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ON THE EQUIVARIANT K-THEORY OF THE NILPOTENT CONE

VIKTOR OSTRIK

ABSTRACT. In this note we construct a "Kazhdan-Lusztig type" basis in equivariant K-theory of the nilpotent cone of a simple algebraic group G. This basis conjecturally is very close to the basis of this K-group consisting of irreducible bundles on nilpotent orbits. As a consequence we get a natural (conjectural) construction of Lusztig's bijection between dominant weights and pairs {nilpotent orbit $\mathcal{O}$, irreducible G-bundle on $\mathcal{O}$}.

1. INTRODUCTION

Let G be a simple simply connected algebraic group over the complex numbers. Let $\mathfrak{g}$ be the Lie algebra of G. Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone. Let $\mathcal{G} = G \times \mathbb{C}^*$. Let us consider the action of $\mathcal{G}$ on $\mathfrak{g}$ given by the rule $(g, z)x = z^{-2}\mathrm{Ad}(g)x$. The nilpotent cone $\mathcal{N}$ is invariant under this action. The aim of this paper is to make some conjectures on the equivariant K-group $K_{\mathcal{G}}(\mathcal{N})$; see e.g. [7]. Namely, we introduce a "Kazhdan-Lusztig type" canonical basis of $K_{\mathcal{G}}(\mathcal{N})$ over the representation ring of $\mathbb{C}^*$, parametrized by dominant weights of G. We conjecture that this basis is close to the basis consisting of irreducible G-equivariant bundles on nilpotent orbits. This would give us a bijection between two sets: {dominant weights for G} and {pairs consisting of a nilpotent orbit $\mathcal{O}$ and an irreducible G-equivariant bundle on $\mathcal{O}$}. Such a bijection appeared in the work of G. Lusztig on the asymptotic affine Hecke algebra; see [14], IV, 10.8. We conjecture that our (conjectural) bijection coincides with Lusztig's. We also conjecture that some specific elements of $K_{\mathcal{G}}(\mathcal{N})$ closely related with irreducible local systems on nilpotent orbits belong to our basis. All our conjectures are motivated by the study of the cohomology of *quantized tilting modules*. So this paper should be considered as a generalization of Humphreys' Conjecture [12].

This paper is based on an idea of George Lusztig. I learned of this idea from Michael Finkelberg. I am deeply grateful to Roman Bezrukavnikov for extremely useful conversations while preparing this paper. I wish to thank David Vogan who pointed out a gap in the first version of this paper. Thanks is also due to Jim Humphreys for valuable suggestions. Finally, I would like to acknowledge Harvard University for its hospitality while this paper was being written.

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2. Canonical basis of $K_{\mathcal{G}}(\mathcal{N})$

2.1. Let $B \subset G$ be a Borel subgroup and let $\mathcal{B} = G/B$ be the flag variety. Let $\mathfrak{b} = \mathrm{Lie}(B) \subset \mathfrak{g}$ and let $\mathfrak{n} \subset \mathfrak{g}$ be the nilpotent radical of $\mathfrak{b}$. It is well known that the cotangent bundle $T^*\mathcal{B}$ is naturally isomorphic to $G \times_B \mathfrak{n}$ and the map (*Springer resolution*) $s : G \times_B \mathfrak{n} \to \mathcal{N},\ (g, n) \mapsto \mathrm{Ad}(g)n$ is a resolution of singularities of $\mathcal{N}$; see e.g. [7]. This map is $\mathcal{G}$-equivariant with respect to the $\mathcal{G}$-action on $G \times_B \mathfrak{n}$ given by $(g, z)(g_1, n) = (gg_1, z^{-2}n)$.

Let $X = \mathrm{Hom}(B, \mathbb{C}^*)$ be the weight lattice of G and let X_+ be the set of dominant weights. For any $\lambda \in X_+$ let V_λ denote the irreducible representation of G with highest weight λ. We will also consider V_λ as a $\mathcal{G}$-module via projection $pr_1 : G \times \mathbb{C}^* \to G$.

For any $\lambda \in X$ one associates the line bundle $\mathcal{L}_\lambda$ on $\mathcal{B}$ (see [7]). Let $\pi : G \times_B \mathfrak{n} \to \mathcal{B}$ be the natural projection. For any $\lambda \in X_+$ and $i > 0$ we have $R^i s_* \pi^* \mathcal{L}_\lambda = 0$ (Andersen-Jantzen vanishing; see [2, 4]); in this case we will call the sheaf $\widetilde{AJ}(\lambda) = s_*\pi^*\mathcal{L}_\lambda$ an *Andersen-Jantzen sheaf* (or AJ-*sheaf*). Any AJ-sheaf $\widetilde{AJ}(\lambda)$ is endowed with the natural structure of an equivariant $\mathcal{G}$-sheaf.

Let $R(\mathcal{G})$ be the representation ring of $\mathcal{G}$. The ring $R(\mathcal{G})$ acts on $K_{\mathcal{G}}(V)$ for any $\mathcal{G}$-variety V. Let $v \in R(\mathcal{G})$ correspond to the one-dimensional representation of $\mathcal{G}$ given by the second projection $pr_2 : \mathcal{G} = G \times \mathbb{C}^* \to \mathbb{C}^*$. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}] \subset R(\mathcal{G})$ be the subring of $R(\mathcal{G})$ generated by v and v^{-1}.

Let $\tilde{R}(\mathcal{G})$ denote the $\mathcal{A}$-module of formal linear combinations $\sum_{\lambda \in X_+} k_{lambda} V_\lambda, k_\lambda \in \mathcal{A}$ (in general an infinite sum). The space $\tilde{R}(\mathcal{G})$ is endowed with an obvious $R(\mathcal{G})$-structure. For any $\mathcal{G}$-equivariant sheaf $\mathcal{S}$ on $\mathcal{N}$, its global sections $\Gamma(\mathcal{S})$ form a $\mathcal{G}$-module in a natural way and $\mathrm{Hom}_G(V_\lambda, \Gamma(\mathcal{S}))$ is finite dimensional for any $\lambda \in X_+$. Since $\mathcal{N}$ is affine the functor Γ is exact and we obtain a well-defined map $\Gamma : K_{\mathcal{G}}(\mathcal{N}) \to \tilde{R}(\mathcal{G})$. Clearly this map is $R(\mathcal{G})$-linear. Note that for any $\lambda \in X_+$

(a)
$$\Gamma(\widetilde{AJ}(\lambda)) \in V_\lambda + \sum_{\lambda' \in X_+} v\mathbb{Z}[v] V_{\lambda'}.$$

2.2.

Lemma (R. Bezrukavnikov)**.** *The classes $[\widetilde{AJ}(\lambda)] \in K_{\mathcal{G}}(\mathcal{N}), \lambda \in X_+$ form an $\mathcal{A}$-basis of $K_{\mathcal{G}}(\mathcal{N})$.*

Proof. Let us prove that $[\widetilde{AJ}(\lambda)]$ are linearly independent. Suppose that

$$\sum_{\lambda \in X_+} a_\lambda [\widetilde{AJ}(\lambda)] = 0, \quad a_\lambda \in \mathcal{A}.$$

We may assume that $a_\lambda \in \mathbb{Z}[v]$ and $a_\lambda \notin v\mathbb{Z}[v]$ for at least one λ. Applying Γ to both sides of this equality and using 2.1 (a) we obtain a contradiction.

Let us prove that the map $s_* : K_{\mathcal{G}}(T^*\mathcal{B}) \to K_{\mathcal{G}}(\mathcal{N})$ is surjective. To this end let us show that for any $\mathcal{G}$-equivariant sheaf $\mathcal{S}$ on $\mathcal{N}$ there exists $\alpha \in K_{\mathcal{G}}(T^*\mathcal{B})$ such that $[\mathcal{S}] - s_*\alpha$ has a strictly smaller support than $[\mathcal{S}]$ (then we are done by devissage). We can assume that the support of $[\mathcal{S}]$ is the closure $\bar{\mathcal{O}}$ of a nilpotent orbit $\mathcal{O}$. But it is well known (see e.g. [16]) that $\bar{\mathcal{O}}$ admits a resolution of singularities $r : X \to \bar{\mathcal{O}}$ where X is some G-equivariant subbundle of the cotangent bundle of some partial flag variety. Clearly the support of $[\mathcal{S}] - r_*[r^*\mathcal{S}]$ is contained in $\bar{\mathcal{O}} - \mathcal{O}$. Finally, one shows using the Koszul complex that the image of $K_{\mathcal{G}}(X)$ under r_* is contained in the image of $K_{\mathcal{G}}(T^*\mathcal{B})$ under s_*. The surjectivity is proved.

It is well known that the sheaves $\pi^*\mathcal{L}_\lambda, \lambda \in X$ form an $\mathcal{A}$-basis in $K_{\mathcal{G}}(T^*\mathcal{B})$; see e.g. [7]. Now let $\lambda \in X \setminus X_+$. Let α_i be a simple root such that $\langle \lambda, \alpha_i^\vee \rangle < 0$. A simple SL_2-calculation (see e.g. [5], 3.15) shows that

$$[s_*\pi^*\mathcal{L}_\lambda] = \begin{cases} v^2[s_*\pi^*\mathcal{L}_{s_{\alpha_i}\lambda}] & \text{if } \langle \lambda, \alpha_i^\vee \rangle = -1; \\ -[s_*\pi^*\mathcal{L}_{s_{\alpha_i}\lambda-\alpha_i}] + v^2[s_*\pi^*\mathcal{L}_{s_{\alpha_i}\lambda}] + v^2[s_*\pi^*\mathcal{L}_{\lambda+\alpha_i}] & \text{if } \langle \lambda, \alpha_i^\vee \rangle \le -2. \end{cases}$$

The lemma follows. □

2.3. Let W be the Weyl group of G and let ν be the number of positive roots in G. Let $l : W \to \mathbb{N}$ be the length function and let $w_0 \in W$ be the longest element (then $\nu = l(w_0)$). For any $\lambda \in X$ let $W_\lambda \subset W$ be the stabilizer of λ in W and let ν_λ be the length of the longest element w_λ of W_λ.

It follows from the Lemma 2.2 that the classes $AJ(\lambda) = (-v)^{\nu-\nu_\lambda}[\widetilde{AJ}(\lambda)], \lambda \in X_+$ form a $\mathbb{Z}[v, v^{-1}]$-basis of $K_{\mathcal{G}}(\mathcal{N})$. We will call $\{AJ(\lambda)\}$ the *Andersen-Jantzen basis.*

2.4. Let $Z = T^*\mathcal{B} \times_{\mathcal{N}} T^*\mathcal{B}$ be the Steinberg variety; see [15]. Let $\mathcal{H}$ be the affine Hecke algebra (over $\mathbb{Z}[v, v^{-1}]$) associated with G; see [15]. We identify $K_{\mathcal{G}}(Z)$ and $\mathcal{H}$ via the isomorphism constructed in [15] (see also [7]). Let $\hat{W}^a$ be the (extended) affine Weyl group as defined in [15], 1.7. The Weyl group W is embedded in $\hat{W}^a$ and the set of double cosets $W \setminus \hat{W}^a / W$ is identified with the set X_+ (see [14]); for any $w \in \hat{W}^a$ let $\lambda_w \in X_+$ denote its double coset. Let $l(\lambda_w)$ denote the length of the shortest element in λ_w. Let $\tilde{T}_w, w \in \hat{W}^a$ be the basis of $\mathcal{H}$ as defined in [15], 1.8. The natural projection $st : Z \to \mathcal{N}$ induces homomorphism $st_* : K_{\mathcal{G}}(Z) \to K_{\mathcal{G}}(\mathcal{N})$.

Lemma. *We have an equality $st_*(\tilde{T}_w) = (-v)^{-l(w)+l(\lambda_w)}AJ(\lambda_w)$. In particular, the map st_* is surjective.*

Proof. The map st can be factorized in two ways: $st = s \cdot pr_1$ and $st = s \cdot pr_2$ where $pr_i, i = 1, 2$ are two projections $Z \to T^*\mathcal{B}$. It follows (see [15], 7.25 and 7.19) that $st_*(T_{s_iw}) = -v^{-1}st_*(T_w)$ and $st_*(T_{ws_i}) = -v^{-1}st_*(T_w)$ (where s_i denote a simple reflection lying in the finite Weyl group W). Finally, we note that our formula follows from definitions in [15] for any translation by a dominant weight considered as an element of $\hat{W}^a$. □

Remark. Let $Cent(\mathcal{H})$ be the center of $\mathcal{H}$ (see e.g. [7] for its description). It *is not* true that the map $st_* : Cent(\mathcal{H}) \to K_{\mathcal{G}}(\mathcal{N})$ is surjective, in fact its image consists of trivial bundles on $\mathcal{N}$ with possibly nontrivial $\mathcal{G}$-structure. But it is a consequence of Lemmas 2.2 and 2.4 that the map $st_* : Cent(\mathcal{H}) \otimes_{\mathcal{A}} \mathbb{Q}(v) \to K_{\mathcal{G}}(\mathcal{N}) \otimes_{\mathcal{A}} \mathbb{Q}(v)$ is surjective.

2.5. Let $\overline{} : \mathcal{H} \to \mathcal{H}$ be the Kazhdan-Lusztig involution of the ring $\mathcal{H}$; see [15] 1.8. It follows from Lemma 2.4 that the kernel of the map st_* is invariant under $\overline{}$. So, we obtain a well defined involution $\overline{} : K_{\mathcal{G}}(\mathcal{N}) \to K_{\mathcal{G}}(\mathcal{N})$. We call this map Kazhdan-Lusztig involution.

Let $\omega : \mathfrak{g} \to \mathfrak{g}$ be an opposition (see [15], 9.8). Let $D : K_{\mathcal{G}}(\mathcal{N}) \to K_{\mathcal{G}}(\mathcal{N})$ be the Serre-Grothendieck duality map; see [15], 6.10.

Lemma. *The Kazhdan-Lusztig involution $\overline{} : K_{\mathcal{G}}(\mathcal{N}) \to K_{\mathcal{G}}(\mathcal{N})$ equals $v^{-2\nu}D\omega^*$.*

Proof. This immediately follows from [15], 9.12 if we note that restriction of st_* to the center of the Hecke algebra is surjective after tensoring with $\mathbb{Q}(v)$. □

2.6. We say that $x \in K_{\mathcal{G}}(\mathcal{N})$ is selfdual if $\bar{x} = x$.

Lemma. *For any $\lambda \in X_+$ there exists a unique selfdual element $C(\lambda) \in K_{\mathcal{G}}(\mathcal{N})$ such that $C(\lambda) \in AJ(\lambda) + \sum_{\mu \in X_+} v^{-1}\mathbb{Z}[v^{-1}]AJ(\mu)$. The elements $C(\lambda)$ form a basis of $K_{\mathcal{G}}(\mathcal{N})$.*

Proof. Let $c'_w \in \tilde{T}_w + \sum_{w'<w} v^{-1}\mathbb{Z}[v^{-1}]\tilde{T}_{w'}, \bar{c}'_w = c'_w, w \in \hat{W}^a$ be the Kazhdan-Lusztig basis of $\mathcal{H}$; see [15], 1.5, 1.8. For any $\lambda \in X_+ = W \setminus \hat{W}^a/W$ let $m_\lambda \in \lambda$ be the shortest element. We set $C(\lambda) = st_*(c'_{m_\lambda})$. The unicity follows from the existence in a standard way; see e.g. [20], 2.4. □

Remark. Let $C(\lambda') = \sum_\lambda b_{\lambda,\lambda'} AJ(\lambda)$ where $b_{\lambda,\lambda'} \in \mathbb{Z}[v^{-1}]$. The polynomials $b_{\lambda,\lambda'}$ appeared in the work of G.Lusztig [13]. The idea that the matrix $b_{\lambda,\lambda'}$ or rather its inverse should have representation theoretic meaning is due to Ivan Mirković. We believe that this note is a step in this direction.

Corollary. *We have $st_*(c'_w) = 0$ unless $w = m_{\lambda_w}$ in which case $st_*(c'_w) = C(\lambda_w)$.*

3. Conjectures

In this section we formulate a number of conjectures on the basis $\{C(\lambda)\}$.

3.1. For any $C \in K_{\mathcal{G}}(\mathcal{N})$ one defines its *support* as the complement to the union of all open $j : U \hookrightarrow \mathcal{N}$ such that $j^*C = 0$. Clearly, the support of any $C \in K_{\mathcal{G}}(\mathcal{N})$ is a closed G-invariant subset of $\mathcal{N}$.

Conjecture 1. *The support of any $C(\lambda)$ is irreducible.*

In other words, the support of any element $C(\lambda)$ is the closure of some nilpotent orbit $\mathcal{O} = \mathcal{O}_\lambda$. Let $j_{\mathcal{O}} : \mathcal{O} \hookrightarrow \mathcal{N}$ be the inclusion. The element $j^*_{\mathcal{O}_\lambda} C(\lambda) \in K_{\mathcal{G}}(\mathcal{O}_\lambda)$ is well defined.

Conjecture 2. *The class $\pm j^*_{\mathcal{O}_\lambda} C(\lambda)$ is represented by an irreducible $\mathcal{G}$-equivariant bundle $\mathcal{V}_\lambda$ on $\mathcal{O}_\lambda$.*

Let us choose a set $\{e_{\mathcal{O}}\}$ of representatives of all nilpotent orbits. Let $C_{\mathcal{G}}(e_{\mathcal{O}})$ (resp. $C_G(e_{\mathcal{O}})$) be the centralizer of $e_{\mathcal{O}}$ in $\mathcal{G}$ (resp. in G). Irreducible representations of $C_{\mathcal{G}}(e_{\mathcal{O}})$ (resp. $C_G(e_{\mathcal{O}})$) factor through the quotient $C^{red}_{\mathcal{G}}(e_{\mathcal{O}})$ (resp. $C^{red}_G(e_{\mathcal{O}})$) of $C_{\mathcal{G}}(e_{\mathcal{O}})$ (resp. $C_G(e_{\mathcal{O}})$) by its unipotent radical. The exact sequence

$$1 \to C_G(e_{\mathcal{O}}) \to C_{\mathcal{G}}(e_{\mathcal{O}}) \to \mathbb{C}^* \to 1 \tag{1}$$

where the first map is an obvious inclusion and the second one is a restriction of the second projection $pr_2 : \mathcal{G} \to \mathbb{C}^*$ induces an exact sequence

$$1 \to C^{red}_G(e_{\mathcal{O}}) \to C^{red}_{\mathcal{G}}(e_{\mathcal{O}}) \to \mathbb{C}^* \to 1. \tag{2}$$

We remark that the last sequence canonically splits. Namely, let $\phi_{\mathcal{O}} : SL_2(\mathbb{C}) \to G$ be a homomorphism such that $d\phi_{\mathcal{O}} \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = e_{\mathcal{O}}$, which exists by the Jacobson-Morozov Theorem. Then $\psi_{\mathcal{O}} : \mathbb{C}^* \to \mathcal{G}$, $\psi_{\mathcal{O}}(z) = (\phi_{\mathcal{O}} \left(\begin{smallmatrix} z^{-1} & 0 \\ 0 & z \end{smallmatrix}\right), z)$ is a desired splitting. It is not canonical for the sequence (1) since it requires a choice of $\phi_{\mathcal{O}}$ but all choices give the same splitting of the sequence (2).

Taking the stalk at $e_{\mathcal{O}}$ defines an equivalence of categories $\{\mathcal{G}$-equivariant bundles on $\mathcal{O}\}$ and $\{$representations of $C_{\mathcal{G}}(e_{\mathcal{O}})\}$; see e.g. [7]. So Conjecture 2 gives us an irreducible representation ρ'_λ of $C_{\mathcal{G}}(e_{\mathcal{O}_\lambda})$ attached to the bundle $\mathcal{V}_\lambda$. By the above we can consider ρ'_λ as a representation of $C^{red}_{\mathcal{G}}(e_{\mathcal{O}_\lambda})$. Let $a(\mathcal{O}) = \frac{1}{2}\mathrm{codim}_{\mathcal{N}}\mathcal{O}$. We expect that the group $\psi_\lambda(\mathbb{C}^*)$ acts on ρ'_λ by dilatations $z \mapsto z^{-a(\mathcal{O}_\lambda)}Id$. So the representation ρ'_λ is completely defined by its restriction ρ_λ to $C_G^{red}(e_{\mathcal{O}_\lambda})$.

The group $C_G^{red}(e_{\mathcal{O}_\lambda})$ contains a canonical central involution $\phi_{\mathcal{O}_\lambda}\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$. It acts on the irreducible representation ρ_λ by $\pm Id$. We expect that the sign here is the same as the sign in Conjecture 2.

Conjectures 1 and 2 provide a map $\mathfrak{L} : \lambda \mapsto (\mathcal{O}_\lambda, \rho_\lambda)$ from the set of dominant weights to the set of pairs {nilpotent orbit $\mathcal{O}$, irreducible representation of $C_G^{red}(e_{\mathcal{O}})$} .

Conjecture 3. *The map $\mathfrak{L}$ is a bijection. Moreover, this bijection coincides with Lusztig's, defined in* [14]*, IV, 10.8.*

In particular, the orbit $\mathcal{O}_\lambda$ can be determined as follows. We take an element $m_\lambda \in \hat{W}^a$ and find the *two-sided cell* $\underline{c} = \underline{c}_\lambda \subset \hat{W}^a$ containing m_λ; see [14]. The main result of [14], IV is a bijection $\mathfrak{l}$ between the set of two sided cells and the set of nilpotent orbits. Humphreys' Conjecture [12] predicts that $\mathcal{O}_\lambda = \mathfrak{l}(\underline{c}_\lambda)$.

Now let $\mathcal{O}$ be a nilpotent orbit and let $\mathbb{C}_{\mathcal{O}}$ be the trivial one-dimensional bundle on $\mathcal{O}$. The sheaf $\mathbb{C}_{\mathcal{O}}$ has an obvious $\mathcal{G}$-equivariant structure. Let $j : \mathcal{O} \to \mathcal{N}$ be the inclusion. Consider the sheaf $j_*\mathbb{C}_{\mathcal{O}}$.

Conjecture 4. *The element $v^{-a(\mathcal{O})}[j_*\mathbb{C}_{\mathcal{O}}] \in K_{\mathcal{G}}(\mathcal{N})$ is of the form $C(\lambda)$ for some $\lambda = \lambda_{\mathcal{O}}$. Moreover, the element $d_{\mathcal{O}} = m_{\lambda_{\mathcal{O}}}$ is a distinguished involution in $\hat{W}^a$; see* [14]*, II.*

Remark that self-duality of the element $v^{-a(\mathcal{O})}[j_*\mathbb{C}_{\mathcal{O}}]$ follows from the Theorem of Hinich and Panyushev [11, 19] stating that normalization of the closure of any nilpotent orbit has rational singularities. Furthermore, let ρ be an irreducible representation of some $C_G(e_{\mathcal{O}})$ which factors through a finite quotient of $C_G(e_{\mathcal{O}})$. Using the splitting of the exact sequence (2) we extend ρ to a representation of $C_{\mathcal{G}}(e_{\mathcal{O}})$ trivial on $\psi_{\mathcal{O}}(\mathbb{C}^*)$. Let $\mathcal{V}_\rho$ be the corresponding $\mathcal{G}$-equivariant bundle on $\mathcal{O}$. We expect that the class $v^{-a(\mathcal{O})}[j_*\mathcal{V}_\rho] \in K_{\mathcal{G}}(\mathcal{N})$ also is of the form $C(\lambda)$ for some $\lambda = \lambda_\rho$. The self-duality of this element should be a consequence of [6], 6.3. We note that by no means is such a statement true for general ρ; see 4.5 below.

3.2. **Tilting modules.** This subsection is devoted to the explanation of the connection of our conjectures with the theory of tilting modules. In fact, the conjectures above were motivated by the study of cohomology of tilting modules for quantum groups. Tilting modules provide a lifting of these conjectures from the K-theoretical level to the level of categories. We refer the reader to [1] for the definition and basic properties of tilting modules.

Let U be the quantum group over $\mathbb{C}$ with the same root datum as the group G at a primitive l-th root of unity where l is an odd number (prime to 3 if G is of type G_2) greater than the Coxeter number of G. Let $u \subset U$ be the small quantum group. Let $\mathbb{C}$ denote the trivial representation of U. The Ginzburg-Kumar Theorem [9] states that odd cohomology $H^{odd}(u, \mathbb{C})$ vanishes and the graded algebra of even cohomology $H^{ev}(u, \mathbb{C})$ is isomorphic to the algebra $\mathbb{C}[\mathcal{N}]$ of regular functions on the nilpotent cone $\mathcal{N}$ (the grading on the latter algebra comes from the $\mathbb{C}^*$-action). Moreover, the natural G-actions on both algebras are the same under this isomorphism. For any finite dimensional U-module M the cohomology $H^\bullet(u, M)$ is a module over $H^\bullet(u, \mathbb{C})$ via cup-product. This module is finitely generated; see *loc. cit.* So we can identify $H^\bullet(u, M)$ with a $\mathcal{G}$-equivariant coherent sheaf on $\mathcal{N}$; see [9]. Further, we can attach to M the class of its Euler characteristic $\chi(M) = [H^{ev}(u, M)] - [H^{odd}(u, M)] \in K_{\mathcal{G}}(\mathcal{N})$.

Now let $\mathbf{X}$ be the weight lattice of U. Of course this lattice is isomorphic to X but we prefer to distinguish the two lattices (in fact, it is natural to identify X with the sublattice $l\mathbf{X} \subset \mathbf{X}$). Let us define a *dot-action* $(w, x) \mapsto w \cdot x$ of the group $\hat{W}^a$ on $\mathbf{X}$ as follows. Recall that the group $\hat{W}^a$ is canonically isomorphic to the semidirect product of W with X; see e.g. [14], IV, 1.6. Now for any $w = \lambda v \in \hat{W}^a$ with $\lambda \in X,\ v \in W$ we set $w \cdot x = v(x + \varrho) - \varrho + l\lambda$. Here $\varrho \in \mathbf{X}$ is the half sum of positive roots.

For any dominant $x \in \mathbf{X}$ let $T(x)$ denote an indecomposable tilting U-module with highest weight x (it is unique up to nonunique isomorphism). In most cases the cohomology $H^\bullet(u, T(x))$ vanishes. First of all the Linkage Principle (see e.g. [3]) shows that $H^\bullet(u, T(x)) \neq 0$ implies that $x \in \hat{W}_a \cdot 0$. Further, we claim that $H^\bullet(u, T(w \cdot 0)) \neq 0$ implies that w is the minimal length element in its double coset $WwW \subset \hat{W}^a$. Indeed, w has minimal length in its coset wW since $w \cdot 0$ is dominant. If w is not of minimal length in coset Ww, then there exists a simple reflection $s \in W$ such that $l(sw) < l(w)$. Let Θ_s be the wall-crossing functor corresponding to s; see e.g. [20]. It easy to see that $T(w \cdot 0)$ is a direct summand of $\Theta_s T(sw \cdot 0)$. By adjointness of functors we have

$$H^\bullet(u, \Theta_s T(sw \cdot 0)) = \mathrm{Ext}^\bullet_u(\mathbb{C}, \Theta_s T(sw \cdot 0)) = \mathrm{Ext}^\bullet_u(\Theta_s \mathbb{C}, T(sw \cdot 0)) = 0$$

since $\Theta_s \mathbb{C} = 0$. So the condition $H^\bullet(u, T(x)) \neq 0$ implies that $x = m_\lambda \cdot 0$ for some $\lambda \in X_+$. This condition is also sufficient as shown by the following

Observation. *Let $For\colon K_{\mathcal{G}}(\mathcal{N}) \to K_G(\mathcal{N})$ be the forgetting map. Then*

$$For(\chi(T(m_\lambda \cdot 0))) = For(C(\lambda)).$$

This observation is a consequence of the (quantum version of) the main result of [2], Soergel's formula for characters of quantum tilting modules [20], and additivity of Euler characteristic. Indeed, Soergel's formula is in terms of the $\mathcal{H}$-module $\mathcal{H} \otimes_{\mathcal{H}_f} \mathcal{L}(-v^{-1})$ where $\mathcal{H}_f$ is a Hecke algebra of W and $\mathcal{L}(-v^{-1})$ is the one-dimensional $\mathcal{H}_f$-module corresponding to the sign representation of W. Arguing as in the proof of the Lemma 2.4 we identify $\mathcal{H} \otimes_{\mathcal{H}_f} \mathcal{L}(-v^{-1})$ with $K_{\mathcal{G}}(T^*\mathcal{B})$. We omit further details.

Conjectures 1–4 have analogues for tilting modules. We therefore expect that the sheaf $H^\bullet(u, T(m_\lambda \cdot 0))$ has irreducible support. In fact, we have two sheaves $H^{ev}(u, T(m_\lambda \cdot 0))$ and $H^{odd}(u, T(m_\lambda \cdot 0))$ and we expect that their supports are related by strict inclusion. The parity of the biggest cohomology sheaf $H^{big}(u, T(m_\lambda))$ is determined by the sign in Conjecture 2 ($+$ corresponds to H^{ev} and $-$ corresponds to H^{odd}). The sheaf $H^{big}(u, T(m_\lambda))$ restricted to $\mathcal{O}_\lambda$ should be equal to $\mathcal{V}_{\rho_\lambda}$. This picture is a generalization of Humphreys' Conjecture on support varieties of tilting modules; see [12].

For any nilpotent orbit $\mathcal{O}$ there exists a unique distinguished involution $d_{\mathcal{O}}$ such that $d_{\mathcal{O}}$ is of minimal length in the double coset $Wd_{\mathcal{O}}W$ and $d_{\mathcal{O}}$ is contained in the cell $\underline{c}$ with $\mathfrak{l}(\underline{c}) = \mathcal{O}$. The tilting counterpart of Conjecture 4 states that the cohomology $H^\bullet(u, T(d_{\mathcal{O}} \cdot 0))$ vanishes in odd degrees, has a natural structure of graded commutative algebra and is isomorphic as algebra to the algebra $\mathbb{C}[\mathcal{O}]$ of regular functions on $\mathcal{O}$. This is true at least for the regular nilpotent orbit (by Ginzburg-Kumar Theorem), for the trivial nilpotent orbit (by [1]) and the subregular nilpotent orbit (by [18]). We also expect that the parity vanishing holds in all cases when ρ_λ is trivial on the connected component of $C_G(e_{\mathcal{O}_\lambda})$.

4. Examples

In this section we describe various cases where we are able to check some of our conjectures. It will be convenient to use the notations $e^\lambda = [s_*\pi^*\mathcal{L}_\lambda]$ for any $\lambda \in X$ (in particular, if $\lambda \in X_+$, then $e^\lambda = [\widetilde{AJ}(\lambda)]$). The formula $\overline{e^\lambda} = e^{w_0\lambda}$ which is a consequence of [15], 1.22 is very useful.

4.1. SL_2**.** In this case $X = \mathbb{Z}$ and $X_+ = \mathbb{Z}_{\geq 0}$. It is easy to calculate that $C(0) = AJ(0) = e^0, C(1) = AJ(1) = e^1$, $C(2) = AJ(2) + v^{-1}AJ(0) = v^{-1}(e^0 - v^2e^2), C(n) = AJ(n) - v^{-2}AJ(n-2) = v^{-1}(e^{n-2} - v^2e^n)$ for $n \geq 3$. The support of $C(0)$ and $C(1)$ is the full nilpotent cone and the support of $C(n), n \geq 2$ is the trivial nilpotent orbit. The element $C(0)$ (resp. $C(1)$) corresponds to the trivial (resp. unique irreducible nontrivial) bundle on the regular nilpotent orbit. G-equivariant bundles on the point bijectively correspond to representations of G. Under this identification the element $C(n), n \geq 2$ corresponds to the irreducible SL_2-representation with highest weight $n-2$. We see that Conjectures 1–4 hold in this case. Moreover, it is easy to check that all tilting conjectures are true in this case.

4.2. **Regular nilpotent orbit.** The support of an element $C(\lambda)$ is the full nilpotent cone if and only if λ is a *minimal* weight, that is minuscule weight or zero. In this case $C(\lambda) = AJ(\lambda) = (-v)^{\nu-\nu_\lambda}e^\lambda$. This fits nicely with results of Graham [10] (see also [8]) who computed the G-module structure of the ring of functions on universal cover $\tilde{\mathcal{O}}$ of the regular nilpotent orbit. Namely, Graham proved the following equality in $K_{\mathcal{G}}(\mathcal{N})$:

$$[\mathbb{C}[\tilde{\mathcal{O}}]] = \sum_{\lambda \text{ is minimal}} v^{\nu-\nu_\lambda}e^\lambda.$$

4.3. **Lowest cell.** The support of an element $C(\lambda)$ should be a point if and only if $\lambda - 2\varrho$ is dominant. In this case $C(\lambda)$ should correspond to the irreducible representation of G with highest weight $\lambda - 2\varrho$. Using the Koszul complex we see that this is equivalent to the equality

$$(*) \qquad C(\lambda) = v^{-\nu}e^{\lambda-2\varrho}\prod_{\alpha\in R_+}(e^0 - v^2e^\alpha)$$

where R_+ is the set of positive roots. This formula should be understood as follows: first we make all multiplications and then interpret the result as an element of $K_{\mathcal{G}}(\mathcal{N})$. The reader should be aware that the map $s_* : K_{\mathcal{G}}(T^*\mathcal{B}) \to K_{\mathcal{G}}(\mathcal{N})$ *is not* multiplicative (and moreover the group $K_{\mathcal{G}}(\mathcal{N})$ has no natural multiplicative structure). We say that $\lambda \in X_+$ is *very dominant* if for any subset $J \subset R_+$ the weight $\lambda + \sum_{\alpha\in J}\alpha$ is dominant. One can show that the right hand side of (*) is a selfdual element of $K_{\mathcal{G}}(\mathcal{N})$. Now it is clear from the definitions that formula (*) is true for any very dominant λ. It would be interesting to prove the formula (*) in general, the most interesting case being $\lambda = 2\varrho$. I checked this formula for groups of rank 2.

4.4. **McGovern formula.** Conjecture 4 is very easy to check in each particular case thanks to McGovern's formula [16] for G-structure of the ring of functions on nilpotent orbits. We restate this formula as follows. The Dynkin diagram of a nilpotent orbit determines a grading of the set of positive roots (this grading is additive and the gradation of a simple root is its label in the Dynkin diagram). Let $R_{+0} \subset R_+$ (resp. $R_{+1} \subset R_+$) be the set of positive roots with gradation 0 (resp. 1). In notations of Conjecture 4 McGovern proved that

$$[j_* \mathbb{C}_{\mathcal{O}}] = \prod_{\alpha \in R_{+0} \cup R_{+1}} (e^0 - v^2 e^{\alpha}).$$

Combining this with Conjecture 4 we get the following conjectural formula:

$$C(\lambda_{\mathcal{O}}) = v^{-a(\mathcal{O})} \prod_{\alpha \in R_{+0} \cup R_{+1}} (e^0 - v^2 e^{\alpha}).$$

Here $a(\mathcal{O}) = |R_{+0}| + \frac{1}{2}|R_{+1}|$. We remark that the right hand side of this formula is selfdual (as mentioned after Conjecture 4). We checked that this formula works for groups of rank 2, for the subregular nilpotent orbit, and in some other cases. As a consequence of this formula we obtain a conjectural algorithm for calculation $\lambda_{\mathcal{O}}$ and hence for computing dominant distinguished involutions in $\hat{W}^a$. It would be very interesting to find such formulas for other cases mentioned after Conjecture 4.

In the special case of the group SL_n the formula for $\lambda_{\mathcal{O}}$ can be described (following a remark by David Vogan) quite explicitly as follows. Let the sizes of Jordan blocks of an element $e \in \mathcal{O}$ be given by partition $p = p_1 \geq p_2 \geq \dots$. Let $p' = p'_1 \geq p'_2 \geq \dots$ be the dual partition and let $\mathcal{O}'$ be the nilpotent orbit consisting of matrices with Jordan blocks of sizes $p'_1, p'_2, \dots$. Let us consider the Dynkin diagram of $\mathcal{O}'$ as a weight λ for SL_n. Then $\lambda_{\mathcal{O}} = \lambda$. It would be interesting to find similar combinatorial formulas for other classical groups.

4.5. SL_3**.** Let ω_1 and ω_2 denote fundamental weights of SL_3. The weights of SL_3 not covered by the previous discussion are weights "near the walls" $n\omega_i, n \geq 2, i = 1, 2$ and $\omega_1 + \omega_2 + n\omega_i, n \geq 1, i = 1, 2$ corresponding to the subregular nilpotent orbit of SL_3. One checks that in this case all our conjectures are true. Here we only consider the weight $\lambda = 3\omega_1$. One calculates $C(3\omega_1) = AJ(3\omega_1) + (v^{-1} + v^{-3})AJ(\omega_1 + \omega_2) + v^{-2}AJ(0) = v^2 e^{3\omega_1} - (1 + v^2)e^{\omega_1 + \omega_2} + v^{-2}e^0$. Further, $\Gamma(C(3\omega_1)) = v^{-2}V_0 - v^2 V_{3\omega_2} - v^4 \dots$ (cf. with the last paragraph of [18]). In particular, $C(3\omega_1)$ is not of the form $[\mathcal{S}]$ for some sheaf $\mathcal{S}$.

4.6. **Subregular nilpotent orbits.** Let $\mathcal{O}$ be the subregular nilpotent orbit. For any simple root α_i let P_i be the corresponding parabolic subgroup. As it is well known for any *short* simple root α_i the moment map $T^*(G/P_i) \to \mathfrak{g}$ is a resolution of singularities of the closure $\bar{\mathcal{O}}$; see e.g. [4]. We get that $\lambda_{\mathcal{O}}$ is the unique short dominant root and $C(\lambda_{\mathcal{O}}) = v^{-1}e^0 - ve^{\alpha_i}$.

Let G_{ad} be the adjoint group of the same type as G. A nontrivial G_{ad}-equivariant irreducible bundle on $\mathcal{O}$ exists if and only if G is not simply laced. In cases of types $B_n, C_n (n \geq 2), F_4$ such a bundle $\mathcal{V}$ is unique (the case of type G_2 is considered below). For any *long* simple root α_j the image of the moment map $T^*(G/P_j)$ is $\bar{\mathcal{O}}$ and this map is generically two to one. We deduce that the weight $\lambda_{\mathcal{V}}$ corresponding to the bundle $\mathcal{V}$ is the unique long dominant root and $C(\lambda_{\mathcal{V}}) = v(e^{\alpha_i} - e^{\alpha_j})$, where α_i is a short simple root.

4.7. **The subregular nilpotent orbit in G_2.** The fundamental group of the subregular nilpotent orbit $\mathcal{O}$ for the group G of type G_2 is the symmetric group in three letters S_3. It is not hard to guess what weights should correspond to irreducible representations of S_3. Let ω_1 and ω_2 be the fundamental weights for G_2 such that $\dim V_{\omega_1} = 14$ and $\dim V_{\omega_2} = 7$. The weight ω_2 (resp. ω_1, $2\omega_2$) corresponds to the trivial representation of S_3 (resp. the irreducible two-dimensional, the sign representation). We have

$$\begin{aligned} C(\omega_2) &= AJ(\omega_2) + v^{-1}AJ(0) = v^{-1}(e^0 - v^6 e^{\omega_2}),\\ C(\omega_1) &= AJ(\omega_1) - v^{-4}AJ(\omega_2) = v^{-1}(v^2 e^{\omega_2} - v^6 e^{\omega_1}),\\ C(2\omega_2) &= AJ(2\omega_2) - v^{-2}AJ(\omega_1) = v^{-1}(v^4 e^{\omega_1} - v^6 e^{2\omega_2}). \end{aligned}$$

This implies the following formula for the image of the trivial bundle $\mathbb{C}[\tilde{\mathcal{O}}]$ on the universal cover $\tilde{\mathcal{O}}$ in the group $K_{\mathcal{G}}(\mathcal{N})$:

$$\begin{aligned} [\mathbb{C}[\tilde{\mathcal{O}}]] &= C(\omega_2) + 2C(\omega_1) + C(2\omega_2)\\ &= v^{-1}(e^0 + 2v^2 e^{\omega_2} + v^4 e^{\omega_1} - v^6 e^{\omega_2} - 2v^6 e^{\omega_1} - v^6 e^{2\omega_2}). \end{aligned}$$

In particular, we obtain a (conjectural) formula for graded multiplicities of simple G-modules in the function ring of $\tilde{\mathcal{O}}$. Fortunately, another formula for these multiplicities is available in the literature thanks to the work of McGovern [17]. We checked that McGovern's formula and ours are equivalent.

References

[1] H. H. Andersen, Tensor products of quantized tilting modules, Comm. Math. Phys. **149** (1992), 149-159. MR **94b:**17015

[2] H. H. Andersen, J. C. Jantzen, Cohomology of induced representations for algebraic groups, Math. Ann. **269** (1984), 487-525. MR **86g:**20057

[3] H. H. Andersen, J. C Jantzen, W. Soergel, Representations of quantum groups at pth root of unity and of semi-simple groups in characteristic p: independence of p, Asterisque, 220 (1994). MR **95j:**20036

[4] A. Broer, Line bundles on the cotangent bundle of the flag variety, Invent. Math. **113** (1993), 1-20. MR **94g:**14027

[5] A. Broer, Normality of some nilpotent varieties and cohomology of line bundles on the cotangent bundle of the flag variety, in "Lie Theory and Geometry. In Honor of Bertram Kostant", (J.-L. Brylinski et al., Eds.), P.M., Vol. 123, 1-19, Birkhäuser, Boston, 1994. MR **96g:**14042

[6] A. Broer, Decomposition varieties in semisimple Lie algebras, Canad. J. Math. **50** (5) (1998), 929-971. MR **99k:**14077

[7] N. Chriss, V. Ginzburg, Representation Theory and Complex Geometry, Birkhäuser, Boston, 1997. MR **98i:**22021

[8] V. Ginzburg, Perverse sheaves on a loop group and Langlands' duality, preprint alg-geom/9511007.

[9] V. Ginzburg, S. Kumar, Cohomology of quantum groups at roots of unity, Duke Math. J., **69** No. 1 (1993), 179-198. MR **94c:**17026

[10] W. Graham, Functions on the universal cover of principal nilpotent orbit, Invent. Math. **108** (1992), 15-27. MR **93h:**22026

[11] V. Hinich, On the singularities of nilpotent orbits, Israel J. Math. **73** (1991), 297-308. MR **92m:**14005

[12] J. Humphreys, Comparing modular representations of semisimple groups and their Lie algebras, Modular Interfaces (Riverside, CA, 1995), 69-80, AMS/IP Stud. Adv. Math., 4, Amer. Math. Soc., Providence, RI, 1997. MR **98h:**17006

[13] G. Lusztig, Nonlocal finiteness of a W-graph, Representation Theory **1** (1997), 25-30. MR **98c:**20078

[14] G. Lusztig, Cells in affine Weyl groups, Algebraic Groups and Related Topics, Adv. Studies in Pure Math. vol. 6, North Holland and Kinokuniya, Amsterdam and Tokyo, 1985, 255-287; II, J. Algebra **109** (1987), 536-548; III, J. Fac. Sci. Univ. Tokyo (IA) **34** (1987), 223-243; IV, J. Fac. Sci. Univ. Tokyo (IA) **36** (1989), 297-328. MR **87h:**20074; MR **88m:**20103a; MR **88m:**20103b; MR **90k:**20068

[15] G. Lusztig, Bases in equivariant K-theory, Representation Theory **2** (1998), 298-369. MR **99i:**19005

[16] W. McGovern, Rings of regular functions on nilpotent orbits and their covers, Invent. Math. **97** (1989) 209-217. MR **90g:**22022

[17] W. McGovern, A branching law for $Spin(7,\mathbb{C}) \to G_2$ and its applications to unipotent representations, J. Alg. **130** (1990), 165-175. MR **91d:**22010

[18] V. Ostrik, Cohomology of subregular tilting modules for small quantum groups, preprint q-alg/9902094.

[19] D. Panyushev, Rationality of singularities and the Gorenstein property for nilpotent orbits, Funct. Anal. Appl. **25** (1991), 225-226. MR **92i:**14047

[20] W. Soergel, Kazhdan-Lusztig-Polynome und eine Kombinatorik für Kipp-Moduln, Representation Theory **1** (1997), 37-68. MR **99d:**17023

Independent Moscow University, 11 Bolshoj Vlasjevskij per., Moscow 121002 Russia
E-mail address: ostrik@mccme.ru