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# ON LAGUERRE POLYNOMIALS, BESSEL FUNCTIONS, HANKEL TRANSFORM AND A SERIES IN THE UNITARY DUAL OF THE SIMPLY-CONNECTED COVERING GROUP OF $Sl(2,\mathbb{R})$

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Abstract. Analogous to the holomorphic discrete series of $Sl(2,\mathbb{R})$ there is a continuous family $\{\pi_r\}$, $-1 < r < \infty$, of irreducible unitary representations of $G$, the simply-connected covering group of $Sl(2,\mathbb{R})$. A construction of this series is given in this paper using classical function theory. For all $r$ the Hilbert space is $L_2((0,\infty))$. First of all one exhibits a representation, $D_r$, of $\mathfrak{g} = Lie\, G$ by second order differential operators on $C^\infty((0,\infty))$. For $x \in (0,\infty)$, $-1 < r < \infty$ and $n \in \mathbb{Z}_+$ let $\varphi_n^{(r)}(x) = e^{-x}x^{\frac{r}{2}}L_n^{(r)}(2x)$ where $L_n^{(r)}(x)$ is the Laguerre polynomial with parameters $\{n,r\}$. Let $\mathcal{H}_r^{HC}$ be the span of $\varphi_n^{(r)}$ for $n \in \mathbb{Z}_+$. Next one shows, using a famous result of E. Nelson, that $D_r|\mathcal{H}_r^{HC}$ exponentiates to the unitary representation $\pi_r$ of $G$. The power of Nelson's theorem is exhibited here by the fact that if $0 < r < 1$, one has $D_r = D_{-r}$, whereas $\pi_r$ is inequivalent to $\pi_{-r}$. For $r = \frac{1}{2}$, the elements in the pair $\{\pi_{\frac{1}{2}}, \pi_{-\frac{1}{2}}\}$ are the two components of the metaplectic representation. Using a result of G.H. Hardy one shows that the Hankel transform is given by $\pi_r(a)$ where $a \in G$ induces the non-trivial element of a Weyl group. As a consequence, continuity properties and enlarged domains of definition, of the Hankel transform follow from standard facts in representation theory. Also, if $J_r$ is the classical Bessel function, then for any $y \in (0,\infty)$, the function $J_{r,y}(x) = J_r(2\sqrt{xy})$ is a Whittaker vector. Other weight vectors are given and the highest weight vector is given by a limiting behavior at 0.

## 0. Introduction

0.1. Throughout this paper $r$ is a real number where $-1 < r < \infty$. The classical Laguerre polynomials, $\{L_n^{(r)}(x)\}$, are defined for those values of $r$ and non-negative integers $n$. See e.g. [Ja], p. 184 or [Sz], p. 96. We will take the normalization as defined in [Ja]. Let $\mathcal{H}$ be the Hilbert space $L_2((0,\infty))$ with respect to Lebesgue measure $dx$. Let $\varphi_n^{(r)} \in C^\infty((0,\infty))$ be defined by putting $\varphi_n^{(r)}(x) = e^{-x}x^{\frac{r}{2}}L_n^{(r)}(2x)$. We refer to the $\{\varphi_n^{(r)}\}$ as Laguerre functions. Then for each value of $r$ the subset $\{\varphi_n^{(r)}\}$, $n = 0,1,...$, of Laguerre functions is an orthogonal basis of $\mathcal{H}$. In particular, $\mathcal{H}_r^{HC}$ is a dense subspace of $\mathcal{H}$ where $\mathcal{H}_r^{HC}$ is defined to be the linear span of this subset.

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Let $\mathfrak{g} = Lie\, Sl(2,\mathbb{R})$ and let $U(\mathfrak{g})$ be the universal enveloping algebra over $\mathfrak{g}$ with coefficients in $\mathbb{C}$. Let $\{h,e,f\} \subset \mathfrak{g}$ be the $S$-triple basis of $\mathfrak{g}$ where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Let $G$ be a simply-connected Lie group with Lie algebra $\mathfrak{g}$. The center $Z$ of $G$ is infinite cyclic and one identifies $Sl(2,\mathbb{R})$ with $G/Z^2$. The metaplectic group, $Mp(2,\mathbb{R})$, the double cover of $Sl(2,\mathbb{R})$, is identified with $G/Z^4$. Conjugation of $\mathfrak{g}$ by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ defines an outer automorphism $\kappa$ of $G$.

Let $\mathcal{L}$ be a Hilbert space and assume that $\pi\colon G \to U(\mathcal{L})$ is an irreducible unitary representation of $G$. Let $\mathcal{L}^\infty \subset \mathcal{L}$ be the dense subspace of infinitely differentiable vectors in $\mathcal{L}$. From the general theory of unitary representations the "differential" of $\pi$ induces a representation

$$\pi^\infty\colon U(\mathfrak{g}) \to \operatorname{End} \mathcal{L}^\infty.$$

The group $G$ is not in the Harish-Chandra category because $Z$ is infinite. In particular, even though the element $e - f \in \mathfrak{g}$ is elliptic, the subgroup $K$ of $G$ corresponding to $\mathbb{R}(e-f)$ is not compact. Nevertheless, much of the Harish-Chandra theory is still valid. In particular, if $h' = -i(e-f)$, then the linear span, $\mathcal{L}^{HC}$, of the $\pi^\infty(h')$ eigenvectors in $\mathcal{L}^\infty$ is a dense $U(\mathfrak{g})$-submodule of $\mathcal{L}^\infty$. We refer to $\mathcal{L}^{HC}$ as the Harish-Chandra module associated to $\pi$. We will say that $\pi$ is one-sided if the spectrum of $\pi^\infty(h')|\mathcal{L}^{HC}$ is either a set of positive numbers or a set of negative numbers. To distinguish between these two cases, and adopting terminology which is valid for the discrete representations of $Sl(2,\mathbb{R})$, we will say that $\pi$ is holomorphic (resp. anti-holomorphic) if the spectrum of $\pi^\infty(h')|\mathcal{L}^{HC}$ is a set of positive (resp. negative) numbers. It is immediate that $\pi$ is holomorphic if and only if $\pi^\kappa$ is anti-holomorphic, where for $g \in G$, $\pi^\kappa(g) = \pi(\kappa(g))$.

Lajos Pukanszky in [Pu] has determined, among other things, the unitary dual of $G$. See p. 102 in [Pu]. There are 3 series of representations. The one-sided representations are exactly the members of the second series. We are concerned, in the present paper, with a construction, and resulting harmonic analysis, of a model, using Laguerre functions, for the members of this second series. Using $\kappa$ it is enough to give the model for the members of the holomorphic subfamily. Due to a sign difference the members of this subfamily are denoted, in [Pu], by $D_\ell^-$ where $\ell > 0$. In the present paper these representations will appear as $\pi_r$ where, as stated in the first sentence of the introduction, $-1 < r < \infty$. (The letter $D$ will be reserved here for another purpose.) The relation between the parameter $\ell$ and $r$ is $r = 2\ell - 1$.

0.2. Let $Diff\, C^\infty((0,\infty))$ be the algebra of all differential operators on $(0,\infty)$. Let $x \in C^\infty((0,\infty))$ be the natural coordinate function. Identifying elements of $C^\infty((0,\infty))$ with the corresponding multiplication operators, it is easy to see that there is a representation, $D_r : U(\mathfrak{g}) \to Diff\, C^\infty((0,\infty))$ so that

$$\begin{aligned} D_r(h) &= 2x\frac{d}{dx} + 1, \\ D_r(e) &= ix, \\ D_r(f) &= i(x\frac{d^2}{dx^2} + \frac{d}{dx} - \frac{r^2}{4x}). \end{aligned}$$

Next, one finds that $\mathcal{H}_r^{HC}$ is stable under $D_r(U(\mathfrak{g}))$. In fact, $\{h', e', f'\}$ is an $S$-triple in the complexification, $\mathfrak{g}_{\mathbb{C}}$, of $\mathfrak{g}$, where $e' = \frac{1}{2}(-ih+e+f)$, $f' = \frac{1}{2}(ih+e+f)$ and, as above, $h' = -i(e-f)$. When applied to the Laguerre functions one finds (see Theorem 2.6), for $n \in \mathbb{Z}_+$,

$$\begin{aligned} D_r(h')\varphi_n^{(r)} &= (2n+r+1)\varphi_n^{(r)}, \\ D_r(e')\varphi_n^{(r)} &= i\varphi_{n+1}^{(r)}, \\ D_r(f')\varphi_n^{(r)} &= i(nr+n^2)\varphi_{n-1}^{(r)}, \end{aligned} \tag{0.1}$$

where we put $\varphi_{-1}^{(r)} = 0$. In particular, the spectrum of $D_r(h')|\mathcal{H}_r^{HC}$ is positive. Furthermore, $\varphi_0^{(r)}(x) = x^{\frac{r}{2}}e^{-x}$ spans the eigenspace for the minimal eigenvalue, $r+1$, of $D_r(h')|\mathcal{H}_r^{HC}$ and $D_r(f')\varphi_0^{(r)} = 0$.

For $u \in \mathfrak{g}$ the operator $D_r(u)|\mathcal{H}_r^{HC}$ is formally skew-symmetric. But "formal" isn't good enough since the Laguerre functions do not vanish as $x$ approaches 0 for $r \leq 0$. In fact, they become unbounded for $r < 0$. Nevertheless, these operators are skew-symmetric. But much more is true. The hypotheses for Nelson's beautiful Theorem 5 in [Ne] are satisfied so $D_r|\mathcal{H}_r^{HC}$ can be exponentiated for all $-1 < r < \infty$. We prove

**Theorem 0.1.** *There exists a unique irreducible unitary representation $\pi_r : G \to U(\mathcal{H})$ whose Harish-Chandra module is $D_r|\mathcal{H}_r^{HC}$. Furthermore, the family $\{\pi_r\}$, $-1 < r < \infty$ is the same as Pukanszky's holomorphic family $D_\ell^-$, $\ell > 0$.*

The representation $\pi_r$ descends to $Sl(2,\mathbb{R})$ when $r$ is an integer. In such a case $\pi_r$ defines a holomorphic discrete series representation of $Sl(2,\mathbb{R})$ when $r$ is positive and $\pi_0$ is often referred to as a limit of such series. The representation $\pi_r$ descends to the metaplectic group $Mp(2,\mathbb{R})$ when $r \in \mathbb{Z}/2$.

We will say $\pi_r$ is special if $0 < |r| < 1$. Obviously the special representations occur in pairs, $\{\pi_r, \pi_{-r}\}$.

*Remark* 0.2. What seems to be particularly striking about a special pair $\{\pi_{-r}, \pi_r\}$, and what seems to be illustrative of the power of Nelson's theorem, is that, whereas clearly $D_r = D_{-r}$, one has the inequivalence of $\pi_r$ and $\pi_{-r}$. The only special pair which descends to the metaplectic group $Mp(2,\mathbb{R})$ is $\{\pi_{-\frac{1}{2}}, \pi_{\frac{1}{2}}\}$. As one might suspect these two representations are the two irreducible components (even and odd) of the holomorphic metaplectic (or oscillator) representation. See Theorem 3.13. In a number of ways the special pairs can be regarded as a generalization of the metaplectic representation. For example, if one defines an equivalence relation by putting $\pi_r \sim \pi_{r'}$ in case $\pi_r$ and $\pi_{r'}$ have the same infinitesimal character, then the equivalence classes are the special pairs and the singlets $\{\pi_r\}$ where $\pi_r$ is not special. There is, however, an important distinction between the set of special representations where $r$ is positive and the set where $r$ is negative. Pukanszky in [Pu] shows that the former has positive Plancherel measure whereas the latter has zero Plancherel measure.

0.3. Let $\mathcal{H}_r^\infty$ denote the space of $C^\infty$ vectors in $\mathcal{H}$ with respect to $\pi_r$. One establishes

$$C_o^\infty((0,\infty)) \subset \mathcal{H}_r^\infty \subset C^\infty((0,\infty))$$

where subscript $o$ denotes compact support. Furthermore,

$$\pi_r^\infty = D_r|\mathcal{H}_r^\infty. \tag{0.2}$$

See Propositions 4.6 and 4.7. As is standard in representation theory the action of $U(\mathfrak{g})$ induces a Fréchet topology on $\mathcal{H}_r^\infty$ which we will refer to as the $\pi_r$-Fréchet topology. The $\pi_r$-Fréchet topology is in fact strictly finer than the topology induced on $\mathcal{H}_r^\infty$ by its inclusion in $C^\infty((0,\infty))$ when $C^\infty((0,\infty)) = \mathcal{E}((0,\infty))$ is given the familiar Fréchet topology of distribution theory. In fact, $C_o^\infty((0,\infty))$ is not dense in $\mathcal{H}_r^\infty$ with respect to the $\pi_r$-Fréchet topology. See Theorem 5.26.

The representation $\pi_r^\infty$ can be defined on the convolution algebra, $Dist_o(G)$, of distributions of compact support on $G$ and for any $\nu \in Dist_o(G)$, $\pi_r^\infty(\nu)$ is continuous with respect to the $\pi_r$-Fréchet topology. See e.g. [Ca]. We recall that $Dist_o(G)$ contains both $G$ and $U(\mathfrak{g})$. Contragrediently, there is a $Dist_o(G)$-module $\mathcal{H}_r^{-\infty}$ with respect to a representation $\pi_r^{-\infty}$ and one has inclusions

$$\mathcal{H}_r^\infty \subset \mathcal{H} \subset \mathcal{H}_r^{-\infty}$$

together with a sesquilinear form $\{\varphi,\rho\}$ for $\varphi \in \mathcal{H}_r^\infty$ and $\rho \in \mathcal{H}_r^{-\infty}$ such that every continuous linear functional on $\mathcal{H}_r^\infty$ is uniquely of the form $\varphi \mapsto \{\varphi,\rho\}$ for $\rho \in \mathcal{H}_r^{-\infty}$. In addition, $\pi_r^\infty = \pi_r^{-\infty}|\mathcal{H}_r^\infty$. There is a natural embedding of the space, $Dist_o((0,\infty))$, of distributions of compact support on $(0,\infty)$ in $\mathcal{H}_r^{-\infty}$. See Proposition 4.11. In addition, $V_a$ embeds into $\mathcal{H}_r^{-\infty}$ where $a > 0$ and $V_a$ is the space of all Borel measurable functions, $\psi$, on $(0,\infty)$ which are $L_2$ on $(0,a)$ and $O(x^k)$ on $[a,\infty)$, for some $k \in \mathbb{N}$. See Proposition 4.13.

Let $u \in \mathfrak{g}_\mathbb{C}$. Then an element $\rho \in \mathcal{H}_r^{-\infty}$ is called a $u$-weight vector of weight $\lambda \in \mathbb{C}$ in case $\pi_r^{-\infty}(u)(\rho) = \lambda\,\rho$. In case $u = h, e$ or $f$, we prove that the corresponding weight space is at most 1-dimensional. For a more general statement see Theorem 5.15. An $e$-weight (resp. $f$-weight) vector of $\rho$ of weight $\lambda$ will be referred to as an $e$-Whittaker (resp. $f$-Whittaker) vector in case $\lambda \neq 0$. It will be referred to as a highest (resp. lowest) weight vector if $\lambda = 0$. Here we are adopting terminology from finite dimensional representation theory emphasizing $e$ (resp. $f$) behavior and not extremal $h$-weights. In fact, $h$-weights are unbounded. Theorem 5.17 asserts that for any $y \in (0,\infty)$ the Dirac measure $\delta_y$ at $y$ is (up to scalar multiplication) the unique $e$-Whittaker vector of weight $iy$. Now let $J_r(z)$ be the Bessel function of order $r$. For any $y \in (0,\infty)$ let $J_{r,y} \in V_1 \subset \mathcal{H}_r^{-\infty}$ be defined by putting $J_{r,y}(x) = J_r(2\sqrt{yx})$ so that for any $\varphi \in \mathcal{H}_r^\infty$,

$$\{\varphi, J_{r,y}\} = \int_0^\infty J_r(2\sqrt{yx})\varphi(x)dx. \tag{0.3}$$

Then one has (see Theorem 5.17)

**Theorem 0.3.** *The function $J_{r,y}(x)$ is, up to scalar multiplication, the unique $f$-Whittaker vector of weight $-iy$.*

*Remark* 0.4. If $0 < |r| < 1$, then the Bessel functions $J_r(z)$ and $J_{-r}(z)$ are a basis of the space of solutions of the corresponding Bessel (2nd order differential) equation. One sees then that for the special pair $\{\pi_{-r},\pi_r\}$ of unitary representations both solutions become involved in the determination of $f$-Whittaker vectors.

Let $k_o \in K$ be defined by putting $k_o = exp\,\frac{\pi}{2}(e-f)$. Let

$$U_r = c_r\pi_r^{-\infty}(k_o) \tag{0.4}$$

where $c_r = e^{-\frac{r+1}{2}\pi i}$ so that $U_r|\mathcal{H}$ is a unitary operator. It is immediate from the first equation in (0.1) that

$$U_r(\varphi_n^{(r)}) = (-1)^n \varphi_n^{(r)} \tag{0.5}$$

so that $U_r$ is of order 2 and stabilizes the Harish-Chandra module $\mathcal{H}_r^{HC}$. On the other hand, the operator with kernel function $J_r(2\sqrt{yx})$ is classical and is known as the Hankel transform (modulo a slight change in parameters). Furthermore, it is also classical that

$$(-1)^n \varphi_n^{(r)}(y) = \int_0^\infty J_r(2\sqrt{yx})\varphi_n^{(r)}(x)dx, \tag{0.6}$$

a result that Carl Herz in [He] says is implicit in the 19th century work of Sonine. The result (0.6) is stated as Exercise 21, p. 371 in [Sz] where a reference to the paper [Ha] of G.H. Hardy is given. The result is important for us since it implies

**Theorem 0.5.** *$U_r|\mathcal{H}$ is the Hankel transform.*

For completeness we include a proof of (0.6) given to us by John Stalker.

*Remark* 0.6. The domain of the Fourier transform is normally regarded to be the space of tempered distributions. It is not generally appreciated but, since it stabilizes the Harish-Chandra module of Hermite functions, it operates contragrediently on the algebraic dual of this module. This makes its domain much larger than the space of tempered distributions. The same statement is true of the Hankel transform. Nevertheless, we only require here that it operates (see (0.4)) on $\mathcal{H}_r^{-\infty}$.

Let $G_1$ be a semisimple Lie group whose corresponding symmetric space $X$ is Hermitian. The holomorphic discrete series of $G_1$ is normally constructed on a space of square integrable holomorphic sections of a line bundle on $X$. If $X$ is of tube type, then Ding and Gross in [DG] have recognized that the natural generalization of the Hankel transform is given by the action of an element in $G_1$ and have used this fact in [DG] to transfer the discrete series, constructed on $X$, to another model built on a symmetric cone. Applied to the case considered here this constructs our representations $\pi_n$, $n = 1, 2, ....$ This method, however, would not yield $\pi_r$ for $r \le 0$. As pointed out to us by David Vogan, for such values of $r$, $\pi_r$ cannot be realized on a space of square integrable holomorphic sections of a line bundle on $X$ arising from the natural action of $G_1$ on $X$. For $r$ rational this follows from Harish-Chandra's classification of the discrete series for a group with finite center.

0.4. Note that $Ad\, k_o$ defines the non-trivial element of the Weyl group of $G$ with respect to the pair $(\mathbb{R}h, \mathfrak{g})$. That is, $Ad\, k_o(h) = -h$. It follows that the Hankel transform $U_r$ carries $h$-weight vectors of weight $\lambda$ to $h$-weight vectors of weight $-\lambda$. The following result is stated in Theorem 5.22.

**Theorem 0.7.** *Assume $\lambda \in \mathbb{C}$ where $Re\, \lambda > 0$. Let $\mu = \frac{\lambda-1}{2}$ so that $x^\mu \in V_1 \subset \mathcal{H}_r^{-\infty}$. Then, up to scalar multiplication, $x^\mu$ is the unique $h$-weight vector of weight $\lambda$ and $U_r(x^\mu)$ is the unique $h$-weight vector of weight $-\lambda$.*

*Remark* 0.8. Note for $Re\, \lambda > 0$ the $h$-weight vector is independent of $r$ whereas for $Re\, \lambda < 0$ there is an apparent dependence on $r$. This dependence is made explicit in Theorem 5.25 for the case where $\lambda = -(r+1)$. The situation when $\lambda$ is purely imaginary will be explored elsewhere.

Now note that $Ad(k_0)(e) = -f$ so that the Hankel transform $U_r$ carries $e$-weight vectors of weight $\lambda$ to $f$-weight vectors of weight $-\lambda$. In fact, for $y \in (0,\infty)$ one has, for Whittaker vectors,

$$U_r(\delta_y) = J_{r,y}.$$

See Theorem 5.17. More interesting is the question about highest and lowest weight vectors if they exist at all in $\mathcal{H}_r^{-\infty}$. In fact, they do exist in $\mathcal{H}_r^{-\infty}$. The highest (resp. lowest) weight vector, given below, in Theorem 0.8, as $\delta_{r,0}$ (resp. $J_{r,0}$) is the unique element (up to scalar multiplication) in $\mathcal{H}_r^{-\infty}$ which is simultaneously an $h$-weight vector and an $e$-weight (resp. $f$-weight) vector. See Theorem 5.25.

**Theorem 0.9.** *There exists a unique element $\delta_{r,0} \in \mathcal{H}_r^{\infty}$ such that for any $\varphi \in \mathcal{H}_r^{\infty}$ the limit in (0.7) below exists and one has*

$$\{\varphi, \delta_{r,0}\} = \lim_{x\to 0} x^{-\frac{r}{2}}\varphi(x). \tag{0.7}$$

*Moreover, $\delta_{r,0}$ is, up to scalar multiplication, the unique highest weight vector (i.e. $\delta_{r,0} \in Ker\,\pi_r^{-\infty}(e)$). Furthermore, $\delta_{r,0}$ is also an $h$-weight vector of weight $-(r+1)$.*

*The $h$-weight vector, $J_{r,0}$, of weight $r+1$ given by*

$$\{\varphi, J_{r,0}\} = \frac{1}{\Gamma(r+1)}\int_0^\infty \varphi(x)x^{\frac{r}{2}}dx \tag{0.8}$$

*for any $\varphi \in \mathcal{H}_r^{\infty}$, (see Theorem 0.7) is, up to scalar multiplication, the unique lowest weight vector (i.e. $J_{r,0} \in Ker\,\pi_r^{-\infty}(f)$). In addition, with respect to the Hankel transform, one has*

$$U_r(\delta_{r,0}) = J_{r,0}. \tag{0.9}$$

*Remark* 0.10. Note that (0.7) vanishes if $\varphi \in C_o^\infty((0,\infty))$ thereby establishing that $C_o^\infty((0,\infty))$ is not dense in $\mathcal{H}_r^{\infty}$.

0.5. Following this introduction there are five chapters:

1. The representation $D_r$ and the Casimir element
2. Laguerre functions and Harish-Chandra modules
3. The series, $\{\pi_r\}$, $-1 < r < \infty$, of irreducible unitary representations of $G$ on $L_2((0,\infty))$
4. Distribution theory on $(0,\infty)$ and the spaces $\mathcal{H}_r^{\infty}$ and $\mathcal{H}_r^{-\infty}$
5. Whittaker vectors, Bessel functions and the Hankel transform

Sections 1 and 2 should be regarded as a joint work of Nolan Wallach and myself. Its history is as follows: To deal with a problem arising in the theory of global solutions of Maxwell's equations I was motivated to model the holomorphic representations of $G$ on the spectrum, $(0,\infty)$, of the nilpotent element $-ie$. Towards this end Wallach and I came up with the Lie algebra respresentation $D_r$. It was Wallach who first noted that the eigenfunctions of $D_r(e-f)$ were the Laguerre functions.

Besides Nolan Wallach I would like to thank David Vogan, Richard Melrose and John Stalker for valuable conversations.

*Remark* 0.11. It was pointed out by an editor of this journal that the Lie algebra representation $D_r$ on $\mathcal{H}_r^{HC}$ was known to physicists. The citation is [BL], p. 284–287. However, a serious error is made on p. 287 (top paragraph) in [BL] which in effect asserts that $D_r|\mathcal{H}_r^{HC}$ is not a representation by skew-symmetric operators. An implication would be that $D_r|\mathcal{H}_r^{HC}$ could not be integrated to a unitary representation of $G$ and in fact no such group representation is given in [BL]. Although it is not transparent, particularly for $-1 < r < 0$, it is one of the results of our paper (see Lemma 3.2) that $D_r|\mathcal{H}_r^{HC}$ is indeed a representation by skew-symmetric operators. This is a key point since it sets the stage for the use of Nelson's theorem to establish that $D_r|\mathcal{H}_r^{HC}$ integrates to a unitary representation of $G$.

## 1. The representation $D_r$ and the Casimir element

1.1. Let $\mathfrak{g}_{\mathbb{C}} = Lie\, Sl(2,\mathbb{C})$ and let $\mathfrak{g} = Lie\, Sl(2,\mathbb{R})$ so that $\mathfrak{g}$ is a real form of $\mathfrak{g}_{\mathbb{C}}$. Let $\{h,e,f\} \subset \mathfrak{g}$ be the $S$-triple given by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Consider the algebra $\mathit{Diff}\, C^\infty((0,\infty))$ of differential operators on the open interval $(0,\infty)$. Let $x \in C^\infty((0,\infty))$ be the natural coordinate function and let $\mathcal{H}$ be the Hilbert space $L_2((0,\infty))$ with respect to the usual measure defined by $dx$. Unless stated otherwise we will always assume $r \in \mathbb{R}$ where $r > -1$. Let $D_r : \mathfrak{g}_{\mathbb{C}} \to Diff\; C^\infty((0,\infty))$ be the linear map, where identifying elements in $C^\infty((0,\infty))$ with the corresponding multiplication operators,

$$\begin{aligned} D_r(e) &= ix, \\ D_r(h) &= 2x\frac{d}{dx} + 1, \\ D_r(f) &= i(x\frac{d^2}{dx^2} + \frac{d}{dx} - \frac{r^2}{4x}). \end{aligned} \tag{1.1}$$

Noting that $[x, x\frac{d^2}{dx^2}] = -2x\frac{d}{dx}$ one readily establishes that $D_r$ is a Lie algebra homomorphism. Thus if $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, then $D_r$ extends to an algebra homomorphism $D_r : U(\mathfrak{g}) \to Diff\, C^\infty((0,\infty))$. Let $Cas \in U(\mathfrak{g})$ be the quadratic Casimir element corresponding to the Killing form on $\mathfrak{g}_{\mathbb{C}}$. Clearly,

$$Cas = \frac{1}{4}(\frac{h^2}{2} + ef + fe). \tag{1.2}$$

Computation yields

**Proposition 1.1.** *The differential operator $D_r(Cas)$ is the scalar operator given by*

$$D_r(Cas) = \frac{1}{8}(r^2 - 1). \tag{1.3}$$

*Proof.* We first note that

$$\begin{aligned} D_r(h^2/2) &= (2x\frac{d}{dx} + 1)^2/2 \\ &= 2(x^2(\frac{d}{dx})^2 + 2x\frac{d}{dx} + 1/4). \end{aligned} \tag{1.4}$$

Next,

$$\begin{aligned} D_r(ef+fe) &= ix(i(x(\frac{d}{dx})^2 + \frac{d}{dx} - \frac{r^2}{4x})) + i(x(\frac{d}{dx})^2 + \frac{d}{dx} - \frac{r^2}{4x})ix \\ &= -2x^2(\frac{d}{dx})^2 - 4x\frac{d}{dx} - 1 + \frac{r^2}{2}. \end{aligned} \tag{1.5}$$

But then (1.3) follows from (1.1), (1.2), (1.4) and (1.5). QED

Let $G$ be a simply-connected covering group with Lie algebra $\mathfrak{g}$.

*Remark* 1.2. Note the scalar value of $D_r(Cas)$ is negative if and only if $|r| < 1$. Later in this paper the representation $D_r$ will lead to an irreducible unitary representation $\pi_r$ of $G$. The representation $\pi_r$ will depend on more than just $D_r$. In fact, for $0 < |r| < 1$ the representations $\pi_r$ and $\pi_{-r}$ will be inequivalent whereas $D_r$ and $D_{-r}$ are clearly the same.

It is convenient to think of the parameter $r$ as an analogue of dimension. In fact, if $r \in \mathbb{N}$ and $\nu_r$ is the finite dimensional irreducible representation of $U(\mathfrak{g})$ having dimension $r$, then $\frac{1}{8}(r^2 - 1)$ is the scalar value of $\nu_r(Cas)$. Indeed, with the usual definition of $\rho$ the "strange formula" of Freudenthal-de Vries asserts that $|\rho|^2 = \frac{1}{8}$ because the formula asserts $|\rho|^2 = \frac{dim\,\mathfrak{g}}{24}$ and in our case $dim\,\mathfrak{g} = 3$. However, $(r-1)\rho$ is the highest weight of $\nu_r$. But then the scalar value of $\nu_r(Cas)$ is $|r\,\rho|^2 - |\rho|^2 = \frac{1}{8}(r^2 - 1)$.

## 2. Laguerre functions and Harish-Chandra modules

2.1. Recall $-1 < r < \infty$. The Laguerre polynomials $L_n^{(r)}(x)$, $n = 0, 1, ...,$ are defined, as in Chapter X, p. 184 of [Ja], by putting

$$L_n^{(r)}(x) = (-1)^n x^{-r} e^x \frac{d^n}{dx^n}(x^{r+n}e^{-x}).$$

One has $deg\, L_n^{(r)} = n$ and the coefficient of $x^n$ is 1. Furthermore, the sequence of Laguerre polynomials is a Gram-Schmidt orthogonal family of polynomials associated to the monomials $1, x, x^2, ...,$ with respect to the finite measure $x^r e^{-x} dx$ on $(0, \infty)$. See §2, Chapter X in [Ja]. But then upon multiplication by the square root of the weighting factor $x^r e^{-x}$ it follows that for $n = 0, 1, 2...,$

$$x^{\frac{r}{2}} e^{-\frac{x}{2}} L_n^{(r)}(x) = (-1)^n x^{-\frac{r}{2}} e^{\frac{x}{2}} \frac{d^n}{dx^n}(x^{r+n}e^{-x}) \tag{2.1}$$

is an orthogonal family of functions in $\mathcal{H}$. If we replace $x$ by $2x$ in (2.1) and divide by $2^{\frac{r}{2}}$, it then follows that

$$\begin{aligned} \varphi_n^{(r)}(x) &= x^{\frac{r}{2}} e^{-x} L_n^{(r)}(2x) \\ &= x^{\frac{r}{2}} e^{-x} (-1)^n x^{-r} e^{2x} \frac{d^n}{dx^n}(x^{r+n}e^{-2x}) \\ &= (-1)^n x^{-\frac{r}{2}} e^{x} \frac{d^n}{dx^n}(x^{r+n}e^{-2x}) \end{aligned} \tag{2.2}$$

is again an orthogonal basis of functions in $\mathcal{H}$. We will refer to the sequence of functions $\varphi_n^{(r)}(x)$, $n = 0, 1, ...,$ as Laguerre functions (of type $r$).

**Proposition 2.1.** *The set of Laguerre functions* $\{\varphi_n^{(r)}(x)\}$, $n = 0, 1, ...,$ *is an orthogonal basis of* $\mathcal{H}$. *Furthermore,*

$$\int_0^\infty |\varphi_n^{(r)}(x)|^2 dx = \frac{1}{2^{r+1}} n!\,\Gamma(n + r + 1). \tag{2.3}$$

*Proof.* By (5.7.1) in Theorem 5.7.1, p. 104, in [Sz] the functions $x^{\frac{r}{2}} e^{-\frac{x}{2}} L_n^{(r)}(x)$, $n = 0, 1, ...,$ span a dense subspace of $\mathcal{H}$ and hence, in the Hilbert space sense, are an orthogonal basis of $\mathcal{H}$. But then the same statement is true after scaling and replacing $x$ by $2x$. Hence the Laguerre functions are also an orthogonal basis of $\mathcal{H}$.

Let $\psi_n$ be the function defined by the left side of (2.1). Now by the computation of $d_n$ on the bottom of p. 185 in [Ja], one has

$$\int_0^\infty |\psi_n(x)|^2 dx = n!\,\Gamma(n+r+1). \tag{2.4}$$

But then (2.3) follows from (2.4) since $\varphi_n^{(r)}(x) = 2^{-\frac{r}{2}}\psi_n(2x)$. QED

2.2. We now consider a new $S$-triple $\{h', e', f'\} \subset \mathfrak{g}_\mathbb{C}$ defined by putting $\{h', e', f'\} = A\{h, e, f\}A^{-1}$ where $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$. One readily has

$$\begin{aligned} h' &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\ &= -i(e-f). \end{aligned} \tag{2.5}$$

Next,

$$\begin{aligned} e' &= \frac{1}{2}\begin{pmatrix} -i & -1 \\ -1 & i \end{pmatrix} \\ &= \frac{1}{2}(-ih + e + f). \end{aligned} \tag{2.6}$$

Also,

$$\begin{aligned} f' &= \begin{pmatrix} i & -1 \\ -1 & -i \end{pmatrix} \\ &= \frac{1}{2}(ih + e + f). \end{aligned} \tag{2.7}$$

The element $e-f$ is elliptic in $\mathfrak{g}$ and hence $h = -i(e-f) \in \mathfrak{g}_\mathbb{C}$ is hyperbolic. We are looking for a $\mathfrak{g}$-submodule of $C^\infty((0,\infty))$, with respect to $D_r$, which is spanned by eigenvectors of $D_r(h')$. By (2.5) $D_r(h') = -(x(\frac{d}{dx})^2 + \frac{d}{dx} - \frac{r^2}{4x} - x)$. Hence if $\varphi \in C^\infty((0,\infty))$ is an eigenvector for $D_r(h')$ with eigenvalue $\lambda$ we have (recalling (1.1)) the differential equation

$$(x(\frac{d}{dx})^2 + \frac{d}{dx} - \frac{r^2}{4x} - x + \lambda)\varphi = 0. \tag{2.8}$$

**Lemma 2.2.** *For $n = 0, 1, \ldots$, one has*

$$D_r(h')\varphi_n^{(r)} = (2n+r+1)\varphi_n^{(r)}. \tag{2.9}$$

*Proof.* By (3) on p. 186 in [Ja] (or (5.1.2) on p. 96 in [Sz]) one has

$$(x(\frac{d}{dx})^2 + (r+1-x)\frac{d}{dx} + n)L_n^{(r)}(x) = 0. \tag{2.10}$$

If $A$ is the operator on $C^\infty((0,\infty))$ defined by putting $Af(x) = f(2x)$, then conjugating the differential operator in (2.10) by $A$ and then multiplying by 2 yields the differential equation

$$(x(\frac{d}{dx})^2 + (r+1-2x)\frac{d}{dx} + 2n)L_n^{(r)}(2x) = 0. \tag{2.11}$$

Conjugating the differential operator in (2.11) by the multiplication operator $M = e^{-x}x^{\frac{r}{2}}$ then establishes the differential equation

$$M(x(\frac{d}{dx})^2 + (r+1-2x)\frac{d}{dx} + 2n)M^{-1}\varphi_n^{(r)} = 0. \tag{2.12}$$

However,

$$M\frac{d}{dx}M^{-1} = \frac{d}{dx} + (-\frac{r}{2x} + 1) \tag{2.13}$$

so that

$$\begin{aligned} M(r+1-2x)\frac{d}{dx}M^{-1} &= (r+1-2x)\frac{d}{dx} + (r+1-2x)(-\frac{r}{2x}+1) \\ &= (r+1-2x)\frac{d}{dx} + 2r + 1 - 2x - \frac{r^2}{2x} - \frac{r}{2x}. \end{aligned} \tag{2.14}$$

But (2.13) also implies

$$\begin{aligned} Mx(\frac{d}{dx})^2M^{-1} &= x(\frac{d}{dx} + (-\frac{r}{2x}+1))^2 \\ &= x\frac{d}{dx}^2 + (-r+2x)\frac{d}{dx} + \frac{r}{2x} + \frac{r^2}{4x} - r + x. \end{aligned} \tag{2.15}$$

But then adding (2.14), (2.15) and $2n$ yields

$$M(x(\frac{d}{dx})^2 + (r+1-2x)\frac{d}{dx} + 2n)M^{-1} = x(\frac{d}{dx})^2 + \frac{d}{dx} - \frac{r^2}{4x} - x + (2n+r+1).$$

But then (2.9) follows from (2.12) and (2.8) with $\lambda = 2n+r+1$ and $\varphi = \varphi_n^{(r)}$. QED

*Remark* 2.3. With our assumptions on $r$ and $n$ note that the eigenvalue $2n+r+1$ is always positive.

2.3. Let $\mathcal{H}_r^{HC}$ be the linear span of the $\{\varphi_n^{(r)}\}$, $n \in \mathbb{Z}_+$. By Proposition 2.1 it follows that $\mathcal{H}_r^{HC}$ is dense in $\mathcal{H}$. We wish to show that $\mathcal{H}_r^{HC}$ is stable under the action of $D_r(\mathfrak{g}_{\mathbb{C}})$. By Lemma 2.2 $\mathcal{H}_r^{HC}$ is clearly stable under $D_r(h')$. But now by (1.1), (2.6) and (2.7) one has

$$D_r(e') = \frac{i}{2}(x\frac{d^2}{dx^2} + (1-2x)\frac{d}{dx} - \frac{r^2}{4x} - 1 + x) \tag{2.16}$$

and

$$D_r(f') = \frac{i}{2}(x\frac{d^2}{dx^2} + (1+2x)\frac{d}{dx} - \frac{r^2}{4x} + 1 + x). \tag{2.17}$$

*Remark* 2.4. Note that $D_r(e')$ and $D_r(f')$ differ only in the sign of 2 terms.

Recall (2.2). We make the following computations:

$$\begin{aligned} x\frac{d^2}{dx^2}\varphi_n^{(r)}(x) = (-1)^n x^{-\frac{r}{2}} e^x ((( &\frac{r}{2})(\frac{r}{2}+1)x^{-1} - r + x)\frac{d^n}{dx^n}(x^{r+n}e^{-2x}) \\ &+ (-r+2x)\frac{d^{n+1}}{dx^{n+1}}(x^{r+n}e^{-2x}) \\ &+ x\frac{d^{n+2}}{dx^{n+2}}(x^{r+n}e^{-2x})). \end{aligned} \tag{2.18}$$

Next,

$$\begin{aligned} (1 \mp 2x)\frac{d}{dx}\varphi_n^{(r)}(x) &= (-1)^n x^{-\frac{r}{2}} e^x ((-\frac{r}{2}x^{-1} \\ &+ 1 \mp (-r) \mp 2x)\frac{d^n}{dx^n}(x^{r+n}e^{-2x})(1 \mp 2x)\frac{d^{n+1}}{dx^{n+1}}(x^{r+n}e^{-2x})) \end{aligned} \tag{2.19}$$

and finally,

$$(2.20)\quad (-\frac{r^2}{4x} \mp 1 + x)\varphi_n^{(r)}(x) = (-1)^n x^{-\frac{r}{2}} e^x ((-\frac{r^2}{4} x^{-1} \mp 1 + x)\frac{d^n}{dx^n}(x^{r+n}e^{-2x})).$$

It follows from (2.16) upon adding (2.18), (2.19) and (2.20) choosing the top signs that

$$(2.21)\qquad D_r(e')\varphi_n^{(r)} = \frac{i}{2}(-1)^n x^{-\frac{r}{2}} e^x ((1-r) + x\frac{d}{dx})(\frac{d}{dx})^{n+1} x^{r+n} e^{-2x}.$$

But

$$(2.22)\qquad [x\frac{d}{dx}, \frac{d^{n+1}}{dx^{n+1}}] = -(n+1)\frac{d^{n+1}}{dx^{n+1}}.$$

Thus,

$$\begin{aligned} x\frac{d}{dx}\frac{d^{n+1}}{dx^{n+1}}(x^{r+n}e^{-2x}) &= \frac{d^{n+1}}{dx^{n+1}} x\frac{d}{dx}(x^{r+n}e^{-2x}) - (n+1)\frac{d^{n+1}}{dx^{n+1}}(x^{r+n}e^{-2x}) \\ &= \frac{d^{n+1}}{dx^{n+1}}((r+n)x^{r+n} - 2x^{r+n+1} - (n+1)x^{r+n})e^{-2x} \\ &= (r-1)\frac{d^{n+1}}{dx^{n+1}}(x^{r+n}e^{-2x}) - 2\frac{d^{n+1}}{dx^{n+1}}(x^{r+n+1}e^{-2x}). \end{aligned}$$

But then by (2.2) and (2.21),

$$(2.23)\qquad \begin{aligned} \delta_r(e')\varphi_n^{(r)}(x) &= i(-1)^{n+1} x^{-\frac{r}{2}} e^x \frac{d^{n+1}}{dx^{n+1}}(x^{r+n+1}e^{-2x}) \\ &= i\varphi_{n+1}^{(r)}. \end{aligned}$$

We have proved

**Lemma 2.5.** *$\mathcal{H}_r^{HC}$ is stable under $D_r(e')$. In fact,*

$$(2.24)\qquad D_r(e')\varphi_n^{(r)} = i\varphi_{n+1}^{(r)}.$$

Now from (2.17) and adding (2.18), (2.19) and (2.20) choosing the bottom signs and taking $n = 0$ one has

$$D_r(f')\varphi_0^{(r)}(x) = \frac{i}{2} x^{-\frac{r}{2}} e^x ((-2r + 2 + 4x) + (1 - r + 4x)\frac{d}{dx} + x(\frac{d}{dx})^2) x^r e^{-2x}.$$

But it is straightforward to verify that $x^r e^{-2x}$ satisfies the differential equation

$$(x(\frac{d}{dx})^2 + (1 - r + 4x)\frac{d}{dx}) + (-2r + 2 + 4x))x^r e^{-2x} = 0$$

and hence,

$$(2.25)\qquad D_r(f')\varphi_0^{(r)} = 0.$$

Let $\mathfrak{k} = \mathbb{R}(e - f) = i\mathbb{R}h'$ and let $K \subset G$ be the subgroup corresponding to $\mathfrak{k}$. Since the center $G$ is not finite, the group $G$ is not in the Harish-Chandra class and $K$ is not compact. Accordingly, we will modify the definition of a Harish-Chandra module to suit the case at hand. In this paper a module for $U(\mathfrak{g})$ will be called a Harish-Chandra module if it is spanned by 1-dimensional submodules under the action of $\mathfrak{k}$. Partly summarizing some of the results above one has

**Theorem 2.6.** *As always $-1 < r < \infty$. The dense subspace $\mathcal{H}_r^{HC}$, in $\mathcal{H} = L_2((0,\infty))$, spanned by the Laguerre functions $\varphi_n^{(r)}(x) = x^{\frac{r}{2}} e^{-x} L_n^{(r)}(2x)$, $n = 0,1,\ldots,$ is stable under the action of $D_r(\mathfrak{g}_{\mathbb{C}})$ and, with respect to $D_r$, defines an irreducible Harish-Chandra module for $U(\mathfrak{g})$. In fact,*

$$\begin{aligned} D_r(h')\varphi_n^{(r)} &= (2n+r+1)\varphi_n^{(r)}, \\ D_r(e')\varphi_n^{(r)} &= i\varphi_{n+1}^{(r)}. \end{aligned} \tag{2.26}$$

*In particular, the spectrum of $D_r(h')$ is positive. Furthermore, $\varphi_0^{(r)}(x) = x^{\frac{r}{2}} e^{-x}$ spans the "minimal $\mathfrak{k}$-type" and*

$$D_r(f')\varphi_0^{(r)} = 0. \tag{2.27}$$

*Moreover, for $n > 0$ one has*

$$D_r(f')\varphi_n^{(r)} = i(nr+n^2)\varphi_{n-1}^{(r)}. \tag{2.28}$$

*Proof.* The equations (2.26) have been previously established. See Lemmas 2.2 and 2.5. Also (2.27) is just (2.25). We will prove (2.28) by induction on $n$. The result has actually been established for $n = 0$ by putting $\varphi_{-1}^{(r)} = 0$. Inductively, assume (2.28) for $n$. Then

$$\begin{aligned} (2n+r+1)\varphi_n^{(r)} &= D_r(h')\varphi_n^{(r)} \\ &= (D_r(e')D_r(f') - D_r(f')D_r(e'))\varphi_n^{(r)} \\ &= i(nr+n^2)D_r(e')\varphi_{n-1}^{(r)} - iD_r(f')\varphi_{n+1}^{(r)} \\ &= -(nr+n^2)\varphi_n^{(r)} - iD_r(f')\varphi_{n+1}^{(r)}. \end{aligned}$$

Thus,

$$\begin{aligned} D_r(f')\varphi_{n+1}^{(r)} &= i((2n+r+1) + nr + n^2)\varphi_n^{(r)} \\ &= i((n+1)r + (n+1)^2))\varphi_n^{(r)}. \end{aligned}$$

This establishes (2.28) for all $n$. Thus $\mathcal{H}_r^{HC}$ is stable under $D_r(\mathfrak{g}_{\mathbb{C}})$. It is then clearly a Harish-Chandra module for $\mathfrak{g}$ where all the eigenvalues of $D_r(h')$ are positive and have multiplicity 1. But since $nr + n^2 = n(r+1)$ is positive for all $n = 1, ...,$ it follows from (2.28) that any nonzero $\mathfrak{g}_{\mathbb{C}}$-submodule of $\mathcal{H}_r^{HC}$ must contain $\varphi_0^{(r)}$ and hence, by the second equation in (2.26), must be equal to $\mathcal{H}_r^{HC}$. This establishes irreducibility. QED

## 3. The series, $\{\pi_r\}$, $-1 < r < \infty$, of irreducible unitary representations of $G$ on $L_2((0,\infty))$

3.1. If $\psi$ and $\varphi$ are measurable functions on $(0,\infty)$ and $\psi\overline{\varphi}$ is integrable, we will write

$$\{\psi,\varphi\} = \int_0^\infty \psi(x)\overline{\varphi}(x)dx.$$

If $A$ is a densely defined operator on $\mathcal{H}$, we will denote its domain by $Dom(A) \subset \mathcal{H}$. Given such an operator we recall that $A$ has a closure, denoted by $\overline{A}$, if and only if $A$ admits a densely defined adjoint operator $A^*$. In such a case

$$\{A\psi,\varphi\} = \{\psi, A^*\varphi\}$$

for all $\psi \in Dom(A)$ and $\varphi \in Dom(A^*)$. If $A$ is densely defined and is symmetric (resp. skew-symmetric) on its domain, then $A$ admits a closure, $Dom(A) \subset Dom(A^*)$ and $A = A^*|Dom(A)$ (resp. $-A = A^*|Dom(A)$). In such a case $A$ is called essentially self-adjoint (resp. essentially skew-adjoint) if $\overline{A} = A^*$ (resp. $-\overline{A} = A^*$).

*Remark* 3.1. If one considers the Schwartz space, $\mathcal{S}$, of functions on $\mathbb{R}$, it is obvious that $\frac{d}{dx}|\mathcal{S}$ is skew-symmetric. Now recall (1.1). It then follows easily that $D_r(u)$, for any $u \in \mathfrak{g}$, regarded as an operator $\mathcal{S}$, is also skew-symmetric. That is, $D_r(u)$ is formally skew-symmetric. However, $\frac{d}{dx}$ cannot automatically be taken as skew-symmetric when defined on domains in $\mathcal{H}$ that contain functions which do not converge to zero as $x$ tends to zero. In particular, it is not automatic that $D_r(u)|\mathcal{H}_r^{HC}$ is skew-symmetric when $r \leq 0$. Nevertheless, the following lemma asserts that any such operator is indeed skew-symmetric.

**Lemma 3.2.** *For any $u \in \mathfrak{g}$ the operator $D_r(u)|\mathcal{H}_r^{HC}$ is skew-symmetric.*

*Proof.* Let $v \mapsto v^*$ be the conjugate linear map on $\mathfrak{g}_{\mathbb{C}}$ defined so that $v^* = -v$ for $v \in \mathfrak{g}$. Let $\psi, \varphi \in \mathcal{H}_r^{HC}$. It suffices to prove

$$\{D_r(v)\psi, \varphi\} = \{\psi, D_r(v^*)\varphi\} \tag{3.1}$$

for any $v \in \mathfrak{g}_{\mathbb{C}}$. However, since $v \mapsto \{\psi, D_r(v^*)\varphi\}$ is complex linear, it suffices only to prove (3.1) for a basis of $\mathfrak{g}_{\mathbb{C}}$. We will establish (3.1) for the basis $\{h', e', f'\}$. But $(h')^* = h'$. However, since the Laguerre functions $\varphi_n^{(r)}$ are mutually orthogonal, (3.1) follows from the first equation of (2.26) when $v = h'$. Next it follows from (2.6) and (2.7) that $(e')^* = -f'$. By symmetry it suffices then only to prove

$$\{D_r(e')\psi, \varphi\} = -\{\psi, D_r(f')\varphi\}.$$

But then from the second equation of (2.26) and (2.28), we have only to prove that

$$\{D_r(e')\varphi_n^{(r)}, \varphi_{n+1}^{(r)}\} = -\{\varphi_n^{(r)}, D_r(f')\varphi_{n+1}^{(r)}\}$$

for $n = 0, 1, \ldots$. But then again, from the second equation of (2.26) and (2.28) we must show that

$$i\{\varphi_{n+1}^{(r)}, \varphi_{n+1}^{(r)}\} = i((n+1)r + (n+1)^2)\{\varphi_n^{(r)}, \varphi_n^{(r)}\}. \tag{3.2}$$

But the left side of (3.2) equals $\frac{i}{2^{r+1}}(n+1)!\,\Gamma(n+r+2)$ by (2.3) whereas the right side of (3.2) equals $\frac{i}{2^{r+1}}((n+1)r+(n+1)^2)n!\,\Gamma(n+r+1)$. But since $\Gamma(n+r+2) = (n+r+1)\Gamma(n+r+1)$ it follows easily that the left and right sides of (3.2) are equal. QED

3.2. Since $K$ is non-compact some care must be exercised in appealing to the literature about Harish-Chandra modules. We will refer instead to [Pu], modified by some general representation theory. Pukanszky in [Pu] has determined the unitary dual of $G$. See the bottom of p. 102 in [Pu]. Assume

$$\pi\colon G \to U(\mathcal{L})$$

is an irreducible unitary representation of $G$ on a Hilbert space $\mathcal{L}$. If $u \in \mathfrak{g}$, then $t \mapsto \pi(exp\,tu)$ is a one parameter group of unitary operators. By Stone's theorem there exists a skew-adjoint operator, $\widetilde{\pi}(u)$, which in the sense of the spectral theorem generates this one parameter group. Let $Dom_\pi(u) \subset \mathcal{L}$ be the domain of $\widetilde{\pi}(u)$. Let $\mathcal{L}^\infty$ be the space of infinitely differentiable vectors in $\mathcal{L}$. Then $\mathcal{L}^\infty$ is dense in

$\mathcal{L}$ and $\mathcal{L}^\infty \subset Dom_\pi(u)$ for any $u \in \mathfrak{g}$. Furthermore, if $\pi^\infty(u) = \widetilde{\pi}(u)|\mathcal{L}^\infty$, then $\mathcal{L}^\infty$ is stable under $\pi^\infty(u)$ and $\pi^\infty$ defines a Lie algebra representation of $\mathfrak{g}$ on $\mathcal{L}^\infty$. In addition, $\pi^\infty(u)$ is essentially skew-adjoint. See [Se]. The representation $\pi^\infty$ extends to a representation $\pi^\infty \colon U(\mathfrak{g}) \to End\,\mathcal{L}^\infty$. It follows easily that if $\mathcal{L}_o \subset \mathcal{L}$ is a subspace such that $\mathcal{L}_o \subset Dom_\pi(v_i)$ for a basis $\{v_i\}$ of $\mathfrak{g}$ and $\mathcal{L}_o$ is stable under all $\widetilde{\pi}(v_i)$, then $\mathcal{L}_o$ is a $U(\mathfrak{g})$-submodule of $\mathcal{L}^\infty$ with respect to $\pi^\infty$. Now Pukanszky, in [Pu], observes that there exists a character $\chi$ on $K$ such that $\chi\,\pi|K$ descends to a representation of the circle group. Hence, if we let $\mathcal{L}_o$ be the span of the eigenvectors of $\widetilde{\pi}(e-f)$, then $\mathcal{L}_o$ is dense in $\mathcal{L}$. He then proves (see pp. 99 and 100 in [Pu]) that, for a basis $\{v_i\}$ of $\mathfrak{g}$, $\mathcal{L}_o \subset Dom_\pi(v_i)$ and $\mathcal{L}_o$ is stable under $\widetilde{\pi}(v_i)$. Thus $\mathcal{L}_o$ is a $U(\mathfrak{g})$ submodule of $\mathcal{L}^\infty$. Consequently, if we define $\mathcal{L}^{HC}$ to be the span of the eigenvectors of $\pi^\infty(h')$, then one has $\mathcal{L}^{HC} = \mathcal{L}_o$. In particular, $\mathcal{L}^{HC}$ is a Harish-Chandra module for $U(\mathfrak{g})$ with respect to $\pi^\infty$ and we refer to $\mathcal{L}^{HC}$ as the Harish-Chandra module associated to $\pi$.

Pukanszky lists three families of irreducible unitary representations. From the list one readily notes that $\pi^\infty(Cas) = C_\pi$ for a scalar $C_\pi$ and if we denote the spectrum of $\pi^\infty(h')|\mathcal{L}^{HC}$ by $S_\pi$, then each $\lambda \in S_\pi$ has multiplicity 1. Furthermore by inspection one has

**Proposition 3.3.** *Let $\pi^i$, $i = 1,2$, be two irreducible unitary representations of $G$. Then $\pi^1$ is equivalent to $\pi^2$ if and only if there exists $\lambda_i \in S_{\pi^i}$ such that, as pairs,*

$$(C_{\pi^1}, \lambda_1) = (C_{\pi^2}, \lambda_2). \tag{3.3}$$

*In particular, any irreducible unitary representation of $G$ is determined by its Harish-Chandra module.*

To align the notation of [Pu] with that used here one notes that if $H_0$ and the Casimir eigenvalue $q$ are defined as in [Pu], then

$$H_0|\mathcal{L}^{HC} = -\frac{1}{2}\pi^\infty(h')|\mathcal{L}^{HC} \tag{3.4}$$

and

$$q = -2C_\pi. \tag{3.5}$$

The members of the second family (denoted by II) of irreducible unitary representations are characterized by the property that the numbers in $S_\pi$ are either all positive or all negative. Correspondingly, this family, respectively, breaks up into 2 series, $D_\ell^-$ and $D_\ell^+$ both for $0 < \ell < \infty$ (note the sign change in (3.4)). For $\pi = D_\ell^-$ one has

$$S_\pi = \{2\ell, 2\ell+2, 2\ell+4, ...\}$$

and

$$C_\pi = \frac{1}{2}\ell(\ell-1). \tag{3.6}$$

Although it is an abuse of terminology, for suggestive reasons, we will refer to the set of representations, $\{D_\ell^-\}$, for $0 < \ell < \infty$, as the holomorphic series of representations of $G$.

3.3. Let $u_1 = e - f, u_2 = e + f$ and $u_3 = h$ so that $\{u_i\}$, $1 = 1, 2, 3$, is a basis of $\mathfrak{g}$. Moreover, the basis is an orthogonal basis with respect to the Killing form. Clearly, $8\,Cas = -u_1^2 + u_2^2 + u_3^2$. But $u_1^2 = -(h')^2$ by (2.5) so that

$$8\,Cas - 2(h')^2 = \sum_{i=1}^{3} u_i^2. \tag{3.7}$$

On the other hand,

$$\begin{aligned} D_r(8\,Cas - 2(h')^2)\varphi_n^{(r)} &= ((r^2-1) - 2(2n+r+1)^2)\varphi_n^{(r)} \\ &= -(8n(n+r+1) + (r+3)(r+1))\varphi_n^{(r)} \end{aligned} \tag{3.8}$$

by Lemma 1.1 and (2.9). The power of Nelson's theorem on analytic vectors enables us to exponentiate $D_r(\mathfrak{g})|\mathcal{H}_r^{HC}$ to a unitary representation of $G$ on $\mathcal{H} = L_2((0,\infty))$.

**Theorem 3.4.** *For any $-1 < r < \infty$ there exists an irreducible (necessarily unique) unitary representation, $\pi_r$, of $G$ on $L_2((0,\infty))$ where the corresponding Harish-Chandra module is $\mathcal{H}_r^{HC}$ with respect to the action of $\mathfrak{g}$ defined by $u \mapsto D_r(u)|\mathcal{H}_r^{HC}$. Furthermore, using the notation of Pukanszky, $\pi_r$ is equivalent to $D^-_{\frac{r+1}{2}}$. In particular $\{\pi_r\}$, $-1 < r < \infty$, is the holomorphic series.*

*Proof.* Let $u_i$, $i = 1, 2, 3$ be the basis of $\mathfrak{g}$ defined in (3.7). By (3.7) and (3.8) the elements of the orthogonal basis, $\{\varphi_n^{(r)}\}$, of $\mathcal{H}$ are eigenvectors of $D_r(\sum_{i=1}^3 u_i^2)$ and the eigenvalues are real. It is immediate then that $D_r(\sum_{i=1}^3 u_i^2)|\mathcal{H}_r^{HC}$ is essentially self-adjoint. On the other hand, $D_r(u)|\mathcal{H}_r^{HC}$ is skew-symmetic for any $u \in \mathfrak{g}$ by Lemma 3.2. It follows then from Theorem 5, p. 602 in [Ne], that there exists a unitary representation, $\pi_r$ of $G$ on $\mathcal{H}$ such that, in the notation of §3.2, $\mathcal{H}_r^{HC}$ is in the domain of $\widetilde{\pi_r}(u)$ for all $u \in \mathfrak{g}$ and

$$\widetilde{\pi_r}(u)|\mathcal{H}_r^{HC} = D_r(u)|\mathcal{H}_r^{HC}. \tag{3.9}$$

But now $u_1 = ih'$ by (2.5) and hence by (2.9) and (3.9) the elements of the orthogonal basis, $\{\varphi_n^{(r)}\}$, of $\mathcal{H}$ are eigenvectors of $\widetilde{\pi_r}(u_1)$ with pure imaginary eigenvalues. But then $\widetilde{\pi_r}(u_1)|\mathcal{H}_r^{HC}$ is essentially skew-adjoint. Hence $\mathcal{H}_r^{HC}$ is clearly the Harish-Chandra module of $\pi_r$ and, by (3.9), the action of $u \in \mathfrak{g}$ on $\mathcal{H}_r^{HC}$ is given by $D_r(u)|\mathcal{H}_r^{HC}$.

Let $\mathcal{H}^1$ be any nonzero closed subspace of $\mathcal{H}$ which is stable under $\pi_r(G)$. Using the spectral theorem for $\widetilde{\pi_r}(u_1)$ it follows that there exists some $n \in \mathbb{Z}_+$ such that $\varphi_n^{(r)} \in \mathcal{H}^1$. But then using (3.9) it follows that $D_r(U(\mathfrak{g}))(\varphi_n^{(r)}) \subset \mathcal{H}^1$. Thus $\mathcal{H}_r^{HC} \subset \mathcal{H}^1$ by Theorem 2.6. Hence $\mathcal{H}^1 = \mathcal{H}$. Thus $\pi_r$ is irreducible. But now if we define $\ell$ by

$$r = 2\ell - 1, \tag{3.10}$$

then $\ell$ is arbitrary in the interval $(0,\infty)$. Furthermore, $S_{\pi_r} = \{2\ell, 2\ell+2, 2\ell+4, ...\}$ by (2.26). But $C_{\pi_r} = \frac{1}{8}(r^2-1)$ by (1.3). But $\frac{1}{8}(r^2-1) = \frac{1}{2}\ell(\ell-1)$. Thus, recalling (3.5), (3.6) and Proposition 3.3, $\pi_r$ is equivalent to $D_\ell^-$ in the notation of [Pu]. QED

Let $Q$ be the set of representations $\{\pi_r\}$ where $0 < |r| < 1$. A representation $\pi_r$ will be called special if $\pi_r \in Q$. It is clear from (1.3) that $\pi_r$ is special if and only if the infinitesimal character of $\pi_r$ takes a negative value on $Cas$. It may be noted from Pukanszky's list (see p. 102 in [Pu]) that this infinitesimal character is shared only by elements in the family (Pukanszky's notation) $E_q^\tau$. The family $E_q^\tau$ is the analogue for $G$ of the complementary series of $Sl(2,\mathbb{R})$. Obviously, $Q$ is a union of pairs

$$Q = \bigcup_{0<r<1} \{\pi_{-r}, \pi_r\}. \tag{3.11}$$

The pairs in (3.11) will be referred to as special pairs.

*Remark* 3.5. As noted earlier, for $0 < |r| < 1$, it is rather striking that whereas $D_r = D_{-r}$ one has that $\pi_r$ is not equivalent to $\pi_{-r}$. That is, if in one case we restrict $D_r$ to $\mathcal{H}_r^{HC}$ and in a second case restrict $D_r$ to $\mathcal{H}_{-r}^{HC}$ and apply Nelson's theorem in both cases, we obtain a special pair of 2 inequivalent irreducible unitary representations of $G$ on $L_2((0,\infty))$.

Write

$$Q = Q_- \cup Q_+$$

where $Q_- = \{\pi_r\}$, $-1 < r < 0$ (i.e. in the notation of [Pu], $\{D_\ell^-\}$, $0 < \ell < \frac{1}{2}$, (see (3.10)) and where $Q_+ = \{\pi_r\}$, $0 < r < 1$ (i.e. in the notation of [Pu], $\{D_\ell^-\}$, $\frac{1}{2} < \ell < 1$).

*Remark* 3.6. The subset, $Q_-$, of the unitary dual of $G$ differs in a fundamental way from the subset $Q_+$. As established by Pukanszky, $Q_-$ has zero Plancherel measure whereas $Q_+$ has positive Plancherel measure. See [Pu], formula (2.19), p. 117. See also the bottom of p. 102 in [Pu].

3.4. Let $exp : \mathfrak{g} \to G$ be the exponential map for $G$. Since $G$ is simply-connected, one knows that

$$R \to K, \quad t \mapsto exp\, t(e-f) \tag{3.12}$$

is a diffeomorphism. Furthermore, if $Z = Cent\, G$, then $Z \subset K$. Since $Z$ is the kernel of the adjoint representation and (see (2.5)) $e - f = ih'$, it follows that $Z \cong \mathbb{Z}$ and

$$Z = \{exp\, m\pi(e-f) \mid m \in \mathbb{Z}\}. \tag{3.13}$$

Return to the representation $\pi_r$. For any element $exp\, t(e-f) \in K$ one has

$$\pi_r(exp\, t(e-f))\varphi_n^{(r)} = e^{it(2n+r+1)}\varphi_n^{(r)} \tag{3.14}$$

by (2.26). The scalar value on $\mathcal{H}$ of any central element is then given by

$$\pi_r(exp\, m\pi(e-f)) = e^{im\pi(r+1)} I. \tag{3.15}$$

We have proved

**Proposition 3.7.** *For any $m \in \mathbb{Z}$ one has $exp\, m\pi(e-f) \in Ker\, \pi_r$ if and only if $m(r+1) \in 2\mathbb{Z}$.*

For $k = 1, 2, \ldots$, clearly $G/Z^k$ is the $k$-fold covering of the adjoint group $PSl(2,\mathbb{R})$. As an immediate consequence of Proposition 3.7 one has

**Proposition 3.8.** *The unitary representation $\pi_r$ descends to a unitary representation of $G/Z^k$ if and only if $k(\frac{r+1}{2}) \in \mathbb{Z}$.*

Three cases stand out, namely, when $k = 1, 2, 4$. We may identify

$$\begin{aligned} PSl(2,\mathbb{R}) &= G/Z, \\ Sl(2,\mathbb{R}) &= G/Z^2, \\ Mp(2,\mathbb{R}) &= G/Z^4, \end{aligned} \tag{3.16}$$

where $Mp(2,\mathbb{R})$ is the 2-fold covering of $Sl(2,\mathbb{R})$, the so-called metaplectic group. But then Proposition 3.8 implies

**Proposition 3.9.** *The unitary representation $\pi_r$ descends to $PSl(2,\mathbb{R})$ if and only if $r$ is an odd integer.*

*It descends to $Sl(2,\mathbb{R})$ if and only if $r$ is an integer. From the spectrum (see (2.26)), in the latter case, $\pi_r$ is a holomorphic discrete series representation if $r = 1, 2, ...$ and is the so-called limit of such series when $r = 0$.*

*The unitary representation $\pi_r$ descends to the metaplectic group $Mp(2,\mathbb{R})$ if and only if $r \in \frac{1}{2}\mathbb{Z}$.*

3.5. It is clear from Proposition 3.9 that $\{\pi_{-\frac{1}{2}}, \pi_{\frac{1}{2}}\}$ is the only special pair for the metaplectic group $Mp(2,\mathbb{R})$. From the spectrum (2.26) it is clear that these representations are the two components of the "holomorphic" metaplectic representation $\mu$ of $Mp(2,\mathbb{R})$ on $L_2(\mathbb{R})$. Recall $\mu = \mu_{even} \oplus \mu_{odd}$ where $\mu_{even}$ (resp. $\mu_{odd}$) is the subrepresentation of $\mu$ defined by the even (resp. odd) square integrable functions on $\mathbb{R}$. In this section we will be explicit about the equivalences $\pi_{-\frac{1}{2}} \cong \mu_{even}$ and $\pi_{\frac{1}{2}} \cong \mu_{odd}$.

We recall some well-known properties of the metaplectic representation. Let $w \in C^\infty(\mathbb{R})$ denote the linear coordinate function. Then

$$D_\mu : U(\mathfrak{g}) \to \mathit{Diff}\ C^\infty(\mathbb{R})$$

is a representaion of $U(\mathfrak{g})$ by differential operators where

$$\begin{aligned} D_\mu(e) &= \frac{i}{2}w^2, \\ D_\mu(h) &= w\frac{d}{dw} + \frac{1}{2}, \\ D_\mu(f) &= \frac{i}{2}(\frac{d}{dw})^2. \end{aligned} \tag{3.17}$$

Let the Hermite polynomials $H_m(w)$, $m = 0, 1, ...$, on $\mathbb{R}$ be defined as in [Sz], p. 101 (not [Ja]). Thus these polynomials are the Gram-Schmidt orthogonalization of the monomials $1, w, w^2, ...$, with respect to the measure $e^{-w^2}dw$ on $\mathbb{R}$. It then follows from (5.7.2) in Theorem 5.7.1, p. 104 in [Sz] that $\psi_m(w) = e^{-\frac{w^2}{2}}H_m(w)$, for $m = 0, 1, ...$, define an orthogonal basis of the Hilbert space $\mathcal{L} = L_2(\mathbb{R})$ with respect to Lebesgue measure $dw$. But now by (5.5.2), p. 102 in [Sz] one has (recalling (2.5))

$$D_\mu(h')\psi_m = (m + \frac{1}{2})\psi_m. \tag{3.18}$$

For $m = 0, 1, ...$, let $\mathcal{L}^{HC}_{even}$ be the span $\psi_{2m}$ and let $\mathcal{L}^{HC}_{odd}$ be the span of $\psi_{2m+1}$. Let $\mathcal{L}^{HC} = \mathcal{L}^{HC}_{even} \oplus \mathcal{L}^{HC}_{odd}$. Now by the second equation in (5.5.10) in [Sz], p. 102 one has

$$(-\frac{d}{dw} + 2w)H_m = H_{m+1}. \tag{3.19}$$

But then this readily implies ("creation operator")

$$(-\frac{d}{dw} + w)\psi_m = \psi_{m+1}. \tag{3.20}$$

Squaring the operator on the left side of (3.20) implies

$$(\frac{d}{dw})^2 - 2w\frac{d}{dw} + w^2 - 1)\psi_m = \psi_{m+2}. \tag{3.21}$$

**Lemma 3.10.** *For $m = 0, 1, ...,$ one has*

$$D_\mu(e')\psi_m = \frac{i}{4}\psi_{m+2}. \tag{3.22}$$

*Proof.* It is immediate from (2.6) and (3.7) that the differential operator on the left side of (3.21) equals $\frac{4}{i}D_\mu(e')$. But then (3.22) follows from (3.21). QED

Squaring the "annihilation operator" yields

**Lemma 3.11.** *One has*

$$\begin{aligned} 0 &= D_\mu(f')\psi_0 \\ &= D_\mu(f')\psi_1. \end{aligned} \tag{3.23}$$

*Proof.* Obviously, $\psi_0(w) = e^{-\frac{w^2}{2}}$. But

$$\frac{d}{dw}(e^{-\frac{w^2}{2}}) = -w(e^{-\frac{w^2}{2}}). \tag{3.24}$$

Hence,

$$(\frac{d}{dw} + w)\psi_0 = 0. \tag{3.25}$$

But (3.20) and (3.24) imply that $\psi_1(w) = 2we^{-\frac{w^2}{2}}$. But then

$$(\frac{d}{dw} + w)\psi_1 = 2\psi_0. \tag{3.26}$$

Hence, $(\frac{d}{dw} + w)^2$ annihilates $\psi_0$ and $\psi_1$. But $(\frac{d}{dw} + w)^2 = (\frac{d}{dw})^2 + 2w\frac{d}{dw} + w^2 + 1$. But then by (2.7) one has $(\frac{d}{dw} + w)^2 = \frac{2}{i}D_\mu(f')$. This implies (3.23). QED

The argument establishing (2.28) in the proof of Theorem 2.6 when applied in the present case easily yields

$$D_\mu(f')\psi_m = i\, m(m-1)\psi_{m-2} \tag{3.27}$$

for $m = 2, 3, 4, ....$ It follows then from (3.18), (3.22) and (3.27) that $\mathcal{L}^{HC}_{even}$ and $\mathcal{L}^{HC}_{odd}$ are both stable and irreducible under $D_\mu(U(\mathfrak{g}))$. Next note that, by (3.17), $D_\mu(u)$ is manifestly skew-symmetric for any $u \in \mathfrak{g}$. In addition, one readily computes that

$$D_\mu(Cas) = -\frac{3}{32}. \tag{3.28}$$

Consequently, as in the proof of Theorem 3.4 one has that $D_\mu(\sum_{i=1}^3 u_i^2)|\mathcal{L}^{HC}$ is essentially self-adjoint. Furthermore, $D_\mu(u_1)|\mathcal{L}^{HC}$ is essentially skew-adjoint by (3.18). Let $\mathcal{L}_{even}$ (resp. $\mathcal{L}_{odd}$) be the closure of $\mathcal{L}^{HC}_{even}$ (resp. $\mathcal{L}^{HC}_{odd}$) so that $\mathcal{L}_{even}$ (resp. $\mathcal{L}_{odd}$) is the subspace of even (resp. odd) functions in $\mathcal{L} = L_2(\mathbb{R})$ and of course

$$\mathcal{L} = \mathcal{L}_{even} \oplus \mathcal{L}_{odd}.$$

As in the proof of Theorem 3.4, Nelson's Theorem 5, p. 602 in [Ne], applies and now recovers (without needing the intervention of the Heisenberg group) the following well-known result about the metaplectic representation.

**Proposition 3.12.** *The representation $D_\mu|\mathcal{L}^{HC}$ exponentiates to define a unitary representation $\mu$ of $G$ on $\mathcal{L} = L_2(\mathbb{R})$. Furthermore, $\mu$ descends to the metaplectic group $Mp(2,\mathbb{R})$. Moreover, $\mathcal{L}_{even}$ and $\mathcal{L}_{odd}$ are stable subspaces and respectively define irreducible unitary representations $\mu_{even}$ and $\mu_{odd}$ of $Mp(2,\mathbb{R})$. Finally, $\mathcal{L}^{HC}_{even}$ (resp. $\mathcal{L}^{HC}_{odd}$) is the Harish-Chandra module of $\mu_{even}$ (resp. $\mu_{odd}$) where any $u \in \mathfrak{g}_\mathbb{C}$ operates as $D_\mu(u)|\mathcal{L}^{HC}_{even}$ (resp. $D_\mu(u)|\mathcal{L}^{HC}_{odd}$).*

*Proof.* The only matter to be checked is the descent to $Mp(2,\mathbb{R})$. Let $g \in Z$, using the notation of (3.13), so that $g = exp\, k\pi(e-f)$ for $k \in \mathbb{Z}$. But then

$$\mu(g)\psi_m = e^{ik\pi(m+\frac{1}{2})}\psi_m \tag{3.29}$$

by (3.18). It follows then that $g \in Ker\,\mu$ if and only if $g \in Z^4$. But then the result follows from (3.16). QED

Our definition of Laguerre polynomials $L_n^{(r)}$ is taken from [Ja] and not from [Sz]. Denoting the latter by $L_n^{(r)}(Szego)$ it follows from a comparison of (1), p. 184 in [Ja] and (5.1.5), p. 97 in [Sz] that

$$L_m^{(r)}(Szego) = (-1)^m \frac{1}{m!} L_m^{(r)}. \tag{3.30}$$

But then equations (5.6.1), p. 102 in [Sz], writing Hermite polynomials in terms of Laguerre polynomials, simplifies to

$$H_{2m}(w) = 2^{2m} L_m^{(-\frac{1}{2})}(w^2),$$
$$H_{2m+1}(w) = 2^{2m+1} w L_m^{(\frac{1}{2})}(w^2).$$

But then

$$\begin{aligned} \psi_{2m}(w) &= 2^{2m} e^{-\frac{w^2}{2}} L_m^{(-\frac{1}{2})}(w^2), \\ \psi_{2m+1}(w) &= 2^{2m+1} w e^{-\frac{w^2}{2}} L_m^{(\frac{1}{2})}(w^2). \end{aligned} \tag{3.31}$$

But on the other hand, the maps

$$\eta_{even} : L_2((0,\infty)) \to \mathcal{L}_{even},$$
$$\eta_{odd} : L_2((0,\infty)) \to \mathcal{L}_{odd}$$

are manifestly unitary isomorphisms where, for $\varphi \in L_2((0,\infty))$,

$$\begin{aligned} (\eta_{even}\varphi)(w) &= |\frac{w}{2}|^{\frac{1}{2}}\, \varphi(\frac{w^2}{2}), \\ (\eta_{odd}\varphi)(w) &= |\frac{w}{2}|^{\frac{1}{2}}\, \varphi(\frac{w^2}{2})\, sign\, w. \end{aligned} \tag{3.32}$$

But by definition

$$\varphi_m^{(-\frac{1}{2})}(x) = x^{-\frac{1}{4}} e^{-x} L_m^{(-\frac{1}{2})}(2x),$$
$$\varphi_m^{(\frac{1}{2})}(x) = x^{\frac{1}{4}} e^{-x} L_m^{(\frac{1}{2})}(2x).$$

But then by (3.31) and (3.32)

$$\begin{aligned}(\eta_{even}(\varphi_m^{(-\frac{1}{2})}))(w) &= |\frac{w}{r2}|^{\frac{1}{2}} 2^{\frac{1}{4}} |w|^{-\frac{1}{2}} e^{-\frac{w^2}{2}} L_m^{(-\frac{1}{2})}(w^2) \\ &= 2^{-\frac{1}{4}} e^{-\frac{w^2}{2}} L_m^{(-\frac{1}{2})}(w^2) \\ &= 2^{-2m-\frac{1}{4}} \psi_{2m}(w).\end{aligned} \tag{3.33}$$

Again by (3.31) and (3.32)

$$\begin{aligned}(\eta_{odd}(\varphi_m^{(\frac{1}{2})}))(w) &= |\frac{w}{2}|^{\frac{1}{2}} 2^{-\frac{1}{4}} |w|^{\frac{1}{2}}\, sign\, w\; e^{-\frac{w^2}{2}} L_m^{(\frac{1}{2})}(w^2) \\ &= 2^{-\frac{3}{4}} w e^{-\frac{w^2}{2}} L_m^{(\frac{1}{2})}(w^2) \\ &= 2^{-2m-\frac{7}{4}} \psi_{2m+1}(w).\end{aligned} \tag{3.34}$$

**Theorem 3.13.** *The unitary isomorphism $\eta_{even}\colon L_2((0,\infty)) \to \mathcal{L}_{even}$ (see (3.32)) intertwines the irreducible unitary representations $\pi_{-\frac{1}{2}}$ and $\mu_{even}$ of the metaplectic group $Mp(2,\mathbb{R})$. Also the unitary isomorphism $\eta_{odd}\colon L_2((0,\infty)) \to \mathcal{L}_{odd}$ intertwines the irreducible unitary irreducible representations $\pi_{\frac{1}{2}}$ and $\mu_{odd}$ of $Mp(2,\mathbb{R})$.*

*Proof.* It is clear from (3.33) and (3.34) that

$$\begin{aligned}\eta_{even}(\mathcal{H}^{HC}_{-\frac{1}{2}}) &= \mathcal{L}^{HC}_{even}, \\ \eta_{odd}(\mathcal{H}^{HC}_{\frac{1}{2}}) &= \mathcal{L}^{HC}_{odd}.\end{aligned} \tag{3.35}$$

Since all the unitary representations in question arise by application of Nelson's theorem, it suffices to show, for all $u \in \mathfrak{g}_{\mathbb{C}}$, (1) that $\eta_{even}|\mathcal{H}^{HC}_{-\frac{1}{2}}$ intertwines $D_{-\frac{1}{2}}(u)|\mathcal{H}^{HC}_{-\frac{1}{2}}$ and $D_\mu(u)|\mathcal{L}^{HC}_{even}$ and (2) that $\eta_{odd}|\mathcal{H}^{HC}_{\frac{1}{2}}$ intertwines $D_{\frac{1}{2}}(u)|\mathcal{H}^{HC}_{\frac{1}{2}}$ and $D_\mu(u)|\mathcal{L}^{HC}_{odd}$. But since $e$ and $e'$ clearly generate the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ it suffices to prove (1) and (2) only for $u = e$ and $u = e'$. In the case $u = e$, (1) and (2) are obvious from (1.1), (3.17) and (3.32). Now assume $u = e'$. By (2.26), (3.22) it follows from (3.33) that

$$\begin{aligned}\eta_{even}(D_{-\frac{1}{2}}(e')\varphi_m^{(-\frac{1}{2})}) &= \eta_{even}(i\varphi_{m+1}^{(-\frac{1}{2})}) \\ &= i2^{-2(m+1)-\frac{1}{4}} \psi_{2(m+1)} \\ &= 2^{-2m-\frac{1}{4}} (\frac{i}{4}) \psi_{2(m+1)} \\ &= D_\mu(e') 2^{-2m-\frac{1}{4}} \psi_{2m} \\ &= D_\mu(e') \eta_{even}(\varphi_m^{(-\frac{1}{2})}).\end{aligned}$$

This establishes (1). The argument establishing (2), using (3.34) is identical. QED

## 4. Distribution theory on $(0,\infty)$ and the spaces $\mathcal{H}_r^\infty$ and $\mathcal{H}_r^{-\infty}$

4.1. As usual $-1 < r < \infty$. A function $\varphi \in L_2((0,\infty))(= \mathcal{H})$ is called $\pi_r$-smooth if $G \to \mathcal{H}$, $g \mapsto \pi_r(g)(\varphi)$ is a $C^\infty$-map. Let $\mathcal{H}_r^\infty$ be the space of all $\pi_r$-smooth vectors in $\mathcal{H}$. We recall some properties of $\mathcal{H}_r^\infty$. See [Ca]. Also see §4.4, p. 252 in [Wa]. First of all $\mathcal{H}_r^\infty$ is contained in the domain of $\widetilde{\pi_r}(u)$ for any $u \in \mathfrak{g}$. The restriction of $\widetilde{\pi_r}(u)$ to $\mathcal{H}_r^\infty$ stabilizes $\mathcal{H}_r^\infty$ and defines a representation of $\mathfrak{g}$ on $\mathcal{H}_r^\infty$. This then extends to a representation

$$U(\mathfrak{g}) \to End\, \mathcal{H}_r^\infty. \tag{4.1}$$

Also, $\mathcal{H}_r^\infty$ is stable under $\pi_r(g)$ for any $g \in G$. But now if $Dist_o(G)$ denotes the convolution algebra of distributions of compact support on $G$, then we may regard $G \subset Dist_o(G)$ when $G$ is identified with the group of Dirac measures. In addition, $U(\mathfrak{g}) \subset Dist_o(G)$ when $U(\mathfrak{g})$ is identified with the algebra of distributions having support at the identity. But one has a representation

$$\pi_r^\infty\colon Dist_o(G) \to End\,\mathcal{H}_r^\infty \tag{4.2}$$

where if $\nu \in Dist_o(G)$ and $\varphi \in \mathcal{H}_r^\infty$, then

$$\pi_r^\infty(\nu)\varphi = \int_G \pi_r(g)\varphi\,\nu(g)dg. \tag{4.3}$$

The homomorphism $\pi_r^\infty$ extends the domain of (4.1) and in addition includes $\pi_r(g)|\mathcal{H}_r^\infty$ for any $g \in G$. If $\|\varphi\| = \{\varphi,\varphi\}^{\frac{1}{2}}$ for any $\varphi \in \mathcal{H}$, one defines a Fréchet topology on $\mathcal{H}_r^\infty$ by taking as semi-norms $\|\varphi\|_v = \|\pi_r(v)\varphi\|$ for $v \in U(\mathfrak{g})$ and $\varphi \in \mathcal{H}_r^\infty$. We will refer to this topology as the $\pi_r$-Fréchet topology. The image of (4.2) are continuous operators and hence, by transpose, define operators on the continuous dual to $\mathcal{H}_r^\infty$. This may be put in a neater form. One defines a $*$-operation on $Dist_o(G)$, $\nu \mapsto \nu^*$, by defining $\nu^*$ so that

$$\int_G \psi(g)\nu^*(g)dg = \overline{\int_G \overline{\psi}(g^{-1})\nu(g)dg}$$

for any $\psi \in C_o^\infty(G)$. Here $C_o^\infty(G)$ is the space of $C^\infty$ functions of compact support on $G$. It is easy to see that this $*$-operation is an extension of the $*$-operation defined on $\mathfrak{g}_\mathbb{C}$ considered in (3.1). Clearly, there now exists a vector space ($\pi_r$-tempered distributions) $\mathcal{H}_r^{-\infty}$ and a sesquilinear form $\{\varphi,\rho\}$ for $\varphi \in \mathcal{H}_r^\infty$ and $\rho \in \mathcal{H}_r^{-\infty}$ such that $\varphi \mapsto \{\varphi,\rho\}$ is a continuous linear functional on $\mathcal{H}_r^\infty$ and every continuous linear functional is uniquely of this form. See e.g. [Ca]. Clearly, one has a representation

$$\pi_r^{-\infty}\colon Dist_o \to End\,\mathcal{H}_r^{-\infty} \tag{4.4}$$

so that for $\varphi \in \mathcal{H}_r^\infty$, $\rho \in \mathcal{H}_r^{-\infty}$ and $\nu \in Dist_o(G)$,

$$\{\pi_r^\infty(\nu^*)\varphi,\rho\} = \{\varphi,\pi_r^{-\infty}(\nu)\rho\}.$$

The map $\varphi \mapsto \{\varphi,\psi\}$ for $\varphi \in \mathcal{H}_r^\infty$ and $\psi \in \mathcal{H}$, where $\{\varphi,\psi\}$ is the ordinary inner product in $\mathcal{H}$, is clearly a linear functional on $\mathcal{H}_r^\infty$ which is continuous with respect to the $\pi_r$-Fréchet topology. In this way one has an embedding of $L_2((0,\infty))$ in $\mathcal{H}_r^{-\infty}$ and hence inclusions

$$\mathcal{H}_r^\infty \subset L_2((0,\infty)) \subset \mathcal{H}_r^{-\infty}. \tag{4.5}$$

Furthermore,

$$\pi_r^{-\infty}(\nu)|\mathcal{H}_r^\infty = \pi_r^\infty(\nu) \tag{4.6}$$

for any $\nu \in Dist_o(G)$. Thus $\mathcal{H}_r^\infty$ as a $Dist_o(G)$-module defined by (4.2) is just a submodule of $\mathcal{H}_r^{-\infty}$ (defined by (4.4)). In particular, both $G$ and $U(\mathfrak{g})$ operate on $\mathcal{H}_r^{-\infty}$ and

$$\pi_r^{-\infty}(g)|L_2((0,\infty)) = \pi_r(g).$$

For any $u \in \mathfrak{g}$ let $Dom_r(u)$ be the domain of $\widetilde{\pi_r}(u)$. If $u \in \mathfrak{g}$, note that

$$\widetilde{\pi_r}(u) = \pi_r^{-\infty}(u)|Dom_r(u). \tag{4.7}$$

Indeed, this is clear since if $\varphi \in \mathcal{H}_r^\infty$ and $\psi \in Dom_r(u)$, one has

$$\{-\pi_r^\infty(u)\varphi, \psi\} = \{\varphi, \widetilde{\pi_r}(u)\psi\}. \tag{4.8}$$

The equality (4.8) follows from the obvious fact that $\mathcal{H}_r^\infty \subset Dom_r(u)$ and

$$\pi_r^\infty(u) = \widetilde{\pi_r}(u)|\mathcal{H}_r^\infty. \tag{4.9}$$

For completeness we should note that, as a consequence of Theorem 3.4, one has

$$\mathcal{H}_r^{HC} \subset \mathcal{H}_r^\infty \tag{4.10}$$

and by (4.9) and (3.9)

$$D_r(u)|\mathcal{H}_r^{HC} = \pi_r^\infty(u)|\mathcal{H}_r^{HC}. \tag{4.11}$$

A vector $\varphi \in \mathcal{H}$ will be called $\pi_r$-analytic if the map $G \to \mathcal{H}$, $g \mapsto \pi_r(g)\varphi$ is analytic. In such a case obviously $\varphi \in \mathcal{H}^\infty$ and one knows (Harish-Chandra, see Theorem 4.4.5.4, p. 278 in [Wa]) that there exists a neighborhood, $\mathcal{U}$ of 0 in $\mathfrak{g}$ such that for any $u \in \mathcal{U}$ the left side of (4.12), below, converges in $L_2$ and

$$\sum_{m=0}^{\infty} \frac{1}{m!}\pi_r^\infty(u^m)\varphi = \pi_r(exp\, u)\varphi. \tag{4.12}$$

**Proposition 4.1.** *Any $\varphi \in \mathcal{H}_r^{HC}$ is $\pi_r$-analytic. Furthermore, if $u \in \mathfrak{g}$, then $D_r(u)|\mathcal{H}_r^{HC}$ is essentially skew-adjoint. That is, $\widetilde{\pi_r}(u)$ is the operator closure of $D_r(u)|\mathcal{H}_r^{HC}$.*

*Proof.* One has, restating (3.9),

$$D_r(u)|\mathcal{H}_r^{HC} = \widetilde{\pi_r}(u)|\mathcal{H}_r^{HC}. \tag{4.13}$$

It is a theorem of I.E. Segal (see [Se]) that $\widetilde{\pi_r}(u)$ is the operator closure of the restriction of $\pi_r^\infty(u)$ to the Garding space. But then $\widetilde{\pi_r}(u)$ is the operator closure of $D_r(u)|\mathcal{H}_r^{HC}$ by the last statement of Nelson's Theorem 5 in [Ne]. That is, $D_r(u)|\mathcal{H}_r^{HC}$ is essentially skew-adjoint. This and more also follows directly from [Ne]. Indeed, put $A = D_r(\sum_{j=1}^3 u_i^2)$, using the notation of (3.7). Let $\varphi \in \mathcal{H}_r^{HC}$. But $\varphi$ is a finite sum of eigenvectors for $A$. Obviously, $\varphi$ is then an analytic vector for $A - 1$, in the terminology of [Ne]. But then, by Lemma 6.2 in [Ne], $\varphi$ is an analytic vector for $D_r(u)$ for any $u \in \mathfrak{g}$. This implies that $D_r(u)|\mathcal{H}_r^{HC}$ is essentially skew-adjoint by Lemma 5.1 in [Ne]. Finally, any $\varphi \in \mathcal{H}_r^{HC}$ is a $\pi_r$-analytic vector by Goodman's theorem. Stated as Theorem 4.4.6.1 in [Wa] one notes, in the notation of that theorem, $\varphi$ is an analytic vector for $B$ since it is, clearly, a finite sum of eigenvectors for $B$. QED

*Remark* 4.2. As a consequence of the phenomenon noted in Remark 3.5, one cannot have the equality of $\pi_r^{-\infty}(u)$ with the differential operator $D_r(u)$ on all infinitely differentiable functions in $L_2((0,\infty))$ for all $u \in \mathfrak{g}$ when $0 < |r| < 1$. Indeed, for such values of $r$ the equality of $\pi_r^{-\infty}(u)$ and $D_r(u)$ for all $u \in \mathfrak{g}$ must already fail on $\mathcal{H}_{-r}^{HC}$. In fact, by Remark 3.5, (2.26), and using the notation of (3.7), the operator $D_r(u_1)|\mathcal{H}_{-r}^{HC}$ has eigenvalues which cannot be eigenvalues of $\pi_r^{-\infty}(u_1)|\mathcal{H}$. If $\pi_r^{-\infty}(u_1)$ had such an eigenvalue, the corresponding nonzero eigenvector, by necessity, would have to be orthogonal to each element of the orthogonal basis $\{\varphi_n^r\}$ of $\mathcal{H}$. This is of course a contradiction.

4.2. Recall that $D_r(e) = ix$.

**Proposition 4.3.** *If $\phi \in \mathcal{H}_r^\infty$, then $(ix)^k\phi(x) \in \mathcal{H}_r^\infty$ for any $k \in \mathbb{Z}_+$. Furthermore,*

$$\pi_r^\infty(e^k)\phi(x) = (ix)^k\phi(x). \tag{4.14}$$

*Also, for any $t \in \mathbb{R}$,*

$$\pi_r(exp\, te)\psi(x) = e^{itx}\psi(x) \tag{4.15}$$

*for any $\psi \in L_2((0,\infty))$.*

*Proof.* Let $\phi \in \mathcal{H}_r^\infty$. Then $\phi$ is in the domain of the closure of $D_r(e)|\mathcal{H}_r^{HC}$ by Proposition 4.1. Thus there is a sequence $\phi_n \in \mathcal{H}_r^{HC}$ such that $\phi_n$ converges to $\phi$ and $ix\phi_n$ converges to $\pi_r^\infty(e)(\phi)$ in $L_2((0,\infty))$. But then, almost everywhere, $\phi_{n_j}$ converges to $\phi$ and $ix\phi_{n_j}$ converges to $\pi_r^\infty(e)(\phi)$ for some subsequence $\phi_{n_j}$ of $\phi_n$. But this implies $\pi_r^\infty(e)(\phi) = ix\phi$. In particular, $ix\phi \in \mathcal{H}_r^\infty$. By iteration one obtains (4.14) and the first statement of Proposition 4.3.

Let $\psi \in \mathcal{H}_r^{HC}$. Then since $\psi$ is a $\pi_r$ analytic vector by Proposition 4.1, there exists a neighborhood $\mathcal{V}$ of 0 in $\mathbb{R}$ such that for $t \in \mathcal{V}$ the left side of (4.16) below converges in $L_2$ and

$$\sum_{m=0}^{\infty}\frac{1}{m!}t^m D_r(e^m)\psi = \pi_r(exp\, te)\psi. \tag{4.16}$$

In particular, a subsequence of the sequence of cut-off sums on the left side of (4.16) converges almost everywhere to the right side of (4.15). But the left side of (4.16) converges everywhere to $e^{itx}\psi(x)$. Thus $\pi_r(exp\, te)\psi(x) = e^{itx}\psi(x)$. But then if $A$ is the skew-adjoint infinitesimal generator (given by Stone's theorem) of the 1-parameter group, $t \mapsto e^{itx}$, of unitary operators on $\mathcal{H}$, one has that $\mathcal{H}_r^{HC}$ is in the domain of $A$ and $A|\mathcal{H}_r^{HC} = D_r(e)$. But $D_r(e)|\mathcal{H}_r^{HC}$ is essentially skew-adjoint by Proposition 4.1 and hence $A = \widetilde{\pi_r}(e)$. This proves (4.15). QED

4.3. If $X$ is a manifold, let $M(X)$ denote the space of Borel measurable functions on $X$ and if $\mu$ is a Borel measure on $X$, let $L_2(x,\mu)$ be the Hilbert space of square integrable functions on $X$ with respect to $\mu$. Let $\alpha : M((0,\infty)) \to M((0,\infty))$ be the linear isomorphism defined by putting

$$\alpha(\varphi)(x) = x^{\frac{1}{2}}\varphi(x). \tag{4.17}$$

It is immediate that

$$\alpha\colon \mathcal{H} \to L_2((0,\infty), \frac{dx}{x}) \tag{4.18}$$

is a unitary isomorphism and one notes that $\frac{dx}{x}$ is Haar measure on $(0,\infty)$ as a multiplicative group. Next note that

$$\alpha\colon C^\infty((0,\infty)) \to C^\infty((0,\infty)) \tag{4.19}$$

is a linear isomorphism and $2x\frac{d}{dx}$ is a left- (or right)-invariant vector field on $(0,\infty)$ with respect to the multiplicative group structure. Recall that $D_r(h) = 2x\frac{d}{dx} + 1$. One easily has

$$2x\frac{d}{dx} \circ \alpha = \alpha \circ D_r(h) \tag{4.20}$$

on $C^\infty((0,\infty))$. Now $\varepsilon : \mathbb{R} \to (0,\infty)$ is an isomorphism of Lie groups where $\varepsilon(w) = e^w$. Let $\beta\colon M((0,\infty)) \to M(\mathbb{R})$ be defined by putting $\beta(\psi) = \psi \circ \varepsilon$. Then clearly

$\beta : L_2((0,\infty), \frac{dx}{x}) \to L_2(\mathbb{R}, dw)$ is a unitary isomorphism and $\beta\colon C^\infty((0,\infty)) \to C^\infty(\mathbb{R})$ is an algebra isomorphism. Furthermore, $\beta \circ 2x\frac{d}{dx} = 2\frac{d}{dw}\circ\beta$ on $C^\infty((0,\infty))$. Finally, let $\gamma = \beta\circ\alpha$. Then by composition

$$\gamma : \mathcal{H} \to L_2(\mathbb{R}, dw) \tag{4.21}$$

is a unitary isomorphism. Furthermore,

$$\gamma\colon C^\infty((0,\mathbb{R})) \to C^\infty(\mathbb{R}) \tag{4.22}$$

is a linear isomorphism

$$2\frac{d}{dw}\circ\gamma = \gamma\circ D_r(h) \tag{4.23}$$

on $C^\infty((0,\mathbb{R}))$. We are now ready to establish that the functions in $\mathcal{H}_r^\infty$ are (infinitely) smooth.

**Proposition 4.4.** *One has*

$$\mathcal{H}_r^\infty \ \subset\ C^\infty((0,\infty)). \tag{4.24}$$

*Also, the 1-parameter group of unitary operators on $\mathcal{H}$, $t\mapsto \pi_r(exp\ th)$, is given by*

$$\pi_r(exp\ th)(\varphi)(x) = e^t\varphi(e^{2t}x) \tag{4.25}$$

*for any $\varphi\in\mathcal{H}$.*

*Proof.* Let $W$ be the space of all functions in $L_2(\mathbb{R}, dw)$ which are absolutely continuous on every finite interval and whose first derivative is again in $L_2(\mathbb{R}, dw)$. Then it is classical that $2\frac{d}{dw}$ operating on $W$ defines a skew-adjoint operator $B$. (See e.g. Example 3, p. 198 in [Yo].) Let $V = \gamma(\mathcal{H}_r^{HC})$ so that by (4.21), (4.22) and (4.23) one has

$$V \ \subset\ C^\infty(\mathbb{R})\cap L_2(\mathbb{R}, dw)$$

and that furthermore, $V$ is stable under $2\frac{d}{dw}$. But clearly $V \ \subset\ W$ and

$$B|V = 2\frac{d}{dw}|V.$$

But $2\frac{d}{dw}|V$ is essentially skew-adjoint by Proposition 4.1, and hence $B$ must be the closure of $2\frac{d}{dw}|V$. Also, one must have

$$W = \gamma(Dom_r(h))$$

(using the notation of Remark 4.1) and

$$B\circ\gamma = \gamma\circ\widetilde{\pi_r}(h) \tag{4.26}$$

on $Dom_r(h)$. But since, obviously, $\mathcal{H}_r^\infty \ \subset\ Dom_r(h)$, one has

$$V \ \subset\ Y \ \subset\ W \tag{4.27}$$

where $Y = \gamma(\mathcal{H}_r^\infty)$ and

$$B\circ\gamma = \gamma\circ\pi_r^\infty(h) \tag{4.28}$$

on $\mathcal{H}_r^\infty$. But $\mathcal{H}_r^\infty$ is stable under $\pi_r^\infty(h)$. Hence, $Y$ is stable under $B$ by (4.28). But then, by the definition of $B$, one must have $Y \ \subset\ C^\infty(\mathbb{R})$. Thus $\mathcal{H}_r^\infty \ \subset\ C^\infty((0,\infty))$ by inverting (4.23).

It is classical and immediate that the 1-parameter group of unitary operators, $e^{tB}$, on $L_2(\mathbb{R}, dw)$ generated by $B$, is the translation group given by $e^{tB}\varphi(w) = \varphi(w+2t)$ for any $\varphi\in L_2(\mathbb{R}, dw)$. It follows then from (4.26) and the definition of

$\widetilde{\pi_r}(h)$ that $e^{tB} \circ \gamma = \gamma \circ \pi_r(exp\ th)$. But if $t \mapsto \pi(t)$ is the 1-parameter group of unitary operators on $\mathcal{H}$ defined by the right side of (4.25), then, by the definition of $\gamma$, clearly $e^{tB} \circ \gamma = \gamma \circ \pi(t)$. This proves (4.25). QED

4.4. Let $C_o^\infty((0,\infty))$, $Dist_o((0,\infty))$ and $Dist((0,\infty))$ be, respectively, the spaces, on $(0,\infty)$, of all $C^\infty$ functions of compact support, all distributions of compact support and the space of all distributions.

*Remark* 4.5. As one easily sees, the differential operators, $D_r(u)$, for $u \in \mathfrak{g}$ (see (1.1)) are "formally skew-symmetric" in the sense that they are skew-symmetric for pairings when $\frac{d}{dx}$ operates as a skew-symmetric operator. This is certainly the case of a pairing of $\phi \in C^\infty((0,\infty))$ with $\psi \in C_o^\infty((0,\infty))$. That is, for any $u \in \mathfrak{g}$,

$$\int_0^\infty D_r(u)\psi(x)\overline{\phi(x)}dx = -\int_0^\infty \psi(x)\overline{D_r(u)\phi(x)}dx. \tag{4.29}$$

Another theorem of Nelson enables us to establish

**Proposition 4.6.** *One has*

$$C_o^\infty((0,\infty)) \ \subset \ \mathcal{H}_r^\infty \tag{4.30}$$

*and for any* $u \in \mathfrak{g}$ *one has*

$$D_r(u)|C_o^\infty((0,\infty)) = \pi_r^\infty(u)|C_o^\infty((0,\infty)). \tag{4.31}$$

*Proof.* By (4.29) one has

$$\{D_r(u)\psi,\phi\} = -\{\psi, D_r(u)\phi\} \tag{4.32}$$

for any $\phi \in \mathcal{H}_r^{HC}$ and $\psi \in C_o^\infty((0,\infty))$. Let $\Delta = \sum_{i=1}^3 u_i^2$ using the notation of (3.7). Then (4.32) implies

$$\{D_r(\Delta)\psi,\phi\} = \{\psi, D_r(\Delta)\phi\}. \tag{4.33}$$

But, as established in the proof of Theorem 3.4, $D_r(\Delta)|\mathcal{H}_r^{HC}$ is essentially self-adjoint. Let $A$ be the closure of $D_r(\Delta)|\mathcal{H}_r^{HC}$. Then (4.33) implies

$$C_o^\infty((0,\infty)) \ \subset \ Dom(A) \tag{4.34}$$

and

$$A|C_o^\infty((0,\infty)) = D_r(\Delta)|C_o^\infty((0,\infty)). \tag{4.35}$$

But since $\pi_r^\infty(\Delta)$ is clearly a symmetric extension of $D_r(\Delta)|\mathcal{H}_r^{HC}$, it follows that $A$ is the closure of $\pi_r^\infty(\Delta)$. But $C_o^\infty((0,\infty))$ is stable under $A$ by (4.35). Thus $C_o^\infty((0,\infty)) \ \subset \ Dom(A^k)$ for all $k \in \mathbb{Z}_+$. But then $C_o^\infty((0,\infty)) \ \subset \ \mathcal{H}_r^\infty$ by Theorem 4.4.4.5, p. 270 in [Wa] (another result of Nelson). But then (4.32) implies (4.31) since $\mathcal{H}_r^{HC}$ is Hilbert space dense in $\mathcal{H}$. QED

4.5. We can now prove that (4.31) is true when we replace $C_o^\infty((0,\infty))$ by $\mathcal{H}_r^\infty$. Recall that $\mathcal{H}_r^\infty \ \subset \ C^\infty((0,\infty))$ by (4.24). One should keep in mind the possible non-equality of $D_r(u)$ with $\pi_r^{-\infty}(u)$ on $C^\infty((0,\infty)) \cap \mathcal{H}$ when $0 < |r| < 1$. See Remark 4.2.

**Proposition 4.7.** *For any* $u \in \mathfrak{g}$ *one has*

$$\pi_r^\infty(u) = D_r(u)|\mathcal{H}_r^\infty. \tag{4.36}$$

*See (4.24). Furthermore, the injection map*

$$\mathcal{H}_r^\infty \to C^\infty((0,\infty)) \tag{4.37}$$

*is continuous where $\mathcal{H}_r^\infty$ has the $\pi_r$-Fréchet topology (see §4.1) and $C^\infty((0,\infty))$ (written as $\mathcal{E}((0,\infty))$ in distribution theory) has the Fréchet toplogy defined as in distribution theory. That is, uniform convergence of all derivatives on compact sets.*

*Proof.* Let $\psi \in C_o^\infty((0,\infty))$, $\phi \in \mathcal{H}_r^\infty$ and $u \in \mathfrak{g}$. Then, by (4.31), $\{D_r(u)\psi,\phi\} = -\{\psi,\pi_r^\infty(u)\phi\}$. But then, by (4.29),

$$\int_0^\infty \psi(x)\overline{(D_r(u)\phi(x) - \pi_r^\infty(u)\phi(x))}dx = 0.$$

Since $\psi \in C_o^\infty((0,\infty))$ is arbitrary, this clearly implies $D_r(u)\phi(x) = \pi_r^\infty(u)\phi$ establishing (4.36).

Now assume that a sequence $\phi_n$ converges to zero in $\mathcal{H}_r^\infty$ with respect to the $\pi_r$-Fréchet topology. Then, recalling (1.1), for any $k \in \mathbb{Z}_+$, with the norm in $\mathcal{H}$, $\|(2x\frac{d}{dx}+1)^k\phi_n(x)\|$ converges to zero. With the notation of (4.21) one has that $\varphi_n \in C^\infty(\mathbb{R})$ and $(2\frac{d}{dw})^k\varphi_n$ converges to zero in $L_2(\mathbb{R},dw)$. That is, using the notation of §8, p. 55 in [Sc], $\varphi_n$ converges to 0 in $\mathcal{D}_{L^2}$. But then it is classical that $\varphi_n$ converges to zero in $\mathcal{E}(\mathbb{R})$. (See the inclusion on p. 166 in [Sc]. One multiplies by a suitable function in $\mathcal{D}(\mathbb{R})$ and uses integration by parts.) But now since $x$ is bounded away from 0 and also from above on any closed subinterval $[a,b] \subset (0,\infty)$, it follows by induction, that $(\frac{d}{dx})^k\phi_n$ converges to 0 uniformly on $[a,b]$. This proves the continuity of (4.37). QED

4.6. Let $\rho \in \mathcal{H}_r^{-\infty}$. Then, by (4.30),

$$\psi \mapsto \{\psi,\rho\} \tag{4.38}$$

defines a linear functional on $C_o^\infty((0,\infty))$.

**Lemma 4.8.** *Let $\mathcal{D}((0,\infty))$ denote $C_o^\infty((0,\infty))$ when endowed with the LF-topology of distribution theory. Then (4.38) is continuous. That is, (4.38) defines an element in $Dist((0,\infty))$.*

*Proof.* We have only to show that if $\psi_n$ is a sequence in $C_o^\infty((0,\infty))$ with support in some closed interval $[a,b] \subset (0,\infty)$ such that $(\frac{d}{dx})^k\psi_n$ converges uniformly to 0 in $[a,b]$ for any $k \in \mathbb{Z}_+$, then

$$\{\psi_n,\rho\} \text{ converges to } 0. \tag{4.39}$$

But, recalling (4.31), the assumption on $\psi_n$ clearly implies that if $B$ is any differential operator on $(0,\infty)$ with coefficients in the ring $\mathbb{C}[x,x^{-1}]$, then, with the $\mathcal{H}$-norm, $\|B(\psi_n)\|$ converges to zero. But then, recalling (1.1), $\|D_r(v)(\psi_n)\|$ converges to zero for any $v \in U(\mathfrak{g})$. But by (4.31) this implies $\psi_n$ converges to 0 in the $\pi_r$-Fréchet topology. But then one has (4.39). QED

4.7. As a consequence of Lemma 4.8 one has a linear map

$$\zeta_r\colon \mathcal{H}_r^{-\infty} \to Dist\,((0,\infty)) \tag{4.40}$$

such that for any $\rho \in \mathcal{H}_r^{-\infty}$ and $\psi \in C_o^\infty((0,\infty))$,

$$\{\psi,\rho\} = \overline{\int_0^\infty \overline{\psi}(x)\zeta_r(\rho)(x)dx}. \tag{4.41}$$

It will be established later, in Theorem 5.26, that $C_o^\infty((0,\infty))$ is not dense in $\mathcal{H}_r^\infty$ with respect to the $\pi_r$-Fréchet topology. In particular, using the Hahn-Banach theorem, the kernel of $\zeta_r$ is always non-trivial. For the case $0<|r|<1$ this can be established now.

**Proposition 4.9.** *If $0<|r|<1$, then $C_o^\infty((0,\infty))$ is not dense in $\mathcal{H}_r^\infty$ with respect to the Fréchet topology of §4.1. In particular, (via the Hahn-Banach theorem) the kernel of $\zeta_r$ is non-trivial.*

*Proof.* Let $\psi\in\mathcal{H}_r^\infty$, $\varphi\in\mathcal{H}_{-r}^{HC}$ and $u\in\mathfrak{g}$. Assume $C_o^\infty((0,\infty))$ is dense in $\mathcal{H}_r^\infty$. Then there exists a sequence $\psi_n\in C_o^\infty((0,\infty))$ such that (see (4.31)) $\psi_n$ and $D_r(u)\psi_n$ respectively converge to $\psi$ and $\pi_r^\infty(u)\psi$ in $\mathcal{H}$. But $\{D_r(u)\psi_n,\varphi\} = -\{\psi_n, D_r(u)\varphi\}$ by (4.29). Taking the limit implies $\{\pi_r^\infty(u)\psi,\varphi\} = -\{\psi, D_r(u)\varphi\}$. But $\{\pi_r^\infty(u)\psi,\varphi\} = -\{\psi,\pi_r^{-\infty}(u)\varphi\}$. Hence, one has the equality $D_r(u)\varphi = \pi_r^{-\infty}(u)\varphi$. This contradicts the statement of Remark 4.2. QED

*Remark* 4.10. For the case (metaplectic case) where $r=\pm\frac{1}{2}$ one can already exhibit elements in $Ker\,\zeta_r$. Indeed, let $\eta_{-\frac{1}{2}}=\eta_{even}$ and $\eta_{\frac{1}{2}}=\eta_{odd}$ using the notation of (3.32). Then clearly, from (3.32), $\eta_r(\varphi)$ vanishes in a neighborhood of 0 for any $\varphi\in C_o^\infty((0,\infty))$. However, one must have that $\eta(\mathcal{H}_r^\infty)$ equals the space of even Schwartz functions or odd Schwartz functions according to whether $r=-\frac{1}{2}$ or $r=\frac{1}{2}$. Upon extending the domain of $\eta_r$ to $\mathcal{H}_r^{-\infty}$ the image of $\mathcal{H}_r^{-\infty}$ is, accordingly, the space of even or odd tempered distributions. But then $(\eta_{-\frac{1}{2}})^{-1}(\delta_0)\in Ker\,\zeta_{-\frac{1}{2}}$ and $(\eta_{\frac{1}{2}})^{-1}(\delta_0')\in Ker\,\zeta_{\frac{1}{2}}$ where $\delta_0$ is the Dirac measure at the origin and $\delta_0'$ is its first derivative.

If $V\subset Dist\,((0,\infty))$ is a subspace, then a linear map

$$\xi_V\colon V\to\mathcal{H}_r^{-\infty}$$

will be called a cross-section of $\zeta_r$ in case $\zeta_r\circ\xi_V$ is the identity on $V$. When $\xi_V$ is a specified cross-section and there is no possibility of ambiguity we will, when suitable, identify $V$ with $\xi_V(V)$. We have already done so when $V=L_2((0,\infty))$. We now do so when $V=Dist_o((0,\infty))$.

**Proposition 4.11.** *We may (and will) embed $Dist_o((0,\infty))$ in $\mathcal{H}_r^{-\infty}$ so that for any $\varphi\in\mathcal{H}_r^\infty$ and $\nu\in Dist_o((0,\infty))$ one has*

$$\{\varphi,\nu\} = \overline{\int_0^\infty \overline{\varphi}(x)\,\nu(x)dx}. \tag{4.42}$$

*Proof.* We have only to establish that the linear functional on $\mathcal{H}_r^\infty$ defined by the right side of (4.42) is continuous with respect to the $\pi_r$-Fréchet topology on $\mathcal{H}_r^\infty$. But this is immediate from the continuity of (4.37). Recall that $Dist_o((0,\infty))$ is the dual space to $\mathcal{E}((0,\infty))$. QED

*Remark* 4.12. A very important consequence, for us, of Proposition 4.11 is the now established fact that $\delta_y\in\mathcal{H}_r^{-\infty}$ for any $y\in(0,\infty)$ where $\delta_y$ is the Dirac measure at $y$.

Next, let $V_1\subset Dist(\mathbb{R})$ be the space of all Borel measurable functions $\rho$ on $(0,\infty)$ such that (A),

$$\rho|(0,1)\in L_2((0,1),dx) \tag{4.43}$$

and (B), there exists a positive constant $C$ and $k \in \mathbb{Z}_+$ (with $k$ and $C$ dependent on $\rho$) such that, on $[1,\infty)$,

$$|\rho(x)| < Cx^k. \tag{4.44}$$

We now find that $V_1$ also embeds in $\mathcal{H}_r^{-\infty}$.

**Proposition 4.13.** *We may (and will) embed $V_1$ in $\mathcal{H}_r^{-\infty}$ such that for any $\varphi \in \mathcal{H}_r^{-\infty}$ and $\rho \in V_1$ the integrand on the right side of (4.45) below is in $L_1((0,\infty),dx)$ and*

$$\{\varphi,\rho\} = \int_0^\infty \varphi(x)\overline{\rho(x)}dx \tag{4.45}$$

*Proof.* Since $\varphi \in \mathcal{H}$ one has $\varphi|(0,1) \in L_2((0,1),dx)$. But then

$$\varphi\overline{\rho}|(0,1) \in L_1((0,1),dx) \tag{4.46}$$

by (4.43). Let $C$ and $k$ be as in (4.44). Then, by (4.44),

$$x^{-(k+1)}\rho(x)|[1,\infty) \in L_2([1,\infty),dx). \tag{4.47}$$

On the other hand, $\pi_r^\infty(e^{k+1})(\varphi) \in \mathcal{H}_r^\infty$. In particular, $\pi_r^\infty(e^{k+1})(\varphi) \in \mathcal{H}$. But $\pi_r^\infty(e^{k+1})(\varphi)(x) = (ix)^{k+1}\varphi(x)$ by (1.1) and (4.36). Thus

$$(ix)^{k+1}\varphi(x)|[1,\infty) \in L_2([1,\infty),dx).$$

Multiplying by the conjugate of the function in (4.47), together with (4.46), proves that the integrand on the right side of (4.45) is absolutely integrable on $(0,\infty)$. Now assume that $\varphi_n \in \mathcal{H}_r^\infty$ converges to zero in the $\pi_r$-Fréchet topology. We have only to show that

$$\int_0^\infty \varphi_n(x)\overline{\rho(x)}dx \text{ converges to } 0. \tag{4.48}$$

But now, among other conditions, $\varphi_n$ converges to 0 in $\mathcal{H}$. Thus $\varphi_n|(0,1)$ converges to zero in $L_2((0,1),dx)$. Hence, by (4.43),

$$\int_0^1 \varphi_n(x)\overline{\rho(x)}dx \text{ converges to } 0. \tag{4.49}$$

But also $\|\pi_r^\infty(e^{k+1})\varphi_n\|$ converges to zero. Thus, in particular, $x^{k+1}\varphi_n(x)|[1,\infty)$ converges to 0 in $L_2([1,\infty),dx)$. But then, upon multiplication with the conjugate of the function in (4.47),

$$\int_1^\infty \varphi_n(x)\overline{\rho(x)}dx \text{ converges to } 0.$$

But then (4.48) follows from (4.49). QED

4.8. Let $\psi_n^{(r)} = (\frac{1}{2^{r+1}}n!\,\Gamma(n+r+1))^{-\frac{1}{2}}\varphi_n^{(r)}$ so that, by (2.3), $\{\psi_n^{(r)}\}$, $n = 0,1,...,$ is an orthonormal basis of $L_2((0,\infty),dx)$. By classical Hilbert space theory any $\varphi \in \mathcal{H}$ has the expansion

$$\varphi = \sum_{n=0}^\infty a_n\psi_n^{(r)} \tag{4.50}$$

where $a_n = \{\varphi, \psi_n^{(r)}\}$. We will refer to (4.50) as the Fourier-Laguerre expansion (of type $r$) of $\varphi$. For $j \in \mathbb{Z}_+$ let $\varphi_j = \sum_{n=0}^{j} a_n \psi_n^{(r)}$ so that $\varphi_j \in \mathcal{H}_r^{HC}$ and

$$\lim_{j\to\infty} \varphi_j = \varphi \tag{4.51}$$

in $\mathcal{H}$. Now assume $\varphi \in \mathcal{H}_r^\infty$. As in the proof of Proposition 4.6 let $\Delta = \sum_{i=1}^{3} u_i^2$. Then, for any $m \in \mathbb{Z}_+$, clearly

$$\{\pi_r^\infty((\Delta-1)^m)\varphi, \psi_n^{(r)}\} = \{\varphi, \pi_r^\infty((\Delta-1)^m)\psi_n^{(r)}\}.$$

But then since the $\psi_n^{(r)}$ are eigenvectors for $\pi_r^\infty((\Delta-1)^m)$, by (3.8), one has

$$\pi_r^\infty((\Delta-1)^m)(\varphi_j) = (\pi_r^\infty((\Delta-1)^m)(\varphi))_j. \tag{4.52}$$

But by (4.51) this implies

$$\lim_{j\to\infty} \pi_r^\infty((\Delta-1)^m)(\varphi_j) = \pi_r^\infty((\Delta-1)^m)(\varphi) \tag{4.53}$$

in $\mathcal{H}$.

**Lemma 4.14.** *Let $v \in U(\mathfrak{g})$. Then $\pi_r^\infty(v)(\varphi_j)$ is a Cauchy sequence in $\mathcal{H}$.*

*Proof.* Let $U_{\mathbb{R}}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, defined using only coefficients in $\mathbb{R}$. Since $U(\mathfrak{g}) = U_{\mathbb{R}}(\mathfrak{g}) + iU_{\mathbb{R}}(\mathfrak{g})$, it clearly suffices to assume $v \in U_{\mathbb{R}}(\mathfrak{g})$. But then $v \in U_{\mathbb{R}}(\mathfrak{g})_{2m}$ for some $m \in \mathbb{Z}_+$, using the notation of the standard filtration of $U_{\mathbb{R}}(\mathfrak{g})$. But by (6.7), p. 588 in [Ne] there exists a constant $k$ such that for any $i, j \in \mathbb{Z}_+$ with $i < j$,

$$\begin{aligned}\|\pi_r^\infty(v)\varphi_j - \pi_r^\infty(v)\varphi_i\| &= \|\pi_r^\infty(v)(\varphi_j - \varphi_i)\| \\ &\le k\|\pi_r^\infty((\Delta-1)^m)(\varphi_j - \varphi_i)\| \\ &= k\|\pi_r^\infty((\Delta-1)^m)(\varphi_j) - \pi_r^\infty((\Delta-1)^m)(\varphi_i)\|.\end{aligned}$$

This proves the lemma since $\pi_r^\infty((\Delta-1)^m)(\varphi_j)$ is Cauchy by (4.53). QED

In contradistinction to the non-density statement of Theorem 4.9 for $C_o^\infty((0,\infty))$ one has

**Proposition 4.15.** *The subspace $\mathcal{H}_r^{HC}$ is dense in $\mathcal{H}_r^\infty$ with respect to the $\pi_r$-Fréchet topology on $\mathcal{H}_r^\infty$. In fact, if $\varphi \in \mathcal{H}_r^{HC}$ and $\varphi_j \in \mathcal{H}_r^{HC}$ is defined as in (4.51), then $\varphi_j$ converges to $\varphi$ in the $\pi_r$-Fréchet topology of $\mathcal{H}_r^\infty$.*

*Proof.* Let $\varphi \in \mathcal{H}_r^\infty$ and let $v \in U(\mathfrak{g})$. Then $\varphi$ is contained in the domain of the closure of $\pi_r^\infty(v)|\mathcal{H}_r^{HC}$ by (4.51) and Lemma 4.14. On the other hand, $\pi_r^\infty(v)$ is an extension of $\pi_r^\infty(v)|\mathcal{H}_r^{HC}$. But, using (4.9), it is clear that $\pi_r^\infty(v)$ itself admits a closure (the dense subspace $\mathcal{H}_r^\infty$ is in the domain of its adjoint). Hence, one must have, in the notation of Lemma 4.14,

$$\lim_{j\to\infty} \pi_r^\infty(v)(\varphi_j) = \pi_r^\infty(v)\varphi. \tag{4.54}$$

That is $\|\varphi_j - \varphi\|_v$ converges to 0. QED

4.9. As a consequence of Proposition 4.15 we may introduce a vector space $\mathcal{H}_r^{-HC}$ where

$$\mathcal{H}_r^{-\infty} \subset \mathcal{H}_r^{-HC} \tag{4.55}$$

and there is a sesquilinear form $\{\psi,\rho\}$ for $\psi\in\mathcal{H}_r^{HC}$ and $\rho\in\mathcal{H}_r^{-HC}$ which extends the sesquilinear pairing of $\mathcal{H}_r^{HC}$ and $\mathcal{H}_r^{-\infty}$ and is such that every (algebraic) linear functional on $\mathcal{H}_r^{HC}$ is uniquely of the form $\psi\mapsto\{\psi,\rho\}$ for a unique $\rho\in\mathcal{H}_r^{-HC}$. Furthermore, if $(U(\mathfrak{g}),K)$ is the subalgebra of $Dist_o(G)$ generated by $U(\mathfrak{g})$ and $K$ one has a representation

$$\pi_r^{-HC}\colon (U(\mathfrak{g}),K)\to End\,\mathcal{H}_r^{-HC} \tag{4.56}$$

such that in the notation defining (4.4), for $\psi\in\mathcal{H}_r^{HC}$, $\rho\in\mathcal{H}_r^{-HC}$ and $\nu\in(U(\mathfrak{g}),K)$,

$$\{\pi_r^{\infty}(\nu^*)\psi,\rho\}=\{\psi,\pi_r^{-HC}(\nu)\rho\}.$$

It is immediate that

$$\pi_r^{-\infty}(\nu)=\pi_r^{-HC}(\nu)|\mathcal{H}_r^{-\infty} \tag{4.57}$$

## 5. Whittaker vectors, Bessel functions and the Hankel transform

5.1. We will consider the classical Bessel functions $J_r(x)$ restricted to $(0,\infty)$ where, as usual in this paper, $r\in(-1,\infty)$. By definition $J_r(x)$ is the solution of the differential equation

$$(x^2(\frac{d}{dx})^2+x\frac{d}{dx}+x^2-r^2)J(x)=0 \tag{5.1}$$

which is given explicitly by the well-known power series (1.17.1), p. 14 in [Sz]. It has the property that one can write

$$J_r(x)=x^rJ_r^*(x) \tag{5.2}$$

where $J_r^*$ is an entire function on $\mathbb{C}$ which is real on $(0,\infty)$ and is such that $J_r^*(0)=2^{-r}$.

*Remark* 5.1. For the special values of $r$ where $0<|r|<1$ the functions $J_r(x)$ and $J_{-r}(x)$ are 2 linearly independent solutions of (5.1). Hence $J_r(x)$ and $J_{-r}(x)$ span the space of all solutions. See e.g. bottom paragraph "To sum up..." of p. 43 in [Wt].

One also knows that as, $x\longrightarrow\infty$,

$$J_r(x)=O(x^{-\frac{1}{2}}). \tag{5.3}$$

See e.g. (1.71.7), p. 15 or (1.71.11), p. 16 in [Sz]. Now for any $y\in(0,\infty)$ let $J_{r,y}$ be the function on $(0,\infty)$ defined by putting

$$J_{r,y}(x)=J_r(2\sqrt{yx}). \tag{5.4}$$

**Proposition 5.2.** *Using the notation of Proposition 4.13 one has, for any $y\in(0,\infty)$, $J_{r,y}\in V_1$ so that $J_{r,y}\in\mathcal{H}_r^{-\infty}$.*

*Proof.* Clearly, by (5.2), $J_{r,y}(x)=O(x^{\frac{r}{2}})$ as $x$ approaches 0. Since $J_{r,y}$ is smooth, it follows that $J_{r,y}|(0,1)\in L_2((0,1),dx)$. On the other hand, by (5.3), there exists a positive constant $C$ such that $|J_{r,y}(x)|<Cx^{-\frac{1}{4}}$ on $[1,\infty)$. In particular,

$$|J_{r,y}(x)|<C \tag{5.5}$$

on $[1,\infty)$. This proves that $J_{r,y}\in V_1$. QED

5.2. Now for any $\varphi \in \mathcal{H}_r^\infty$ let $T_r(\varphi)$ be the function on $(0,\infty)$ defined so that for $y \in (0,\infty)$,

$$T_r(\varphi)(y) = \{\varphi, J_{r,y}\}. \tag{5.6}$$

Dropping the factor 2 in $J_{r,y}$ let $J'_{r,y} \in V_1$ be the function defined by putting $J'_{r,y}(x) = J_r(\sqrt{yx})$ and let $T'_r(\varphi)(y) = \frac{1}{2}\{\varphi, J'_{r,y}\}$. Regard $T_r$ and $T'_r$ as linear maps from $\mathcal{H}_r^\infty$ to the space $\Phi$ of all functions on $(0,\infty)$. Let $R$ be the operator on $\Phi$ defined so that, for $\phi \in \Phi$ and $y \in (0,\infty)$, $R\phi(y) = \frac{1}{\sqrt{2}}\phi(\frac{y}{2})$. Note that, by (4.25), if $a_o \in G$ is given by $a_o = exp\; t_o h$ for $t_o = -log\sqrt{2}$, then

$$R|\mathcal{H} = \pi_r(a_o). \tag{5.7}$$

In particular, $R^{-1}$ and $R$ stabilize $\mathcal{H}_r^\infty$ and one notes that

$$RT_rR^{-1} = T'_r. \tag{5.8}$$

Now recall that the Laguerre functions $\varphi_n^{(r)}$ are given by $\varphi_n^{(r)}(x) = x^{\frac{r}{2}}e^{-x}L_n^{(r)}(2x)$. But then

$$R\varphi_n^{(r)}(x) = 2^{-\frac{(r+1)}{2}}\varphi_n^{[r]}(x) \tag{5.9}$$

where we define $\varphi_n^{[r]} \in \mathcal{H}_r^\infty$ by putting

$$\varphi_n^{[r]}(x) = x^{\frac{r}{2}}e^{-\frac{x}{2}}L_n^{(r)}(x). \tag{5.10}$$

Let $\mathcal{H}_r^{[HC]}$ be the span of the functions $\{\varphi_n^{[r]}\}$, $n = 0, 1, \ldots$ .

*Remark* 5.3. Note that, by (5.7), the functions $\{\varphi_n^{[r]}\}$, $n = 0, 1, ...$, are again an orthogonal basis of $\mathcal{H}$ and that $\mathcal{H}_r^{[HC]} \subset \mathcal{H}$ is the Harish-Chandra module for $\pi_r$ when $K$ is replaced by the conjugate group $K' = a_o\, K\, a_o^{-1}$.

The following is a famous result with a long and interesting history.

**Theorem 5.4.** *For $n = 0, 1, ...,$ one has*

$$T'_r\varphi_n^{[r]} = (-1)^n\varphi_n^{[r]}. \tag{5.11}$$

*Remark* 5.5. Carl Herz in [He] says that the result is implicit in work of Sonine. He also cites an early $20^{th}$ century reference but that reference deals only with the case $r = 0$. In [He], Herz establishes the result in great generality where $(0,\infty)$ is replaced by a space of matrices. G.H. Hardy claims the result in [Ha]. However, he gives a proof only for a restricted set of values for $r$. The complete result is given as Exercise 21, p. 371 in [Sz] with some hints and a reference to [Ha].

For an understanding of Theorem 5.4 we will present a proof given to us by John Stalker. It begins with the following equality

$$\int_0^\infty J_r(at)e^{-st^2}t^{r+1}dt = \frac{a^r}{(2s)^{r+1}}e^{-\frac{a^2}{4s}} \tag{5.12}$$

where $s \in \mathbb{C}$ and $Re\, s > 0$. Equation (5.12) is established in [Wt], §13.3 (4), p. 394 where we have written $s = p^2$. Watson remarks that (5.12) is the basis of investigations of Sonine in an 1880 paper. Writing $t^2 = x$ and $a^2 = y$, (5.12) becomes the following Laplace transform equality

$$\frac{1}{2}\int_0^\infty J_r(\sqrt{yx})x^{\frac{r}{2}}e^{-sx}dx = \frac{y^{\frac{r}{2}}}{(2s)^{r+1}}e^{-\frac{y}{4s}}. \tag{5.13}$$

*Remark* 5.6. Note that Theorem 5.4, for the case $n = 0$, is established by (5.13) by putting $s = \frac{1}{2}$. The proof of (5.12) in [Wt] is straightforward as soon as Watson justifies the term-by-term integration of the power series expansion of $J'_{r,y}(x)$.

*Proof of Theorem* 5.4. (Stalker). Now it is well-known (see e.g. [Ja], §4 of Chapter X, p. 187) that

$$\sum_{n=0}^{\infty}(-1)^n L_n^{(r)}(x)\frac{t^n}{n!} = (1-t)^{-(r+1)}e^{\frac{-xt}{1-t}} \tag{5.14}$$

as an equality of two analytic functions in $(x, t)$ where $x$ is arbitrary and $|t| < 1$. Multiplying both sides of (5.14) by $x^{\frac{r}{2}}e^{-\frac{x}{2}}$ one has

$$\sum_{n=0}^{\infty}(-1)^n \varphi_n^{[r]}(x)\frac{t^n}{n!} = (1-t)^{-(r+1)}x^{\frac{r}{2}}e^{-\frac{x}{2}\frac{1+t}{1-t}}. \tag{5.15}$$

Now put $s = \frac{1}{2}\frac{1+t}{1-t}$ and note that $Re\ s > 0$ when $|t| < 1$. Substituting this expression for $s$ in (5.13) one has

$$\frac{1}{2}\int_0^{\infty} J_r(\sqrt{yx})x^{\frac{r}{2}}e^{-\frac{x}{2}\frac{1+t}{1-t}}dx = y^{\frac{r}{2}}(\frac{1-t}{1+t})^{r+1}e^{-\frac{y}{2}\frac{1-t}{1+t}}$$

and hence

$$\frac{1}{2}\int_0^{\infty} J_r(\sqrt{yx})(1-t)^{-(r+1)}x^{\frac{r}{2}}e^{-\frac{x}{2}\frac{1+t}{1-t}}dx = (1+t)^{(r+1)}y^{\frac{r}{2}}e^{-\frac{y}{2}\frac{1-t}{1+t}}. \tag{5.16}$$

But $J_r(\sqrt{yx})x^{\frac{r}{2}}$, as a function of $x$, is integrable at 0 by (5.2) and is bounded by a constant multiple $x^k$, for $k$ sufficiently large, as $x$ tends to $\infty$ by (5.3). Thus if one differentiates (5.13) $n$-times for any $n \in \mathbb{Z}_+$, with respect to $s$, and then sets $s = \frac{1}{2}$, the computation on the left side can be carried out under the integral sign. But since $s$ is an analytic function of $t$ for $|t| < 1$, the same differentiation statement is true for (5.16) where $t$ replaces $s$ and we set $t = 0$. But then Theorem 5.4 follows from (5.15). QED

Since the elements $\{\varphi_n^{[r]}\}$ are an orthogonal basis of $\mathcal{H}$ (see Remark 5.3), it follows immediately from Theorem 5.4 that there exists a unique unitary operator $U'_r$ on $\mathcal{H}$ such that $U'_r|\mathcal{H}_r^{[HC]} = T'_r|\mathcal{H}_r^{[HC]}$ and that furthermore $U'_r$ has order 2. The operator $U'_r$ is often referred to as the Hankel transform. However, it is more convenient here to make a slight change (conjugating by $\pi_r(a_o^{-1})$) and refer to this altered operator as the Hankel transform. That is, first of all, by applying $\pi_r(a_o^{-1})$ to (5.11) it follows from (5.7), (5.8) and (5.9) that Theorem 5.4 can be rewritten as

**Theorem 5.7.** *For $n = 0, 1, ...,$ one has*

$$T_r\varphi_n^{(r)} = (-1)^n\varphi_n^{(r)}. \tag{5.17}$$

We will let $U_r$ be the unique unitary operator (necessarily of order 2) on $\mathcal{H}$ such that

$$U_r|\mathcal{H}_r^{HC} = T_r|\mathcal{H}_r^{HC}.$$

Clearly,

$$\pi_r(a_o)U_r\pi_r(a_o)^{-1} = U'_r.$$

We refer to $U_r$ as the Hankel transform.

5.3. If $G_1$ is a linear semisimple Lie group and $X = G_1/K_1$ is a Hermitian symmetric space, where $K_1$ is a maximal compact subgroup of $G_1$, then the holomorphic discrete series of $G_1$ is generally constructed on spaces of square integrable holomorphic sections of line bundles on $X$. In case $X$ is of tube type, Ding and Gross in [DG] use this construction to create another model, for the holomorphic discrete series, which is built on the symmetric cone $\Omega$ associated to $X$. In case $G_1 = Sl(2,\mathbb{R})$, then $\Omega = (0,\infty)$ and the new model is (essentially), $\pi_r$, where $r$ is a positive integer. The main point in their construction of the new model is their recognition of the fact that the Hankel transform can be represented, up to a scalar, by a group element in $K_1$. For $G_1 = Sl(2,\mathbb{R})$ this means that, if $r$ is a positive integer, then for some $c_r \in \mathbb{C}$ and $k \in K_1$,

$$U_r = c_r\pi_r(k). \tag{5.18}$$

Even if one went to the covering group, $G$, their construction does not lead to $\pi_r$ for $-1 < r \leq 0$ since in this range, as pointed out to us by David Vogan, $\pi_r$ cannot be represented on $X$ by spaces of square-integrable holomorphic sections of line bundles. (If $r$ is rational, this follows from Harish-Chandra's classification of discrete series represenations since, in that case, $\pi_r$ descends to a group in the Harish-Chandra class.) Our construction of $\pi_r$, using Nelson's theorem, and our knowledge of the Harish-Chandra module, immediately (see Theorem 5.8 below) yields (5.18) for $G$ and all $-1 < r < \infty$. For another representation theoretic (for the "ax+b" group) treatment of the Hankel transform see Chapter 7 in [Ta].

It should be mentioned that if the rank of $G_1$ is greater than 1 there are new analogues of $\pi_0$ discovered by Wallach and referred to as Wallach points (see [W$\ell$-1] and [W$\ell$-2]). Symmetric cone models of this finite set of representations have been constructed by Rossi and Vergne in [R-V].

Let $k_o \in K$ be defined by putting

$$k_o = exp\,\frac{\pi}{2}(e-f).$$

Under the quotient map (see (3.16)) $G \to Sl(2,\mathbb{R})$ note that $k_o \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is immediate then that $Ad\,k_o$ defines the non-trivial element of the Weyl group, with respect to $(\mathbb{R}h, \mathfrak{g})$. One has that $Ad\,k_o$ is of order 2. In fact,

$$\begin{aligned} Ad\,k_o(h) &= -h, \\ Ad\,k_o(e) &= -f, \\ Ad\,k_o(f) &= -e. \end{aligned} \tag{5.19}$$

In particular, $k_o^2$ is central in $G$. See also (3.13).

**Theorem 5.8.** *Let* $c_r = e^{-\frac{r+1}{2}\pi i}$. *Then*

$$U_r = c_r\pi_r(k_o). \tag{5.20}$$

*Proof.* By (2.5) and (2.26) one has

$$\pi_r(k_o)\varphi_n^{(r)} = e^{\frac{\pi}{2}(2n+r+1)i}.$$

Hence,

$$\begin{aligned} c_r\pi_r(k_o)\varphi_n^{(r)} &= e^{n\pi p}\varphi_n^{(r)} \\ &= (-1)^n\varphi_n^{(r)}. \end{aligned} \tag{5.21}$$

But then, since both $c_r\pi_r(k_o)$ and $U_r$ are unitary operators on $\mathcal{H}$, (5.20) follows from (5.21) and (5.17). QED

One immediate consequence of Theorem 5.8 is the following property of the Hankel transform $U_r$. This property is an analogue of the fact that the Fourier transform stabilizes the Schwartz space and is continuous with respect to its Fréchet topology.

**Theorem 5.9.** *For the Hankel transform one has*

$$U_r\colon \mathcal{H}_r^\infty \to \mathcal{H}_r^\infty \tag{5.22}$$

*and (5.22) is continuous with respect to the $\pi_r$-Fréchet topology on $\mathcal{H}_r^\infty$.*

Since $\pi_r(k_o) = \pi_r^{-\infty}(k_o)|\mathcal{H}$ we can initially extend the domain of the Hankel transform to $\mathcal{H}_r^{-\infty}$ by putting $U_r = c_r\pi_r^{-\infty}(k_o)$. But in fact, recalling (4.57), one can go further and extend the domain of the Hankel transform $U_r$ to $\mathcal{H}_r^{-HC}$ by putting

$$U_r = c_r\pi_r^{-HC}(k_o). \tag{5.23}$$

One of course still has $U_r^2 = \text{Identity}$. See Theorem 5.7.

*Remark* 5.10. Note that if $0 < |r| < 1$ and $\rho \in \mathcal{H}_r^{-\infty}$, the problem of computing $U_r(\rho)$, by Proposition 4.9, seems to be complicated by the fact that $\rho$ may vanish as a distribution on $(0,\infty)$.

Even if $\varphi \in \mathcal{H}$, as in the case of the Fourier transform, it is not transparent when $U_r(\varphi)$ can be determined by integrating $\varphi(x)$ against the kernel $J_r(2\sqrt{yx})$. The following result says that if $\varphi \in \mathcal{H}_r^\infty$ everything is fine.

**Theorem 5.11.** *Let $\varphi \in \mathcal{H}_r^\infty$ so that (by Theorem 5.9) $U_r(\varphi) \in \mathcal{H}_r^\infty$ and hence $U_r(\varphi)(y)$ is a $C^\infty$-function of $y$. Furthermore, recalling Proposition 5.2, $U_r(\varphi)(y)$ is given by*

$$U_r(\varphi)(y) = \{\varphi, J_{r,y}\} \tag{5.24}$$

*for any $y \in (0,\infty)$.*

*Proof.* There exists, by Proposition 4.15, a sequence $\psi_n \in \mathcal{H}_r^{HC}$ that converges to $\varphi$ in the $\pi_r$-Fréchet topology of $\mathcal{H}_r^\infty$. But then $ix\,\psi_n$ converges to $ix\,\varphi$ in this topology since $\pi_r(e)$ is certainly continuous in the $\pi_r$-Fréchet topology. In particular, $\psi_n$ and $ix\,\psi_n$ converge respectively to $\varphi$ and $ix\,\varphi$ in $\mathcal{H}$. Let $y \in (0,\infty)$. Then since $J_{r,y}(x)$ is in $L_2$ at 0 one has

$$\lim_{n\to\infty}\int_0^1 \psi_n(x)J_r(2\sqrt{yx})dx = \int_0^1 \varphi(x)J_r(2\sqrt{yx})dx. \tag{5.25}$$

On the other hand, $ix^{-1}J_{r,y}(x)|[1,\infty) \in L_2([1,\infty),dx)$ by (5.3). But then

$$\int_1^\infty \psi_n(x)J_{r,y}(x)dx = \int_1^\infty ix\psi_n(x)(ix)^{-1}J_{r,y}(x)dx$$

converges to

$$\int_1^\infty ix\varphi(x)(ix)^{-1}J_{r,y}(x)dx = \int_1^\infty \varphi(x)J_{r,y}(x)dx$$

Hence, by (5.25)

$$\lim_{n\to\infty}\int_0^\infty \psi_n(x)J_{r,y}(x)dx = \int_0^\infty \varphi(x)J_{r,y}(x)dx. \tag{5.26}$$

But the left side of (5.26) is just $U_r(\psi_n)(y)$, by definition of $U_r$. However, by Theorem 5.9, $U_r(\psi_n)$ converges to $U_r(\varphi)$ in the $\pi_r$-Fréchet topology of $\mathcal{H}_r^\infty$. But then, by Proposition 4.7, $U_r(\psi_n)$ certainly converges pointwise to $U_r(\varphi)$. But then $U_r(\varphi)(y)$ is given by the right side of (5.26). QED

5.4. Let $u \in \mathfrak{g}_\mathbb{C}$. A vector $\rho \in \mathcal{H}_r^{-HC}$ will be called a $u$-weight vector of weight $\lambda \in \mathbb{C}$ if $\pi_r^{-HC}(u)\rho = \lambda\rho$. The span of the set of all such vectors will be referred to as the corresponding $u$-weight space, or simply weight space if $u$ is understood. If $u = e$ (resp. $u = f$) and $\lambda \neq 0$, then the $e$-weight (resp. $f$-weight) vector will also be referred to as an $e$-Whittaker (resp. $f$-Whittaker) vector. An $e$-weight (resp. $f$-weight) vector of weight 0 will be called a highest (resp. lowest) weight vector.

*Remark* 5.12. The term highest and lowest is a convenient misnomer. It is a misnomer since it refers, respectively to the kernels of $\pi_r^{-HC}(e)$ and $\pi_r^{-HC}(f)$ and not, as in finite dimensional representation theory, to $h$-weights. Note that, as a consequence of (5.23), $\rho \in \mathcal{H}_r^{-HC}$ is a $u$-weight vector of weight $\lambda$ if and only if its Hankel transform $U_r(\rho)$ is an $Ad\, k_o(u)$-weight vector of weight $\lambda$. In particular, if $u = h$, then, by (5.19), $\rho$ is an $h$-weight vector of weight $\lambda$ if and only if its Hankel transform $U_r(\rho)$ is an $h$-weight vector of weight $-\lambda$. Furthermore, again by (5.19), $\rho$ is an $e$-weight vector of weight $\lambda$ if and only if its Hankel transform $U_r(\rho)$ is an $f$-weight vector of weight $-\lambda$.

Now if $u \in \mathfrak{g}_\mathbb{C}$, we can write

$$u = \alpha(u)h' + \beta(u)e' + \gamma(u)f'$$

for some unique $\alpha(u), \beta(u), \gamma(u) \in \mathbb{C}$. Let $\mathfrak{g}_\mathbb{C}^{(*)} = \{u \in \mathfrak{g}_\mathbb{C} \mid \gamma(u) \neq 0\}$.

**Theorem 5.13.** *Let $u \in \mathfrak{g}_\mathbb{C}^{(*)}$. Then for any $\lambda \in \mathbb{C}$ the $u$-weight space, in $\mathcal{H}_r^{-HC}$, of weight $\lambda$ is 1-dimensional. Furthermore $h, e, f \in \mathfrak{g}_\mathbb{C}^{(*)}$ so that in particular for $u = h, e, f$ the $u$-weight space, in $\mathcal{H}_r^{-HC}$, of weight $\lambda$ is 1-dimensional.*

*Proof.* Recalling the definition of the $*$-operation in $\mathfrak{g}_\mathbb{C}$ (see (3.1)) it is clear from (2.5), (2.6) and (2.7) that $(e')^* = -f'$, $(h')^* = h'$ and $(f')^* = -e'$. It then follows immediately that $\beta(u^*) = -\overline{\gamma(u)}$. Hence,

$$v \in \mathfrak{g}_\mathbb{C}^{(*)} \iff \beta(v^*) \neq 0 \tag{5.27}$$

However, it is immediate from (2.5), (2.6) and (2.7) that

$$\begin{aligned} h &= i(e' - f'),\\ e &= \frac{1}{2}(ih' + e' + f'),\\ f &= \frac{1}{2}(ih' + e' + f'),\end{aligned}$$

so that $h, e, f \in \mathfrak{g}_\mathbb{C}^{(*)}$. Let $\lambda \in \mathbb{C}$ and let $v = u^* - \overline{\lambda}$. To prove the theorem it clearly suffices to prove that (recalling (4.36)) if $Y = D_r(v)(\mathcal{H}_r^{HC})$, then $Y$ has codimension 1 in $\mathcal{H}_r^{HC}$. For $m \in \mathbb{Z}_+$ let $X_m$ be the span of $\{\varphi_n^{(r)}\}$, $n = 0, 1, ..., m$. But now it is immediate from (2.26), (2.27), (2.28) and (5.27) that

$$\begin{aligned} &(a)\ Ker\, D_r(v)|\mathcal{H}_r^{HC} = 0,\\ &(b)\ X_0 \not\subset Y,\\ &(c)\ D_r(v)(X_m)\ \subset\ X_{m+1}. \end{aligned} \tag{5.28}$$

It follows from (b) of (5.28) that $Y \neq \mathcal{H}_r^{HC}$. But $dim\, X_m = m + 1$. Hence $dim\, D_r(v)(X_m) = m + 1$ by (a) of (5.28). But then

$$X_m = X_0 \oplus D_r(v)(X_{m-1}) \tag{5.29}$$

by (b) and (c) of (5.28). But then clearly $\mathcal{H}_r^{HC} = X_0 \oplus Y$. QED

*Remark* 5.14. A Weyl group of $G$ has order 2. Note that Theorem 5.13 is not a contradiction of Theorem 6.8.1 on p. 182 of [Ko] (which, in this case, asserts the existence of a 2-dimensional space of algebraic Whittaker vectors). Indeed the hypothesis of this result in [Ko] requires that one must add, to $\mathcal{H}_r^{HC}$, the Harish-Chandra module of the anti-holomorphic representation corresponding to $\pi_r$. See (6.8.1) in [Ko].

An element $\rho \in \mathcal{H}_r^{-HC}$ will be called $\pi_r$-tempered in case $\rho \in \mathcal{H}_r^{-\infty}$. We now obtain explicit results on the "temperedness" of certain Whittaker vectors. For $\lambda = -iy$, $y \in (0, \infty)$ the corresponding $f$-Whittaker vector is $\pi_r$-tempered and is given in terms of the Bessel function $J_r$.

**Theorem 5.15.** *Let $y \in (0,\infty)$ so that $\delta_y \in \mathcal{H}_r^{-\infty}$ where $\delta_y$ is the Dirac measure at $y$ (see Remark 4.12). Then $\mathbb{R}\delta_y$ is the weight space of e-Whittaker vectors of weight $iy$. In particular, any such Whittaker vector is $\pi_r$-tempered.*

*The function $J_{r,y}(x) = J_r(2\sqrt{yx})$ is in $\mathcal{H}_r^{-\infty}$ by Proposition 5.2. Furthermore, $\mathbb{R}J_{r,y}$ is the weight space of $f$-Whittaker vectors of weight $-iy$. In particular, any such Whittaker vector is $\pi_r$-tempered. Furthermore,*

$$U_r(\delta_y) = J_{r,y} \tag{5.30}$$

*where, we recall $U_r$ is the Hankel transform.*

*Proof.* Let $\varphi \in \mathcal{H}_r^{-\infty}$. Then (see (4.36))

$$\begin{aligned} \{\varphi, \pi_r^{-\infty}(e)\delta_y\} &= \{ix\varphi, \delta_y\}\\ &= -iy\varphi(y)\\ &= \{\varphi, iy\delta_y\}. \end{aligned}$$

This proves $\pi_r^{-\infty}(e)\delta_y = iy\delta_y$.

Now,

$$\pi_r^{\infty}(f)U_r(\delta_y) = -iyU_r(\delta_y) \tag{5.31}$$

by Remark 5.14. On the other hand, since $U_r$ has order 2, it follows from (5.23) and (5.24) that

$$\begin{aligned} \{\varphi, U_r(\delta_y)\} &= \{U_r(\varphi), \delta_y\}\\ &= \{\varphi, J_{r,y}\}, \end{aligned}$$

This proves (5.30). The theorem then follows from (5.31). QED

5.5. Let $\Delta \in U(\mathfrak{g})$ be as in (4.52). Using the notation of (4.50) we may, by (3.8), define a quadradic polynomial $q$ on $\mathbb{R}$ such that for all $n \in \mathbb{Z}$,

$$\pi_r^\infty(\Delta - 1)(\psi_n^{(r)}) = q(n)\psi_n^{(r)}. \tag{5.32}$$

Now let $\varphi \in \mathcal{H}_r^\infty$ and let $\{a_n\}$ be, as in (4.50), the Fourier-Laguerre coefficients of $\varphi$. Then, by (4.53), the Fourier-Laguerre expansion of $\pi_r^\infty((\Delta-1)^m)(\varphi)$ is given by

$$\pi_r^\infty((\Delta - 1)^m)(\varphi) = \sum_{n=0}^{\infty} q(n)^m a_n \psi_n^{(r)} \tag{5.33}$$

for any $m \in \mathbb{Z}_+$. Since this is an expansion of an element in $\mathcal{H}$, the sum of norm squares of the coefficents is finite. In particular, the norm of the coefficents is bounded. On the other hand, the coefficents of $q$ are positive, since $r > -1$. See (3.8). Thus for all $m \in \mathbb{Z}_+$ there exists a positive constant $C_m$ such that for all $n$

$$|a_n| < C_m(n^2+1)^{-m}. \tag{5.34}$$

Now for any function $\psi$ on $(0,\infty)$ let $\widetilde{\psi}$ be the function on $(0,\infty)$ defined by putting $\widetilde{\psi}(x) = x^{-\frac{r}{2}}\psi(x)$. In case the limit of $\widetilde{\psi}(x)$ exists as $x$ tends to 0, we will put

$$\widetilde{\psi}(0) = \lim_{x\to 0} \widetilde{\psi}(x). \tag{5.35}$$

In such a case if $\psi$ is continuous on $(0,\infty)$, then of course $\widetilde{\psi}$ is continuous on $[0,\infty)$. If $\psi \in \mathcal{H}_r^{HC}$, then $\widetilde{\psi}$ is clearly the restriction to $(0,\infty)$ of an entire function and hence $\widetilde{\psi}(0)$ exists and $\widetilde{\psi}$ is continuous on $[0,\infty)$.

**Theorem 5.16.** *Let $\varphi \in \mathcal{H}_r^\infty$. Then $\widetilde{\varphi}(0)$ exists. Furthermore, for any $\omega > 0$, $\widetilde{\varphi_j}$ converges to $\widetilde{\varphi}$ pointwise and uniformly on the closed interval $[0,\omega]$, using the notation of (4.51).*

*Proof.* In the proof of Proposition 3.12 we noted that if $L_n^{(r)}(Szego)$ is the normalization of $L_n^{(r)}$ as given in (5.1.5), p. 97 of [Sz], then

$$L_n^{(r)}(Szego) = (-1)^n \frac{1}{n!} L_n^{(r)}. \tag{5.36}$$

It follows then from (7.6.11), p. 173 in [Sz] that there exists $a > -\frac{1}{2}$ and positive constants $C$ and $D$ such that for all $x \in [0,\omega]$ and $n \in \mathbb{Z}_+$,

$$|L_n^{(r)}(2x)| < n!\,(Cn^a + D).$$

But then certainly

$$|e^{-\frac{x}{2}} L_n^{(r)}(2x)| < n!\,(Cn^a + D) \tag{5.37}$$

for all $n \in \mathbb{Z}_+$ and $x \in [0,\omega]$. Let $d_n = (\frac{1}{2^{r+1}} n!\,\Gamma(n+r+1))^{-\frac{1}{2}}$. Then $n!\,d_n = (2^{r+1}\frac{n!}{\Gamma(n+r+1)})^{\frac{1}{2}}$. But

$$\begin{aligned}\Gamma(n+r+1) &= (n+r)(n+r-1)\cdots(r+2)\Gamma(r+2)\\ &= (n+r+1)(n+r)\cdots(r+2)(\frac{\Gamma(r+2)}{n+r+1}).\end{aligned}$$

But $(n+r+1)(n+r)\cdots(r+2) > n!$. Hence

$$n!\,d_n < \left(2^{r+1}\frac{(n+r+1)}{\Gamma(r+2)}\right)^{\frac{1}{2}}. \tag{5.38}$$

But recalling the definition of the orthonormal basis $\{\psi_n^{(r)}\}$ (see (4.50)) one has

$$\widetilde{\psi_n^{(r)}} = d_n e^{-\frac{x}{2}} L_n^{(r)}(2x). \tag{5.39}$$

Hence, for all $n \in \mathbb{Z}_+$ and $x \in [0,\omega]$ there exists positive constants $E, F$ and $k \in \mathbb{N}$ such that

$$\begin{aligned} |\widetilde{\psi_n^{(r)}}(x)| &< d_n n!\,(Cn^\alpha + D) \\ &< \left(2^{(r+1)}\frac{(n+r+1)}{\Gamma(r+2)}\right)^{\frac{1}{2}}(Cn^\alpha + D) \\ &< En^k + D. \end{aligned} \tag{5.40}$$

But then if we choose $m = k+1$ in (5.34), one has for some constant $F$,

$$\begin{aligned} |a_n \widetilde{\psi_n^{(r)}}(x)| &< C_{k+1}(n^2+1)^{-(k+1)}(En^k + D) \\ &< \frac{F}{n^2+1}. \end{aligned}$$

This proves that $\widetilde{\varphi_j}$ converges pointwise to a continuous function $\psi$ on $[0,\infty)$ and uniformly so on any closed interval $[0,\omega]$. But, by (4.51), a subsequence $\varphi_{j_i}$ converges almost everywhere to $\varphi$ on $(0,\infty)$. Hence $\widetilde{\varphi_{j_i}}$ converges almost everywhere to $\widetilde{\varphi}$ on $(0,\infty)$. Hence $\widetilde{\varphi} = \psi$ on $(0,\infty)$. But then $\widetilde{\varphi}(0)$ is defined and $\widetilde{\varphi} = \psi$ as functions on $[0,\infty)$. QED

Now by (5.1.7), p. 97 in [Sz] and (5.45) one has

$$L_n^{(r)}(0) = (-1)^n n! \binom{n+r}{n}. \tag{5.41}$$

But by standard properties of the Γ-function

$$\frac{\Gamma(n+r+1)}{n!} = \binom{n+r}{n}\Gamma(r+1). \tag{5.42}$$

But using the notation of (5.39) one has, using (5.41)

$$\begin{aligned} \widetilde{\psi_n^{(r)}}(0) &= d_n(-1)^n n! \binom{n+r}{n} \\ &= \left(\frac{2^{r+1}}{\Gamma(r+1)}\right)^{\frac{1}{2}}(-1)^n \binom{n+r}{n}^{\frac{1}{2}}. \end{aligned} \tag{5.43}$$

As an immediate consequence of (5.43) and Theorem 5.16 one has

**Theorem 5.17.** *Let $\varphi \in \mathcal{H}_r^\infty$ and let (4.50) be its Fourier-Laguerre expansion. Then the sum on the right side of (5.44), below, absolutely converges and*

$$\widetilde{\varphi}(0) = \left(\frac{2^{r+1}}{\Gamma(r+1)}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty}(-1)^n a_n \binom{n+r}{n}^{\frac{1}{2}}. \tag{5.44}$$

We can determine the highest weight vector (unique up to scalar multiplication by Theorem 5.13) of $\pi_r$ and establish that it is $\pi_r$-tempered. Let $\delta_{r,0} \in \mathcal{H}_r^{-HC}$ be defined so that $\{\psi, \delta_{r,0}\} = \widetilde{\psi}(0)$ for any $\psi \in \mathcal{H}_r^{HC}$.

**Theorem 5.18.** *The linear functional $\delta_{r,0}$ is a highest weight vector and spans the highest weight space. Furthermore, $\delta_{r,0}$ is $\pi_r$-tempered (i.e. $\delta_{r,0} \in \mathcal{H}_r^{-\infty}$) and for any $\varphi \in \mathcal{H}_r^\infty$ one has*

$$\begin{aligned} \{\varphi, \delta_{r,0}\} &= \widetilde{\varphi}(0) \\ &= \lim_{x\to 0} x^{-\frac{r}{2}}\varphi(x). \end{aligned} \tag{5.45}$$

*(See Theorm 5.8.) In particular, if (4.50) is the Fourier-Laguerre expansion of $\varphi$, then $\{\varphi, \delta_{r,0}\}$ is given by the convergent sum (5.44).*

*Proof.* Let $\psi \in \mathcal{H}_r^{HC}$. Then

$$\begin{aligned} \{\psi, \pi_r^{-HC}(e)(\delta_{r,0})\} &= \{\pi_r^\infty(e)(\psi), \delta_{r,0}\} \\ &= \{ix\psi, \delta_{r,0}\} \\ &= \lim_{x\to 0} ixx^{-\frac{r}{2}}\psi(x). \end{aligned} \tag{5.46}$$

But $\lim_{x\to 0} x^{-\frac{r}{2}}\psi(x) = \widetilde{\psi}(0)$ exists. Hence, the limit in (5.46) (with the extra $x$-factor) must be zero. Thus $\pi_r^{-HC}(e)(\delta_{r,0}) = 0$ so that $\delta_{r,0}$ is a (clearly nonzero) highest weight vector. Now let $\delta'_{r,0}$ be the linear functional on $\mathcal{H}_r^\infty$ defined so that $\delta'_{r,0}(\varphi) = \widetilde{\varphi}(0)$. To prove that $\delta_{r,0}$ is $\pi_r$-tempered and all the remaining statements of the theorem, it clearly suffices only to prove that $\delta'_{r,0}$ is continuous in the $\pi_r$-Fréchet topology on $\mathcal{H}_r^\infty$. Indeed one would then have that $\delta_{r,0} \in \mathcal{H}_r^{-\infty}$ and $\{\varphi, \delta_{r,0}\} = \delta'_{r,0}(\varphi)$ for all $\varphi \in \mathcal{H}_r^\infty$.

Let $k \in \mathbb{N}$ be fixed such that $k \geq r$. Then

$$\begin{aligned} \binom{n+r}{n} &\leq \binom{n+k}{n} \\ &= \binom{n+k}{k} \\ &\leq \frac{(n+k)^k}{k!}. \end{aligned}$$

Thus, there exists positive constants $A, B$, such that $\binom{n+r}{n} \leq An^k + B$ and hence there exists positive constants $C, D$ such that

$$\binom{n+r}{n}^{\frac{1}{2}} \leq Cn^{\frac{k}{2}} + D \tag{5.47}$$

Now recalling (5.33) let $m \in \mathbb{N}$ be such that $2m \geq \frac{k}{2} + 2$. But since the coefficients of the quadratic polynomial $q$ are positive, there exist positive constants $E, F$ such that for all $n \in \mathbb{Z}_+$, $q(n)^m \geq En^{2m} + F$. But clearly $\sum_{n=0}(Cn^{\frac{k}{2}} + D)/(En^{2m} + F)$ converges. Hence, by (5.47) one has a convergent sum

$$\sum_{n=0}^{\infty} \binom{n+r}{n}^{\frac{1}{2}} / q(n)^{2m} = M. \tag{5.48}$$

Now in the notation of (5.33) let $v = (\Delta - 1)^m \in U(\mathfrak{g})$. Let $\varphi \in \mathcal{H}_r^\infty$ so that, by definition, $\|\pi_r^\infty(v)(\varphi)\| = \|\varphi\|_v$. Let (4.50) be the Fourier-Laguerre expansion of $\varphi$. Then, by (5.33), $|q(n)^m a_n| \leq \|\varphi\|_v$ for all $n \in \mathbb{Z}_+$. That is, for all $n \in \mathbb{Z}_+$,

$$|a_n| \leq \|\varphi\|_v / q(n)^m. \tag{5.49}$$

But then if $N$ is the constant preceding the sum in (5.44), one has by (5.44), (5.48), and (5.49)

$$|\delta'_{r,0}(\varphi)| \leq N \sum_{n=0}^{\infty} |a_n \binom{n+r}{n}^{\frac{1}{2}}| \tag{5.50}$$
$$\leq \|\varphi\|_v N \sum_{n=0}^{\infty} | \binom{n+r}{n}^{\frac{1}{2}} / q(n)^m |$$
$$\leq NM \|\varphi\|_v.$$

But this proves that $\delta'_{r,0}$ is continuous with respect to the seminorm $\|\varphi\|_v$ and hence $\delta'_{r,0}$ is a continuous linear functional on $\mathcal{H}_r^{\infty}$ with respect to the $\pi_r$-Fréchet topology. QED

5.6. We now turn to the lowest weight vector. Let $J_{r,0} = U_r(\delta_{r,0})$ so that, see Remark 5.12, $J_{r,0}$ is a lowest weight vector. Furthermore, $\mathbb{R}\, J_{r,0}$ is the lowest weight space and since $U_r$ stabilizes $\mathcal{H}_r^{\infty}$ (see (5.23)), one has that $J_{r,0}$ is smooth. As an easy consequence of Theorem 5.18 one has

**Theorem 5.19.** *Let $\varphi \in \mathcal{H}_r^{\infty}$. Then*

$$\{\varphi, J_{r,0}\} = \lim_{y \to 0} y^{-\frac{r}{2}} \int_0^{\infty} J_r(2\sqrt{yx})\varphi(x)dx \tag{5.51}$$

*noting, in particular, that the limit on the right side of (5.51) exists.*

*Proof.* Since the Hankel transform, $U_r$, is of order 2, it follows from (5.23), (5.24), (5.30) and (5.45) that

$$\begin{aligned}\{\varphi, J_{r,0}\} &= \{U_r(\varphi), U_r(J_{r,0})\} \\ &= \{U_r(\varphi), \delta_{r,0}\} \\ &= \lim_{y \to 0} y^{-\frac{r}{2}} U_r(\varphi)(y) \\ &= \lim_{y \to 0} y^{-\frac{r}{2}} \{\varphi, J_{r,y}\}.\end{aligned}$$

But this is just the statement of (5.51). QED QED

Now consider $h$-weight vectors. By Theorem 5.13 any weight is possible and all have multiplicity 1. But of course there is the question as to whether the weight vector is $\pi_r$-tempered.

**Theorem 5.20.** *Let $\mu \in \mathbb{C}$ be such that $Re\,\mu > -\frac{1}{2}$ so that $x^{\mu} \in V_1$, in the notation of §4.7, and hence, by Proposition 4.13, $x^{\mu} \in \mathcal{H}_r^{-\infty}$. Let $\lambda = 2\mu + 1$ so that $Re\,\lambda > 0$. Then $x^{\mu}$ (uniquely up to a scalar multiple) is an $h$-weight vector of weight $\lambda$. In particular, $x^{\frac{r}{2}}$ is a smooth $h$-weight vector of weight $r+1$.*

*On the other hand, $U_r(x^{\mu})$ is a smooth $h$-weight of weight $-\lambda$ for $Re\,\lambda < 0$. In particular, $U_r(x^{\frac{r}{2}})$ is a smooth $h$-weight vector of weight $-(r+1)$.*

*Proof.* Let $\varphi \in \mathcal{H}_r^{\infty}$. By (4.36)

$$\{-(2x\frac{d}{dx} + 1)(\varphi), x^{\mu}\} = \{\varphi, \pi_r^{-\infty}(h)(x^{\mu})\} \tag{5.52}$$

But now, for any $k \in \mathbb{Z}$, $x^k\varphi(x) \in \mathcal{H}_r^{\infty}$, since $\mathcal{H}_r^{\infty}$ is stable under $\pi_r^{\infty}(e^k)$ and hence $\mathbb{R} \to \mathcal{H}$ is a $C^{\infty}$ map, where $t \mapsto \pi_r(exp\,(-t)h)(x^k\varphi(x))$. But, by (4.25), $\pi_r(exp\,(-t)h)(x^k\varphi(x)) = e^{-2tk}x^k\pi_r(exp\,(-t)h)(\varphi(x))$. We may multiply by $e^{2tk}$

and conclude that $t \mapsto x^k\pi_r(exp\,(-t)h)(\varphi(x))$ also defines a differentiable map from $\mathbb{R}$ to $\mathcal{H}$. Furthermore, one must have

$$\frac{d}{dt}(x^k\pi_r(exp\,(-t)h)(\varphi))_{t=0} = x^k(-(2x\frac{d}{dx}+1))(\varphi) \tag{5.53}$$

in $\mathcal{H}$ since the difference quotient whose limit in $\mathcal{H}$ defines the left side of (5.53) clearly converges pointwise to the right side of (5.53), by (4.25) and the smoothness of $\varphi$. But by a change of variables one has

$$\begin{aligned}\{\pi_r(exp\,(-t)h)(\varphi), x^\mu\} &= \int_0^\infty \varphi(e^{-2t}x)x^{\overline{\mu}}e^{-t}dt\\ &= \{\varphi, e^{t(2\mu+1)}x^\mu\}.\end{aligned}$$

Hence,

$$\frac{d}{dt}(\{\pi_r(exp\,(-t)h)(\varphi), x^\mu\})_{t=0} = \{\varphi, (2\mu+1)x^\mu\}. \tag{5.54}$$

For any $\psi \in \mathcal{H}_r^\infty$ and $i = 1,2$, let $\{\psi, x^{\mu_i}\}_i = \int_{a_i}^{b_i}\psi(x)x^{\overline{\mu_i}}dx$ where $a_1 = 0$, $b_1 = 1$, $a_2 = 1$, $b_2 = \infty$, $Re\,\mu_1 > -\frac{1}{2}$ and $\mu_2 \in \mathbb{C}$ is arbitrary. Clearly, $\{\psi, x^\mu\} = \{\psi, x^\mu\}_1 + \{\psi, x^\mu\}_2$. But now

$$\frac{d}{dt}(\{\pi_r(exp\,(-t)h)(\varphi), x^\mu\}_1)_{t=0} = \{-(2x\frac{d}{dx}+1)(\varphi), x^\mu\}_1 \tag{5.55}$$

by (5.53) where we put $k = 0$. On the other hand, if we choose $k$ so that $x^{\mu-k}|[1,\infty) \in L_2((1,\infty), dx)$, then by (5.53),

$$\begin{aligned}\frac{d}{dt}(\{\pi_r(exp\,(-t)h)(\varphi), x^\mu\}_2)_{t=0} &= \frac{d}{dt}(\{x^k\pi_r(exp\,(-t)h)(\varphi), x^{\mu-k}\}_2)_{t=0}\\ &= \{-x^k(2x\frac{d}{dx}+1)(\varphi), x^{\mu-k}\}_2\\ &= \{-(2x\frac{d}{dx}+1)(\varphi), x^\mu\}_2.\end{aligned} \tag{5.56}$$

But then by (5.54), (5.55) and (5.56) one has

$$\{-(2x\frac{d}{dx}+1)(\varphi), x^\mu\} = \{\varphi, (2\mu+1)x^\mu\}.$$

But then $\pi_r^{-\infty}(h)(x^\mu) = (2\mu+1)x^\mu$ by (5.52). The last statements follow from Remark 5.12 and the fact that $U_r$ stabilizes $\mathcal{H}_r^{-\infty}$. See (5.23). QED

*Remark* 5.21. It is clear in Theorem 5.20 that the $h$-weight vectors for weights $\lambda$ when $Re\,\lambda$ is positive is independent of $r$. On the other hand, there is an apparent dependence on $r$ when $Re\,\lambda$ is negative. Since $\delta_{r,0}$ has an obvious dependence on $r$ the dependence of the $h$-weight vector of weight $-(r+1)$, on $r$, will be verified in Theorem 5.23.

From the commutation relations in $\mathfrak{g}$ it is obvious that highest weight vector $\delta_{r,0}$ and the lowest weight vector $J_{r,0}$ must be $h$-weight vectors. The question is: what are the $h$-weights? This is determined in Theorem 5.23 below. Contrary to one's experience for finite dimensional representations, it turns out that the $h$-weight of the highest weight vector (recall definitions in §5.4) is smaller than the $h$-weight of the lowest weight vector. In fact, the latter is positive and the former is just its negative. The lowest weight vector $J_{r,0}$ is given in Theorem 5.19 as a limit. In Theorem 5.23 it will be given as an integral.

*Remark* 5.22. By the multiplicity 1 statement of Theorem 5.13 and the commutation relations in $\mathfrak{g}$ note that (up to scalar multiplication) $\delta_{r,0}$ is the only nonzero element in $\mathcal{H}_r^{-HC}$, and a fortiori in $\mathcal{H}_r^{-\infty}$, which is simultaneously an $h$-weight vector and an $e$-weight vector. The same is true of $J_{r,0}$ when $f$ replaces $e$.

**Theorem 5.23.** *One has*

$$\pi_r^\infty(h)(\delta_{r,0}) = -(r+1)(\delta_{r,0}) \tag{5.57}$$

*and*

$$\pi_r^\infty(h)(J_{r,0}) = (r+1)(J_{r,0}). \tag{5.58}$$

*Furthermore,* $J_{r,0}$ *given by (5.51), can also be given by*

$$\{\varphi, J_{r,0}\} = \frac{1}{\Gamma(r+1)}\int_0^\infty \varphi(x)x^{\frac{r}{2}}dx \tag{5.59}$$

*for any* $\varphi\in\mathcal{H}_r^\infty$ *so that*

$$U_r(\delta_{r,0}) = \frac{1}{\Gamma(r+1)}x^{\frac{r}{2}}. \tag{5.60}$$

*Proof.* From the commutation relations $2\pi_r^\infty(e) = [\pi_r^\infty(h),\pi_r^\infty(e)]$ and $-2\pi_r^\infty(f) = [\pi_r^\infty(h),\pi_r^\infty(f)]$ together with the multiplicity 1 statement of Theorem 5.13 it is immediate that there exists scalars $\mu,\nu\in\mathbb{C}$ such that

$$\pi_r^\infty(h)(\delta_{r,0}) = \mu\,\delta_{r,0},$$
$$\pi_r^\infty(h)(J_{r,0}) = \nu\,J_{r,0}.$$

But then

$$\nu = -\mu \tag{5.61}$$

by Remark 5.12. Thus, for any $\varphi\in\mathcal{H}_r^\infty$ one has

$$\begin{aligned}\{-(2x\frac{d}{dx}+1)(\varphi),\delta_{r,0}\} &= \{\varphi,\pi_r^\infty(h)(\delta_{r,0})\}\\ &= \overline{\mu}\{\varphi,\delta_{r,0}\}.\end{aligned}$$

Hence,

$$\lim_{x\to 0} x^{-\frac{r}{2}}(-(2x\frac{d}{dx}+1)(\varphi))(x) = \overline{\mu}\lim_{x\to 0}x^{-\frac{r}{2}}\varphi(x). \tag{5.62}$$

Now put $\varphi = \varphi_0^{(r)}$ so that $\varphi(x) = e^{-x}x^{\frac{r}{2}}$. Then $x^{-\frac{r}{2}}(-(2x\frac{d}{dx}+1)(\varphi))(x) = -(r+1)e^{-x} + 2e^{-x}x$. Thus the limit on the right side of (5.62) is $-(r+1)$. But $x^{-\frac{r}{2}}\varphi(x) = e^{-x}$. Hence, the limit on the right side of (5.62) is $\overline{\mu}$. This proves $\mu = -(r+1)$. But then $\nu = (r+1)$ by (5.61). But then by the multiplicity 1 statement of Theorem 5.13 one must have $J_{r,0} = cx^{\frac{r}{2}}$ for some constant $c$ by Theorem 5.20. Thus

$$\lim_{y\to 0} y^{-\frac{r}{2}}\{\varphi, J_{r,y}\} = \overline{c}\int_0^\infty \varphi(x)x^{\frac{r}{2}}dx \tag{5.63}$$

for any $\varphi\in\mathcal{H}_r^\infty$. Now as above choose $\varphi = \varphi_0^{(r)}$. But $\{\varphi_0^{(r)}, J_{r,y}\} = \varphi_0^{(r)}(y)$ by Theorem 5.7. Since $\varphi_0^{(r)}(y) = e^{-y}y^{\frac{r}{2}}$ one has

$$\lim_{y\to 0} e^{-y} = \overline{c}\int_0^\infty e^{-x}x^r dx. \tag{5.64}$$

But the left side of (5.64) is 1 and, by definition of the Γ-function, the right side is $\overline{c}\Gamma(r+1)$. Hence $c = \frac{1}{\Gamma(r+1)}$. QED

Recall the map $\zeta_r : \mathcal{H}_r^{-\infty} \to Dist((0,\infty))$. See (4.40).

**Theorem 5.24.** *The subspace $C_o^\infty((0,\infty))$ is not dense in $\mathcal{H}_r^\infty$ with respect to the $\pi_r$-Fréchet topology. In fact, the highest weight vector, $\delta_{r,0}$, and the Category-O module, $\pi_r^{-\infty}(U(\mathfrak{g}))(\delta_{r,0})$, it generates, vanishes on $C_o^\infty((0,\infty))$. That is,*

$$\pi_r^{-\infty}(U(\mathfrak{g}))(\delta_{r,0}) \ \subset\ Ker\,\zeta_r. \tag{5.65}$$

*Furthermore, (see (4.37)) the injection map $\mathcal{H}_r^\infty \to C^\infty((0,\infty))$ is continuous but is not a homeomorphism onto its image where $\mathcal{H}_r^\infty$ has the $\pi_r$-Fréchet topology and $C^\infty((0,\infty)) = \mathcal{E}((0,\infty))$ has the Fréchet topology of distribution theory.*

*Proof.* It is obvious from (5.45) that $\delta_{r,0} \in Ker\,\zeta_r$. But this implies (5.65) since $C_o^\infty((0,\infty))$ is stable under $\pi_r^\infty(U(\mathfrak{g}))$ by (4.36). The continuity of (4.37) is established in Proposition 4.7. It is not a homeomorphism onto its image since, as one knows, $C_o^\infty((0,\infty))$ is dense in $\mathcal{E}((0,\infty))$ with respect to the distribution theory Fréchet topology on $\mathcal{E}((0,\infty))$. QED

*Remark* 5.25. In contrast to the highest weight vector, $\delta_{r,0}$, note that, by (5.59), the lowest weight vector, $J_{r,0}$, is not in the kernel of $\zeta_r$. See (4.40).

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