $m \geq 2$,

$$\begin{aligned}\mu^*(\pi) \leq \quad & \mu^*(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_{m-1}}\rho,\nu^{a_{m-1}}\rho]) \\ & \qquad \rtimes \delta([\nu^{-b_m}\rho,\nu^{a_m}\rho],\ldots[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t) \\ & = M^*(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_{k-1}}\rho,\nu^{a_{k-1}}\rho])) \\ & \quad \rtimes \mu^*(\delta([\nu^{-b_m}\rho,\nu^{a_m}\rho],\ldots[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t).\end{aligned}$$

As in Claim 1, since $\mu^*_{\delta([\nu^{-a_m}\rho,\nu^{c_r}\rho])}(\pi) \neq 0$, we must have

$$\mu^*_{\delta([\nu^{-a_m}\rho,\nu^{x}\rho])}(\delta([\nu^{-b_m}\rho,\nu^{a_m}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t) \neq 0,$$

for some $x \geq -a_m$. In order to avoid contradicting the Casselman criteria for the square-integrability of $\delta([\nu^{-b_m}\rho,\nu^{a_m}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$, we must have $x > a_m$. Since any term in $r_{min}(\delta([\nu^{-b_m}\rho,\nu^{a_m}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)$ has the form $\nu^x\rho\otimes\ldots$ with $x \in \{b_m,\ldots,b_k,a_m,\ldots,a_k\}$, we see that $x = a_s$ for $s \geq m+1$.

Finally,

$$\begin{aligned}&\mu^*(\delta([\nu^{-b_m}\rho,\nu^{a_m}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t) \\ &\qquad \leq \mu^*(\delta([\nu^{-b_m}\rho,\nu^{a_m}\rho]) \times \cdots \times \delta([\nu^{-b_{s-2}}\rho,\nu^{a_{s-2}}\rho]) \\ &\qquad\qquad \rtimes \delta([\nu^{-b_{s-1}}\rho,\nu^{a_{s-1}}\rho],\ldots[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t) \\ &\qquad = M^*(\delta([\nu^{-b_m}\rho,\nu^{a_m}\rho]) \times \cdots \times \delta([\nu^{-b_{s-2}}\rho,\nu^{a_{s-2}}\rho])) \\ &\qquad\qquad \rtimes \mu^*(\delta([\nu^{-b_{s-1}}\rho,\nu^{a_{s-1}}\rho],\ldots[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t).\end{aligned}$$

Since $b_m,\ldots,b_{s-2},a_m,\ldots,a_{s-2} < a_{s-2}+1$, in order to have $\mu^*_{\delta([\nu^{-a_m}\rho,\nu^{a_s}\rho])} \neq 0$, we must have

$$\begin{gathered}\mu^*_{\delta([\nu^{a_{s-2}+1}\rho,\nu^{a_s}\rho])}(\delta([\nu^{-b_{s-1}}\rho,\nu^{a_{s-1}}\rho],\ldots[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t) \neq 0, \\ \Downarrow \\ \mu^*_{\delta([\nu^{a_{s-1}+1}\rho,\nu^{a_s}\rho])}(\delta([\nu^{-b_{s-1}}\rho,\nu^{a_{s-1}}\rho],\ldots[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t) \neq 0.\end{gathered}$$

However, by the inductive hypothesis, this contradicts Corollary 7.9. Thus Claim 2 holds.

<u>Claim 3:</u> $a_1 \neq d_i$ for any i.

Suppose this were not the case—say $d_r = a_1$. As before,

$$\begin{aligned}\pi \hookrightarrow \quad & \delta([\nu^{-d_1}\rho,\nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{r-1}}\rho,\nu^{c_{r-1}}\rho]) \times \delta([\nu^{-a_1}\rho,\nu^{c_r}\rho]) \\ & \quad \times \delta([\nu^{-d_{r+1}}\rho,\nu^{c_{r+1}}\rho]) \times \cdots \times \delta([\nu^{-d_h}\rho,\nu^{c_h}\rho]) \rtimes \sigma \\ \cong \quad & \delta([\nu^{-a_1}\rho,\nu^{c_r}\rho]) \times \delta([\nu^{-d_1}\rho,\nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{r-1}}\rho,\nu^{c_{r-1}}\rho]) \\ & \quad \times \delta([\nu^{-d_{r+1}}\rho,\nu^{c_{r+1}}\rho]) \times \cdots \times \delta([\nu^{-d_h}\rho,\nu^{c_h}\rho]) \rtimes \sigma \\ & \qquad\qquad \Downarrow \\ & \qquad \mu^*_{\delta([\nu^{-a_1}\rho,\nu^{c_r}\rho])}(\pi) \neq 0.\end{aligned}$$

Now, $c_r \in \{b_1,\ldots,b_k,a_2,\ldots,a_k\}$. Suppose $c_r \leq b_2$. Then, $c_r \leq b_1, a_1$. Therefore, since

$$\mu^*(\pi) \leq \mu^*(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t),$$

it follows from Lemma 2.6 that

$$\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{c_r}\rho])}(\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t) \neq 0,$$

which is in contradiction to the Casselman criteria for the square-integrability of $\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$. Thus $c_r > b_2$.

Next, suppose $c_r = b_1$. Then, $\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{b_1}\rho])}(\pi) \neq 0$. A straightforward calculation (or Lemma 3.4, [Jan2]) tells us

$$\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{b_1}\rho])}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t) \\ = \delta([\nu^{-a_1}\rho,\nu^{b_1}\rho]) \otimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t.$$

Therefore,

$$\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{b_1}\rho])}(\pi) = \delta([\nu^{-a_1}\rho,\nu^{b_1}\rho]) \otimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t.$$

We claim that this contradicts the minimality of $\delta_0(\pi)$; in particular, that it has $t.e.$ lower than $t.e.(\delta_0(\pi))$. To see this, observe that by Lemma 6.1, $a_1, a_2, \dots, a_k$ contribute the same amount to $t.e.(\delta([\nu^{-a_1}\rho,\nu^{b_1}\rho]) \otimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho]) \otimes \dots \otimes \delta([\nu^{-b_k}\rho,\nu^{a_k}\rho]))$ as to $\delta_0(\pi)$ (since $a_m = c_{m+h-k}$ for $m \geq 2$ appear as upper segment ends in both and $a_1 = d_r$ appears as a lower segment end in both). The same holds for $b_1 = c_r$, which is an upper segment end in both. Denote the total contribution of these terms to the $t.e.$ for both by T. Let $N = \{-\beta+1, -\beta+2, \dots, -\beta+(h-k)\} \cup \{\beta-2, \beta-3, \dots, \beta-(h-k)-1\}$. Also, let $L = \{i \geq 2 | b_i \text{ appears as } d_j \text{ for some } j\}$ and $U = \{i \geq 2 | b_i \text{ appears as } c_j \text{ for some } j\}$. Then comparing $t.e.$, we want

$$T - \sum_{i\in L\cup U} \frac{b_i(b_i+1)}{2} < T + \sum_{i\in U} \frac{b_i(b_i+1)}{2} - \sum_{i\in L} \frac{b_i(b_i+1)}{2} - \sum_{x\in N} \frac{x(x+1)}{2}$$

or

$$2\sum_{i\in U} \frac{b_i(b_i+1)}{2} > \sum_{x\in N} \frac{x(x+1)}{2}.$$

An easy calculation gives $|N| = 2|U|$. Since $b_i \geq x$ for all $i \in U$ and $x \in N$ with at most one equality, the $t.e.$ claim holds. Since this contradicts the minimality of $\delta_0(\pi)$, we could not have had $c_r = b_1$. Thus, $c_r = a_i$ for some $i \geq 2$. By Lemma 3.2, there is an irreducible θ such that

1. $s_{app}(\theta) = \delta([\nu^{-d_1}\rho,\nu^{c_1}\rho]) \otimes \dots \otimes \delta([\nu^{-d_{h-k}}\rho,\nu^{c_{h-k}}\rho]) \otimes \theta$.
2. $\delta_0(\theta) = \delta([\nu^{-d_{h-k+1}}\rho,\nu^{a_2}\rho]) \otimes \dots \otimes \delta([\nu^{-d_h}\rho,\nu^{a_k}\rho]) \otimes \sigma$.

By Theorem 4.2.1, [Jan4], θ is square-integrable. Therefore, by the inductive hypothesis, $[\nu^{-d_{h-k+1}}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_h}\rho,\nu^{a_k}\rho]$ must satisfy condition 4 of Theorem 1.1. In particular, this forces $a_k > \dots > a_2 > d_{h-k+1} > \dots > d_h$. Thus, $c_r = a_2$.

Next, we show that

$$\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{a_2}\rho])}(\pi) = \delta([\nu^{-a_1}\rho,\nu^{a_2}\rho]) \\ \otimes \delta([\nu^{-b_2}\rho,\nu^{b_1}\rho],[\nu^{-b_3}\rho,\nu^{a_3}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t.$$

To do this, we first show $\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{a_2}\rho])}(\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t) = 0$. Write $\mu^*(\delta([\nu^{-b_3}\rho,\nu^{a_3}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t) = \sum_f \tau_f \otimes \theta_f$. Then,

$$\mu^*(\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho]) \rtimes \delta([\nu^{-b_3}\rho,\nu^{a_3}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t) \\ = \sum_f \sum_{i=-b_2}^{a_2+1} \sum_{j=i}^{a_2+1} \delta([\nu^{-i+1}\rho,\nu^{b_2}\rho]) \times \delta([\nu^{j}\rho,\nu^{a_2}\rho]) \times \tau_f \otimes \delta([\nu^{i}\rho,\nu^{j-1}\rho]) \rtimes \theta_f.$$

Observe that for $\tau_f \neq 1$, any term in $r_{min}(\tau_f)$ has the form $\nu^x\rho \otimes \dots$ with $x \in \{b_3, \dots, b_k, a_3, \dots, a_k\}$. Since none of these is between b_2 and a_2, to contribute to $\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{a_2}\rho])}$, we must have $j \leq b_2 + 1$. Therefore, $i \leq b_2 + 1$. In particular,

$-i+1, j \geq -b_2 > -a_1$, which implies that the copies of $\nu^{-a_1}\rho, \ldots, \nu^{-b_2-1}\rho$ required, if we are to have $\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{a_2}\rho])} \neq 0$, must come from τ_f. Thus,

$$\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{x}\rho])}(\delta([\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) \neq 0$$

for some $a_2 \geq x \geq -b_2-1$. Again, $x \in \{b_3, \ldots, b_k, a_3, \ldots, a_k\}$; certainly, we cannot have $x \in \{a_3, \ldots, a_k\}$. If $x \in \{b_3, \ldots, b_k\}$, then this is in violation of the Casselman criteria for the square-integrability of $\delta([\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$. Thus, we must have

$$\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{a_2}\rho])}(\delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) = 0,$$

as claimed.

Now we are ready to check that

$$\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{a_2}\rho])}(\pi) = \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \otimes \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t.$$

Recycling the notation from the preceding paragraph in a slightly different context, write $\mu^*(\delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) = \sum_f \tau_f \otimes \theta_f$. Then,

$$\mu^*(\pi) \leq \sum_f \sum_{i=-b_1}^{a_1+1} \sum_{j=i}^{a_1+1} \delta([\nu^{-i+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{j}\rho, \nu^{a_1}\rho]) \times \tau_f \otimes \delta([\nu^{i}\rho, \nu^{j-1}\rho]) \rtimes \theta_f.$$

To calculate the contribution to $\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{a_2}\rho])}$, we focus on the $\nu^{-a_1}\rho$ which must appear. Now, τ_f cannot contribute a copy of $\nu^{-a_1}\rho$. If it did, in order to avoid contradicting the Casselman criteria, we would need to have $\tau_f = \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho])$. However, we just showed that this is not the case. Therefore, to have a copy of $\nu^{-a_1}\rho$, we must have $i = a_1+1$. This implies $\tau_h = \delta([\nu^{b_1+1}\rho, \nu^{a_2}\rho])$. By Corollary 7.8 and the inductive hypothesis, we then have $\theta_h = \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$. Thus $\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{a_2}\rho])}(\pi)$ is as claimed.

In a moment, we shall show b_1 is not one of the d_i's. Once we have established this, we are done: $t.e.(\delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \otimes \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma) < t.e.(\delta_0(\pi))$ as above. Thus we get a contradiction; Claim 3 then follows immediately.

First, suppose $b_1 = d_i$ with $i < h-k+1$. Then, the usual commuting argument tells us $\mu^*_{\delta([\nu^{-b_1}\rho,\nu^{c_i}\rho])}(\pi) \neq 0$. Since $c_i < b_1$, Lemma 2.6 implies

$$\mu^*_{\delta([\nu^{-b_1}\rho,\nu^{c_i}\rho])}(\delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) \neq 0,$$

contradicting the Casselman criteria. Of course, we cannot have $b_1 = d_{h-k+1}$ (it is a_1). Suppose $b_1 = d_i$ for $i > h-k+1$. It then follows easily from Lemma 3.1, Theorem 4.2.1, [Jan4], and the inductive hypothesis that $d_{h-k+2} > \cdots > d_h$. Therefore, $b_1 = d_{h-k+2}$. Note that $c_{h-k+2} = a_3$. Then, since $[\nu^{-b_1}\rho, \nu^{a_3}\rho], [\nu^{-a_1}\rho, \nu^{a_2}\rho] \supset$

$[\nu^{-d_j}\rho, \nu^{c_j}\rho]$ for $j < h-k+1$, a commuting argument gives

$$\begin{aligned}
\pi \hookrightarrow \quad & \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{h-k}}\rho, \nu^{c_{h-k}}\rho]) \times \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \\
& \times \delta([\nu^{-b_1}\rho, \nu^{a_3}\rho]) \times \delta([\nu^{-d_{h-k+3}}\rho, \nu^{a_4}\rho]) \times \dots \\
\cong \quad & \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \delta([\nu^{-b_1}\rho, \nu^{a_3}\rho]) \times \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \\
& \times \cdots \times \delta([\nu^{-d_{h-k}}\rho, \nu^{c_{h-k}}\rho]) \times \delta([\nu^{-d_{h-k+3}}\rho, \nu^{a_4}\rho]) \times \dots \\
& \Downarrow \\
& s_{\delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \otimes \delta([\nu^{-b_1}\rho, \nu^{a_3}\rho])}(\pi) \neq 0 \\
& \Downarrow \\
& \mu^*_{\delta([\nu^{-b_1}\rho, \nu^{a_3}\rho])}(\delta([\nu^{-b_2}\rho, \nu^{b_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \dots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) \neq 0,
\end{aligned}$$

contradicting Corollary 7.9. Thus, we do not have $b_1 = d_i$ for any i, finishing Claim 3.

<u>Claim 4:</u> a_1 does not appear as a c_i.

Suppose a_1 did appear as c_i for some i. Then, $i = h-k+1$.

We begin by showing that $b_1 = c_{h-k}$. Since the argument is fairly similar to the argument that $a_1 = c_{h-k+2}$ (Claim 3), we will be somewhat sketchy here. First, we argue that $b_1 \neq d_{h-k+1}, \dots, d_h$. As above, since $d_{h-k+1} > \cdots > d_h$, if b_1 were one of these, it would have to be d_{h-k+1}. Then, the usual commuting argument would tell us $\mu^*_{\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho])}(\pi) \neq 0$. However, the same argument as in the proof of Lemma 4.2 then tells us

$$\begin{aligned}
\mu^*_{\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho])}(\pi) \leq \quad & \mu^*_{\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho])}(\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \\
& \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \dots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) \\
= \quad & c \cdot \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \dots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t.
\end{aligned}$$

As in the proof of Claim 3 above, we have $t.e.(\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma) < t.e.(\delta_0(\pi))$, contradicting the minimality of $\delta_0(\pi)$. Thus we cannot have $b_1 = d_{h-k+1}, \dots, d_h$. Next, we argue that $b_1 \neq d_1, \dots, d_{h-k}$. If we had $b_1 = d_i$ with $i \leq h-k$, the usual commuting argument would tell us $\mu^*_{\delta([\nu^{-b_1}\rho, \nu^{c_i}\rho])}(\pi) \neq 0$. Note that $c_i = b_j$ for some $j \geq 2$. But, by Lemma 2.6, this requires

$$\mu^*_{\delta([\nu^{-b_1}\rho, \nu^{b_j}\rho])}(\delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \dots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) \neq 0,$$

in contradiction to the Casselman criteria. Therefore, $b_1 \neq d_1, \dots, d_{h-k}$. It follows that $b_1 = c_{h-k}$, as needed.

We now finish the proof of Claim 4 in three cases. For the first case, suppose $h = k+1$ and $b_2 \geq 0$ (since $\beta \geq 2$, we must then have $b_2 > 0$). Then,

$$\delta_0(\pi) = \delta([\nu^{-d_1}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{-d_2}\rho, \nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_{k+1}}\rho, \nu^{a_k}\rho]) \otimes \sigma.$$

By Theorem 4.2.1, [Jan4], this π is square-integrable. By the results of the preceding two sections, we know that conditions 3 and 4 of Theorem 1.1 must be satisfied. Since $b_1 > max(d_1, \dots, d_{k+1})$, the only way this can happen is if $d_1 > \cdots > d_{k+1}$. Since $b_2 \geq 0$, we have $b_2 \geq d_1, \dots, d_{k+1}$. Therefore, $b_2 = d_1$. Since $\pi \hookrightarrow \mathrm{Ind}(\delta_0(\pi))$, it follows that easily $s_{\delta([\nu^{-b_2}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{b_2+1}\rho, \nu^{a_1}\rho])}(\pi) \neq 0$. Let us start by considering $\mu^*_{\delta([\nu^{-b_2}\rho, \nu^{b_1}\rho])}(\pi)$. Writing

$$\mu^*(\pi) \leq \sum_f \sum_{i=-b_1}^{a_1+1} \sum_{j=i}^{a_1+1} \delta([\nu^{-i+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^j\rho, \nu^{a_1}\rho]) \times \tau_f \otimes \delta([\nu^i\rho, \nu^{j-1}\rho]) \rtimes \theta_f$$

as in Claim 3 above, we see that to contribute to $\mu^*_{\delta([\nu^{-b_2}\rho,\nu^{b_1}\rho])}(\pi)$, we must have $j = a_1 + 1$. Therefore, $\tau_f = \delta([\nu^{-b_2}\rho, \nu^{-i-1}\rho])$. If $\tau_f \neq 1$, we must have $b_2 < -i-1 \leq b_1$ (to avoid contradicting the Casselman criteria for $\delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$). Since this cannot happen, we have $\tau_f = 1$, $i = b_2 + 1$, and

$$\begin{gathered}\mu^*_{\delta([\nu^{-b_2}\rho,\nu^{b_1}\rho])}(\pi) \leq \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho]) \\ \otimes \delta([\nu^{b_2+1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t \\ \Downarrow \\ s_{\delta([\nu^{-b_2}\rho,\nu^{b_1}\rho])\otimes\delta([\nu^{b_2+1}\rho,\nu^{a_1}\rho])}(\pi) = \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho]) \\ \otimes \delta([\nu^{b_2+1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t.\end{gathered}$$

This, as above, gives rise to a *t.e.* lower than that of $\delta_0(\pi)$, a contradiction. Thus we have eliminated this case.

The case $h = k+1$ with $b_2 < 0$ is covered by Lemma 6.5 (note that in this case, $\beta = \alpha - k + 2 > 1$). The case $h = k + 1$ and no b_2 (i.e., $k = 1$) is also covered by Lemma 6.5.

The last case is $h > k + 1$ (which forces $\beta > 2$ and $b_2 > 0$). We first argue that $b_2 = c_{h-k-1}$. If b_2 were one of $d_{h-k}, \ldots, d_h$, it would have to be d_{h-k}. We could then commute $\delta([\nu^{-b_2}\rho, \nu^{b_1}\rho])$ forward as above to conclude that $\mu^*_{\delta([\nu^{-b_2}\rho,\nu^{b_1}\rho])}(\pi) \neq 0$. However, we have already ruled out this possibility in our discussion of the case $b_2 \geq 0$ with $h = k + 1$. Having b_2 equal to one of $d_1, \ldots, d_{h-1}$ would require $\mu^*_{\delta([\nu^{-b_2}\rho,\nu^{c_i}\rho])}(\delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) \neq 0$, in contradiction to the Casselman criteria. Therefore, $b_2 = c_{h-k-1}$.

Next, arguing as above, we see that either $b_3 = c_{h-k-2}$ or d_{h-k-1}. If $b_3 = d_{h-k-1}$, we take $i = 2$. Otherwise, we look at b_4; in this case, either $b_4 = c_{h-k-3}$ or d_{h-k-2}. If $b_4 = d_{h-k-2}$, we set $i = 3$; otherwise we continue iteratively. Eventually, one of the following happens:

(a) there is an $i \geq 2$ such that $\delta([\nu^{-b_{i+1}}\rho, \nu^{b_i}\rho])$, $b_{i+1} > 0$ appears in $\delta_0(\pi)$,

(b) every $b_i > 0$ appears as a c_j.

In case (a), the usual commuting argument tells us $\mu^*_{\delta([\nu^{-b_{i+1}}\rho,\nu^{b_i}\rho])}(\pi) \neq 0$. However, since $b_{i-1} > b_i > b_{i+1}$, we also get (noting that $\mu^*_{\delta([\nu^{-b_{i+1}}\rho,\nu^{b_i}\rho])}(\pi) \neq 0$ implies $\mu^*_{\delta([\nu^{b_{i+1}+1}\rho,\nu^{b_i}\rho])}(\pi) \neq 0$)

$$\begin{gathered}\mu^*_{\delta([\nu^{b_{i+1}+1}\rho,\nu^{b_i}\rho])}(\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_{i-1}}\rho, \nu^{a_{i-1}}\rho]) \\ \rtimes \delta([\nu^{-b_i}\rho, \nu^{a_i}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) \quad \neq 0 \\ \Downarrow \\ \mu^*_{\delta([\nu^{b_{i+1}+1}\rho,\nu^{b_i}\rho])}(\delta([\nu^{-b_i}\rho, \nu^{a_i}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) \neq 0\end{gathered}$$

by Lemma 2.6. However, by Corollary 7.9 and the inductive hypothesis, this is not the case. Thus, we can rule out (a). If (b) occurs,

$$\begin{gathered}\delta_0(\pi) = \delta([\nu^{-d_1}\rho, \nu^{b_i}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_i}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{-d_{i+1}}\rho, \nu^{a_1}\rho]) \\ \otimes \cdots \otimes \delta([\nu^{-d_h}\rho, \nu^{a_k}\rho]) \otimes \sigma.\end{gathered}$$

By Theorem 4.2.1, [Jan4], π is square-integrable. Therefore, by the previous section, we know that conditions 3 and 4 of Theorem 1.1 hold. Again, since $a_1, \ldots, a_k, b_1, \ldots, b_i \geq \beta - 1 > d_1, \ldots, d_h$, the only way for this to happen is to have $d_1 > \cdots > d_h$. Therefore, $d_1 = \beta - 2$. Corollary 6.7 below then finishes the proof. ☐

Lemma 6.4. *With notation as above, suppose $b_1 \geq 0$ and $\alpha - k + 2 > 1$. Let*

$$\pi' = \delta([\nu^{-\alpha+k}\rho, \nu^{b_1}\rho]) \rtimes \delta([\nu^{\alpha-k+1}\rho, \nu^{a_1}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma),$$
$$\pi'' = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)$$

(if $k = 1$, we take $\delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma) = \sigma$). Then, there is a unique irreducible representation π_0 such that both of the following conditions hold: $\pi_0 \leq \pi', \pi''$ and $\mu^*_{\delta([\nu^{\alpha-k+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{\alpha-k+1}\rho, \nu^{a_1}\rho])}(\pi_0) \neq 0$. Further, $\pi_0 \hookrightarrow \pi', \pi''$.

Proof. Let

$$\begin{aligned} \pi^* = \quad & \delta([\nu^{-\alpha+k}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{\alpha-k+1}\rho, \nu^{a_1}\rho]) \\ & \qquad \rtimes \delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma) \\ = \quad & \delta([\nu^{-b_1}\rho, \nu^{\alpha-k}\rho]) \times \delta([\nu^{\alpha-k+1}\rho, \nu^{a_1}\rho]) \\ & \qquad \rtimes \delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma). \end{aligned}$$

Then $\pi', \pi'' \leq \pi^*$. We claim

$$\begin{aligned} & \mu^*_{\delta([\nu^{\alpha-k+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{\alpha-k+1}\rho, \nu^{a_1}\rho])}(\pi') \\ & \quad = \mu^*_{\delta([\nu^{\alpha-k+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{\alpha-k+1}\rho, \nu^{a_1}\rho])}(\pi'') \\ & \quad = \mu^*_{\delta([\nu^{\alpha-k+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{\alpha-k+1}\rho, \nu^{a_1}\rho])}(\pi^*) \\ & \quad = \delta([\nu^{\alpha-k+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{\alpha-k+1}\rho, \nu^{a_1}\rho]) \\ & \qquad \otimes \delta([\nu^{-\alpha+k}\rho, \nu^{\alpha-k}\rho]) \rtimes \delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma). \end{aligned}$$

The argument is similar to the calculations in Lemma 5.2.5, [Jan4] (and follows from Lemma 2.6 for π'' and π^*); we omit the details.

Next, we claim $\delta([\nu^{-\alpha+k}\rho, \nu^{\alpha-k}\rho]) \rtimes \delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)$ is irreducible. Suppose θ_0 is an irreducible subrepresentation. Then,

$$\begin{aligned} \theta_0 \hookrightarrow \quad & \delta([\nu^{-\alpha+k}\rho, \nu^{\alpha-k}\rho]) \rtimes \delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma) \\ \hookrightarrow \quad & \delta([\nu^{-\alpha+k}\rho, \nu^{\alpha-k}\rho]) \times \delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\ \cong \quad & \delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \times \delta([\nu^{-\alpha+k}\rho, \nu^{\alpha-k}\rho]) \rtimes \sigma \\ & \Downarrow \\ s_{app}(\theta_0) \geq \; & \delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \otimes (\delta([\nu^{-\alpha+k}\rho, \nu^{\alpha-k}\rho]) \rtimes \sigma) \end{aligned}$$

noting that this is irreducible (Theorem 13.2, [Tad3]). By Corollary 7.8 and Lemma 2.6, we see that

$$\begin{aligned} & \mu^*_{\delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho])}(\delta([\nu^{-\alpha+k}\rho, \nu^{\alpha-k}\rho]) \rtimes \delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)) \\ & = \delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho]) \otimes \delta([\nu^{-\alpha+k}\rho, \nu^{\alpha-k}\rho]) \rtimes \delta([\nu^{\alpha-k+3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma). \end{aligned}$$

Iterating this argument, we get

$$\begin{aligned} s_{\delta([\nu^{\alpha-k+2}\rho,\nu^{a_2}\rho])\otimes\cdots\otimes\delta([\nu^{\alpha}\rho,\nu^{a_k}\rho])}&(\delta([\nu^{-\alpha+k}\rho,\nu^{\alpha-k}\rho]) \\ &\rtimes \delta([\nu^{\alpha-k+2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)) \\ = \delta([\nu^{\alpha-k+2}\rho,\nu^{a_2}\rho])\otimes\cdots\otimes\delta([\nu^{\alpha}\rho,\nu^{a_k}\rho])&\otimes(\delta([\nu^{-\alpha+k}\rho,\nu^{\alpha-k}\rho])\rtimes\sigma). \end{aligned}$$

Thus, θ_0 is the only irreducible subrepresentation, hence (by unitarity) we have irreducibility, as claimed.

At this point, it is clear that there is a unique irreducible π_0 such that $\pi_0 \leq \pi', \pi''$ and $\mu^*_{\delta([\nu^{\alpha-k+1}\rho,\nu^{b_1}\rho])\times\delta([\nu^{\alpha-k+1}\rho,\nu^{a_1}\rho])}(\pi_0) \neq 0$.

Next, we argue that $\pi_0 \hookrightarrow \pi'$. Observe that $\mu^*_{\delta([\nu^{\alpha-k+1}\rho,\nu^{b_1}\rho])}(\pi_0) \neq 0$. Now, $\pi_0 \leq \pi'$ and

$$\begin{aligned} \mu^*_{\delta([\nu^{\alpha-k+1}\rho,\nu^{b_1}\rho])}(\pi') = \delta([\nu^{\alpha-k+1}\rho,\nu^{b_1}\rho]) \\ \otimes \delta([\nu^{-\alpha+k}\rho,\nu^{\alpha-k}\rho]) \rtimes \delta([\nu^{\alpha-k+1}\rho,\nu^{a_1}\rho],\ldots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma). \end{aligned}$$

Therefore, $\mu^*_{\delta([\nu^{\alpha-k+1}\rho,\nu^{b_1}\rho])}(\pi_0) \geq \delta([\nu^{\alpha-k+1}\rho,\nu^{b_1}\rho]) \otimes S$ for some irreducible $S \leq \delta([\nu^{-\alpha+k}\rho,\nu^{\alpha-k}\rho]) \rtimes \delta([\nu^{\alpha-k+1}\rho,\nu^{a_1}\rho],\ldots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)$ (cf. Lemma 5.5, [Jan2]). By unitarity, $S \hookrightarrow \delta([\nu^{-\alpha+k}\rho,\nu^{\alpha-k}\rho]) \rtimes \delta([\nu^{\alpha-k+1}\rho,\nu^{a_1}\rho],\ldots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)$. Thus, by central character considerations,

$$\pi_0 \hookrightarrow \nu^{b_1}\rho \times \cdots \times \nu^{\alpha-k+1}\rho \rtimes S$$

for some such S. Therefore, by Lemma 5.5, [Jan2], $\pi_0 \hookrightarrow \tau \rtimes S$ for some irreducible $\tau \leq \nu^{b_1}\rho \times \cdots \times \nu^{\alpha-k+1}\rho$. Any subquotient of $\nu^{b_1}\rho \times \cdots \times \nu^{\alpha-k+1}\rho$ other than $\delta([\nu^{\alpha-k+1}\rho,\nu^{b_1}\rho])$ would give $r_{min}(\tau)$, and therefore $s_{min}(\pi_0)$, containing a term of the form $\nu^x\rho \otimes \ldots$ with $\alpha - k + 1 \leq x < b_1$, a contradiction. Thus,

$$\begin{aligned} \pi_0 \hookrightarrow \quad & \delta([\nu^{\alpha-k+1}\rho,\nu^{b_1}\rho]) \rtimes S \\ \hookrightarrow \quad & \delta([\nu^{\alpha-k+1}\rho,\nu^{b_1}\rho]) \times \delta([\nu^{-\alpha+k}\rho,\nu^{\alpha-k}\rho]) \\ & \qquad \rtimes \delta([\nu^{\alpha-k+1}\rho,\nu^{a_1}\rho],\ldots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma) \\ & \Downarrow \text{ (Lemma 5.5, [Jan2])} \\ \pi_0 \hookrightarrow \quad & \delta([\nu^{-\alpha+k}\rho,\nu^{b_1}\rho]) \rtimes \delta([\nu^{\alpha-k+1}\rho,\nu^{a_1}\rho],\ldots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma) \\ & \qquad \text{or} \\ & \mathcal{L}(\delta([\nu^{-\alpha+k}\rho,\nu^{\alpha-k}\rho]),\delta([\nu^{\alpha-k+1}\rho,\nu^{b_1}\rho])) \\ & \qquad \rtimes \delta([\nu^{\alpha-k+1}\rho,\nu^{a_1}\rho],\ldots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma). \end{aligned}$$

The latter would give $s_{min}(\pi_0) \geq \nu^{\alpha-k}\rho \otimes \ldots$, a contradiction. Thus the former, which says $\pi \hookrightarrow \pi'$–holds, as claimed.

Finally, we show $\pi \hookrightarrow \pi''$. Write

$$\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho]) \rtimes \delta([\nu^{\alpha-k+2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma) = \bigoplus_i S_i,$$

where the S_i's are inequivalent by [Gol]. (It will turn out that $i = 1, 2$ (cf. Lemma 7.4 below), though that is not needed for the argument.) By Lemma 2.6,

$$\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}(\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \rtimes S_i) = \delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \otimes S_i.$$

Thus $\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \rtimes S_i$ has a unique irreducible subrepresentation; denote it π_{S_i}. If we set

$$\pi^{**} = \delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \times \delta([\nu^{-b_1}\rho,\nu^{b_1}\rho]) \rtimes \delta([\nu^{\alpha-k+2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma),$$

we have $\pi_{S_i}, \pi'' \leq \pi^{**}$ and $\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}(\pi'') = \mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}(\pi^{**})$. Therefore, $\pi_{S_i} \leq \pi''$ for all i. Further, π_{S_i} appears with multiplicity one in π''. Since $\pi_{S_i}, \pi'' \hookrightarrow \pi^{**}$, we get $\pi_{S_i} \hookrightarrow \pi''$. (To see this, consider the subspace $V_{\pi''} + V_{\pi_{S_i}} \subset V_{\pi^{**}}$.) Therefore, any irreducible subquotient of π'' with $\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])} \neq 0$ is a π_{S_i}, hence a subrepresentation. In particular, $\pi_0 \hookrightarrow \pi''$, as needed. $\square$

Lemma 6.5. *With notation as above, suppose $b_1 \geq 0$ and*

$$\pi \leq \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma),$$

with $\alpha - k + 2 > 1$. Then,

$$\delta_0(\pi) \neq \delta([\nu^{-\alpha+k}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{\alpha-k+1}\rho, \nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \otimes \sigma.$$

Proof. Suppose $\delta_0(\pi)$ had this form.

First, with π', π'' as in the preceding lemma, Lemma 2.6 tells us

$$\mu^*_{\delta([\nu^{-\alpha+k}\rho,\nu^{b_1}\rho])}(\pi') = c \cdot \delta([\nu^{-\alpha+k}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{\alpha-k+1}\rho, \nu^{a_1}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma),$$

where $c = 1$ if $\alpha - k < 0$; $c = 2$ if $\alpha - k \geq 0$. Consider $\mu^*_{\delta([\nu^{-\alpha+k}\rho,\nu^{b_1}\rho])}(\pi'')$. Write $\mu^*(\delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)) = \sum_h \tau_f \otimes \theta_f$. Then,

$$\mu^*(\pi'') = \sum_f \sum_{i=-b_1}^{a_1+1} \sum_{j=i}^{a_1+1} [\delta([\nu^{-i+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{j}\rho, \nu^{a_1}\rho]) \times \tau_f] \otimes \delta([\nu^{i}\rho, \nu^{j-1}\rho]) \rtimes \theta_f.$$

To contribute to $\mu^*_{\delta([\nu^{-\alpha+k}\rho,\nu^{b_1}\rho])}$, we must have $j = a_1 + 1$. Also, if $\tau_f \neq 1$, the $r_{min}(\tau_f)$ consists of terms of the form $\nu^x\rho \otimes \ldots$ with $x \in \{a_2, \ldots, a_k\}$. Therefore, $\tau_f = 1$. Thus,

$$\mu^*_{\delta([\nu^{-\alpha+k}\rho,\nu^{b_1}\rho])}(\pi'') = \delta([\nu^{-\alpha+k}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{\alpha-k+1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma).$$

Therefore, $\mu^*_{\delta([\nu^{-\alpha+k}\rho,\nu^{b_1}\rho])}(\pi'')$ contains $\delta([\nu^{-\alpha+k}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{\alpha-k+1}\rho, \nu^{a_1}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)$ with multiplicity exactly one (by Theorem 7.7 and the inductive hypothesis). Therefore, by Frobenius reciprocity, there is at most one irreducible subrepresentation of π' which is also a subquotient of π''; by the preceding lemma, there is at least one, namely π_0. Thus, $\pi = \pi_0$. However, by the preceding lemma,

$$\pi_0 \hookrightarrow \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{\alpha-k+2}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \rtimes \sigma,$$

which (by Frobenius reciprocity) contradicts the minimality of $\delta_0(\pi)$. The lemma follows. $\square$

Lemma 6.6. *We continue to assume $a_k > \cdots > a_1 > b_1 > \cdots > b_k$ satisfy conditions 1–4 of Theorem 1.1. We also assume $\beta > 2$ and $b_2 \geq 0$. We take i so that $b_{i+1} = -\beta$ (and $b_i \geq 0$). Let*

$$\delta_t = \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_i}\rho, \nu^{a_i}\rho], [\nu^{\beta}\rho, \nu^{a_{i+1}}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)_t,$$

noting that $t = 1$ unless $i = k$. Then,

1. *if $t = 1$,*

$$\mu^*_{\delta([\nu^{-\beta+2}\rho,\nu^{b_i}\rho])}(\delta_1) \leq 2 \cdot \delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho]) \otimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_{i-1}}\rho, \nu^{a_{i-1}}\rho], [\nu^{\beta-1}\rho, \nu^{a_i}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)_1.$$

2. *if* $t=2$,

$$\mu^*_{\delta([\nu^{-\beta+2}\rho,\nu^{b_i}\rho])}(\delta_2)=0.$$

Proof. Let us verify 2 first. Since $t=2$, we must have $i=k$, and therefore $\beta=\alpha+1$. We have

$$\delta_2\hookrightarrow\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])\times\cdots\times\delta([\nu^{-b_{k-1}}\rho,\nu^{a_{k-1}}\rho])\rtimes\delta([\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_2.$$

Since $b_2,\dots,b_{k-1},a_2,\dots,a_{k-1}>b_k$, Lemma 2.6 tells us that

$$\begin{gathered}\mu^*_{\delta([\nu^{-\alpha+1}\rho,\nu^{b_k}\rho])}(\delta_2)\neq0\\ \Downarrow\\ \mu^*_{\delta([\nu^{-\alpha+1}\rho,\nu^{b_k}\rho])}(\delta([\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_2)\neq0.\end{gathered}$$

This is not the case (cf. Theorem 4.5, [Tad6], noting that $\beta>2$ requires $\alpha>1$); 2 follows.

We now turn to 1. By Theorem 7.7 and the inductive hypothesis, we have

$$\begin{aligned}&\delta([\nu^{-b_i}\rho,\nu^{a_i}\rho],[\nu^{\beta}\rho,\nu^{a_{i+1}}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)_1\\ &\qquad\hookrightarrow\delta([\nu^{-\beta+2}\rho,\nu^{b_i}\rho])\rtimes\delta([\nu^{\beta-1}\rho,\nu^{a_i}\rho],[\nu^{\beta}\rho,\nu^{a_{i+1}}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)_t.\end{aligned}$$

If $i=2$, then 1 now follows immediately from Lemma 2.6. Suppose $i\geq3$. Then, we have

$$\begin{aligned}\delta_1\hookrightarrow\;&\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])\times\cdots\times\delta([\nu^{-b_{i-1}}\rho,\nu^{a_{i-1}}\rho])\\ &\rtimes\delta([\nu^{-b_i}\rho,\nu^{a_i}\rho],[\nu^{\beta}\rho,\nu^{a_{i+1}}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)_1\\ \hookrightarrow\;&\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])\times\cdots\times\delta([\nu^{-b_{i-1}}\rho,\nu^{a_{i-1}}\rho])\times\delta([\nu^{-\beta+2}\rho,\nu^{b_i}\rho])\\ &\rtimes\delta([\nu^{\beta-1}\rho,\nu^{a_i}\rho],[\nu^{\beta}\rho,\nu^{a_{i+1}}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)\\ \cong\;&\delta([\nu^{-\beta+2}\rho,\nu^{b_i}\rho])\times\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])\times\cdots\times\delta([\nu^{-b_{i-1}}\rho,\nu^{a_{i-1}}\rho])\\ &\rtimes\delta([\nu^{\beta-1}\rho,\nu^{a_i}\rho],[\nu^{\beta}\rho,\nu^{a_{i+1}}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma).\end{aligned}$$

Write $\delta([\nu^{-b_{i-1}}\rho,\nu^{a_{i-1}}\rho])\rtimes\delta([\nu^{\beta-1}\rho,\nu^{a_i}\rho],[\nu^{\beta}\rho,\nu^{a_{i+1}}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)$ $=\theta_{i-1}+\theta'_{i-1}+\theta''_{i-1}$, where (by Theorem 7.7 and the inductive hypothesis)

$$\begin{aligned}\delta_0(\theta_{i-1})=\;&\delta([\nu^{-b_{i-1}}\rho,\nu^{a_{i-1}}\rho])\otimes\delta([\nu^{\beta-1}\rho,\nu^{a_i}\rho])\otimes\delta([\nu^{\beta}\rho,\nu^{a_{i+1}}\rho])\\ &\otimes\cdots\otimes\delta([\nu^{\alpha}\rho,\nu^{a_k}\rho])\otimes\sigma\end{aligned}$$

$$\begin{aligned}\delta_0(\theta'_{i-1})=\;&\delta([\nu^{\beta-1}\rho,\nu^{b_{i-1}}\rho])\otimes\delta([\nu^{-a_{i-1}}\rho,\nu^{a_i}\rho])\otimes\delta([\nu^{\beta}\rho,\nu^{a_{i+1}}\rho])\\ &\otimes\cdots\otimes\delta([\nu^{\alpha}\rho,\nu^{a_k}\rho])\otimes\sigma\end{aligned}$$

$$\begin{aligned}\delta_0(\theta''_{i-1})=\;&\delta([\nu^{-a_{i-1}}\rho,\nu^{b_{i-1}}\rho])\otimes\delta([\nu^{\beta-1}\rho,\nu^{a_i}\rho])\otimes\delta([\nu^{\beta}\rho,\nu^{a_{i+1}}\rho])\\ &\otimes\cdots\otimes\delta([\nu^{\alpha}\rho,\nu^{a_k}\rho])\otimes\sigma.\end{aligned}$$

By Lemma 5.5, [Jan2],

$$\delta_1\hookrightarrow\delta([\nu^{-\beta+2}\rho,\nu^{b_i}\rho])\times\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])\times\cdots\times\delta([\nu^{-b_{i-2}}\rho,\nu^{a_{i-2}}\rho])\rtimes\theta$$

for some $\theta\in\{\theta_{i-1},\theta'_{i-1},\theta''_{i-1}\}$. We show that $\theta=\theta_{i-1}$ is the only possibility.

Suppose $\theta = \theta'_{i-1}$. Then

$$\begin{aligned}
\delta_1 \hookrightarrow\ & \delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_{i-2}}\rho, \nu^{a_{i-2}}\rho]) \rtimes \theta'_{i-1} \\
\hookrightarrow\ & \delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_{i-2}}\rho, \nu^{a_{i-2}}\rho]) \\
& \times \delta([\nu^{-\beta+2}\rho, \nu^{b_{i-1}}\rho]) \times \delta([\nu^{-a_{i-1}}\rho, \nu^{a_i}\rho]) \\
& \rtimes \delta([\nu^{\beta}\rho, \nu^{a_{i+1}}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma) \\
\cong\ & \delta([\nu^{-a_{i-1}}\rho, \nu^{a_i}\rho]) \times \delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \\
& \times \cdots \times \delta([\nu^{-b_{i-2}}\rho, \nu^{a_{i-2}}\rho]) \times \delta([\nu^{-\beta+2}\rho, \nu^{b_{i-1}}\rho]) \\
& \rtimes \delta([\nu^{\beta}\rho, \nu^{a_{i+1}}\rho], \ldots [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)
\end{aligned}$$

$$\Downarrow$$

$$\mu^*_{\delta([\nu^{-a_{i-1}}\rho, \nu^{a_i}\rho])}(\delta_1) \neq 0.$$

Now, since

$$\begin{aligned}
\delta_1 \hookrightarrow\ & \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_{i-2}}\rho, \nu^{a_{i-2}}\rho]) \\
& \rtimes \delta([\nu^{-b_{i-1}}\rho, \nu^{a_{i-1}}\rho], [\nu^{-b_i}\rho, \nu^{a_i}\rho], [\nu^{\beta}\rho, \nu^{a_{i+1}}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)_1,
\end{aligned}$$

we have

$$\mu^*_{\delta([\nu^{-a_{i-1}}\rho, \nu^{a_i}\rho])}(\delta_1) \neq 0$$

$$\Downarrow$$

$$\begin{aligned}
\mu^*_{\delta([\nu^{a_{i-2}+1}\rho, \nu^{a_i}\rho])}(\delta([\nu^{-b_{i-1}}\rho, \nu^{a_{i-1}}\rho], [\nu^{-b_i}\rho, \nu^{a_i}\rho], \\
[\nu^{\beta}\rho, \nu^{a_{i+1}}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)_1) \neq 0,
\end{aligned}$$

contradicting Corollary 7.9. Thus we cannot have $\theta = \theta'_{i-1}$.

Suppose $\theta = \theta''_{i-1}$. Then

$$\begin{aligned}
\delta_1 \hookrightarrow\ & \delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_{i-2}}\rho, \nu^{a_{i-2}}\rho]) \rtimes \theta''_{i-1} \\
\hookrightarrow\ & \delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_{i-2}}\rho, \nu^{a_{i-2}}\rho]) \\
& \times \delta([\nu^{-a_{i-1}}\rho, \nu^{b_{i-1}}\rho]) \rtimes \delta([\nu^{\beta-1}\rho, \nu^{a_i}\rho], [\nu^{\beta}\rho, \nu^{a_{i+1}}\rho], \ldots [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma).
\end{aligned}$$

Now,

$$\begin{aligned}
\delta([\nu^{-b_{i-2}}\rho, \nu^{a_{i-2}}\rho]) & \times \delta([\nu^{-a_{i-1}}\rho, \nu^{b_{i-1}}\rho]) \\
& = \delta([\nu^{-a_{i-1}}\rho, \nu^{a_{i-2}}\rho]) \times \delta([\nu^{-b_{i-2}}\rho, \nu^{b_{i-1}}\rho]) \\
& + \mathcal{L}(\delta([\nu^{-a_{i-1}}\rho, \nu^{b_{i-1}}\rho]), \delta([\nu^{-b_{i-2}}\rho, \nu^{a_{i-2}}\rho])),
\end{aligned}$$

a sum of irreducible representations. By Lemma 5.5, [Jan2], either

$$\begin{aligned}
\delta_1 \hookrightarrow\ & \delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_{i-3}}\rho, \nu^{a_{i-3}}\rho]) \\
& \times \delta([\nu^{-a_{i-1}}\rho, \nu^{a_{i-2}}\rho]) \times \delta([\nu^{-b_{i-2}}\rho, \nu^{b_{i-1}}\rho]) \\
& \rtimes \delta([\nu^{\beta-1}\rho, \nu^{a_i}\rho], [\nu^{\beta}\rho, \nu^{a_{i+1}}\rho], \ldots [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)_1
\end{aligned}$$

or

$$\begin{aligned}
\delta_1 \hookrightarrow\ & \delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-b_{i-3}}\rho, \nu^{a_{i-3}}\rho]) \\
& \times \mathcal{L}(\delta([\nu^{-a_{i-1}}\rho, \nu^{b_{i-1}}\rho]), \delta([\nu^{-b_{i-2}}\rho, \nu^{a_{i-2}}\rho])) \\
& \rtimes \delta([\nu^{\beta-1}\rho, \nu^{a_i}\rho], [\nu^{\beta}\rho, \nu^{a_{i+1}}\rho], \ldots [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)_1.
\end{aligned}$$

In the first case, an easy commuting argument gives $\mu^*_{\delta([\nu^{-a_{i-1}}\rho,\nu^{a_{i-2}}\rho])}(\delta_1)\neq 0$, contradicting the Casselman criteria for the square-integrability of δ_1. In the second case, we have

$$\begin{aligned}\delta_1\hookrightarrow\quad&\delta([\nu^{-\beta+2}\rho,\nu^{b_i}\rho])\times\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])\times\cdots\times\delta([\nu^{-b_{i-3}}\rho,\nu^{a_{i-3}}\rho])\\&\times\delta([\nu^{-a_{i-1}}\rho,\nu^{b_{i-1}}\rho])\times\delta([\nu^{-b_{i-2}}\rho,\nu^{a_{i-2}}\rho])\\&\rtimes\delta([\nu^{\beta-1}\rho,\nu^{a_i}\rho],[\nu^{\beta}\rho,\nu^{a_{i+1}}\rho],\dots[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)_1.\end{aligned}$$

We now observe that

$$\begin{aligned}&\delta([\nu^{-b_{i-3}}\rho,\nu^{a_{i-3}}\rho])\times\delta([\nu^{-a_{i-1}}\rho,\nu^{b_{i-1}}\rho])\\&\qquad=\delta([\nu^{-a_{i-1}}\rho,\nu^{a_{i-3}}\rho])\times\delta([\nu^{-b_{i-3}}\rho,\nu^{b_{i-1}}\rho])\\&\qquad+\mathcal{L}(\delta([\nu^{-a_{i-1}}\rho,\nu^{b_{i-1}}\rho]),\delta([\nu^{-b_{i-3}}\rho,\nu^{a_{i-3}}\rho])).\end{aligned}$$

Thus, as above, we get either the contradiction $\mu^*_{\delta([\nu^{-a_{i-1}}\rho,\nu^{a_{i-3}}\rho])}(\delta_1)\neq 0$ or

$$\begin{aligned}\delta_1\hookrightarrow\quad&\delta([\nu^{-\beta+2}\rho,\nu^{b_i}\rho])\times\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])\times\cdots\times\delta([\nu^{-b_{i-4}}\rho,\nu^{a_{i-4}}\rho])\\&\times\delta([\nu^{-a_{i-1}}\rho,\nu^{b_{i-1}}\rho])\times\delta([\nu^{-b_{i-3}}\rho,\nu^{a_{i-3}}\rho])\times\delta([\nu^{-b_{i-2}}\rho,\nu^{a_{i-2}}\rho])\\&\rtimes\delta([\nu^{\beta-1}\rho,\nu^{a_i}\rho],[\nu^{\beta}\rho,\nu^{a_{i+1}}\rho],\dots[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)_1.\end{aligned}$$

Iterating this argument, we must eventually get

$$\mu^*_{\delta([\nu^{-a_{i-1}}\rho,\nu^{x}])}(\delta_1)\neq 0$$

for some $x\in\{a_2,\dots,a_{i-2},b_{i-1}\}$, contradicting the Casselman criteria for the square-integrability of δ_1. Thus, $\theta\neq\theta''_{i-1}$.

We now have only the possibility

$$\begin{aligned}\delta_1\hookrightarrow\;&\delta([\nu^{-\beta+2}\rho,\nu^{b_i}\rho])\times\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])\times\cdots\times\delta([\nu^{-b_{i-3}}\rho,\nu^{a_{i-3}}\rho])\\&\rtimes(\delta([\nu^{-b_{i-2}}\rho,\nu^{a_{i-2}}\rho])\rtimes\theta_{i-1}).\end{aligned}$$

If $i=3$, then 1 now follows immediately from Lemma 2.6. If $i\geq 4$, combining Lemma 5.5, [Jan2] and Theorem 7.7 (by the inductive hypothesis), we get

$$\delta_1\hookrightarrow\delta([\nu^{-\beta+2}\rho,\nu^{b_i}\rho])\times\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])\times\cdots\times\delta([\nu^{-b_{i-3}}\rho,\nu^{a_{i-3}}\rho])\rtimes\theta,$$

for some $\theta\in\{\theta_{i-2},\theta'_{i-2},\theta''_{i-2}\}$, where

$$\begin{aligned}\delta_0(\theta_{i-2})=\;&\delta([\nu^{-b_{i-2}}\rho,\nu^{a_{i-2}}\rho])\otimes\delta([\nu^{-b_{i-1}}\rho,\nu^{a_{i-1}}\rho])\otimes\delta([\nu^{\beta-1}\rho,\nu^{a_i}\rho])\\&\otimes\delta([\nu^{\beta}\rho,\nu^{a_{i+1}}\rho])\otimes\cdots\otimes\delta([\nu^{\alpha}\rho,\nu^{a_k}\rho])\otimes\sigma\end{aligned}$$

$$\begin{aligned}\delta_0(\theta'_{i-2})=\;&\delta([\nu^{-b_{i-1}}\rho,\nu^{b_{i-2}}\rho])\otimes\delta([\nu^{-a_{i-2}}\rho,\nu^{a_{i-1}}\rho])\otimes\delta([\nu^{\beta-1}\rho,\nu^{a_i}\rho])\\&\otimes\delta([\nu^{\beta}\rho,\nu^{a_{i+1}}\rho])\otimes\cdots\otimes\delta([\nu^{\alpha}\rho,\nu^{a_k}\rho])\otimes\sigma\end{aligned}$$

$$\begin{aligned}\delta_0(\theta''_{i-2})=\;&\delta([\nu^{-a_{i-2}}\rho,\nu^{b_{i-2}}\rho])\otimes\delta([\nu^{-b_{i-1}}\rho,\nu^{a_{i-1}}\rho])\otimes\delta([\nu^{\beta-1}\rho,\nu^{a_i}\rho])\\&\otimes\delta([\nu^{\beta}\rho,\nu^{a_{i+1}}\rho])\otimes\cdots\otimes\delta([\nu^{\alpha}\rho,\nu^{a_k}\rho])\otimes\sigma.\end{aligned}$$

We can rule out the possibilities $\theta=\theta'_{i-2},\theta''_{i-2}$ using the same arguments as above. Therefore,

$$\delta_1\hookrightarrow\delta([\nu^{-\beta+2}\rho,\nu^{b_i}\rho])\times\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])\times\cdots\times\delta([\nu^{-b_{i-3}}\rho,\nu^{a_{i-3}}\rho])\rtimes\theta_{i-2}.$$

We continue iterating this argument. Eventually, we get

$$\delta_1\hookrightarrow\delta([\nu^{-\beta+2}\rho,\nu^{b_i}\rho])\rtimes\theta_2,$$

with

$$\theta_2 = \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \dots, [\nu^{-b_{i-1}}\rho, \nu^{a_{i-1}}\rho], [\nu^{\beta-1}\rho, \nu^{a_i}\rho], [\nu^{\beta}\rho, \nu^{a_{i+1}}\rho], \dots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)_1.$$

It then follows from Lemma 2.6 that

$$\mu^*_{\delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho])}(\delta_1) \leq 2 \cdot \delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho]) \otimes \theta_2,$$

as needed. □

Now, suppose $a_k > \cdots > a_1 > b_1 > \dots b_k$ with $b_2, \dots, b_k, a_2, \dots, a_k$ as in the preceding lemma. Suppose π is an irreducible representation with

$$\pi \leq \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \dots, [\nu^{-b_i}\rho, \nu^{a_i}\rho], [\nu^{\beta}\rho, \nu^{a_{i+1}}\rho], \dots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)_t.$$

Corollary 6.7. *With assumptions as in Lemma 6.6, we cannot have*

$$\delta_0(\pi) = \delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho]) \otimes \cdots \otimes \delta([\nu^{-\beta+i+1}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{\beta-i}\rho, \nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \otimes \sigma.$$

Proof. Suppose $\delta_0(\pi)$ had this form.

The usual argument (cf. proof of Lemma 4.2) with the preceding lemma gives

$$\mu^*_{\delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho])}(\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \dots, [\nu^{-b_i}\rho, \nu^{a_i}\rho], [\nu^{\beta}\rho, \nu^{a_{i+1}}\rho], \dots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)_t),$$

$$\leq \begin{cases} 0 \quad \text{if } t = 2, \\ c \cdot \delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho]) \otimes \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \dots, \\ \quad [\nu^{-b_{i-1}}\rho, \nu^{a_{i-1}}\rho], [\nu^{\beta-1}\rho, \nu^{a_i}\rho], [\nu^{\beta}\rho, \nu^{a_{i+1}}\rho], \dots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)_t, \\ \quad \text{if } t = 1. \end{cases}$$

Therefore, if $t = 2$ we already see that we cannot have $\mu^*_{\delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho])}(\pi) \neq 0$.

If $t = 1$, we have (by either δ_0 or central character considerations)

$$\pi \hookrightarrow \delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho]) \rtimes \theta$$

for some irreducible θ. By Lemma 5.5, [Jan2] and Theorem 7.7 (by the inductive hypothesis),

$$\delta_0(\theta) = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_{i-1}}\rho, \nu^{a_{i-1}}\rho]) \otimes \delta([\nu^{\beta-1}\rho, \nu^{a_i}\rho]) \otimes \cdots \otimes \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \otimes \sigma,$$

$$\delta([\nu^{-b_2}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \otimes \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_{i-1}}\rho, \nu^{a_{i-1}}\rho]) \otimes \delta([\nu^{\beta-1}\rho, \nu^{a_i}\rho]) \otimes \cdots \otimes \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \otimes \sigma,$$

or

$$\delta([\nu^{-a_1}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_{i-1}}\rho, \nu^{a_{i-1}}\rho]) \otimes \delta([\nu^{\beta-1}\rho, \nu^{a_i}\rho]) \otimes \cdots \otimes \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \otimes \sigma.$$

However, for any of these,

$$s_{app}(\pi) \geq \delta([\nu^{-\beta+2}\rho, \nu^{b_i}\rho]) \otimes \delta_0(\theta)$$

contradicts the minimality of $\delta_0(\pi)$. □

7. Construction of certain discrete series

In this section, we construct certain discrete series needed in this paper. In particular, if $a_k > \cdots > a_1 > b_1 > \cdots > b_k$ satisfy the conditions in Theorem 1.1, we construct an irreducible representation, which we denote by $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$ (more on t in a moment), which has

$$\delta_0(\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) \\ = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma.$$

Further, these constitute all the irreducible representations with this δ_0. The construction is inductive in nature; to obtain $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho], \ldots [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$ we analyze the representation $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$. Using the results of the last section, we can determine the composition series for this induced representation; one of the irreducible subrepresentations has the desired δ_0. The composition series is described in Theorem 7.7, the main theorem in this section. Additional properties we need are given in Corollaries 7.8 and 7.9.

Our situation with respect to the inductive argument is the following: by the inductive hypothesis, we may assume Theorem 7.7 (and Corollaries 7.8 and 7.9) hold when the parabolic rank of the supercuspidal support is less than *p.r.* By Theorems 2.4, 3.7, 4.4, and 5.4, we may assume that Theorem 1.1 holds when the parabolic rank of the supercuspidal support is less than or equal to *p.r.* (Again, we note that Theorem 2.4 holds in general and is not part of the inductive argument.) As a consequence, we may also assume Proposition 6.3 holds when the parabolic rank of the supercuspidal support is less than or equal to *p.r.*

We make two additional remarks before starting in on the results for this section. First, we note that if $b_k < 0$, then $t = 1$; if $b_k \geq 0$, then $t = 1$ or 2. The reason for this is the following: when $k = 1$, then $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \sigma$ has one irreducible subrepresentation (which has $\delta_0 = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \sigma$) when $b_1 < 0$ (in which case $-b_1 = \alpha$) and two irreducible subrepresentations (both having $\delta_0 = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \sigma$) when $b_1 \geq 0$. (This claim follows from the results in [Tad6].) When $k \geq 2$, there is always a unique irreducible subrepresentation of $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$ having $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma$ as its δ_0.

Lemma 7.1. *Suppose π is an irreducible representation with*

$$\delta_0(\pi) = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma.$$

Further, suppose that $a_k > \cdots > a_1 > b_1 > \cdots > b_k$. Suppose that $[\nu^d\rho, \nu^c\rho]$ is such that $b_1 + 1 < d \leq c \leq a_1$ with $d \geq \beta$ (and $d \equiv c \equiv \beta \bmod 1$). Then, $\delta([\nu^d\rho, \nu^c\rho]) \rtimes \pi$ is irreducible.

Proof. We use a Langlands classification argument similar to one in [Tad1] (also, cf. [Jan1], [Jan3]). Write

$$\pi = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho], \ldots, [\nu^{-b_s}\rho, \nu^{a_s}\rho], [\nu^{\beta}\rho, \nu^{a_{s+1}}\rho], \ldots, [\nu^{\alpha}\rho, \nu^{a_k}\rho]; \sigma)_t.$$

Suppose $a_1 \geq x > b_1 + 1$ with $x \geq \beta$. Note that these imply $x \geq 1$. Our first goal is to show $\nu^x\rho \rtimes \pi$ is irreducible. Suppose $\pi_0 \hookrightarrow \nu^x\rho \rtimes \pi$. Then, (with the obvious interpretation if $x > \alpha$)

$$\begin{aligned}\pi_0 \hookrightarrow\ & \nu^x\rho \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_s}\rho, \nu^{a_s}\rho]) \times \delta([\nu^{\beta}\rho, \nu^{a_{s+1}}\rho]) \\ & \times \cdots \times \delta([\nu^{x-1}\rho, \nu^{a_{r-1}}\rho]) \times \delta([\nu^{x}\rho, \nu^{a_{r}}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{a_{r+1}}\rho]) \\ & \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\ \cong\ & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_s}\rho, \nu^{a_s}\rho]) \times \delta([\nu^{\beta}\rho, \nu^{a_{s+1}}\rho]) \\ & \times \cdots \times \delta([\nu^{x-1}\rho, \nu^{a_{r-1}}\rho]) \times \delta([\nu^{x}\rho, \nu^{a_{r}}\rho]) \times \nu^x\rho \times \delta([\nu^{x+1}\rho, \nu^{a_{r+1}}\rho]) \\ & \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\ \hookrightarrow\ & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_s}\rho, \nu^{a_s}\rho]) \times \delta([\nu^{\beta}\rho, \nu^{a_{s+1}}\rho]) \\ & \times \cdots \times \delta([\nu^{x-1}\rho, \nu^{a_{r-1}}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{a_{r}}\rho]) \times \nu^x\rho \times \nu^x\rho \\ & \times \delta([\nu^{x+2}\rho, \nu^{a_{r+1}}\rho]) \times \nu^{x+1}\rho \times \delta([\nu^{x+2}\rho, \nu^{a_{r+2}}\rho]) \times \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\ \cong\ & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_s}\rho, \nu^{a_s}\rho]) \times \delta([\nu^{\beta}\rho, \nu^{a_{s+1}}\rho]) \\ & \times \cdots \times \delta([\nu^{x-1}\rho, \nu^{a_{r-1}}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{a_{r}}\rho]) \times \delta([\nu^{x+2}\rho, \nu^{a_{r+1}}\rho]) \\ & \times \nu^x\rho \times \nu^x\rho \times \nu^{x+1}\rho \times \delta([\nu^{x+2}\rho, \nu^{a_{r+2}}\rho]) \times \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\ & \Downarrow \text{ (Lemma 5.5, [Jan2])} \\ \pi_0 \hookrightarrow\ & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_s}\rho, \nu^{a_s}\rho]) \times \delta([\nu^{\beta}\rho, \nu^{a_{s+1}}\rho]) \\ & \times \cdots \times \delta([\nu^{x-1}\rho, \nu^{a_{r-1}}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{a_{r}}\rho]) \times \delta([\nu^{x+2}\rho, \nu^{a_{r+1}}\rho]) \\ & \times \xi([\nu^x\rho, \nu^{x+1}\rho]) \times \nu^x\rho \times \delta([\nu^{x+2}\rho, \nu^{a_{r+2}}\rho]) \times \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \rtimes \sigma,\end{aligned}$$

where $\xi([\nu^x\rho, \nu^{x+1}\rho])$ is one of the two irreducible subquotients of $\nu^x\rho \times \nu^{x+1}\rho$ (the irreducibility of $\nu^x\rho \times \xi([\nu^x\rho, \nu^{x+1}\rho])$ follows from Theorem 4.2, [Zel] and Theorem 9.7, [Zel]). Continuing,

$$\begin{aligned}\pi_0 \hookrightarrow\ & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_s}\rho, \nu^{a_s}\rho]) \times \delta([\nu^{\beta}\rho, \nu^{a_{s+1}}\rho]) \\ & \times \cdots \times \delta([\nu^{x-1}\rho, \nu^{a_{r-1}}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{a_{r}}\rho]) \times \delta([\nu^{x+2}\rho, \nu^{a_{r+1}}\rho]) \\ & \times \xi([\nu^x\rho, \nu^{x+1}\rho]) \times \delta([\nu^{x+2}\rho, \nu^{a_{r+2}}\rho]) \times \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \times \nu^x\rho \rtimes \sigma.\end{aligned}$$

We now use this to show that $\mu^*_{\nu^{-x}\rho}(\pi_0) \neq 0$. We do this in three cases.

First, suppose $x > \alpha$. Then, since $\nu^x\rho \rtimes \sigma \cong \nu^{-x}\rho \rtimes \sigma$ (irreducible),

$$\begin{aligned}\pi_0 \hookrightarrow\ & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \times \nu^x\rho \rtimes \sigma \\ \cong\ & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \times \nu^{-x}\rho \rtimes \sigma \\ \cong\ & \nu^{-x}\rho \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma\end{aligned}$$

(where the commuting argument works since $x > b_1 + 1$). Thus, $\mu^*_{\nu^{-x}\rho}(\pi_0) \neq 0$.

Next, suppose $x < \alpha$. Then, since $\nu^x\rho \rtimes \sigma \cong \nu^{-x}\rho \rtimes \sigma$ (irreducible),

$$\begin{aligned}\pi_0 \hookrightarrow\ & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_s}\rho, \nu^{a_s}\rho]) \times \delta([\nu^{\beta}\rho, \nu^{a_{s+1}}\rho]) \\ & \times \cdots \times \delta([\nu^{x-1}\rho, \nu^{a_{r-1}}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{a_{r}}\rho]) \times \delta([\nu^{x+2}\rho, \nu^{a_{r+1}}\rho]) \\ & \times \xi([\nu^x\rho, \nu^{x+1}\rho]) \times \delta([\nu^{x+2}\rho, \nu^{a_{r+2}}\rho]) \times \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \times \nu^{-x}\rho \rtimes \sigma \\ \cong\ & \nu^{-x}\rho \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_s}\rho, \nu^{a_s}\rho]) \times \delta([\nu^{\beta}\rho, \nu^{a_{s+1}}\rho]) \\ & \times \cdots \times \delta([\nu^{x-1}\rho, \nu^{a_{r-1}}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{a_{r}}\rho]) \times \delta([\nu^{x+2}\rho, \nu^{a_{r+1}}\rho]) \\ & \times \xi([\nu^x\rho, \nu^{x+1}\rho]) \times \delta([\nu^{x+2}\rho, \nu^{a_{r+2}}\rho]) \times \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \rtimes \sigma\end{aligned}$$

(the commuting argument works since $x > b_1 + 1$ and $x \geq 1$). Thus, $\mu^*_{\nu^{-x}\rho}(\pi_0) \neq 0$.

Now, suppose $x = \alpha$. Then,

$$\begin{aligned} \pi_0 \hookrightarrow\ & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_s}\rho, \nu^{a_s}\rho]) \times \delta([\nu^{\beta}\rho, \nu^{a_{s+1}}\rho]) \times \ldots \\ & \times\delta([\nu^{\alpha-1}\rho, \nu^{a_{k-1}}\rho]) \times \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \times \nu^{\alpha}\rho \rtimes \sigma \\ \hookrightarrow\ & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_s}\rho, \nu^{a_s}\rho]) \times \delta([\nu^{\beta}\rho, \nu^{a_{s+1}}\rho]) \times \ldots \\ & \times\delta([\nu^{\alpha-1}\rho, \nu^{a_{k-1}}\rho]) \times \delta([\nu^{\alpha+1}\rho, \nu^{a_k}\rho]) \times \nu^{\alpha}\rho \times \nu^{\alpha}\rho \rtimes \sigma \\ & \Downarrow \text{ Lemma 5.5, [Jan2]} \\ \pi_0 \hookrightarrow\ & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_s}\rho, \nu^{a_s}\rho]) \times \delta([\nu^{\beta}\rho, \nu^{a_{s+1}}\rho]) \times \ldots \\ & \times\delta([\nu^{\alpha-1}\rho, \nu^{a_{k-1}}\rho]) \times \delta([\nu^{\alpha+1}\rho, \nu^{a_k}\rho]) \rtimes \theta_{\alpha}, \end{aligned}$$

for some irreducible $\theta_{\alpha} \leq \nu^{\alpha}\rho \times \nu^{\alpha}\rho \rtimes \sigma$. Since $\alpha = x \geq 1$, we know that for such a θ_{α}, $\mu^{*}_{\nu^{-\alpha}\rho}(\theta_{\alpha}) \neq 0$ (this follows from Theorem 13.1, [Tad3] and [Aub], [S-S]). Therefore, by central character considerations, $\theta_{\alpha} \hookrightarrow \nu^{-\alpha}\rho \rtimes \eta_{\alpha}$ for some η_{α}. Thus,

$$\begin{aligned} \pi_0 \hookrightarrow\ & \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_s}\rho, \nu^{a_s}\rho]) \times \delta([\nu^{\beta}\rho, \nu^{a_{s+1}}\rho]) \times \ldots \\ & \times\delta([\nu^{\alpha-1}\rho, \nu^{a_{k-1}}\rho]) \times \delta([\nu^{\alpha+1}\rho, \nu^{a_k}\rho]) \times \nu^{-\alpha}\rho \rtimes \eta_{\alpha} \\ \cong\ & \nu^{-\alpha}\rho \times \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_s}\rho, \nu^{a_s}\rho]) \times \delta([\nu^{\beta}\rho, \nu^{a_{s+1}}\rho]) \times \ldots \\ & \times\delta([\nu^{\alpha-1}\rho, \nu^{a_{k-1}}\rho]) \times \delta([\nu^{\alpha+1}\rho, \nu^{a_k}\rho]) \rtimes \eta_{\alpha} \end{aligned}$$

since $\alpha = x > b_1 + 1 > b_i + 1$ for all $i > 1$ and $x \geq 1$. By Frobenius reciprocity, $\mu^{*}_{\nu^{-\alpha}\rho}(\pi_0) \neq 0$, as needed.

Next, we note that for $b_1 < x < a_1$ with $c > 1$,

$$\mu^{*}_{\nu^{-x}\rho}(\nu^{-x}\rho \rtimes \pi) = \nu^{-x}\rho \otimes \pi.$$

Therefore, by central character considerations,

$$\pi_0 \hookrightarrow \nu^{-x}\rho \rtimes \pi,$$

and is the unique irreducible subrepresentation (by Frobenius reciprocity or the Langlands classification). However, since $\pi_0 \hookrightarrow \nu^{x}\rho \rtimes \pi$, this contradicts multiplicity one in the Langlands classification unless $\nu^{x}\rho \rtimes \pi$ is irreducible.

We are now ready to show the irreducibility of $\delta([\nu^{d}\rho, \nu^{c}\rho]) \rtimes \pi$. The argument is similar to the case $c = d = x$ above; suppose $\pi_0 \hookrightarrow \delta([\nu^{d}\rho, \nu^{c}\rho]) \rtimes \pi$. Then,

$$\begin{aligned} \pi_0 \hookrightarrow\ & \delta([\nu^{d}\rho, \nu^{c}\rho]) \rtimes \pi \\ \hookrightarrow\ & \delta([\nu^{d+1}\rho, \nu^{c}\rho]) \times \nu^{d}\rho \rtimes \pi \\ \cong\ & \delta([\nu^{d+1}\rho, \nu^{c}\rho]) \times \nu^{-d}\rho \rtimes \pi \\ \cong\ & \nu^{-d}\rho \times \delta([\nu^{d+1}\rho, \nu^{c}\rho]) \rtimes \pi \\ & \vdots \text{ iterating} \\ \pi_0 \hookrightarrow\ & \nu^{-d}\rho \times \nu^{-d-1}\rho \times \cdots \times \nu^{-c}\rho \rtimes \pi \\ & \Downarrow \\ \pi_0 \hookrightarrow\ & \tau \rtimes \pi \end{aligned}$$

for some irreducible $\tau \leq \nu^{-d}\rho \times \nu^{-d-1}\rho \times \cdots \times \nu^{-c}\rho$. Anything other than $\tau = \delta([\nu^{-c}\rho, \nu^{-d}\rho])$ gives $r_{min}(\tau)$, and therefore $s_{min}(\pi_0)$, containing a term of the form $\nu^{z}\rho \otimes \ldots$ with $z \notin \{-d, b_1, \ldots, b_k, a_1, \ldots, a_k\}$, a contradiction. Thus, $\pi_0 \hookrightarrow \delta([\nu^{-c}\rho, \nu^{-d}\rho]) \rtimes \pi$. Again, this contradicts multiplicity one in the Langlands classification unless $\delta([\nu^{-c}\rho, \nu^{-d}\rho]) \rtimes \pi$ is irreducible. □

We continue to assume $a_k > \cdots > a_1 > b_1 > \cdots > b_k$. Suppose π_1 is an irreducible subquotient of $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$ Our next step is to identify the possibilities for $\delta_0(\pi_1)$.

Write

$$\delta_0(\pi_1) = \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_h}\rho, \nu^{c_h}\rho]) \otimes \sigma.$$

By Proposition 6.3, $h \leq k$. It follows from Lemma 3.1 that

$$\{d_1, \ldots, d_h, c_1, \ldots, c_h\} = \{b_1, \ldots, b_k, a_1, \ldots, a_k\} \setminus \left(\bigcup_{x \in X} \{-x, x-1\} \right).$$

By allowing, e.g., $d_1 = -\beta$ and $c_1 = \beta - 1$, we can write

$$\delta_0(\pi_1) = \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_k}\rho, \nu^{c_k}\rho]) \otimes \sigma,$$

with $d_1, \ldots, d_k, c_1, \ldots c_k$ a permutation of $b_1, \ldots, b_k, a_1, \ldots, a_k$. (We remark that it will turn out that we can have $h = k-1$ in some cases, but never anything smaller.)

Certainly,

$$\begin{aligned} t.e.(\delta_0(\pi_1)) &\geq t.e.(\delta_0(\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t))) \\ &= t.e.(\delta([\nu^{-a_1}\rho, \nu^{b_1}\rho])) + t.e.(\delta_0(\delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t)). \end{aligned}$$

Now, one can check that since $a_k > \cdots > a_1 > b_1 > \ldots b_k$, this implies $c_1, \ldots, c_k$ must be either $a_1, \ldots, a_k$ or $b_1, a_2, \ldots, a_k$. In particular, $c_i = a_i$ for $i \geq 2$. By Lemma 6.1, there is an irreducible representation θ such that

1. $\delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \otimes \theta \leq \mu^*(\pi_1)$,
2. $\delta_0(\theta) \geq \delta([\nu^{-d_2}\rho, \nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_k}\rho, \nu^{a_k}\rho]) \otimes \sigma$.

Therefore, since $a_i > d_i$ for all $i \geq 2$, Theorem 4.2.1, [Jan4] tells us that θ is square-integrable. Therefore, $[\nu^{-d_i}\rho, \nu^{a_i}\rho], [\nu^{-d_j}\rho, \nu^{a_j}\rho]$ satisfy conditions 3 and 4 of Theorem 1.1 for all $i, j \geq 2$ with $i \neq j$ (by Theorems 4.4 and 5.4). Since $a_k > \cdots > a_2 > d_2, \ldots, d_k$, we must have $d_2 > \cdots > d_k$. Therefore, $d_i = b_i$ for $i \geq 3$.

We now have

$$\begin{aligned} \delta_0(\pi_1) = \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) &\otimes \delta([\nu^{-d_2}\rho, \nu^{a_2}\rho]) \otimes \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \\ &\otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma, \end{aligned}$$

with d_1, d_2, c_1 equal to b_1, b_2, a_1 in some order and $c_1 = b_1$ or a_1. We consider the following possibilities:

1. $c_1 = a_1$.
   Then, π_1 is square-integrable by Theorem 4.2.1, [Jan4]. By Theorems 4.4 and 5.4, condition 4 of Theorem 1.1 holds. Since $a_k > \cdots > a_1 > d_1, \ldots, d_k$, this forces $d_1 > \cdots > d_k$. Thus $d_i = b_i$ for all i.
2. $c_1 = b_1$.
   (a) $d_1 = a_1$.
   By Lemma 6.1, there is an irreducible θ with $\delta_0(\theta) = \delta([\nu^{-d_2}\rho, \nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_k}\rho, \nu^{a_k}\rho]) \otimes \sigma$. By Theorem 4.2.1, [Jan4], θ is square-integrable. Again, we get $d_2 > \cdots > d_k$, implying $d_i = b_i$ for $i \geq 2$.
   (b) $d_1 \neq a_1$.
   With θ as in (a), we see that $d_2 > \cdots > d_k$, implying $d_2 = a_1$. Also, by Theorem 4.2.1, [Jan4], π_1 is square-integrable. Again, by Theorems 4.4 and 5.4, condition 4 of Theorem 1.1 holds. This forces $d_1 > d_3 > \cdots > d_k$, so $d_1 = b_2$ and $d_i = b_i$ for $i \geq 3$.

We summarize:

Proposition 7.2. *Suppose $a_k > \cdots > a_1 > b_1 > \cdots > b_k$ satisfy conditions 1–4 of Theorem 1.1. If π_1 is an irreducible subquotient of $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$, then $\delta_0(\pi_1)$ must be one of the following:*

1. $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \otimes (\delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma)$,
2. $\delta([\nu^{-a_1}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \otimes (\delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma)$,
3. $\delta([\nu^{-b_2}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \otimes (\delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma)$.

Corollary 7.3. *$\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$ has a unique irreducible (Langlands) quotient. All other irreducible subquotients are square-integrable.*

Proof. The corollary follows immediately from the preceding proposition, the Langlands classification, Theorem 4.2.1, [Jan4] and Lemma 3.4, [Jan2]. □

Lemma 7.4. *We continue to assume $a_k > \cdots > a_1 > b_1 > \cdots > b_k$ satisfy conditions 1–4 of Theorem 1.1. Suppose $b_1 \geq 0$. Then, $\delta([\nu^{-b_1}\rho, \nu^{b_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$ decomposes as the direct sum of (exactly) two inequivalent irreducible subrepresentations.*

To fix notation when $k = 1$, write

$$\delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \rtimes \sigma = \bigoplus_{t=1}^{2} T_t([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]; \sigma).$$

If $\alpha \geq \frac{1}{2}$, we let $T_1([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]; \sigma)$ be the component which has the larger Jacquet module (cf. Theorem 2.5, [Tad6]*). Then, we may write*

$$\delta([\nu^{-b_1}\rho, \nu^{b_1}\rho]) \rtimes \sigma = \bigoplus_{t=1}^{2} T_t([\nu^{-b_1}\rho, \nu^{b_1}\rho]; \sigma),$$

where $T_t([\nu^{-b_1}\rho, \nu^{b_1}\rho]; \sigma)$ is the component characterized by

$$\begin{aligned}\mu^*_{\delta([\nu^{\alpha+1}\rho, \nu^{b_1}\rho];\sigma) \times \delta([\nu^{\alpha+1}\rho, \nu^{b_1}\rho];\sigma)}&(T_t([\nu^{-b_1}\rho, \nu^{b_1}\rho]; \sigma)) \\ &= \delta([\nu^{\alpha+1}\rho, \nu^{b_1}\rho]; \sigma) \times \delta([\nu^{\alpha+1}\rho, \nu^{b_1}\rho]; \sigma) \otimes T_t([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]; \sigma).\end{aligned}$$

(We note that $T_1([\nu^{-b_1}\rho, \nu^{b_1}\rho]; \sigma)$ is the representation denoted $\delta([\nu^{-b_1}\rho, \nu^{b_1}\rho], \sigma)$ in Theorem 2.5, [Tad6]*.)*

Proof. First, let us address the case $k = 1$. In this case, the reducibility result follows from Theorem 13.2, [Tad3]. To justify the notational convention, it suffices to show that $\mu^*_{\delta([\nu^{\alpha+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{\alpha+1}\rho, \nu^{b_1}\rho])}$ is nonzero for components of

$\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho])\rtimes\sigma$. If T is such a component, then

$$\begin{aligned} T &\hookrightarrow \delta([\nu^{-b_1}\rho,\nu^{b_1}\rho])\rtimes\sigma \\ &\hookrightarrow \delta([\nu^{-\alpha}\rho,\nu^{b_1}\rho])\times\delta([\nu^{-b_1}\rho,\nu^{-\alpha-1}\rho])\rtimes\sigma \\ &\cong \delta([\nu^{-\alpha}\rho,\nu^{b_1}\rho])\times\delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho])\rtimes\sigma \\ &\cong \delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho])\times\delta([\nu^{-\alpha}\rho,\nu^{b_1}\rho])\rtimes\sigma \\ &\hookrightarrow \delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho])\times\delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho])\times\delta([\nu^{-\alpha}\rho,\nu^{\alpha}\rho])\rtimes\sigma, \end{aligned}$$

where the irreducibility of $\delta([\nu^{-b_1}\rho,\nu^{-\alpha-1}\rho])\rtimes\sigma$ follows from Theorem 13.2, [Tad3]. By Frobenius reciprocity, $\mu^*_{\delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho])\times\delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho])}(T)\neq 0$, as needed.

We now address the case $k\geq 2$. First, let

$$\begin{aligned} \pi_t' &= \delta([\nu^{-b_1}\rho,\nu^{b_1}\rho])\rtimes\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t, \\ \pi_t'' &= \delta([\nu^{-b_1}\rho,\nu^{a_2}\rho])\rtimes\delta([\nu^{-b_2}\rho,\nu^{b_1}\rho],[\nu^{-b_3}\rho,\nu^{a_3}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t, \\ \pi_t^* &= \delta([\nu^{-b_1}\rho,\nu^{b_1}\rho])\times\delta([\nu^{b_1+1}\rho,\nu^{a_2}\rho]) \\ &\quad\rtimes\delta([\nu^{-b_2}\rho,\nu^{b_1}\rho],[\nu^{-b_3}\rho,\nu^{a_3}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t. \end{aligned}$$

It follows from an argument like that in Lemma 5.2.5, [Jan4] that

$$\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_2}\rho])}(\pi_t')=\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_2}\rho])}(\pi_t'')=\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_2}\rho])}(\pi_t^*)\neq 0.$$

Therefore, π_t' and π_t'' have at least one irreducible subquotient in common.

Next, we claim

$$\mu^*_{\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho])}(\pi_t')=2\cdot\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho])\otimes\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$

and

$$\begin{aligned} \mu^*_{\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho])}(\pi_t'') &= \delta([\nu^{-b_1}\rho,\nu^{b_1}\rho])\otimes\delta([\nu^{b_1+1}\rho,\nu^{a_2}\rho]) \\ &\qquad\rtimes\delta([\nu^{-b_2}\rho,\nu^{b_1}\rho],[\nu^{-b_3}\rho,\nu^{a_3}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t. \end{aligned}$$

We check the first claim; the second is similar. Write $\mu^*(\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)=\sum_h\tau_h\otimes\theta_h$. Then,

$$\mu^*(\pi_t')=\sum_h\sum_{i=-b_1}^{b_1+1}\sum_{j=1}^{b_1+1}\delta([\nu^{-i+1}\rho,\nu^{b_1}\rho])\times\delta([\nu^{j}\rho,\nu^{b_1}\rho])\times\tau_h\otimes\delta([\nu^{i}\rho,\nu^{j-1}\rho])\rtimes\theta_h.$$

To get a contribution to $\mu^*_{\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho])}$, we must have a copy of $\nu^{-b_1}\rho$ appearing in $\delta([\nu^{-i+1}\rho,\nu^{b_1}\rho])\times\delta([\nu^{j}\rho,\nu^{b_1}\rho])\times\tau_h$. If it appears in $\delta([\nu^{-i+1}\rho,\nu^{b_1}\rho])$, we must have $i=b_1+1$, and therefore $j=j_1+1$ and $\tau_h=1$. This contributes one copy of $\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho])\otimes\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ to $\mu^*_{\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho])}(\pi_t')$. Similarly, we get a second copy when $j=-b_1$ (i.e., $\nu^{-b_1}\rho$ appears in $\delta([\nu^{j}\rho,\nu^{b_1}\rho])$). Finally, if τ_h contributes $\nu^{-b_1}\rho$, we must have $\tau_h=\delta([\nu^{-b_1}\rho,\nu^{-i}\rho])$ or $\delta([\nu^{-b_1}\rho,\nu^{j-1}\rho])$, whichever is appropriate. However, by Frobenius reciprocity, either of these would contradict the Casselman criteria for the square-integrability of $\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$. The claim follows.

We can now verify the lemma. By Frobenius reciprocity (or [Gol]), π_t' decomposes as the direct sum of at most two irreducible components. Further, π_t' and π_t'' have an irreducible subquotient in common. Therefore, to show that π'_t is reducible, it is enough to show that

$$\mu^*_{\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho])}(\pi'_t) \not\leq \mu^*_{\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho])}(\pi''_t),$$

or equivalently,

$$\begin{aligned} &2\cdot\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t \\ &\qquad \not\leq \delta([\nu^{b_1+1}\rho,\nu^{a_2}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{b_1}\rho],[\nu^{-b_3}\rho,\nu^{a_3}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t. \end{aligned}$$

However, a quick look at $\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_2}\rho])}$ for each of these (using Corollary 7.8 and Lemma 2.6, respectively) gives the desired result. Thus π'_t is reducible. That the two components are inequivalent follows from [Gol]. □

Lemma 7.5. *Suppose $a_k > \cdots > a_1 > b_1 > \cdots > b_k$ satisfy conditions 1–4 of Theorem 1.1. Suppose $b_1 \geq 0$. Then, $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ admits exactly two irreducible subrepresentations, and they are inequivalent. Furthermore, an irreducible subquotient π_1 of $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ appears as a subrepresentation if and only if $\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}(\pi_1) \neq 0$.*

Proof. First, it follows from Lemma 2.6 that

$$\begin{aligned} &\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t) \\ &\qquad = \delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \otimes \delta([\nu^{-b_1}\rho,\nu^{b_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t. \end{aligned}$$

Similarly, if we write $\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t = T_1 \oplus T_2$, then Lemma 2.6 also tells us

$$\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}(\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \rtimes T_i) = \delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \otimes T_i.$$

Therefore, by Frobenius reciprocity, $\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \rtimes T_i$ has a unique irreducible subrepresentation–call it S_i. Further, since $T_1 \not\cong T_2$, we have $S_1 \not\cong S_2$. Since

$$\begin{aligned} &\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t \\ &\quad \hookrightarrow \delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \rtimes (\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho]) \\ &\qquad \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t) \\ &\quad \cong \delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \rtimes (T_1 \oplus T_2) \end{aligned}$$

and S_1, S_2 appear with multiplicity one in $\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \rtimes (\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)$ (by $\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}$ considerations), we see that $S_1, S_2 \hookrightarrow \delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ (just consider the subspace of the space of $\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \rtimes (\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)$ generated by the sum of the subspaces for $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$, S_1, and S_2).

Now, it is an easy consequence of Frobenius reciprocity that $\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}$ must be nonzero for an irreducible subrepresentation of $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$. Therefore, S_1 and S_2 are the only irreducible subrepresentations. Further, since

$$\begin{aligned} &\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t) \\ &\qquad = \delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \otimes (T_1 \oplus T_2), \end{aligned}$$

we see that any irreducible subquotient of $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho], \dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ with $\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}$ nonzero must be S_1 or S_2, hence a subrepresentation. The lemma follows. □

Lemma 7.6. $\delta([\nu^{-a_1}\rho,\nu^{a_2}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{b_1}\rho],[\nu^{-b_3}\rho,\nu^{a_3}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ *admits exactly two irreducible subrepresentations, and they are inequivalent. Furthermore, an irreducible subquotient* π_1 *of* $\delta([\nu^{-a_1}\rho,\nu^{a_2}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{b_1}\rho], [\nu^{-b_3}\rho,\nu^{a_3}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ *appears as a subrepresentation if and only if* $\mu^*_{\delta([\nu^{a_1+1}\rho,\nu^{a_2}\rho])}(\pi_1) \neq 0$.

Proof. An argument like that in Lemma 7.4 tells us that $\pi'_t = \delta([\nu^{-a_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{b_1}\rho],[\nu^{-b_3}\rho,\nu^{a_3}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ decomposes as the direct sum of (exactly) two inequivalent irreducible representations. (To make the analogy precise, we take

$$\pi''_t = \delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_1}\rho],[\nu^{-b_3}\rho,\nu^{a_3}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$

and

$$\begin{aligned}\pi^*_t = \delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) &\times \delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])\\ &\rtimes \delta([\nu^{-b_2}\rho,\nu^{a_1}\rho],[\nu^{-b_3}\rho,\nu^{a_3}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t;\end{aligned}$$

we use $\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])\times\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}$ to show that π'_t,π''_t have a subquotient in common.) The proof then parallels that of Lemma 7.5 □

Theorem 7.7. *Suppose* **(H)***. Suppose that Theorem 7.7 (and Corollaries 7.8 and 7.9) is proved when the parabolic rank of the supercuspidal support is less than p.r. and that Theorem 1.1 is proved when the parabolic rank of the supercuspidal support is less than or equal to p.r. We continue to assume* $a_k > \cdots > a_1 > b_1 > \cdots > b_k$ *satisfy conditions 1–4 of Theorem 1.1. Then, we have the following:*

1. *For* $b_1 < 0$*:* $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)$ *has exactly two irreducible subquotients; call them* $\pi^{(0)},\pi^{(1)}$*. We have the following:*
   (a) $\pi^{(0)}$ *is the unique irreducible quotient (Langlands quotient). It is nontempered and has* $\delta_0(\pi^{(0)}) = \delta([\nu^{-a_1}\rho,\nu^{b_1}\rho]) \otimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho,\nu^{a_k}\rho]) \otimes \sigma$*.*
   (b) $\pi^{(1)}$ *is the unique irreducible subrepresentation. It is square-integrable and has* $\delta_0(\pi^{(1)}) = \delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \otimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho,\nu^{a_k}\rho]) \otimes \sigma$*.*

   In this case, we define $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma) = \pi^{(1)}$*.*
2. *For* $b_1 \geq 0$*,* $k=1$*:* $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \sigma$ *has three irreducible subquotients which we denote* $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho];\sigma)_1$*,* $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho];\sigma)_2$*, and* $L(\delta([\nu^{-a_1}\rho,\nu^{b_1}\rho]);\sigma)$*.* $L(\delta([\nu^{-a_1}\rho,\nu^{b_1}\rho]);\sigma)$ *is the unique irreducible quotient (Langlands quotient). It is nontempered and has* $\delta_0(L(\delta([\nu^{-a_1}\rho,\nu^{b_1}\rho]);\sigma)) = \delta([\nu^{-a_1}\rho,\nu^{b_1}\rho]) \otimes \sigma$*. Both* $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho];\sigma)_1$ *and* $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho];\sigma)_2$ *are subrepresentations and are square-integrable. We have* $\delta_0(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho];\sigma)_t) = \delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \otimes \sigma$ *for* $t=1,2$*. If* $\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho]) \rtimes \sigma = T_1([\nu^{-b_1}\rho,\nu^{b_1}\rho];\sigma) \oplus T_2([\nu^{-b_1}\rho,\nu^{b_1}\rho];\sigma)$ *(cf. Lemma 7.4), we may choose notation so that*

$$\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho];\sigma)_t) = \delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \otimes T_t([\nu^{-b_1}\rho,\nu^{b_1}\rho];\sigma).$$

(We note that in Theorem 4.7, [Tad6]*, $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho];\sigma)_1$ is the representation denoted $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],\sigma)$.)*

3. *For $b_1 \geq 0$, $k \geq 2$: $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ has exactly three irreducible subquotients; call them $\pi_t^{(0)},\pi_t^{(1)},\pi_t^{(2)}$. We have the following:*
   (a) *$\pi_t^{(0)}$ is the unique irreducible quotient (Langlands quotient). It is non-tempered and has $\delta_0(\pi_t^{(0)}) = \delta([\nu^{-a_1}\rho,\nu^{b_1}\rho]) \otimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho,\nu^{a_k}\rho]) \otimes \sigma$.*
   (b) *$\pi_t^{(1)}$ is a subrepresentation. It is square-integrable and has $\delta_0(\pi_t^{(1)}) = \delta([\nu^{-b_2}\rho,\nu^{b_1}\rho]) \otimes \delta([\nu^{-a_1}\rho,\nu^{a_2}\rho]) \otimes \delta([\nu^{-b_3}\rho,\nu^{a_3}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho,\nu^{a_k}\rho]) \otimes \sigma$. (Note that if $b_2 < 0$, then $b_2 = \beta$. In this case, if $b_1 = \beta - 1$, the first representation in $\delta_0(\pi_1)$ disappears.)*
   (c) *$\pi_t^{(2)}$ is a subrepresentation. It is square-integrable and has $\delta_0(\pi_t^{(2)}) = \delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \otimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho,\nu^{a_k}\rho]) \otimes \sigma$.*

   Furthermore, we note that if both are defined, $\pi_1^{(2)} \not\cong \pi_2^{(2)}$. We define $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t = \pi_t^{(2)}$.

Proof. We address the case $b_1 < 0$ first. In this case, we may write the induced representation as $\delta([\nu^{\beta}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{\beta+1}\rho,\nu^{a_2}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)$, with $\beta > 0$ and $k = \alpha - \beta + 1$. Observe that the third possibility in Proposition 7.2 cannot occur in this case. Thus, the only possibilities for $\delta_0(\pi_1)$ are those listed. Further, both $\delta([\nu^{\beta}\rho,\nu^{a_1}\rho]) \otimes \delta([\nu^{\beta+1}\rho,\nu^{a_2}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)$ and $\delta([\nu^{-a_1}\rho,\nu^{-\beta}\rho]) \otimes \delta([\nu^{\beta+1}\rho,\nu^{a_2}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)$ appear with multiplicity one in $\mu^*(\delta([\nu^{\beta}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{\beta+1}\rho,\nu^{a_2}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma))$. Thus there is at most one subquotient having $\delta([\nu^{\beta}\rho,\nu^{a_1}\rho]) \otimes \delta([\nu^{\beta+1}\rho,\nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{\alpha}\rho,\nu^{a_k}\rho]) \otimes \sigma$ as its δ_0, and similarly for $\delta([\nu^{-a_1}\rho,\nu^{-\beta}\rho]) \otimes \delta([\nu^{\beta+1}\rho,\nu^{a_2}\rho]) \otimes \cdots \otimes \delta([\nu^{\alpha}\rho,\nu^{a_k}\rho]) \otimes \sigma$. Once we show that $\delta([\nu^{\beta}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{\beta+1}\rho,\nu^{a_2}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)$ is reducible, the $b_1 < 0$ case will follow. If $k = 1$, this follows from Theorem 13.2, [Tad3]; so suppose $k \geq 2$.

Let

$$\begin{aligned}
\pi' &= \delta([\nu^{\beta}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{\beta+1}\rho,\nu^{a_2}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma),\\
\pi'' &= \delta([\nu^{\beta}\rho,\nu^{a_2}\rho]) \rtimes \delta([\nu^{\beta+1}\rho,\nu^{a_1}\rho],[\nu^{\beta+2}\rho,\nu^{a_3}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma),\\
\pi^* &= \delta([\nu^{\beta}\rho,\nu^{a_1}\rho]) \times \delta([\nu^{a_1+1}\rho,\nu^{a_2}\rho])\\
&\quad \rtimes \delta([\nu^{\beta+1}\rho,\nu^{a_1}\rho],[\nu^{\beta+2}\rho,\nu^{a_3}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma).
\end{aligned}$$

It follows easily from Corollary 7.8 (using the inductive hypothesis) that $\pi',\pi'' \leq \pi^*$. Next, we claim

$$\begin{aligned}
\mu^*_{\delta([\nu^{\beta}\rho,\nu^{a_2}\rho])}(\pi') &= \mu^*_{\delta([\nu^{\beta}\rho,\nu^{a_2}\rho])}(\pi'') = \mu^*_{\delta([\nu^{\beta}\rho,\nu^{a_2}\rho])}(\pi^*)\\
&= \delta([\nu^{\beta}\rho,\nu^{a_2}\rho]) \otimes \delta([\nu^{\beta+1}\rho,\nu^{a_1}\rho],[\nu^{\beta+2}\rho,\nu^{a_3}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma).
\end{aligned}$$

Clearly, $\mu^*_{\delta([\nu^{\beta}\rho,\nu^{a_2}\rho])}(\pi'') \neq 0$; it follows easily from Corollary 7.8 (and the inductive hypothesis) that $\mu^*_{\delta([\nu^{\beta}\rho,\nu^{a_2}\rho])}(\pi') \neq 0$. Since $\pi',\pi'' \leq \pi^*$, it then suffices to verify the claim for π^*. Write $\mu^*(\delta([\nu^{\beta+1}\rho,\nu^{a_1}\rho],[\nu^{\beta+2}\rho,\nu^{a_3}\rho],\dots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)) =$

$\sum_h \tau_h \otimes \theta_h$. Then,

$$\mu^*(\pi^*) = \sum_h \sum_{i_1=\beta}^{a_1+1} \sum_{j_1=i_1}^{a_1+1} \sum_{i_2=a_1+1}^{a_2+1} \sum_{j_2=i_2}^{a_2+1} [\delta([\nu^{-i_1+1}\rho, \nu^{-\beta}\rho]) \times \delta([\nu^{j_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-i_2+1}\rho, \nu^{-a_1-1}\rho]) \times \delta([\nu^{j_2}\rho, \nu^{a_2}\rho]) \times \tau_h] \otimes \delta([\nu^{i_1}\rho, \nu^{j_1-1}\rho]) \times \delta([\nu^{i_2}\rho, \nu^{j_2-1}\rho]) \rtimes \theta_h.$$

To contribute to $\mu^*_{\delta([\nu^\beta\rho,\nu^{a_2}\rho])}$, we must certainly have $i_1 = \beta$, $i_2 = a_1+1$. Also, since neither $\delta([\nu^{j_2}\rho, \nu^{a_2}\rho])$ nor τ_h can contain $\nu^\beta\rho$ in their supercuspidal support, we must have $j_1 = \beta$. Since $\nu^x\rho\otimes \leq r_{min}(\tau_h)$ has $x \in \{a_1, a_3, \ldots, a_k\}$—in particular, we do not have $a_1 + 1 \leq x \leq a_2$—we see that $\tau_h = 1$. Therefore, $j_2 = a_1 + 1$; the claim follows. As a consequence, π' and π'' have an irreducible subquotient in common. On the other hand, it is not difficult to show that

$$\mu^*_{\delta([\nu^\beta\rho,\nu^{a_1}\rho])}(\pi'') = 0,$$

so that $\pi' \not\leq \pi''$. Therefore, π' is reducible. The $b_1 < 0$ case is now done.

We now turn to the case $b_1 \geq 0$. For $k = 1$, the reducibility and δ_0 claims follow from Proposition 7.2, Corollary 7.3, and Lemma 7.5 of [Tad6]. Suppose $k \geq 2$. Let

$$\begin{aligned}
\pi'_t &= \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t, \\
\pi''_t &= \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t, \\
\pi^*_t &= \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{b_1+1}\rho, \nu^{a_2}\rho]) \\
&\quad \rtimes \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t.
\end{aligned}$$

First, we observe that the argument from the proof of Lemma 4.2 tells us

$$\mu^*_{\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho])}(\pi'_t) = 2 \cdot \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t.$$

Therefore, there are at most two irreducible subquotients having δ_0 as in 1. of Proposition 7.2.

Next, we claim that

$$\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{a_2}\rho])}(\pi'_t) = \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \otimes \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t.$$

Again, with $\mu^*(\delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) = \sum_h \tau_h \otimes \theta_h$, we have

$$\mu^*(\pi'_t) = \sum_h \sum_{i=-b_1}^{a_1+1} \sum_{j=i}^{a_1+1} \delta([\nu^{-i+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^j\rho, \nu^{a_1}\rho]) \times \tau_h \otimes \delta([\nu^i\rho, \nu^{j-1}\rho]) \rtimes \theta_h.$$

To get a contribution to $\mu^*_{\delta([\nu^{-a_1}\rho,\nu^{a_2}\rho])}$, we need to have a copy of $\nu^{-a_1}\rho$ in either $\delta([\nu^{-i+1}\rho, \nu^{b_1}\rho])$, $\delta([\nu^j\rho, \nu^{a_1}\rho])$, or τ_h. Since $j \geq -b_1$, it cannot come from $\delta([\nu^j\rho, \nu^{a_1}\rho])$. Suppose τ_h contributed the $\nu^{-a_1}\rho$. Then, we must have $\mu^*_{\delta([\nu^{-a_1}\rho,\nu^x\rho])}(\delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) \neq 0$ for some $a_2 \geq x \geq -a_1$. In order to avoid contradicting the Casselman criteria for the square-integrability of $\delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$, we must have $x > a_1$. Since we must also have $x \in \{b_2, \ldots, b_k, a_2, \ldots, a_k\}$, we have a contradiction. Thus, τ_h cannot contribute the $\nu^{-a_1}\rho$. Finally, to have a $\nu^{-a_1}\rho$ in $\delta([\nu^{-i+1}\rho, \nu^{b_1}\rho])$, we need $i = a_1 + 1$. Then, $j = a_1 + 1$ and $\tau_h = \delta([\nu^{b_1+1}\rho, \nu^{a_2}\rho])$. Therefore, by Corollary 7.8 (and the inductive hypothesis), $\theta_h = \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho], [\nu^{-b_3}\rho, \nu^{b_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$. The claim follows.

Now, we check that if π_1 is an irreducible representation which has $\delta_0(\pi_1) = \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \otimes \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \otimes \cdots \otimes \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma$, then $\mu^*_{\delta([\nu^{-a_1}\rho, \nu^{a_2}\rho])}(\pi_1) \neq 0$. Since $\delta([\nu^{-b_2}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho])$ is irreducible, the usual commuting argument tells us

$$\begin{aligned}\pi_1 \quad &\hookrightarrow \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \\ &\qquad \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma \\ &\cong \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{-b_3}\rho, \nu^{a_3}\rho]) \\ &\qquad \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma.\end{aligned}$$

The claim follows. It now follows that there is at most one irreducible subquotient of $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$ having δ_0 as in 3 of Proposition 7.2. Thus $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$ has at most four irreducible subquotients.

A straightforward argument like that in Lemma 5.2.5, [Jan4] (or Lemma 5.1 above) tells us $\pi'_t, \pi''_t \leq \pi^*_t$ and

$$\begin{aligned}\mu^*_{\delta([\nu^{b_1+1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{b_1+1}\rho, \nu^{a_2}\rho])}(\pi'_t) &= \mu^*_{\delta([\nu^{b_1+1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{b_1+1}\rho, \nu^{a_2}\rho])}(\pi''_t) \\ &= \mu^*_{\delta([\nu^{b_1+1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{b_1+1}\rho, \nu^{a_2}\rho])}(\pi^*_t) \\ &\neq 0.\end{aligned}$$

Thus π'_t and π''_t have an irreducible subquotient in common. Let $\pi^{(1)}_t$ denote such a representation (there will turn out to be only one possibility). By Lemmas 7.5 and 7.6, $\pi^{(1)}_t$ is a subrepresentation of both π'_t and π''_t. Then, by Frobenius reciprocity, $\pi^{(1)}_t$ contains both a copy of $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$ and $\delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \otimes \delta([\nu^{-b_2}\rho, \nu^{b_1}\rho], [\nu^{-b_3}\rho, \nu^{a_3}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$. Thus, there are at most three irreducible subquotients. By Lemma 7.5, there are exactly three irreducible subquotients. The claims about δ_0 follow.

Finally, the fact that $\pi^{(2)}_1 \not\cong \pi^{(2)}_2$ follows immediately from the observation that $\mu^*_{\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho])}(\pi^{(2)}_t) = \delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t$. □

Corollary 7.8. *With assumptions as in Theorem 7.7, we have the following:*

1. *For* $max\{-b_1 - 1, b_1\} < c < a_1$,

$$\begin{aligned}&\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho], [\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t \\ &\qquad \hookrightarrow \delta([\nu^{c+1}\rho, \nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_1}\rho, \nu^{c}\rho], [\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t.\end{aligned}$$

   Further, we have

$$\begin{aligned}&\mu^*_{\delta([\nu^{c+1}\rho, \nu^{a_1}\rho])}(\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho], [\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t) \\ &\quad = \delta([\nu^{c+1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b_1}\rho, \nu^{c}\rho], [\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t.\end{aligned}$$

2. *For* $b_2 < d < b_1$ *with* $d \geq \beta - 1$,

$$\begin{aligned}&\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho], [\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t \\ &\qquad \hookrightarrow \delta([\nu^{d+1}\rho, \nu^{b_1}\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^{a_1}\rho], [\nu^{-b_2}\rho, \nu^{a_2}\rho], \ldots, [\nu^{-b_k}\rho, \nu^{a_k}\rho]; \sigma)_t.\end{aligned}$$

(If $k=1$, the conditions on d reduce to $\alpha \leq d < b_1$.) Further, we have

$$\mu^*_{\delta([\nu^{d+1}\rho,\nu^{b_1}\rho])}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)$$
$$=\delta([\nu^{d+1}\rho,\nu^{b_1}\rho])\otimes\delta([\nu^{-d}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t.$$

Further, we may extend 1 to the case where $c=b_1$ as follows: by Lemma 7.4 write

$$\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho])\rtimes\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$
$$\cong T_1([\nu^{-b_1}\rho,\nu^{b_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$
$$\oplus T_2([\nu^{-b_1}\rho,\nu^{b_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t.$$

Then, there is a component

$$\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$
$$=T_i([\nu^{-b_1}\rho,\nu^{b_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$

such that

$$\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$
$$\hookrightarrow\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])\rtimes\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t.$$

We note that $\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ is well-defined (i.e., the choice of components does not depend on a_1). Further, we have

$$\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)$$
$$=\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])\otimes\delta([\nu^{-b_1}\rho,\nu^{b_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\dots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t.$$

Proof. First, we note that if we let

$$\tau=\nu^{a_1}\rho\otimes\nu^{a_1-1}\rho\otimes\cdots\otimes\nu^{b_1+1}\rho\otimes(\nu^{b_1}\rho\otimes\nu^{b_1}\rho)\otimes(\nu^{b_1-1}\rho\otimes\nu^{b_1-1}\rho)$$
$$\otimes\cdots\otimes(\nu^{\alpha+1}\rho\otimes\nu^{\alpha+1}\rho),$$

then $s_\tau(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho];\sigma)_t)$ is a nonzero multiple of $\tau\otimes T_t([\nu^{-\alpha}\rho,\nu^{\alpha}\rho];\sigma)$. To see this, observe that from Lemma 7.5, the definition of $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho];\sigma)_t$, and the proof of Lemma 7.4, we have

$$\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho];\sigma)_t\hookrightarrow\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])\rtimes T_t([\nu^{-b_1}\rho,\nu^{b_1}\rho];\sigma)$$
$$\hookrightarrow\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])\times\delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho])\times\delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho])\rtimes T_t([\nu^{-\alpha}\rho,\nu^{\alpha}\rho];\sigma).$$

We may now conclude that $s_\tau(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho];\sigma)_t)$ contains a nonzero multiple of $\tau\otimes T_t([\nu^{-\alpha}\rho,\nu^{\alpha}\rho];\sigma)$ from the fact that $r_{min}(\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])\otimes\delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho])\times\delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho]))$ contains τ and Frobenius reciprocity. On the other hand, the same arguments as in the proof of Lemma 2.6 (part 1) tell us $s_\tau(\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])\times\delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho])\times\delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho])\rtimes T_t([\nu^{-\alpha}\rho,\nu^{\alpha}\rho];\sigma))$ can only contain multiples of $\tau\otimes T_t([\nu^{-\alpha}\rho,\nu^{\alpha}\rho];\sigma)$.

For 1, we have

$$\begin{aligned}
&\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t\\
&\quad\hookrightarrow \delta([\nu^{-b_1}\rho,\nu^{a_1}\rho])\rtimes\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t\\
&\quad\hookrightarrow \delta([\nu^{c+1}\rho,\nu^{a_1}\rho])\times\delta([\nu^{-b_1}\rho,\nu^{c}\rho])\rtimes\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t\\
&\quad\Downarrow\\
&\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t\hookrightarrow\delta([\nu^{c+1}\rho,\nu^{a_1}\rho])\rtimes\theta
\end{aligned}$$

for some irreducible $\theta\leq\delta([\nu^{-b_1}\rho,\nu^{c}\rho])\rtimes\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ (cf. Lemma 5.5, [Jan2]). The possibilities for θ are given in the preceding theorem. For $k\geq 2$, we see that since

$$\delta([\nu^{c+1}\rho,\nu^{a_1}\rho])\otimes\delta_0(\theta)\leq s_{app}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t),$$

only $\theta=\delta([\nu^{-b_1}\rho,\nu^{c}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ does not end up contradicting the minimality of $\delta_0(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)$ ($t.e.$ considerations are enough). When $k=1$, we see that $\theta=\delta([\nu^{-b_1}\rho,\nu^{c}\rho];\sigma)_{t'}$ for some t'. From the definition of $\delta([\nu^{-b_1}\rho,\nu^{c}\rho];\sigma)_{t'}$ (cf. Theorem 7.7), we see that

$$\begin{aligned}
&\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho];\sigma)_t\hookrightarrow\delta([\nu^{c+1}\rho,\nu^{a_1}\rho])\rtimes\delta([\nu^{-b_1}\rho,\nu^{c}\rho];\sigma)_{t'}\\
&\qquad\Downarrow\\
&\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{c}\rho])\times\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho];\sigma)_t)\\
&\qquad=\delta([\nu^{b_1+1}\rho,\nu^{c}\rho])\times\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])\otimes T_{t'}([\nu^{-b_1}\rho,\nu^{b_1}\rho];\sigma).
\end{aligned}$$

By Lemma 7.4,

$$\begin{aligned}
&s_{\delta([\nu^{b_1+1}\rho,\nu^{c}\rho])\times\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])\otimes\delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho])\times\delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho])}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho];\sigma)_t)\\
&\quad\geq\delta([\nu^{b_1+1}\rho,\nu^{c}\rho])\times\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])\otimes\delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho])\\
&\qquad\times\delta([\nu^{\alpha+1}\rho,\nu^{b_1}\rho])\otimes T_{t'}([\nu^{-\alpha}\rho,\nu^{\alpha}\rho];\sigma),
\end{aligned}$$

so that $s_\tau(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho];\sigma)_t)\geq\tau\otimes T_{t'}([\nu^{-\alpha}\rho,\nu^{\alpha}\rho];\sigma)$. From the discussion above, this forces $t'=t$. The claim about $\mu^*_{\delta([\nu^{c+1}\rho,\nu^{a_1}\rho])}$ is now straightforward; by Frobenius reciprocity,

$$\begin{aligned}
&\mu^*_{\delta([\nu^{c+1}\rho,\nu^{a_1}\rho])}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)\\
&\qquad\geq\delta([\nu^{c+1}\rho,\nu^{a_1}\rho])\otimes\delta([\nu^{-b_1}\rho,\nu^{c}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t.
\end{aligned}$$

On the other hand, it follows from Lemma 2.6 that

$$\begin{aligned}
&\mu^*_{\delta([\nu^{c+1}\rho,\nu^{a_1}\rho])}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)\\
&\qquad\leq\delta([\nu^{c+1}\rho,\nu^{a_1}\rho])\otimes\delta([\nu^{-b_1}\rho,\nu^{c}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t.
\end{aligned}$$

The claim follows.

For 2, we have

$$\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$

$$\hookrightarrow \delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$

$$\hookrightarrow \delta([\nu^{-d}\rho,\nu^{a_1}\rho]) \times \delta([\nu^{-b_1}\rho,\nu^{-d-1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$

(by Lemma 7.1)

$$\cong \delta([\nu^{-d}\rho,\nu^{a_1}\rho]) \times \delta([\nu^{d+1}\rho,\nu^{b_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$

$$\cong \delta([\nu^{d+1}\rho,\nu^{b_1}\rho]) \times \delta([\nu^{-d}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$

$$\Downarrow$$

$$\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t \hookrightarrow \delta([\nu^{d+1}\rho,\nu^{b_1}\rho]) \rtimes \theta$$

for some irreducible $\theta \leq \delta([\nu^{-d}\rho,\nu^{a_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ (cf. Lemma 5.5, [Jan2]). The possibilities for θ are given in the preceding theorem. For $k \geq 2$, we see that since

$$\delta([\nu^{d+1}\rho,\nu^{b_1}\rho]) \otimes \delta_0(\theta) \leq s_{app}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t),$$

only $\theta = \delta([\nu^{-d}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ does not end up contradicting the minimality of $\delta_0(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)$ (total exponent considerations are enough). For $k=1$, an argument like that used in the proof of part 1 above tells us $\theta = \delta([\nu^{-d}\rho,\nu^{a_1}\rho];\sigma)_t$ The claim about $\mu^*_{\delta([\nu^{c+1}\rho,\nu^{a_1}\rho])}$ is now straightforward: by Frobenius reciprocity,

$$\mu^*_{\delta([\nu^{d+1}\rho,\nu^{b_1}\rho])}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)$$

$$\geq \delta([\nu^{d+1}\rho,\nu^{b_1}\rho]) \otimes \delta([\nu^{-d}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t.$$

On the other hand, it follows from Lemma 2.6 that

$$\mu^*_{\delta([\nu^{d+1}\rho,\nu^{b_1}\rho])}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)$$

$$\leq \delta([\nu^{d+1}\rho,\nu^{b_1}\rho]) \otimes \delta([\nu^{-d}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t.$$

The claim follows.

Finally, the existence of a unique

$$T_i([\nu^{-b_1}\rho,\nu^{b_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$

$$\leq \delta([\nu^{-b_1}\rho,\nu^{b_1}\rho]) \rtimes \delta([\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$

such that

$$\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$

$$\hookrightarrow \delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho]) \rtimes T_i([\nu^{-b_1}\rho,\nu^{b_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$

follows from the proof of Lemma 7.5. To see that the choice of $T_i([\nu^{-b_1}\rho,\nu^{b_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ does not depend on a_1, observe that from 1,

$$\mu^*(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)$$

$$\geq \delta([\nu^{b_1+2}\rho,\nu^{a_1}\rho]) \otimes \delta([\nu^{-b_1}\rho,\nu^{b_1+1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t.$$

So, whichever choice of $T_i([\nu^{-b_1}\rho,\nu^{b_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$ works for $a_1=b_1+1$ works in general. The $\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])}$ claim now follows from the proof of Lemma 7.5. ☐

Corollary 7.9. *With assumptions as in Theorem 7.7, we have the following:*

1. $\mu^*_{\delta([\nu^{a_1+1}\rho,\nu^{a_2}\rho])}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)=0$,
2. $\mu^*_{\delta([\nu^{b_2+1}\rho,\nu^{b_1}\rho])}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t)=0$.

Proof. Write $\pi=\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho],[\nu^{-b_2}\rho,\nu^{a_2}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$. Let us focus on part 2 of the corollary. Observe that if $b_1\le 0$, the fact that $\mu^*_{\delta([\nu^{b_2+1}\rho,\nu^{b_1}\rho])}(\pi)=0$ (when $\delta([\nu^{b_2+1}\rho,\nu^{b_1}\rho])$ makes sense) follows immediately from the Casselman criteria. Thus we may assume $b_1>0$.

By Lemma 5.5, [Jan2],

$$\pi\hookrightarrow\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho])\times\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])\rtimes\delta([\nu^{-b_3}\rho,\nu^{a_3}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$
$$\Downarrow$$
$$\pi\hookrightarrow\delta([\nu^{-b_1}\rho,\nu^{a_2}\rho])\times\delta([\nu^{-b_2}\rho,\nu^{a_1}\rho])\rtimes\delta([\nu^{-b_3}\rho,\nu^{a_3}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$$

or

$$\pi\hookrightarrow\mathcal{L}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]),\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho]))\rtimes\delta([\nu^{-b_3}\rho,\nu^{a_3}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t.$$

We now show that the first of these does not occur. Suppose it did. Then Lemma 5.5, [Jan2] tells us $\pi\hookrightarrow\delta([\nu^{-b_1}\rho,\nu^{a_2}\rho])\rtimes\pi_i$ for some $\pi_i\le\delta([\nu^{-b_2}\rho,\nu^{a_1}\rho])\rtimes\delta([\nu^{-b_3}\rho,\nu^{a_3}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t$. By Theorem 7.7, we know $\delta_0(\pi_i)$ for such π_i. Since

$$\mu^*(\pi)\ge\delta([\nu^{-b_1}\rho,\nu^{a_2}\rho])\otimes\pi_i,$$

we see that only $\pi_i=\delta([\nu^{-b_2}\rho,\nu^{a_1}\rho],[\nu^{-b_3}\rho,\nu^{a_3}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_{t'}$ does not produce a contradiction to the minimality of $\delta_0(\pi)$. Note that if $k>2$, we have $t'=t$ automatically. (If $k=2$, we could argue that $t'=t$, but it is not needed.) By the same argument as in Lemma 5.2.5, [Jan4] (also cf. Lemma 6.4 above), there is a unique irreducible representation having $\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])\times\delta([\nu^{b_1+1}\rho,\nu^{a_2}\rho])}\ne 0$ and which is a subquotient of both

$$\delta([\nu^{-b_1}\rho,\nu^{a_2}\rho])\rtimes\delta([\nu^{-b_2}\rho,\nu^{a_1}\rho],[\nu^{-b_3}\rho,\nu^{a_3}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_{t'}$$

and

$$\delta([\nu^{-a_1}\rho,\nu^{a_2}\rho])\rtimes\delta([\nu^{-b_2}\rho,\nu^{b_1}\rho],[\nu^{-b_3}\rho,\nu^{a_3}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_{t'}$$

(use the fact that both induced representations are $\le\pi^*_{t'}=\delta([\nu^{-b_1}\rho,\nu^{a_2}\rho])\times\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])\rtimes\delta([\nu^{-b_2}\rho,\nu^{b_1}\rho],[\nu^{-b_3}\rho,\nu^{a_3}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_{t'}$). Then, since $\mu^*_{\delta([\nu^{-b_1}\rho,\nu^{a_2}\rho])\times\delta([\nu^{-b_2}\rho,\nu^{a_1}\rho])}(\pi)\ne 0$ implies $\mu^*_{\delta([\nu^{b_1+1}\rho,\nu^{a_1}\rho])\times\delta([\nu^{b_1+1}\rho,\nu^{a_2}\rho])}(\pi)\ne 0$, we see that this subquotient must be π. Further, since $\mu^*_{\delta([\nu^{-b_1}\rho,\nu^{a_2}\rho])}(\pi)\ne 0$ implies $\mu^*_{\delta([\nu^{a_1+1}\rho,\nu^{a_2}\rho])}(\pi)\ne 0$, Lemma 7.6 tells us

$$\pi\hookrightarrow\delta([\nu^{-a_1}\rho,\nu^{a_2}\rho])\rtimes\delta([\nu^{-b_2}\rho,\nu^{b_1}\rho],[\nu^{-b_3}\rho,\nu^{a_3}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_{t'},$$

contradicting the minimality of $\delta_0(\pi)$. Therefore, the first possibility above cannot occur, and we must have

$$\pi\hookrightarrow\mathcal{L}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]),\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho]))\rtimes\delta([\nu^{-b_3}\rho,\nu^{a_3}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t.$$

Now, let us focus on the case when $b_2 \geq 0$. In fact, we show a bit more; we show that

$$\mu^*_{\delta([\nu^{b_2+1}\rho,\nu^{b_1}\rho])}(\mathcal{L}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]),\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])) \rtimes \delta([\nu^{-b_3}\rho,\nu^{a_3}\rho],\ldots,[\nu^{-b_k}\rho,\nu^{a_k}\rho];\sigma)_t) = 0.$$

By Lemma 2.6, it is enough to show that

$$M^*_{\delta([\nu^{b_2+1}\rho,\nu^{b_1}\rho])}(\mathcal{L}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]),\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho]))) = 0.$$

Now, consider any term in $r_{min}(\mathcal{L}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]),\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho])))$. Observe that there are two copies of $\nu^{b_2+1}\rho$ in such a term; both will always have copies of $\nu^{a_1}\rho$ appearing to their left. There is one copy of $\nu^{-b_2-1}\rho$; it has a copy of $\nu^{-b_2}\rho$ to its right. Thus, any term in $r_{min}(M^*(\mathcal{L}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]),\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho]))))$ can have up to three copies of $\nu^{b_2+1}\rho$, but each must have either $\nu^{a_1}\rho$ or $\nu^{b_2}\rho$ to its left. In particular, there are no terms of the form $(\nu^{b_1}\rho\otimes\nu^{b_1-1}\rho\otimes\cdots\otimes\nu^{b_2+1}\rho)\otimes\ldots$ in $r_{min}(M^*(\mathcal{L}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]),\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho]))))$. Therefore,

$$M^*_{\delta([\nu^{b_2+1}\rho,\nu^{b_1}\rho])}(\mathcal{L}(\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho]),\delta([\nu^{-b_2}\rho,\nu^{a_2}\rho]))) = 0.$$

Part 2 of the corollary follows. The argument when $b_2 < 0$ is similar.

Part 1 of the corollary when $b_1 \geq 0$ is similar to the argument above. When $b_1 < 0$, we still have

$$\pi \hookrightarrow \delta([\nu^{\beta}\rho,\nu^{a_2}\rho])\times\delta([\nu^{\beta+1}\rho,\nu^{a_1}\rho])\rtimes\delta([\nu^{\beta+2}\rho,\nu^{a_3}\rho],\ldots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)$$

or

$$\pi \hookrightarrow \mathcal{L}(\delta([\nu^{\beta}\rho,\nu^{a_1}\rho]),\delta([\nu^{\beta+1}\rho,\nu^{a_2}\rho]))\rtimes\delta([\nu^{\beta+2}\rho,\nu^{a_3}\rho],\ldots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma).$$

In this case, we can eliminate the first possibility more directly:

$$\begin{aligned}
\pi \hookrightarrow\ & \delta([\nu^{\beta+1}\rho,\nu^{a_1}\rho])\times\delta([\nu^{\beta}\rho,\nu^{a_2}\rho])\rtimes\delta([\nu^{\beta+2}\rho,\nu^{a_3}\rho],\ldots,[\nu^{\alpha}\rho,\nu^{a_k}\rho];\sigma)\\
\hookrightarrow\ & \delta([\nu^{\beta+1}\rho,\nu^{a_1}\rho])\times\delta([\nu^{\beta+1}\rho,\nu^{a_2}\rho])\times\nu^{\beta}\rho\times\delta([\nu^{\beta+2}\rho,\nu^{a_3}\rho])\\
&\qquad\times\cdots\times\delta([\nu^{\alpha}\rho,\nu^{a_k}\rho])\rtimes\sigma\\
\cong\ & \delta([\nu^{\beta+1}\rho,\nu^{a_1}\rho])\times\delta([\nu^{\beta+1}\rho,\nu^{a_2}\rho])\times\delta([\nu^{\beta+2}\rho,\nu^{a_3}\rho])\\
&\qquad\times\cdots\times\delta([\nu^{\alpha}\rho,\nu^{a_k}\rho])\times\nu^{\beta}\rho\rtimes\sigma\\
\cong\ & \delta([\nu^{\beta+1}\rho,\nu^{a_1}\rho])\times\delta([\nu^{\beta+1}\rho,\nu^{a_2}\rho])\times\delta([\nu^{\beta+2}\rho,\nu^{a_3}\rho])\\
&\qquad\times\cdots\times\delta([\nu^{\alpha}\rho,\nu^{a_k}\rho])\times\nu^{-\beta}\rho\rtimes\sigma,
\end{aligned}$$

which contradicts the minimality of $\delta_0(\pi)$ (by Frobenius reciprocity and *t.e.* considerations). The rest of this case is similar to the argument above. □

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