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ON THE SPANNING VECTORS OF LUSZTIG CONES

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ABSTRACT. For each reduced expression $\mathbf{i}$ of the longest element w_0 of the Weyl group W of a Dynkin diagram Δ of type A, D or E, Lusztig defined a cone $\mathcal{C}_{\mathbf{i}}$ such that there corresponds a monomial in the quantized enveloping algebra $\mathbf{U}$ of Δ to each element of $\mathcal{C}_{\mathbf{i}}$ and he asked under what circumstances these monomials belong to the canonical basis of $\mathbf{U}$. In this paper, we consider the case where $\mathbf{i}$ is a reduced expression adapted to a quiver Ω whose graph is Δ and we describe $\mathcal{C}_{\mathbf{i}}$ as the set of non-negative integral combination of spanning vectors. These spanning vectors are themselves described by using the Auslander-Reiten quiver of Ω and homological algebra.

0. INTRODUCTION

Let C be the Cartan matrix of a complex finite dimensional simple simply laced Lie algebra of rank n. We can attach to C its quantized enveloping algebra $\mathbf{U}$ over $\mathbf{Q}(v)$. Recall that $\mathbf{U}$ is an associative algebra with generators E_i, F_i, K_i, K_i^{-1} $(1 \leq i \leq n)$ and relations (see 4.1 for the notations and a precise presentation of $\mathbf{U}$). Let $\mathbf{U}^+$ be the subalgebra of $\mathbf{U}$ generated by the E_i $(1 \leq i \leq n)$. Using different methods, both Kashiwara [8] and Lusztig [9] have constructed a canonical basis $\mathbf{B}$ of $\mathbf{U}^+$ with remarkable properties. Lusztig has shown in [10] that both methods give the same basis $\mathbf{B}$.

A monomial $E_{i_1}^{(c_1)} E_{i_2}^{(c_2)} \cdots E_{i_m}^{(c_m)}$, where $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ is a sequence of elements of $\{1, 2, \ldots, n\}$ and $c_1, c_2, \ldots, c_m \in \mathbf{N}$, is said to be tight (respectively semi-tight) if it belongs to $\mathbf{B}$ (respectively it is a linear combination of elements in $\mathbf{B}$ with constant coefficients). Lusztig gave in [11] a criterion involving the positivity of a non-homogeneous quadratic form $\mathcal{Q}$ for a monomial to be tight or semi-tight.

In [11], Lusztig defined a subset $\mathcal{C}_{\mathbf{i}}$ of $\mathbf{N}^\nu$ for each reduced expression $\mathbf{i} = (i_1, i_2, \ldots, i_\nu)$ of the longest element w_0 (i.e. $w_0 = s_{i_1} s_{i_2} \cdots s_{i_\nu}$) of the finite Weyl group (W, S) associated to C where $S = \{s_1, s_2, \ldots, s_n\}$ and asked under what circumstances is the monomial $E_{i_1}^{(c_1)} E_{i_2}^{(c_2)} \cdots E_{i_\nu}^{(c_\nu)}$ tight or semi-tight for $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathcal{C}_{\mathbf{i}}$. We will recall the definition of $\mathcal{C}_{\mathbf{i}}$ in 3.1. $\mathcal{C}_{\mathbf{i}}$ is related to the linear part of the non-homogeneous quadratic form $\mathcal{Q}$. In the case that C is of type A_n for $n = 1, 2, 3, 4$ and $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathcal{C}_{\mathbf{i}}$, then $E_{i_1}^{(c_1)} E_{i_2}^{(c_2)} \cdots E_{i_m}^{(c_m)}$ is tight. This result has been proved by Lusztig in the case where $n = 1, 2, 3$ (see [11]) and by Marsh in the case where $n = 4$ (See [12]).

R. Marsh has described in [13] these subsets $\mathcal{C}_{\mathbf{i}}$ as the non-negative integer span of ν independent integral vectors (called its spanning vectors) for all reduced expressions $\mathbf{i}$ of w_0 when the Cartan matrix C is of type A_n. He also called them

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Lusztig cones. The combinatorics that he uses to describe these ν spanning vectors involves the chamber diagram of the reduced expression $\mathbf{i}$. These spanning vectors were used by Marsh in [14] and by Carter and Marsh in [5] in relation with parametrizations of the canonical basis using strings of root operators and with piecewise linear functions defined by Lusztig.

In this paper, we will describe $\mathcal{C}_{\mathbf{i}}$ as the non-negative integer span of ν independent integral vectors when $\mathbf{i}$ is a reduced expression of w_0 adapted to a quiver Ω of the Dynkin graph associated to a Cartan matrix C of type A_n $(n \geq 1)$, D_n $(n \geq 4)$ or E_n $(n = 6, 7, 8)$. This description is done using the Auslander-Reiten quiver Γ_Ω of Ω and homological algebra. This is done in section 3 of this article. In the first two chapters, we will recall the basic facts about representations of algebras, Auslander-Reiten quivers, almost split sequences and reduced expressions of w_0 adapted to a quiver. The main theorem of the paper is Theorem 3.8 where the spanning vectors are described. In the last section, we consider monomials in $\mathbf{U}^+$ corresponding to elements of Lusztig cones and show that some of them are independent of a quiver Ω.

1. Notations and basic facts

1.1. Fix an $(n \times n)$ positive definite symmetric matrix $C = (a_{ij})_{1 \leq i,j \leq n}$ such that $a_{ii} = 2$ for $1 \leq i \leq n$ and $a_{ij} = a_{ji} \in \{0, -1\}$ if $1 \leq i \neq j \leq n$. Let Q be the free abelian group with basis $\alpha_1, \alpha_2, \ldots, \alpha_n$. Define an inner product $(\ \mid\ )$ on Q by $(\alpha_i \mid \alpha_j) = a_{ij}$. Let $R = \{\alpha \in Q \mid (\alpha \mid \alpha) = 2\}$, $R^+ = \{\alpha \in R \mid \alpha = \sum b_i \alpha_i$ with $b_i \in \mathbf{N}\}$ and $R^- = -R^+$. R is a simply laced root system with basis $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and R^+ is the corresponding set of positive roots. We will assume from now on that R is irreducible.

For each $\alpha \in R$, we will denote the corresponding reflection by $s_\alpha \colon Q \to Q$. Recall that $s_\alpha(z) = z - (\alpha \mid z)\alpha$ for all $z \in Q$. We will denote s_{α_i} by s_i. Thus $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ for all $1 \leq i, j \leq n$. Let W be the Weyl group of R. Recall that W is the subgroup of $\mathrm{Aut}(Q)$ generated by $S = \{s_1, s_2, \ldots, s_n\}$. We will denote by $\ell(w)$ the length of $w \in W$ relative to S.

We will denote the Dynkin graph associated to the Cartan matrix C by Δ. Recall that the set of vertices of Δ is $\{1, 2, \ldots, n\}$ where i is identified with the simple root $\alpha_i \in B$ and there is an edge between the vertices i and j if and only if $a_{ij} = -1$.

1.2. It is well known that there exists a unique element w_0 of the Weyl group W that is of maximal length and, in this case, $\ell(w_0) = \#(R^+)$. We will also denote this length by ν.

Let σ be the unique permutation of the vertices of Δ such that $w_0(\alpha_i) = -\alpha_{\sigma(i)}$. In other words, if Δ is of type D_n with n even or of type A_1, E_7 or E_8, then σ is the identity; while if Δ is of type A_n with $n > 1$, D_n with n odd or E_6, then σ is the unique non-trivial automorphism of the graph Δ. Denote by h, the Coxeter number of Δ. In other words, h is $(n+1)$, $2(n-1)$, 12, 18 or 30, if Δ is respectively of type A_n, D_n, E_6, E_7 or E_8.

If $s_{i_1} s_{i_2} \cdots s_{i_\nu} = w_0$ is a reduced expression of w_0, then we will abbreviate it by writing $\mathbf{i} = (i_1, i_2, \ldots, i_\nu)$. It is well known that if $\mathbf{i} = (i_1, i_2, \ldots, i_\nu)$ is a reduced expression of w_0, then the sequence $\alpha^{(1)}(\mathbf{i}), \alpha^{(2)}(\mathbf{i}), \ldots, \alpha^{(\nu)}(\mathbf{i})$ defined by $\alpha^{(j)}(\mathbf{i}) = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}(\alpha_{i_j})$ for $j = 1, 2, \ldots, \nu$ contains each root of R^+ once and exactly once.

1.3. If $\mathbf{i} = (i_1, i_2, \ldots, i_\nu)$ and $\mathbf{i}' = (i'_1, i'_2, \ldots, i'_\nu)$ are two reduced expressions of w_0, we say that $\mathbf{i}'$ is related to $\mathbf{i}$ by a short braid relation if $\mathbf{i}'$ is obtained from $\mathbf{i}$ by replacing two consecutive entries x, y in $\mathbf{i}$ (with $a_{xy} = 0$) by y, x; while we say that $\mathbf{i}'$ is related to $\mathbf{i}$ by a long braid relation if $\mathbf{i}'$ is obtained from $\mathbf{i}$ by replacing three consecutive entries x, y, x in $\mathbf{i}$ (with $a_{xy} = -1$) by y, x, y.

It is known that given two reduced expressions $\mathbf{i}$ and $\mathbf{j}$ of w_0, there is a sequence $\mathbf{i} = \mathbf{i}_0, \mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_p = \mathbf{j}$ of reduced expressions of w_0 starting with $\mathbf{i}$, ending with $\mathbf{j}$ and such that, for each $q = 0, 1, 2, \ldots, (p-1)$, $\mathbf{i}_{q+1}$ is related to $\mathbf{i}_q$ by either a short braid relation or by a long braid relation. This is part of a theorem of Tits (see [17]).

We say that two reduced expressions $\mathbf{i}$ and $\mathbf{j}$ of w_0 are commutation-equivalent if there exists a sequence $\Pi\colon \mathbf{i} = \mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_p = \mathbf{j}$ of reduced expressions of w_0 starting with $\mathbf{i}$, ending with $\mathbf{j}$ and such that $\mathbf{i}_{q+1}$ is related to $\mathbf{i}_q$ by a short braid relation for $q = 0, 1, \ldots, (p-1)$. We will write in this case $\mathbf{i} \sim \mathbf{j}$ and an equivalence class for this relation is called a commutation class. We also denote by $[\mathbf{i}]$ the commutation class containing $\mathbf{i}$.

1.4. Given a graph G whose edges are oriented, we say that a vertex i is a sink (respectively a source) if and only if each edge $\{i, j\}$ having i as one of its vertices is oriented as follows: $i \leftarrow j$, the arrow pointing toward i (respectively $i \rightarrow j$, the arrow pointing away from i).

1.5. We will recall the notations of section 4 of [9] for the theory of representations of a quiver. Let Ω be a quiver with underlying graph Δ. In other words, we have oriented the edges of Δ. Let F be an algebraically closed fixed field. The category $\mathrm{Mod}(\Omega)$ of modules or representations of the quiver Ω is given as follows. An object is a collection of finite-dimensional F-vector space V_i $(i \in \{1, 2, \ldots, n\})$ and of F-linear maps $f_{ij} : V_i \to V_i$ defined for each arrow $i \to j$ in Ω and a morphism from the object $\mathbf{V} = ((V_i)_{1 \le i \le n}, (f_{ij})_{i \to j})$ to the object $\mathbf{V}' = ((V'_i)_{1 \le i \le n}, (f'_{ij})_{i \to j})$ is a collection of F-linear maps $g_i : V_i \to V'_i$ $(i \in \{1, 2, \ldots, n\})$ such that $f'_{ij} \circ g_i = g_j \circ f_{ij}$ for all arrows $i \to j$ in Ω. This category is in an obvious way an abelian category.

Recall that if i is a sink (respectively a source) of Ω, then

(a) $s_i(\Omega)$ denotes the quiver obtained from Ω by reversing the orientation of each arrow that ends (respectively starts) at i;
(b) Φ_i^+ (respectively Φ_i^-) denotes the corresponding reflection functor from the category of modules of Ω to the category of modules of $s_i(\Omega)$. (The precise definition of these functors is given in 4.3 of [9]).

1.6. A reduced expression $\mathbf{i} = (i_1, i_2, \ldots, i_\nu)$ of w_0 is said to be adapted to the quiver Ω if and only if i_k is a sink of $s_{i_{k-1}} s_{i_{k-2}} \cdots s_{i_1}(\Omega) = \Omega_k$ for all $k = 1, 2, \ldots, \nu$.

For example, in the case A_3, the reduced expression $\mathbf{i} = (2, 1, 3, 2, 1, 3)$ of w_0 is adapted to the quiver $1 \to 2 \leftarrow 3$, while the reduced expression $\mathbf{j} = (2, 1, 2, 3, 2, 1)$ of w_0 is not adapted to any quiver.

The following facts are known:

(a) A reduced expression $\mathbf{i}$ of w_0 is adapted to at most one quiver Ω of Δ.
(b) For each quiver Ω with graph Δ, there is a reduced expression $\mathbf{i}$ of w_0 adapted to Ω.
(c) Let $\mathbf{i}$, $\mathbf{j}$ be two reduced expressions of w_0 such that $\mathbf{j} \sim \mathbf{i}$. If $\mathbf{i}$ is adapted to the quiver Ω with graph Δ, then so is $\mathbf{j}$.

For (a), see 4.13 in [9]. For (b), see Proposition 4.12 (b) in [9]. Finally, it is easy to verify (c) by simply considering the case where $\mathbf{j}$ is related to $\mathbf{i}$ by a short braid relation.

1.7. Let Ω be a quiver with graph Δ and $\mathbf{i} = (i_1, i_2, \dots, i_\nu)$, a reduced expression of w_0 adapted to Ω. Let e_{i_k} be the simple module $\mathbf{V} = ((V_i)_{1\le i\le n}, (f_{ij} = 0)_{i\to j})$ of Ω_k, as in [9], such that

$$V_i = \begin{cases} F, & \text{if } i = i_k; \\ 0, & \text{otherwise;} \end{cases}$$

and $e_\alpha = \Phi^-_{i_1}\Phi^-_{i_2}\cdots\Phi^-_{i_{k-1}}(e_{i_k})$ for $\alpha = \alpha^{(k)}(\mathbf{i}) = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k})$. Then it is possible to prove that e_α is an indecomposable module of Ω whose dimension is α. Here the dimension $\dim(\mathbf{V})$ of the module $\mathbf{V} = ((V_i)_{1\le i\le n}, (f_{ij})_{i\to j})$ of Ω is defined as $\sum_{i=1}^n (\dim_F V_i)\ \alpha_i$. We will denote the isomorphism class of the module $\mathbf{V}$ of Ω by $[\mathbf{V}]$.

Theorem 1.8 (Gabriel)**.** *The map $[e_\alpha] \to \alpha = \dim(e_\alpha)$ gives a bijection between the set of isomorphism classes of indecomposable modules of Ω with graph Δ and the set R^+ of positive roots of Δ.*

Proof. See Proposition 4.12 in [9] for example. There are also proofs of this result in [6] and [4]. ☐

1.9. For $k \in \{1, 2, \dots, n\}$, denote by $\mathbf{P}(k)$ the following module of Ω: $\mathbf{P}(k)_i$ is the vector space over F with basis the set of paths $k = k_0 \to k_1 \to k_2 \to \cdots \to k_m = i$ from k to i in Ω and for any arrow $i \to j$ in Ω, let $f_{ij}\colon \mathbf{P}(k)_i \to \mathbf{P}(k)_j$ be defined by sending the basis element $k = k_0 \to k_1 \to k_2 \to \cdots \to k_m = i$ to $k = k_0 \to k_1 \to k_2 \to \cdots \to k_m = i \to j$. It is easy to prove that $\mathbf{P}(k) = (\mathbf{P}(k)_i, (f_{ij})_{i\to j}))$ is an indecomposable projective module of Ω and that all indecomposable projective modules are isomorphic to some $\mathbf{P}(k)$ for $k \in \{1, 2, \dots, n\}$.

1.10. We will denote by $\mathcal{P}(\Omega)$ the set of positive roots α such that the indecomposable module e_α of Ω is projective. In other words, $\alpha \in \mathcal{P}(\Omega)$ if and only if α is the dimension $\dim(\mathbf{P}(k))$ of the projective indecomposable module $\mathbf{P}(k)$ for some $k \in \{1, 2, \dots, n\}$.

2. Auslander-Reiten quivers and reduced expressions

2.1. We will also need to recall some notations and results on the Auslander-Reiten quiver Γ_Ω of Ω. For this theory, we refer the reader either to section 6.5 in [7] or to section 2.2 in [15] or the book [2].

The vertices of the Auslander-Reiten quiver Γ_Ω are the isomorphism classes of indecomposable modules of the quiver Ω and two isomorphism classes $[\mathbf{V}]$ and $[\mathbf{W}]$ of indecomposable modules of Ω are linked by an arrow $[\mathbf{V}] \to [\mathbf{W}]$ in Γ_Ω if and only if there exists an irreducible morphism $\mathbf{V} \to \mathbf{W}$.

As seen above, the set of isomorphism classes of indecomposable modules of Ω is in bijection with R^+ and we will represent below each vertex $[e_\alpha]$ of Γ_Ω by simply writing the corresponding positive root $\alpha = \dim(e_\alpha)$. We won't need to explicitly determine the irreducible morphisms between two vertices who are linked together in Γ_Ω, we will just draw the arrow in Γ_Ω corresponding to the fact that there are irreducible morphisms.

The Auslander-Reiten quiver can be computed in a very combinatorial way using the dimension type of the indecomposable projective modules and the additivity property of the dimension types on the Auslander-Reiten sequences.

Let $\mathbf{N}\Omega$ be the following quiver: its set of vertices is $\mathbf{N} \times \{1, 2, \dots, n\}$ and, whenever there is an arrow $i \leftarrow j$ in Ω, we draw one arrow $(z, i) \to (z, j)$ and one arrow $(z, j) \to (z+1, i)$ for each $z \in \mathbf{N}$. Define $A(\Omega)$ as the full subquiver of $\mathbf{N}\Omega$ of all vertices (z, i) such that $1 \le z \le (h + a_i - b_i)/2$ where, for each $i \in \{1, 2, \dots, n\}$, a_i (respectively b_i) is the number of arrows in the unoriented path from i to $\sigma(i)$ that are directed towards i (respectively $\sigma(i)$).

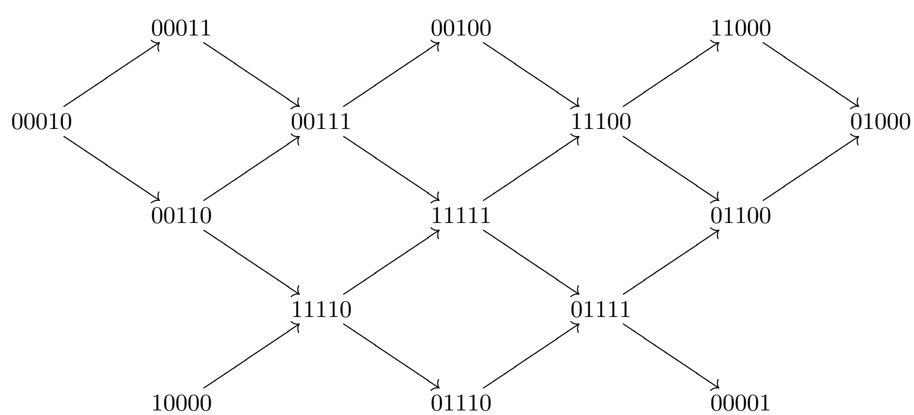
There is a unique isomorphism $\Psi\colon \Gamma_\Omega \to A(\Omega)$ of quivers such that $\Psi([\mathbf{P}(k)]) = (1, k)$ for each $k \in \{1, 2, \dots, n\}$. From the dimension types of the indecomposable projective modules, we can then easily compute Γ_Ω using this isomorphism Ψ and the additivity property of the dimension on the Auslander-Reiten sequences.

We define $\rho_\Omega\colon R^+ \to \{1, 2, \dots n\}$ by $\rho_\Omega(\alpha) = i$ for each $\alpha \in R^+$, where $\Psi([e_\alpha]) = (z, i) \in A(\Omega)$ for some $z \in \mathbf{N}$.

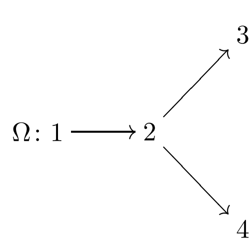
Let $\mathbf{Z}\Delta$ denote the translation quiver associated to the Dynkin graph Δ as presented in Figure 13 of section 6.5 of [7]. Note that this implies a choice of indices for the vertices of Δ. Recall that the set of vertices of $\mathbf{Z}\Delta$ is $\mathbf{Z} \times \{1, 2, \dots, n\}$. The translation τ is the function on the set of vertices of $\mathbf{Z}\Delta$ defined by $\tau(z, i) = (z-1, i)$. There is a unique embedding Ξ of Γ_Ω (or $A(\Omega)$ under the isomorphism Ψ) into $\mathbf{Z}\Delta$ such that $[\mathbf{P}(1)] = \Psi^{-1}(1, 1)$ is mapped to the vertex $(1, 1)$ of $\mathbf{Z}\Delta$.

In the examples below, we write the root $\alpha = \sum_{i=1}^{n} d_i \alpha_i$ by simply displaying the values $(d_1, d_2, \dots, d_n)$ in the same pattern as the Dynkin graph Δ and we have identified $\alpha \in R^+$ with the vertex $[e_\alpha]$ of Γ_Ω.

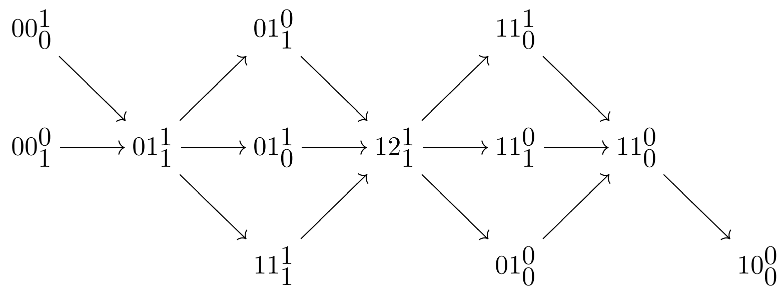
Example 2.2. For the quiver $\Omega\colon 1 \leftarrow 2 \to 3 \to 4 \leftarrow 5$ with an underlying graph of type A_5, the Auslander-Reiten quiver is



Example 2.3. For the quiver



with underlying graph of type D_4, the Auslander-Reiten quiver is



2.4. For two positive roots $\alpha, \beta \in R^+$, we will write $\beta \prec_\Omega \alpha$ if and only if there is a path $\beta = \alpha^0 \to \alpha^1 \to \alpha^2 \to \ldots \to \alpha^k = \alpha$ from β to α in the Auslander-Reiten quiver Γ_Ω. Here we have identified the positive roots with the corresponding isomorphism classes of indecomposable modules as in 2.1. For a quiver Ω of Δ corresponding to our Cartan matrix C, it is known that $\prec_\Omega$ is a partial order.

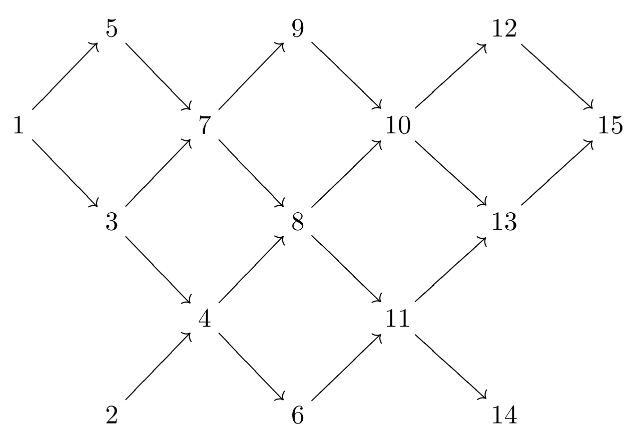
2.5. Let $\mathbf{i} = (i_1, i_2, \ldots, i_\nu)$ be a reduced expression of w_0 adapted to the quiver Ω. We will now describe all the reduced expressions $\mathbf{i}'$ of w_0 in the commutation class $[\mathbf{i}]$.

Theorem. *Let E_Ω be the set of bijections $f : R^+ \to \{1, 2, \ldots, \nu\}$ such that $f(\beta) < f(\alpha)$ whenever α, $\beta \in R^+$ and $\beta \to \alpha$ in Γ_Ω. In other words, E_Ω is the set of total orders on R^+ compatible with $\prec_\Omega$. For a reduced expression $\mathbf{i}'$ of w_0, denote by $\pi_{\mathbf{i}'} : R^+ \to \{1, 2, \ldots, \nu\}$ the function defined by $\pi_{\mathbf{i}'}(\alpha^{(j)}(\mathbf{i}')) = j$ for $j = 1, 2, \ldots, \nu$.*

(a) *If $\mathbf{i}' \sim \mathbf{i}$, then $\pi_{\mathbf{i}'} \in E_\Omega$.*
(b) *The function $[\mathbf{i}] \to E_\Omega$ defined by $\mathbf{i}' \mapsto \pi_{\mathbf{i}'}$ is a bijection whose inverse $E_\Omega \to [\mathbf{i}]$ is given by $f \mapsto (i'_1, i'_2, \ldots, i'_\nu)$ where $i'_k = \rho_\Omega(f^{-1}(k))$ for $k = 1, 2, \ldots, \nu$.*

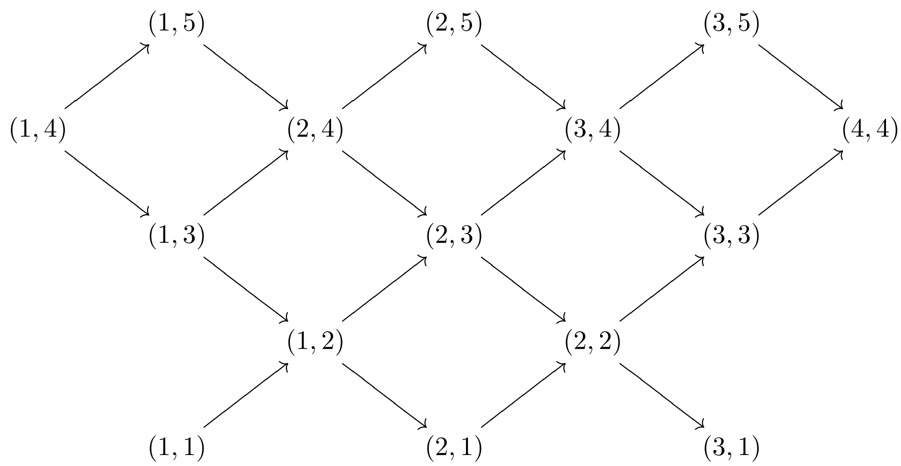
Proof. This is Theorem 2.17 of [3] applied to the case of w_0. □

Example 2.6. If we consider the quiver Ω of Example 2.2 and we represent an element of E_Ω by writing $f(\alpha)$ in the position corresponding to the positive root α in Γ_Ω, then the function f defined by



is an element of E_Ω and the corresponding reduced expression of w_0 adapted to the quiver Ω (under the bijection given in the above theorem) is $(4, 1, 3, 2, 5, 1, 4, 3, 5, 4,$

2, 5, 3, 1, 4). We get this because the quiver $A(\Omega)$ is simply



Here the isomorphism Ψ simply maps corresponding vertices of the two quivers Γ_Ω and $A(\Omega)$.

2.7. To conclude this section, we will recall some results on almost split sequences (also called Auslander-Reiten sequences) and Grothendieck groups of artin algebras. We will describe these results not in full generality as they appeared in [1] and [2], but rather as they are needed for our situation.

Let $\mathbf{V}$, $\mathbf{V}'$ and $\mathbf{V}''$ be three modules of the quiver Ω. A morphism $f : \mathbf{V} \to \mathbf{V}''$ (respectively $g : \mathbf{V}' \to \mathbf{V}$) is said to be right (respectively left) almost split if

(a) it is not a split epimorphism (respectively monomorphism);
(b) any morphism $\mathbf{M} \to \mathbf{V}''$ (respectively $\mathbf{V}' \to \mathbf{M}'$) which is not a split epimorphism (respectively monomorphism) factors through f (respectively g).

An exact sequence $0 \to \mathbf{V}' \xrightarrow{g} \mathbf{V} \xrightarrow{f} \mathbf{V}'' \to 0$ is said to be an almost split sequence if g is left almost split and f is right almost split.

2.8. Let $\mathbf{K}(\Omega, 0)$ be the free abelian group with basis the isomorphism classes $[\mathbf{M}]$ of modules $\mathbf{M}$ of Ω modulo the subgroup generated by the elements of the form $[\mathbf{V}] + [\mathbf{W}] - [\mathbf{V} \oplus \mathbf{W}]$. It is well known that the set $\{[\mathbf{M}] \mid \mathbf{M}$ is an indecomposable module of $\Omega\}$ is a basis of $\mathbf{K}(\Omega, 0)$. Due to Theorem 1.8, this means that $\{[e_\alpha] \mid \alpha \in R^+\}$ is a basis of $\mathbf{K}(\Omega, 0)$.

$\mathbf{K}(\Omega, 0)$ modulo the subgroup generated by the elements of the form $[\mathbf{V}'] + [\mathbf{V}''] - [\mathbf{V}]$ whenever there is an exact sequence $0 \to \mathbf{V}' \to \mathbf{V} \to \mathbf{V}'' \to 0$ is the Grothendieck group $\mathbf{K}(\Omega)$ of the category $\mathrm{Mod}(\Omega)$ of modules of Ω. Denote by $\phi : \mathbf{K}(\Omega, 0) \to \mathbf{K}(\Omega)$ the canonical epimorphism.

Consider the bilinear form $\langle\ ,\ \rangle : \mathbf{K}(\Omega, 0) \times \mathbf{K}(\Omega, 0) \to \mathbf{Z}$ such that, whenever $\mathbf{V}$ and $\mathbf{W}$ are modules of Ω, we have $\langle [\mathbf{V}], [\mathbf{W}] \rangle = \dim_F \mathrm{Hom}_\Omega(\mathbf{V}, \mathbf{W})$, where $\mathrm{Hom}_\Omega(\mathbf{V}, \mathbf{W})$ is the vector space of morphisms $\mathbf{V} \to \mathbf{W}$ in the category $\mathrm{Mod}(\Omega)$ of modules of Ω.

Let $\mathbf{V}''$ be an indecomposable module of Ω. If $\mathbf{V}''$ is nonprojective, then there is a unique, up to isomorphism, almost split sequence $0 \to \mathbf{V}' \xrightarrow{g} \mathbf{V} \xrightarrow{f} \mathbf{V}'' \to 0$. We then associate to $\mathbf{V}''$, the element $r_{\mathbf{V}''} = [\mathbf{V}'] + [\mathbf{V}''] - [\mathbf{V}]$ in $\mathbf{K}(\Omega, 0)$. If $\mathbf{V}''$

is projective, we define $r_{\mathbf{V}''} = [\mathbf{V}''] - [\underline{r}\mathbf{V}''] \in \mathbf{K}(\Omega, 0)$ where $\underline{r}\mathbf{V}''$ is the unique maximal submodule of $\mathbf{V}''$.

From now on, we will denote the element r_{e_α} of $\mathbf{K}(\Omega, 0)$ by r_α. Here $\alpha \in R^+$.

Proposition 2.9. (a) *For all $\alpha, \beta \in R^+$, we have*

$$\langle [e_\alpha], r_\beta \rangle = \begin{cases} 0, & \text{if } \alpha \neq \beta; \\ 1, & \text{if } \alpha = \beta. \end{cases}$$

(b) $\{r_\alpha \mid \alpha \in R^+\}$ *is a basis of* $\mathbf{K}(\Omega, 0)$.
(c) $\{r_\alpha \mid \alpha \in R^+ \setminus \mathcal{P}(\Omega)\}$ *is a basis of* $Ker(\phi)$
(d) *For each $x \in \mathbf{K}(\Omega, 0)$, we have*

$$x = \sum_{\alpha \in R^+} \langle [e_\alpha], x \rangle \ r_\alpha.$$

Proof. (a) Let $\mathbf{i} = (i_1, i_2, \dots, i_\nu)$ be a reduced expression of w_0 adapted to the quiver Ω. By Lemma 1.1 in [1], we get that

$$\langle [e_\alpha], r_\beta \rangle = \begin{cases} 0, & \text{if } \alpha \neq \beta; \\ \langle [e_\alpha], [e_\alpha] \rangle, & \text{if } \alpha = \beta. \end{cases}$$

Because $e_\alpha = \Phi^-_{i_1} \Phi^-_{i_2} \cdots \Phi^-_{i_{k-1}}(e_{i_k})$ for some k, $1 \leq k \leq \nu$, and the functors Φ^-_i give equivalences between appropriate full subcategories of modules, we get that

$$\langle [e_\alpha], [e_\alpha] \rangle = \dim_F \operatorname{Hom}_{\Omega_k}(e_{i_k}, e_{i_k}) = 1.$$

Thus (a) is proved.

(b) and (d) are simply Proposition 2.1 of [1] applied to our situation.

(c) is Theorem 2.3 of [1]. □

2.10. Γ_Ω comes equipped with a translation $\tau = D \circ Tr$ where Tr is the transpose (see chapter IV of [2] for the definition) and D is a duality (see chapter II of [2] for the definition). We will just list some of the properties of τ that are verified in our situation.

(a) If $\mathbf{P}$ is a projective module, then $\tau(\mathbf{P}) = 0$.
(b) If $\mathbf{V}$ and $\mathbf{W}$ are modules of Ω without projective summands, then $\mathbf{V}$ and $\mathbf{W}$ are isomorphic if and only if $\tau(\mathbf{V})$ and $\tau(\mathbf{W})$ are isomorphic.
(c) $\tau(\bigoplus_{i=1}^m \mathbf{V}(i))$ is isomorphic to $\bigoplus_{i=1}^m \tau(\mathbf{V}(i))$ where $\mathbf{V}(1), \mathbf{V}(2), \dots, \mathbf{V}(m)$ are modules of Ω.
(d) τ induces a bijection $[\mathbf{V}] \mapsto [\tau(\mathbf{V})]$ (also denote by τ) from the set of indecomposable nonprojective modules of Ω into the set of indecomposable noninjective modules of Ω with $Tr \circ D$ as inverse.
(e) If $\mathbf{V}$ is an indecomposable nonprojective module of Ω and $\Xi([\mathbf{V}]) = (k, i)$ for some $i \in \{1, 2, \dots, n\}$ and $k \in \mathbf{Z}$, where Ξ is the unique embedding of Γ_Ω into $\mathbf{Z}\Delta$ defined in 2.1, then $\Xi(\tau[\mathbf{V}]) = (k-1, i)$.
(f) If $\mathbf{W}$ is an indecomposable nonprojective module, then the set of vertices $[\mathbf{V}]$ of Γ_Ω such that $[\mathbf{V}] \to [\mathbf{W}]$ in Γ_Ω is equal to the set of vertices $[\mathbf{V}']$ of Γ_Ω such that $\tau[\mathbf{W}] \to [\mathbf{V}']$ in Γ_Ω.
(g) If $\alpha \in R^+ \setminus \mathcal{P}(\Omega)$, then, for the nonprojective indecomposable module e_α of Ω, there exists an almost split sequence $0 \to \tau(e_\alpha) \to \mathbf{V} \to e_\alpha \to 0$ whose middle term $\mathbf{V}$ is isomorphic to the direct sum $\bigoplus e_\beta$ of indecomposable modules e_β where the sum is over all the positive roots β such that $\beta \to \alpha$ in Γ_Ω and this way we get all almost split sequences of modules of Ω up to isomorphism.

3. The subset $\mathcal{C}_{\mathbf{i}}$

3.1. Let $\mathbf{i} = (i_1, i_2, \ldots, i_\nu)$ be a reduced expression of w_0. As in 16 of [11], consider the set $\mathcal{C}_{\mathbf{i}}$ of sequences $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathbf{N}^\nu$ with the following property: for any two indices $p < p'$ in $\{1, 2, \ldots, \nu\}$ such that $i_p = i_{p'} = i$ and $i_q \neq i$ whenever $p < q < p'$, we have

$$(*) \qquad \sum_q c_q \geq c_p + c_{p'}$$

(sum over all q with $p < q < p'$ such that i_q is joined by an edge to i in the Dynkin graph Δ).

3.2. For the rest of this section, Ω will be a fixed quiver with graph Δ and $\mathbf{i} = (i_1, i_2, \ldots, i_\nu)$ will be a fixed reduced expression of w_0 adapted to the quiver Ω.

3.3. For $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathbf{Z}^\nu$, we will denote by $\lambda_{\mathbf{i},\mathbf{c}}$ the unique homomorphism of $\mathbf{K}(\Omega, 0)$ into $\mathbf{Z}$ such that $\lambda_{\mathbf{i},\mathbf{c}}([e_{\alpha^{(j)}(\mathbf{i})}]) = c_j$ for all $j \in \{1, 2, \ldots, \nu\}$ and $\mathbf{c} \in \mathbf{Z}^\nu$. Note that $\lambda_{\mathbf{i},\mathbf{c}}$ is well defined because $\mathbf{K}(\Omega, 0)$ is a free abelian group with basis $\{[e_\alpha] \mid \alpha \in R^+\}$.

Lemma 3.4. (a) *The function* $\Lambda_{\mathbf{i}} : \mathbf{Z}^\nu \to \mathrm{Hom}(\mathbf{K}(\Omega, 0), \mathbf{Z})$ *defined by* $\mathbf{c} \mapsto \lambda_{\mathbf{i},\mathbf{c}}$ *is a well defined isomorphism of abelian groups (dependent on* $\mathbf{i}$*) whose inverse is given by* $\lambda \mapsto \mathbf{c} = (c_1, c_2, \ldots, c_\nu)$ *where* $c_j = \lambda([e_{\alpha^{(j)}(\mathbf{i})}])$ *for all* $j \in \{1, 2, \ldots, \nu\}$ *and all* $\lambda \in \mathrm{Hom}(\mathbf{K}(\Omega, 0), \mathbf{Z})$.

(b) *The image* $\Lambda_{\mathbf{i}}(\mathcal{C}_{\mathbf{i}})$ *of* $\mathcal{C}_{\mathbf{i}}$ *under* $\Lambda_{\mathbf{i}}$ *is the subset* $\mathcal{C}'_\Omega$ *of* $\mathrm{Hom}(\mathbf{K}(\Omega, 0), \mathbf{Z})$ *consisting of the homomorphisms* $\lambda : \mathbf{K}(\Omega, 0) \to \mathbf{Z}$ *such that*

$$\lambda([e_\alpha]) \geq 0 \text{ for all } \alpha \in R^+ \text{ and } \lambda(r_{\alpha'}) \leq 0 \text{ for all } \alpha' \in R^+ \setminus \mathcal{P}(\Omega).$$

Proof. (a) The proof is left to the reader. It follows easily from the fact that $\mathbf{K}(\Omega, 0)$ is a free abelian group with basis $\{[e_\alpha] \mid \alpha \in R^+\}$.

(b) If $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathcal{C}_{\mathbf{i}}$, we want to prove that $\lambda_{\mathbf{i},\mathbf{c}} \in \mathcal{C}'_\Omega$. If $\alpha \in R^+$, then $\alpha = \alpha^{(j)}(\mathbf{i})$ for some $j \in \{1, 2, \ldots, \nu\}$. Consequently, we get that $\lambda_{\mathbf{i},\mathbf{c}}([e_\alpha]) = \lambda_{\mathbf{i},\mathbf{c}}([e_{\alpha^{(j)}(\mathbf{i})}]) = c_j \geq 0$ for all $\alpha \in R^+$. If $\alpha' \in R^+ \setminus \mathcal{P}(\Omega)$, then $\tau([e_{\alpha'}]) = [e_\alpha]$ for some $\alpha \in R^+$ and $r_{\alpha'} = [e_{\alpha'}] + [e_\alpha] - \sum_{\beta \to \alpha'} [e_\beta]$ where the last summation is over all positive roots β such that $\beta \to \alpha'$ in Γ_Ω. This is due to 2.10 (g). By Theorem 2.5 and 2.10 (e), there are two indices $p < p'$ in $\{1, 2, \ldots, \nu\}$ such that $\alpha = \alpha^{(p)}(\mathbf{i})$, $\alpha' = \alpha^{(p')}(\mathbf{i})$, $i_p = i_{p'} = i$ and $i_q \neq i$ whenever $p < q < p'$. By Theorem 2.5 and 2.10 (f), if $\beta \in R^+$ is such that $\beta \to \alpha'$ in Γ_Ω, then $\alpha \to \beta$ in Γ_Ω, $\beta = \alpha^{(q)}(\mathbf{i})$ for some $p < q < p'$ and $i_q = \rho_\Omega(\beta)$ is joined by an edge to $i = \rho_\Omega(\alpha)$ in Δ. Conversely, if $q \in \{1, 2, \ldots, \nu\}$ is such that $p < q < p'$, i_q is joined by an edge to i in Δ, then $\beta = \alpha^{(q)}(\mathbf{i})$ is such that $\beta \to \alpha'$ in Γ_Ω. Because $\mathbf{c} \in \mathcal{C}_{\mathbf{i}}$, we get that

$$r_{\alpha'} = [e_{\alpha^{(p')}(\mathbf{i})}] + [e_{\alpha^{(p)}(\mathbf{i})}] - \sum_q [e_{\alpha^{(q)}(\mathbf{i})}] \quad \text{and} \quad \lambda_{\mathbf{i},\mathbf{c}}(r_{\alpha'}) = c_{p'} + c_p - \sum_q c_q \leq 0$$

where the summations $\sum_q$ are over all q such that $p < q < p'$ and i_q is joined to i by an edge in Δ. So $\lambda_{\mathbf{i},\mathbf{c}} \in \mathcal{C}'_\Omega$ if $\mathbf{c} \in \mathcal{C}_{\mathbf{i}}$.

Reciprocally if $\lambda \in \mathcal{C}'_\Omega$, we want to prove that $\Lambda_{\mathbf{i}}^{-1}(\lambda) = \mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathbf{Z}^\nu$, defined by $c_j = \lambda([e_{\alpha^{(j)}(\mathbf{i})}])$ for $j \in \{1, 2, \ldots, \nu\}$, is an element of $\mathcal{C}_{\mathbf{i}}$. First because $\lambda \in \mathcal{C}'_\Omega$, we get that $c_j \geq 0$ for all j and $\mathbf{c} \in \mathbf{N}^\nu$. Secondly for any two indices $p < p'$

in $\{1, 2, \ldots, \nu\}$ such that $i_p = i_{p'} = i$ and $i_q \neq i$ whenever $p < q < p'$, we have for $\alpha = \alpha^{(p)}(\mathbf{i})$, $\alpha' = \alpha^{(p')}(\mathbf{i})$ that $e_{\alpha'}$ is nonprojective, $[e_\alpha] = \tau([e_{\alpha'}])$ by 2.10 and

$$r_{\alpha'} = [e_{\alpha'}] + [e_\alpha] - \sum_{\beta \to \alpha'} [e_\beta] = [e_{\alpha^{(p')}(\mathbf{i})}] + [e_{\alpha^{(p)}(\mathbf{i})}] - \sum_q [e_{\alpha^{(q)}(\mathbf{i})}]$$

by the same argument as above. Here the summation $\sum_{\beta \to \alpha'}$ is over all positive roots β such that $\beta \to \alpha'$ in Γ_Ω and the summation $\sum_q$ is over all q such that $p < q < p'$ and i_q is joined by an edge to i in Δ. Thus $c_{p'} + c_p - \sum_q c_q = \lambda(r_{\alpha'}) \leq 0$. So $\mathbf{c} = \Lambda_{\mathbf{i}}^{-1}(\lambda) \in \mathcal{C}_{\mathbf{i}}$ if $\lambda \in \mathcal{C}'_\Omega$.

From now on, we will study $\mathcal{C}'_\Omega$ rather that $\mathcal{C}_{\mathbf{i}}$. By the previous lemma, this is equivalent to studying $\mathcal{C}_{\mathbf{i}}$.

To define $\mathcal{C}'_\Omega$, there are $(2\nu - n)$ inequalities: ν of them of the form $\lambda([e_\alpha]) \geq 0$ for $\alpha \in R^+$ and $(\nu - n)$ of the form $\lambda(r_{\alpha'}) \leq 0$ for $\alpha' \in R^+ \setminus \mathcal{P}(\Omega)$. The next proposition shows that we only need ν inequalities. □

Proposition 3.5. *$\mathcal{C}'_\Omega$ is equal to the subset of elements λ of $\mathrm{Hom}(\mathbf{K}(\Omega, 0), \mathbf{Z})$ such that*

$$\lambda([e_\alpha]) \geq 0 \textit{ for all } \alpha \in B \textit{ and } \lambda(r_{\alpha'}) \leq 0 \textit{ for all } \alpha' \in R^+ \setminus \mathcal{P}(\Omega).$$

Recall that B is the set of simple roots.

Proof. If $\lambda \in \mathrm{Hom}(\mathbf{K}(\Omega, 0), \mathbf{Z})$ is such that $\lambda([e_\alpha]) \geq 0$ for all $\alpha \in B$ and $\lambda(r_{\alpha'}) \leq 0$ for all $\alpha' \in R^+ \setminus \mathcal{P}(\Omega)$, then we want to prove that $\lambda([e_\alpha]) \geq 0$ for all $\alpha \in R^+$. We will do this by induction on the height $ht(\alpha) = \sum_{i=1}^n b_i$ of the positive root $\alpha = \sum_{i=1}^n b_i \alpha_i$.

If $ht(\alpha) = 1$, then $\alpha \in B$ and we have by hypothesis $\lambda([e_\alpha]) \geq 0$. If $ht(\alpha) > 1$, assume that the result is true for all positive roots with height strictly smaller than $ht(\alpha)$. Because $ht(\alpha) > 1$, then α is not a simple root and e_α is not a simple module of Ω. Let $\mathbf{V}'$ be a nonzero proper submodule of e_α and $\mathbf{V}''$ be the quotient $e_\alpha/\mathbf{V}'$. Because e_α is not simple, there exist such a proper submodule $\mathbf{V}' \neq 0$ and we get $\mathbf{V}'' \neq 0$. Both $\mathbf{V}'$ and $\mathbf{V}''$ are sums of indecomposable modules whose dimensions are positive roots with height smaller than $ht(\alpha)$. Consider $x = [\mathbf{V}'] + [\mathbf{V}''] - [e_\alpha] \in \mathbf{K}(\Omega, 0)$. Because $0 \to \mathbf{V}' \to e_\alpha \to \mathbf{V}'' \to 0$ is an exact sequence of modules of Ω, then x belongs to $\mathrm{Ker}(\phi)$ and

$$x = \sum_{\alpha' \in R^+ \setminus \mathcal{P}(\Omega)} \langle [e_{\alpha'}], x \rangle \; r_{\alpha'}.$$

Note that $\langle [e_{\alpha'}], x \rangle \geq 0$ for all $\alpha \in R^+ \setminus \mathcal{P}(\Omega)$. In fact, by applying the functor $\mathrm{Hom}_\Omega(e_{\alpha'}, \cdot)$ to the exact sequence $0 \to \mathbf{V}' \to e_\alpha \to \mathbf{V}'' \to 0$, we get the exact sequence

$$0 \to \mathrm{Hom}_\Omega(e_{\alpha'}, \mathbf{V}') \to \mathrm{Hom}_\Omega(e_{\alpha'}, e_\alpha) \to \mathrm{Hom}_\Omega(e_{\alpha'}, \mathbf{V}'')$$

and consequently

$$\dim_F(\mathrm{Hom}_\Omega(e_{\alpha'}, \mathbf{V}'')) \geq \dim_F(\mathrm{Hom}_\Omega(e_{\alpha'}, e_\alpha)) - \dim_F(\mathrm{Hom}_\Omega(e_{\alpha'}, \mathbf{V}')).$$

This last inequality means that $\langle [e_{\alpha'}], x \rangle \geq 0$.

Thus we have

$$\lambda(x) = \lambda([\mathbf{V}'] + [\mathbf{V}''] - [e_\alpha]) = \lambda\left(\sum_{\alpha' \in R^+ \setminus \mathcal{P}(\Omega)} \langle [e_{\alpha'}], x\rangle \; r_{\alpha'}\right)$$
$$= \sum_{\alpha' \in R^+ \setminus \mathcal{P}(\Omega)} \langle [e_{\alpha'}], x\rangle \; \lambda(r_{\alpha'}) \leq 0,$$

because $\lambda(r_{\alpha'}) \leq 0$ and $\langle [e_{\alpha'}], x\rangle \geq 0$ in the summation. So $\lambda([\mathbf{V}'] + [\mathbf{V}'']) \leq \lambda([e_\alpha])$. Because $\mathbf{V}'$ and $\mathbf{V}''$ are sums of indecomposable modules whose dimensions are positive roots with height smaller than $ht(\alpha)$, we get that $0 \leq \lambda([\mathbf{V}'])$ and $0 \leq \lambda([\mathbf{V}''])$ by induction hypothesis. We can conclude that $0 \leq \lambda([e_\alpha])$. This proves the proposition. □

Lemma 3.6. *For each $\alpha \in R^+$, define the element $x_\alpha \in \mathbf{K}(\Omega, 0)$ by*

$$x_\alpha = \begin{cases} [e_{\alpha_i}], & \text{if } \alpha = \dim(\mathbf{P}(i)) \in \mathcal{P}(\Omega) \text{ for some } i; \\ r_\alpha, & \text{if } \alpha \in R^+ \setminus \mathcal{P}(\Omega). \end{cases}$$

Then $\{x_\alpha \mid \alpha \in R^+\}$ is a basis of $\mathbf{K}(\Omega, 0)$.

Proof. First, note that

$$\dim_F \mathrm{Hom}_\Omega(\mathbf{P}(k), e_{\alpha_i}) = \begin{cases} 1, & \text{if } k = i; \\ 0, & \text{if } k \neq i. \end{cases}$$

The proof of this follows easily from the description of the projective indecomposable modules of Ω in 1.9 and the simple module e_{α_i}. In fact, if $k \neq i$, then either there is no path from k to i in Ω and, in this case $\mathbf{P}(k)_i = 0$ and clearly $\mathrm{Hom}_\Omega(\mathbf{P}(k), e_{\alpha_i}) = 0$, or there is a path from k to i in Ω and, in this case $\mathbf{P}(k)_i \cong F$, $\mathbf{P}(k)_j \cong F$, where j is the unique vertex in this path from k to i such that $j \to i$ in Ω, and $\mathbf{P}(k)_j \to \mathbf{P}(k)_i$ is an invertible homomorphism, but if there is a homomorphism from $\mathbf{P}(k)$ to e_{α_i} such that $\mathbf{P}(k)_i \to (e_{\alpha_i})_i$ is invertible, then we get a contradiction by considering the induced map $\mathbf{P}(k)_j \to (e_{\alpha_i})_i$. It is invertible being the composition $\mathbf{P}(k)_j \to \mathbf{P}(k)_i \to (e_{\alpha_i})_i$ and it is 0 being the composition $\mathbf{P}(k)_j \to (e_{\alpha_i})_j = 0 \to (e_{\alpha_i})_i$. Thus for all $k \neq i$, we have that $\mathrm{Hom}_\Omega(\mathbf{P}(k), e_{\alpha_i}) = 0$.

If $k = i$, then there is a linear map from $\mathbf{P}(i)_i$ to $(e_{\alpha_i})_i$ sending the constant path at i (the basis element of $\mathbf{P}(i)_i$) to $1 \in F = (e_{\alpha_i})_i$ and being $0, \mathbf{P}(i)_j \to (e_{\alpha_i})_j$ for $j \neq i$. This gives a basis of $\mathrm{Hom}_\Omega(\mathbf{P}(i), e_{\alpha_i})$ and we get that $\dim_F \mathrm{Hom}_\Omega(\mathbf{P}(i), e_{\alpha_i}) = 1$.

By Proposition 2.9 (d) and the above remark, we get that

$$[e_{\alpha_i}] = \sum_{\beta \in R^+} \langle [e_\beta], [e_{\alpha_i}]\rangle \; r_\beta$$
$$= r_{\dim(\mathbf{P}(i))} + \sum_{\beta \in R^+ \setminus \mathcal{P}(\Omega)} \langle [e_\beta], [e_{\alpha_i}]\rangle \; r_\beta.$$

From this and Proposition 2.9 (b), we get that $\{[e_\alpha] \mid \alpha \in B\} \cup \{r_\alpha \mid \alpha \in R^+ \setminus \mathcal{P}(\Omega)\}$ is a basis of $\mathbf{K}(\Omega, 0)$. So the lemma is proved. □

3.7. For $\alpha \in R^+$, define ϵ_α to be equal to 1 if $\alpha \in \mathcal{P}(\Omega)$ and -1 if $\alpha \in R^+ \setminus \mathcal{P}(\Omega)$.

Theorem 3.8. (a) *For each $\alpha \in R^+$, there is a unique well defined homomorphism $\lambda_\alpha \in \mathrm{Hom}(\mathbf{K}(\Omega, 0), \mathbf{Z})$ such that*

$$\lambda_\alpha(x_\beta) = \begin{cases} 0, & \text{if } \beta \neq \alpha; \\ \epsilon_\alpha, & \text{if } \beta = \alpha. \end{cases}$$

Moreover, $\{\lambda_\alpha \mid \alpha \in R^+\}$ is a basis of $\mathrm{Hom}(\mathbf{K}(\Omega, 0), \mathbf{Z})$.

(b) *$\lambda_\alpha \in \mathcal{C}'_\Omega$ for all $\alpha \in R^+$ and each $\lambda \in \mathcal{C}'_\Omega$ is a linear combination of the λ_α with non-negative coefficients. In fact, $\lambda = \sum_{\alpha \in R^+} \epsilon_\alpha \lambda(x_\alpha)\lambda_\alpha$ where $\epsilon_\alpha \lambda(x_\alpha) \in \mathbf{N}$ for all $\alpha \in R^+$.*

(c) *If $\alpha \in \mathcal{P}(\Omega)$, then λ_α is the homomorphism $\lambda_\alpha = \langle [e_\alpha], \cdot \rangle : \mathbf{K}(\Omega, 0) \to \mathbf{Z}$ defined by $x \mapsto \langle [e_\alpha], x \rangle$ for all $x \in \mathbf{K}(\Omega, 0)$. In particular, if $\alpha = \dim(\mathbf{P}(i))$ for some $i \in \{1, 2, \dots, n\}$ and $\beta = \sum_{k=1}^n b_k \alpha_k \in R^+$, then $\lambda_\alpha([e_\beta]) = b_i$.*

(d) *If $\alpha \in R^+ \setminus \mathcal{P}(\Omega)$, then λ_α is the homomorphism*

$$\lambda_\alpha = \left(\sum_{i=1}^n \langle [e_\alpha], [e_{\alpha_i}] \rangle \ \langle [\mathbf{P}(i)], \cdot \rangle \right) - \langle [e_\alpha], \cdot \rangle : \mathbf{K}(\Omega, 0) \to \mathbf{Z}$$

defined by

$$x \mapsto \left(\sum_{i=1}^n \langle [e_\alpha], [e_{\alpha_i}] \rangle \ \langle [\mathbf{P}(i)], x \rangle \right) - \langle [e_\alpha], x \rangle$$

for all $x \in \mathbf{K}(\Omega, 0)$.

Proof. (a) Because $\{x_\beta \mid \beta \in R^+\}$ is a basis of $\mathbf{K}(\Omega, 0)$, we easily get that the λ_α are unique well defined homomorphisms. Each $\lambda \in \mathrm{Hom}(\mathbf{K}(\Omega, 0), \mathbf{Z})$ can be written uniquely as the linear combination $\lambda = \sum_{\alpha \in R^+} \epsilon_\alpha \lambda(x_\alpha)\lambda_\alpha$. To see this, we compute

$$\sum_{\alpha \in R^+} \epsilon_\alpha \lambda(x_\alpha)\lambda_\alpha(x_\beta) = \epsilon_\beta \lambda(x_\beta)\epsilon_\beta = \lambda(x_\beta)$$

for all $\beta \in R^+$ and consequently $\lambda = \sum_{\alpha \in R^+} \epsilon_\alpha \lambda(x_\alpha)\lambda_\alpha$ because $\{x_\beta \mid \beta \in R^+\}$ is a basis of $\mathbf{K}(\Omega, 0)$. This proves (a).

(b) First we will prove that $\lambda_\alpha \in \mathcal{C}'_\Omega$ for all $\alpha \in R^+$. If $\alpha = \dim(\mathbf{P}(i)) \in \mathcal{P}(\Omega)$ for some $i \in \{1, 2, \dots, n\}$, then we have, for $\beta \in B$, that

$$\lambda_\alpha([e_\beta]) = \begin{cases} 1, & \text{if } \beta = \alpha_i; \\ 0, & \text{if } \beta \neq \alpha_i; \end{cases}$$

and $\lambda_\alpha(r_{\alpha'}) = 0$ for all $\alpha' \in R^+ \setminus \mathcal{P}(\Omega)$. If $\alpha \in R^+ \setminus \mathcal{P}(\Omega)$, then $\lambda_\alpha([e_\beta]) = 0$ for all $\beta \in B$ and, for $\alpha' \in R^+ \setminus \mathcal{P}(\Omega)$,

$$\lambda_\alpha(r_{\alpha'}) = \begin{cases} -1, & \text{if } \alpha' = \alpha; \\ 0, & \text{if } \alpha' \neq \alpha. \end{cases}$$

Due to Proposition 3.5, $\lambda_\alpha \in \mathcal{C}'_\Omega$ for all $\alpha \in R^+$.

If $\lambda \in \mathcal{C}'_\Omega$, we see in the proof of (a) that $\lambda = \sum_{\alpha \in R^+} \epsilon_\alpha \lambda(x_\alpha)\lambda_\alpha$. We must prove that $\epsilon_\alpha \lambda(x_\alpha) \in \mathbf{N}$. If $\alpha = \dim(\mathbf{P}(i)) \in \mathcal{P}(\Omega)$ for some $i \in \{1, 2, \dots, n\}$, then $\epsilon_\alpha \lambda(x_\alpha) = \lambda([e_{\alpha_i}]) \geq 0$ because $\lambda \in \mathcal{C}'_\Omega$. If $\alpha \in R^+ \setminus \mathcal{P}(\Omega)$, then $\epsilon_\alpha \lambda(x_\alpha) = -\lambda(r_\alpha) \geq 0$, because $\lambda \in \mathcal{C}'_\Omega$. In all cases, we have that $\epsilon_\alpha \lambda(x_\alpha) \in \mathbf{N}$.

(c) If $\alpha = \dim(\mathbf{P}(i)) \in \mathcal{P}(\Omega)$ for some $i \in \{1, 2, \ldots, n\}$, then we will first prove that, for $\beta \in R^+$, we have

$$\lambda_\alpha(r_\beta) = \begin{cases} 1, & \text{if } \beta = \alpha; \\ 0, & \text{if } \beta \neq \alpha. \end{cases}$$

If $\beta \in R^+ \setminus \mathcal{P}(\Omega)$, then we have $\lambda_\alpha(r_\beta) = 0$ by definition of λ_α. If $\beta = \dim(\mathbf{P}(k)) \in \mathcal{P}(\Omega)$ for some $k \in \{1, 2, \ldots, n\}$, then either β is a simple root or it is not. If β is a simple root, i.e. $\beta = \alpha_k$, then we have that $r_\beta = [e_{\alpha_k}]$ and

$$\lambda_\alpha([e_{\alpha_k}]) = \begin{cases} 1, & \text{if } k = i \ (\text{i.e.} \quad \beta = \alpha); \\ 0, & \text{if } k \neq i \ \ (\text{i.e.} \quad \beta \neq \alpha). \end{cases}$$

If β is not a simple root, then we have a short exact sequence

$$0 \to \underline{r}\mathbf{P}(k) \to \mathbf{P}(k) \to e_{\alpha_k} \to 0$$

of modules of Ω where $\underline{r}\mathbf{P}(k)$ is the unique maximal submodule of $\mathbf{P}(k)$ and consequently the element $x = [\underline{r}\mathbf{P}(k)] - [\mathbf{P}(k)] + [e_{\alpha_k}] = [e_{\alpha_k}] - r_\beta$ belongs to $\mathrm{Ker}(\phi)$. Thus x is a sum of the form $x = \sum_{\alpha' \in R^+ \setminus \mathcal{P}(\Omega)} \langle [e_{\alpha'}], x \rangle \ r_{\alpha'}$ and

$$\begin{aligned} \lambda_\alpha(x) = \lambda_\alpha([e_{\alpha_k}] - r_\beta) &= \lambda_\alpha \left(\sum_{\alpha' \in R^+ \setminus \mathcal{P}(\Omega)} \langle [e_{\alpha'}], x \rangle \ r_{\alpha'} \right) \\ &= \sum_{\alpha' \in R^+ \setminus \mathcal{P}(\Omega)} \langle [e_{\alpha'}], x \rangle \ \lambda_\alpha(r_{\alpha'}) = 0. \end{aligned}$$

So

$$\lambda_\alpha(r_\beta) = \lambda_\alpha([e_{\alpha_k}]) = \begin{cases} 1, & \text{if } k = i \ (\text{i.e. } \beta = \alpha); \\ 0, & \text{if } k \neq i \ (\text{i.e. } \beta \neq \alpha). \end{cases}$$

We can now use what we have just proved. By Proposition 2.9 (d), we have that $x = \sum_{\beta \in R^+} \langle [e_\beta], x \rangle \ r_\beta$ for all $x \in \mathbf{K}(\Omega, 0)$. By applying λ_α, we get that

$$\begin{aligned} \lambda_\alpha(x) &= \lambda_\alpha \left(\sum_{\beta \in R^+} \langle [e_\beta], x \rangle \ r_\beta \right) \\ &= \sum_{\beta \in R^+} \langle [e_\beta], x \rangle \ \lambda_\alpha(r_\beta) = \langle [e_\alpha], x \rangle \end{aligned}$$

for all $x \in \mathbf{K}(\Omega, 0)$.

To complete the proof of (c), we need to prove $\lambda_{\dim(\mathbf{P}(i))}([e_\beta]) = \langle [\mathbf{P}(i)], [e_\beta] \rangle = b_i$ where $\beta = \sum_{k=1}^n b_k \alpha_k$. This is proved by Ringel in the lemma of section 2.4 in [16]. This finishes the proof of (c).

(d) Recall that we have shown in the proof of Lemma 3.6 that

$$\langle [\mathbf{P}(k)], [e_{\alpha_i}] \rangle = \begin{cases} 1, & \text{if } k = i; \\ 0, & \text{if } k \neq i. \end{cases}$$

If $\alpha \in R^+ \setminus \mathcal{P}(\Omega)$, then we will first show that

$$\lambda_\alpha(r_\beta) = \begin{cases} -1, & \text{if } \beta \in R^+ \setminus \mathcal{P}(\Omega) \text{ and } \beta = \alpha; \\ 0, & \text{if } \beta \in R^+ \setminus \mathcal{P}(\Omega) \text{ and } \beta \neq \alpha; \\ \langle [e_\alpha], [e_{\alpha_i}] \rangle, & \text{if } \beta = \dim(\mathbf{P}(i)) \in \mathcal{P}(\Omega) \text{ for some } i. \end{cases}$$

For $\beta \in R^+ \setminus \mathcal{P}(\Omega)$, we have by definition of λ_α that

$$\lambda_\alpha(r_\beta) = \begin{cases} -1, & \text{if } \beta = \alpha; \\ 0, & \text{if } \beta \neq \alpha. \end{cases}$$

Now we consider the root $\beta = \dim(\mathbf{P}(i)) \in \mathcal{P}(\Omega)$ for some $i \in \{1, 2, \dots, n\}$. By Proposition 2.9 (d) and the above remark, we get that

$$\begin{aligned} [e_{\alpha_i}] &= \sum_{\alpha' \in R^+} \langle [e_{\alpha'}], [e_{\alpha_i}] \rangle \; r_{\alpha'} = \sum_{\alpha' \in \mathcal{P}(\Omega)} \langle [e_{\alpha'}], [e_{\alpha_i}] \rangle \; r_{\alpha'} + \sum_{\alpha' \in R^+ \setminus \mathcal{P}(\Omega)} \langle [e_{\alpha'}], [e_{\alpha_i}] \rangle \; r_{\alpha'} \\ &= r_{\dim(\mathbf{P}(i))} + \sum_{\alpha' \in R^+ \setminus \mathcal{P}(\Omega)} \langle [e_{\alpha'}], [e_{\alpha_i}] \rangle \; r_{\alpha'}. \end{aligned}$$

By applying λ_α and because $\lambda_\alpha([e_{\alpha_i}]) = 0$, we get that

$$\begin{aligned} 0 &= \lambda_\alpha(r_{\dim(\mathbf{P}(i))}) + \lambda_\alpha \left(\sum_{\alpha' \in R^+ \setminus \mathcal{P}(\Omega)} \langle [e_{\alpha'}], [e_{\alpha_i}] \rangle \; r_{\alpha'} \right) \\ &= \lambda_\alpha(r_{\dim(\mathbf{P}(i))}) + \sum_{\alpha' \in R^+ \setminus \mathcal{P}(\Omega)} \langle [e_{\alpha'}], [e_{\alpha_i}] \rangle \lambda_\alpha(r_{\alpha'}) \\ &= \lambda_\alpha(r_{\dim(\mathbf{P}(i))}) + (-1) \langle [e_\alpha], [e_{\alpha_i}] \rangle. \end{aligned}$$

Thus $\lambda_\alpha(r_{\dim(\mathbf{P}(i))}) = \langle [e_\alpha], [e_{\alpha_i}] \rangle$.

We can now use this. By Proposition 2.9 (d), we have for all $x \in \mathbf{K}(\Omega, 0)$ that $x = \sum_{\beta \in R^+} \langle [e_\beta], x \rangle r_\beta$. By applying λ_α, we get that

$$\begin{aligned} \lambda_\alpha(x) = \lambda_\alpha \left(\sum_{\beta \in R^+} \langle [e_\beta], x \rangle \; r_\beta \right) &= \sum_{\beta \in R^+} \langle [e_\beta], x \rangle \; \lambda_\alpha(r_\beta) \\ &= \left(\sum_{i=1}^{n} \langle [e_\alpha], [e_{\alpha_i}] \rangle \; \langle [\mathbf{P}(i)], x \rangle \right) - \langle [e_\alpha], x \rangle. \end{aligned}$$

This proves (d). □

3.9. The previous theorem can easily be applied to compute the values of λ_α on $[e_\beta]$ for all $\alpha, \beta \in R^+$. In other words, we can easily get $\Lambda_{\mathbf{i}}^{-1}(\lambda_\alpha) = (c_1, c_2, \dots, c_\nu)$, because $c_j = \lambda_\alpha([e_{\alpha^{(j)}(\mathbf{i})}])$.

After recalling the notion of additive functions on $\mathbf{Z}\Delta$ as defined in 6.5 of [7], we will also recall how these functions can be used to compute $\lambda_\alpha([e_\beta])$ for all $\alpha, \beta \in R^+$. We will later illustrate this process in an example.

An integer-valued function δ on the set of vertices of $\mathbf{Z}\Delta$ is said to be additive if, for each vertex x, it satisfies the equation

$$\delta(x) + \delta(\tau(x)) = \sum_{y \to x} \delta(y)$$

where the sum is over all arrows $y \to x$ in $\mathbf{Z}\Delta$.

A slice of $\mathbf{Z}\Delta$ is any connected full subquiver of $\mathbf{Z}\Delta$ which contains a unique representative of the vertices (z,i), $z \in \mathbf{Z}$, for each $i \in \{1,2,\ldots,n\}$. For each vertex x of $\mathbf{Z}\Delta$, there is a unique well determined slice admitting x as its unique source and we will call it the slice starting at x.

It is easy to verify that an additive function δ is uniquely determined by its values on a slice and these values can be chosen arbitrarily. We will denote by δ_x the unique additive function which has value 1 on the slice starting at x and we will call it the additive function starting at x. It is possible to prove that if $\mathcal{S}$ is a slice through x and $y \in \mathcal{S}$, then $\delta_x(y) = 1$ or 0 according to whether or not there is a path from x to y within $\mathcal{S}$.

If $\alpha \in \mathcal{P}(\Omega)$, then the homomorphism $\lambda_\alpha = \langle [e_\alpha], \cdot \rangle \colon \mathbf{K}(\Omega,0) \to \mathbf{Z}$ of Theorem 3.8 (c) has the following values on the basis $\{[e_\beta] \mid \beta \in R^+\}$ of $\mathbf{K}(\Omega,0)$:

$$\lambda_\alpha([e_\beta]) = \langle [e_\alpha],[e_\beta]\rangle = \delta_{\Xi([e_\alpha])}(\Xi([e_\beta])),$$

where Ξ is the unique embedding of Γ_Ω into $\mathbf{Z}\Delta$ given in 2.1 and $\delta_{\Xi([e_\alpha])}$ is the additive function on $\mathbf{Z}\Delta$ starting at $\Xi([e_\alpha])$.

We can see this as follows. Because e_α is projective, $\langle [e_\alpha], \cdot\rangle$ is such that $\langle [e_\alpha],[\mathbf{V}']\rangle + \langle [e_\alpha],[\mathbf{V}'']\rangle - \langle [e_\alpha],[\mathbf{V}]\rangle = 0$ for all short exact sequences $0 \to \mathbf{V}' \to \mathbf{V} \to \mathbf{V}'' \to 0$ of modules of Ω. In particular, this is true for all almost split sequences of modules of Ω and this means that the function f defined on $\{\Xi([e_\beta]) \mid \beta \in R^+\}$ by $f(\Xi([e_\beta])) = \langle [e_\alpha],[e_\beta]\rangle$ is the restriction of an additive function on $\mathbf{Z}\Delta$. By using the description of the projective indecomposable modules of Ω, it is easy to prove that for $i,j \in \{1,2,\ldots,n\}$, we have

$$\langle \mathbf{P}(i),\mathbf{P}(j)\rangle = \begin{cases} 1, & \text{if there is a path in } \Omega \text{ from } j \text{ to } i;\\ 0, & \text{otherwise.}\end{cases}$$

From this we get that the above additive function δ is the additive function $\delta_{\Xi([\mathbf{P}(i)])}$ for $\alpha = \dim(\mathbf{P}(i))$.

If $\alpha \in R^+ \setminus \mathcal{P}(\Omega)$, then the homomorphism $\langle [e_\alpha],\cdot\rangle \colon \mathbf{K}(\Omega,0) \to \mathbf{Z}$ that appears in the formula of λ_α in Theorem 3.8 (d) has the following values on the basis $\{[e_\beta] \mid \beta \in R^+\}$ of $\mathbf{K}(\Omega,0)$:

$$\langle [e_\alpha],[e_\beta]\rangle = \begin{cases} \delta_{\Xi([e_\alpha])}(\Xi([e_\beta])), & \text{if } \alpha \prec_\Omega \beta;\\ 0, & \text{otherwise;}\end{cases}$$

where $\delta_{\Xi([e_\alpha])}$ is the additive function on $\mathbf{Z}\Delta$ starting at $\Xi([e_\alpha])$ and $\prec_\Omega$ has been defined in 2.4.

We can see this as follows. For $\alpha,\beta \in R^+$ and $\alpha \neq \beta$, then any $f \in \mathrm{Hom}_\Omega(e_\alpha,e_\beta)$ is a sum of compositions of irreducible morphisms between indecomposable modules of Ω because of Theorem 7.8 and Exercise 7 of chapter V of [2]. If $\langle [e_\alpha],[e_\beta]\rangle \neq 0$, then there exists a nonzero $f \in \mathrm{Hom}_\Omega(e_\alpha,e_\beta)$ and consequently by expressing this f as a sum of compositions of irreducible morphisms, we see that there is a path in Γ_Ω from $[e_\alpha]$ to $[e_\beta]$ corresponding to one of these nonzero compositions of irreducible morphisms. Thus $\alpha \prec_\Omega \beta$.

If $\alpha \prec_\Omega \beta$, we would like to show that

$$\langle [e_\alpha],[e_\beta]\rangle = \delta_{\Xi([e_\alpha])}(\Xi([e_\beta])).$$

We can fix an irreducible expression $\mathbf{i} = (i_1,i_2,\ldots,i_\nu)$ of w_0 adapted to Ω. Then $\alpha = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k})$, $\beta = s_{i_1}s_{i_2}\cdots s_{i_{k'-1}}(\alpha_{i_{k'}})$ with $k < k'$, because $\alpha \prec_\Omega \beta$ and by using Theorem 2.5. We have $e_\alpha = \Phi^-_{i_1}\Phi^-_{i_2}\cdots\Phi^-_{i_{k-1}}(e_{i_k})$, $e_\beta = \Phi^-_{i_1}\Phi^-_{i_2}\cdots$

$\Phi^{-}_{i_{k'-1}}(e_{i_{k'}})$ and $\mathrm{Hom}_\Omega(e_\alpha, e_\beta) \simeq \mathrm{Hom}_{\Omega_k}(e_{i_k}, \Phi^{-}_{i_k} \cdots \Phi_{i_{k'-1}}(e_{i_{k'}}))$ using the fact that the Φ^{-}_{i} are equivalences between appropriate subcategories of modules. But e_{i_k} is a projective indecomposable module of Ω_k and we can then use the same argument as above when α is projective to see that $\dim_F(\mathrm{Hom}_{\Omega_k}(e_{i_k}, \cdot))$ can be described by an additive function. Analysing the relation between the Auslander-Reiten quivers of Ω and Ω_k (see for example Lemma 2.10 in [3]) we can conclude that $\langle e_\alpha, e_\beta\rangle = \delta_{\Xi([e_\alpha])}(\Xi([e_\beta]))$ when $\alpha \prec_\Omega \beta$.

For $\alpha \in R^+ \setminus \mathcal{P}(\Omega)$, we can use this description of $\langle [e_\alpha], \cdot\rangle$ to determine the values $\langle [e_\alpha], [e_{\alpha_i}]\rangle$ for $i = 1, 2, \ldots, n$. Note that $\alpha \prec_\Omega \alpha_i$ if $\langle [e_\alpha], [e_{\alpha_i}]\rangle \neq 0$. Denote $I_\Omega(\alpha) = \{1 \leq i \leq n \mid \alpha \prec_\Omega \alpha_i\}$. Thus if $\alpha \prec_\Omega \beta$, we have

$$\lambda_\alpha([e_\beta]) = \left(\sum_{i \in I_\Omega(\alpha)} \delta_{\Xi([e_\alpha])}(\Xi([e_{\alpha_i}]))\ \delta_{\Xi([\mathbf{P}(i)])}(\Xi([e_\beta])) \right) - \delta_{\Xi([e_\alpha])}(\Xi([e_\beta])).$$

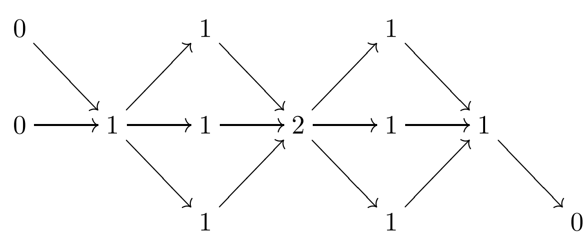
Otherwise, we have

$$\lambda_\alpha([e_\beta]) = \left(\sum_{i \in I_\Omega(\alpha)} \delta_{\Xi([e_\alpha])}(\Xi([e_{\alpha_i}]))\ \delta_{\Xi([\mathbf{P}(i)])}(\Xi([e_\beta])) \right).$$

Example 3.10. Let Ω be the quiver with underlying graph of type D_4 of Example 2.3. If we consider first the root $\alpha = \alpha_2 + \alpha_3 + \alpha_4$ represented by 01^{1}_{1} in the Auslander-Reiten quiver Γ_Ω, then $e_\alpha = \mathbf{P}(2)$ and $\alpha \in \mathcal{P}(\Omega)$. The slice starting at 01^{1}_{1} goes through the vertices

$$11^{0}_{0}, \quad 01^{1}_{1}, \quad 01^{0}_{1}, \quad 01^{1}_{0}.$$

The values $\lambda_\alpha([e_\beta]) = \langle [\mathbf{P}(2)], [e_\beta]\rangle$ given by Theorem 3.8 (c) are written below at the position of β in Γ_Ω:

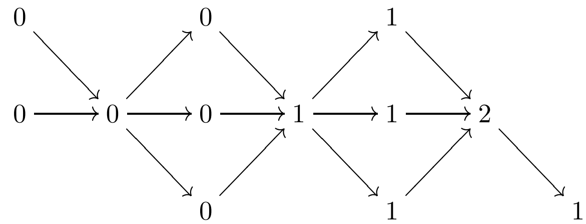


They are given by the restriction of the additive function starting at 01^{1}_{1}.

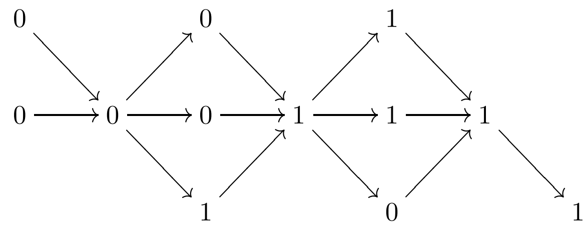
We now consider the root $\alpha = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ represented by 12^{1}_{1} in the Auslander-Reiten quiver Γ_Ω. In this case, $\alpha \in R^+ \setminus \mathcal{P}(\Omega)$. The slice starting at 12^{1}_{1} goes through the vertices

$$01^{0}_{0}, \quad 12^{1}_{1}, \quad 11^{1}_{0}, \quad 11^{0}_{1}.$$

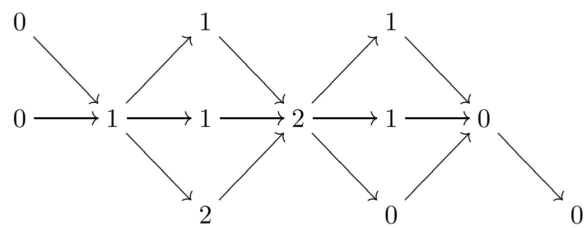
The values $\langle [e_\alpha], [e_\beta] \rangle$ for $\beta \in R^+$ are written below at the position of β in Γ_Ω:



We have $\{\beta \in B \mid \alpha \prec_\Omega \beta\} = \{\alpha_1, \alpha_2\}$. In fact, $\langle [e_\alpha], [e_{\alpha_1}] \rangle = \langle [e_\alpha], [e_{\alpha_2}] \rangle = 1$. Thus $\lambda_\alpha = \langle [\mathbf{P}(1)], \cdot \rangle + \langle [\mathbf{P}(2)], \cdot \rangle - \langle [e_\alpha], \cdot \rangle$. The values $\langle [\mathbf{P}(1)], [e_\beta] \rangle$ for $\beta \in R^+$ are written below at the position of β in Γ_Ω:



We gave above the values $\langle [\mathbf{P}(2)], [e_\beta] \rangle$ for $\beta \in R^+$. Finally, we get that the values $\lambda_\alpha([e_\beta])$ are written below at the position of β in Γ_Ω:



To end this section, we will describe for each $\alpha \in R^+$ the set of i, $1 \leq i \leq n$, such that $\langle e_\alpha, e_{\alpha_i} \rangle \neq 0$ and also give these nonzero values $\langle e_\alpha, e_{\alpha_i} \rangle \neq 0$ in two cases: first, in the case where the underlying graph of the quiver Ω is of type A_n and second, in the case where the quiver is alternating for any underlying graph of type A, D or E. In these two cases, the formula of Theorem 3.8 (d) can then easily be made more explicit.

3.11. Given a quiver Ω with underlying graph Δ and a positive root $\alpha = \sum_{i=1}^n b_i \alpha_i \in R^+$, we recall that the support of α is $\mathrm{supp}(\alpha) = \{1 \leq i \leq n \mid b_i \neq 0\}$. We will denote by $\Omega(\alpha)$ the subquiver of Ω whose underlying graph consists of the full subgraph of Δ whose set of vertices is the support $\mathrm{supp}(\alpha)$ of α.

Proposition 3.12. *Let Ω be a quiver whose underlying graph Δ is of type A_n and a positive root $\alpha \in R^+$. Then $\langle [e_\alpha], [e_{\alpha_i}] \rangle \neq 0$ if and only if i is a source of $\Omega(\alpha)$. In the case that i is a source of $\Omega(\alpha)$, then $\langle [e_\alpha], [e_{\alpha_i}] \rangle = 1$.*

Proof. It is possible to describe e_α. In fact, we easily get that e_α is isomorphic to the module $\mathbf{V} = ((V_j)_{1 \leq j \leq n}, (f_{jk})_{j \to k})$ such that

$$V_j = \begin{cases} F, & \text{if } j \in \mathrm{supp}(\alpha); \\ 0, & \text{otherwise;} \end{cases}$$

and

$$f_{jk} = \begin{cases} Id_F, & \text{if } j,k \in \operatorname{supp}(\alpha) \text{ and } j \to k \text{ in } \Omega; \\ 0, & \text{otherwise.} \end{cases}$$

Also, e_{α_i} is isomorphic to the module $\mathbf{W} = ((W_j)_{1\le j\le n}, (g_{jk})_{j\to k})$ such that

$$W_j = \begin{cases} F, & \text{if } j = i; \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad g_{jk} = 0 \text{ for all } j \to k \text{ in } \Omega.$$

If i is a source of $\Omega(\alpha)$, then it is not difficult to verify that $\Upsilon(a)\colon \mathbf{V} \to \mathbf{W}$ defined by

$$\Upsilon(a)_j = \begin{cases} a\ \mathrm{Id}_F, & \text{if } j = i; \\ 0, & \text{otherwise;} \end{cases}$$

for each $a \in F$ gives a homomorphism of modules of Ω. Because both $\dim_F(V_i) = \dim_F(W_i) = 1$ and $\dim_F(W_j) = 0$ for $j \neq i$, we get that $\langle [e_\alpha], [e_{\alpha_i}] \rangle = 1$.

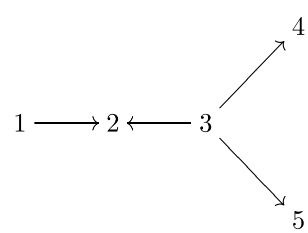
If i is not a source of $\Omega(\alpha)$, then there exists a vertex $i' \in \operatorname{supp}(\alpha)$ such that $i' \to i$ is an edge in $\Omega(\alpha)$. If $\Upsilon : \mathbf{V} \to \mathbf{W}$ is a morphism, then by considering the commuting diagram

$$\begin{array}{ccc} F = V_{i'} & \xrightarrow{\mathrm{Id}_F} & V_i = F \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \Upsilon_i} \\ 0 = W_{i'} & \xrightarrow{0} & W_i = F \end{array}$$

we get that $\Upsilon_i = 0$ and consequently that $\langle [e_\alpha], [e_{\alpha_i}] \rangle = 0$. This proves the proposition. $\square$

3.13. Let Ω be a quiver whose underlying graph is of type A_n with $n \ge 1$, D_n with $n \ge 4$ or E_n with $n = 6, 7, 8$. We say that Ω is alternating if and only if each vertex is either a sink or a source.

For example the quiver



is such an alternating quiver of type D_5.

Proposition 3.14. *Let Ω be an alternating quiver as defined above and a positive root $\alpha = \sum_{i=1}^n b_i\alpha_i \in R^+$. Then $\langle [e_\alpha], [e_{\alpha_i}] \rangle \neq 0$ if and only if i is a source of $\Omega(\alpha)$. In the case that i is a source of $\Omega(\alpha)$, then $\langle [e_\alpha], [e_{\alpha_i}] \rangle = b_i$.*

Proof. Write $e_\alpha = \mathbf{V} = ((V_j)_{1\le j\le n}, (f_{jk})_{j\to k})$. We have $\dim_F(V_j) = b_j$ for all $j = 1, 2, \ldots, n$. Note that e_{α_i} is isomorphic to the module $\mathbf{W} = ((W_j)_{1\le j\le n}, (g_{jk})_{j\to k})$ such that

$$W_j = \begin{cases} F, & \text{if } j = i; \\ 0, & \text{otherwise;} \end{cases} \quad \text{and} \quad g_{jk} = 0 \text{ for all } j \to k \text{ in } \Omega.$$

If i is a source of $\Omega(\alpha)$, then for each $A \in \operatorname{Hom}_F(V_i, W_i)$, we get a homomorphism $\Upsilon_A \colon \mathbf{V} \to \mathbf{W}$ defined by $\Upsilon(A)_i = A$ and $\Upsilon(A)_j = 0$. It is not difficult to check that $A \mapsto \Upsilon(A)$ gives an isomorphism $\operatorname{Hom}_\Omega(e_\alpha, e_{\alpha_i}) \simeq \operatorname{Hom}_F(V_i, W_i)$. From this, we can conclude that $\langle [e_\alpha], [e_{\alpha_i}] \rangle = \dim_F(\operatorname{Hom}_F(V_i, W_i)) = \dim_F(V_i) = b_i$, because $W_i = F$.

If i is not a source of $\Omega(\alpha)$, then i is a sink of Ω. We will first prove that

$$\bigoplus_{i' \to i} f_{i'i} : \bigoplus_{\substack{i' \\ i' \to i}} V_{i'} \to V_i$$

is surjective, where the sum $\bigoplus_{i' \to i} f_{i'i}$ is over all edges in Ω ending at i. Assume that $\bigoplus_{i' \to i} f_{i'i}$ is not surjective. Let $\mathbf{V}' = ((V'_j)_{1 \le j \le n}, (f'_{jk})_{j \to k})$ be defined by

$$V'_j = \begin{cases} \operatorname{image}(\bigoplus_{i' \to i} f_{i'i}), & \text{if } j = i; \\ V_j, & \text{if } j \neq i; \end{cases} \quad \text{and} \quad f'_{jk} = f_{jk} \text{ whenever } j \to k \text{ in } \Omega$$

and also let $\mathbf{V}'' = ((V''_j)_{1 \le j \le n}, (f''_{jk})_{j \to k})$ be defined by

$$V''_j = \begin{cases} V''_i, & \text{if } j = i; \\ 0, & \text{if } j \neq i; \end{cases} \quad \text{and} \quad f''_{jk} = 0 \text{ whenever } j \to k \text{ in } \Omega,$$

where V''_i is any subspace of V_i such that $V_i = V''_i \oplus (\operatorname{image}(\bigoplus_{i' \to i} f_{i'i}))$. Note that we have $\dim_F(V''_i) \neq 0$, because we assume that $\bigoplus_{i' \to i} f_{i'i}$ is not surjective. It is not difficult to verify that both $\mathbf{V}'$ and $\mathbf{V}''$ are submodules of $\mathbf{V}$ and that $\mathbf{V} = \mathbf{V}' \oplus \mathbf{V}''$. But this contradicts the fact that e_α is indecomposable. Thus $\bigoplus_{i' \to i} f_{i'i}$ is surjective.

If $\Upsilon : \mathbf{V} \to \mathbf{W}$ is a morphism, then, by considering the commuting diagram

$$\begin{array}{ccc} \bigoplus_{i'} V_{i'} & \xrightarrow{\bigoplus_{i' \to i} f_{i'i}} & V_i \\ {\scriptstyle 0}\big\downarrow & & \big\downarrow{\scriptstyle \Upsilon_i} \\ 0 = \bigoplus_{i'} W_{i'} & \xrightarrow{0} & W_i = F \end{array}$$

where the direct sum $\bigoplus_{i'} V_{i'}$ is over all the vertices i' joined to i by an edge, we get that $\Upsilon_i \circ (\bigoplus_{i' \to i} f_{i'i}) = 0$. Because $(\bigoplus_{i' \to i} f_{i'i})$ is surjective, we get that $\Upsilon_i = 0$. Because $W_j = 0$ for all $j \neq i$, we can conclude that $\langle [e_\alpha], [e_{\alpha_i}] \rangle = 0$ when i is not a source of $\Omega(\alpha)$. □

4. Monomials

In this last section, we will recall the definition of the quantized enveloping algebra $\mathbf{U}$ associated to the Cartan matrix C and then consider monomials in $\mathbf{U}$ corresponding to elements in the Lusztig cones.

4.1. Let v be an indeterminate. We can attach to C its quantized enveloping algebra $\mathbf{U}$. This is an associative algebra over $\mathbf{Q}(v)$ with generators E_i, F_i, K_i, K_i^{-1} $(1 \le i \le n)$ and relations

(r.1) $K_i K_i^{-1} = K_i^{-1} K_i = 1$, $K_i K_j = K_j K_i$;
(r.2) $K_i E_j = v^{a_{ij}} E_j K_i$, $K_i F_j = v^{-a_{ij}} F_j K_i$;
(r.3) $E_i F_j - F_j E_i = \delta_{ij}((K_i - K_i^{-1})/(v - v^{-1}))$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$;

(r.4) $E_i^2 E_j - (v+v^{-1})E_iE_jE_i + E_jE_i^2 = 0$ if $a_{ij} = -1$, $E_iE_j - E_jE_i = 0$ if $a_{ij} = 0$;
(r.5) $F_i^2 F_j - (v+v^{-1})F_iF_jF_i + F_jF_i^2 = 0$ if $a_{ij} = -1$, $F_iF_j - F_jF_i = 0$ if $a_{ij} = 0$.

We denote by $\mathbf{U}^+$ the subalgebra generated by the elements E_i for all $i \in \{1, 2, \dots, n\}$.

Given an integer $N \geq 0$ and $1 \leq i \leq n$, we define

$$[N]! = \Pi_{k=1}^{N}((v^k - v^{-k})/(v - v^{-1})) \in \mathbf{Q}(v)$$

and we will denote $E_i^N/[N]!$ by $E_i^{(N)}$.

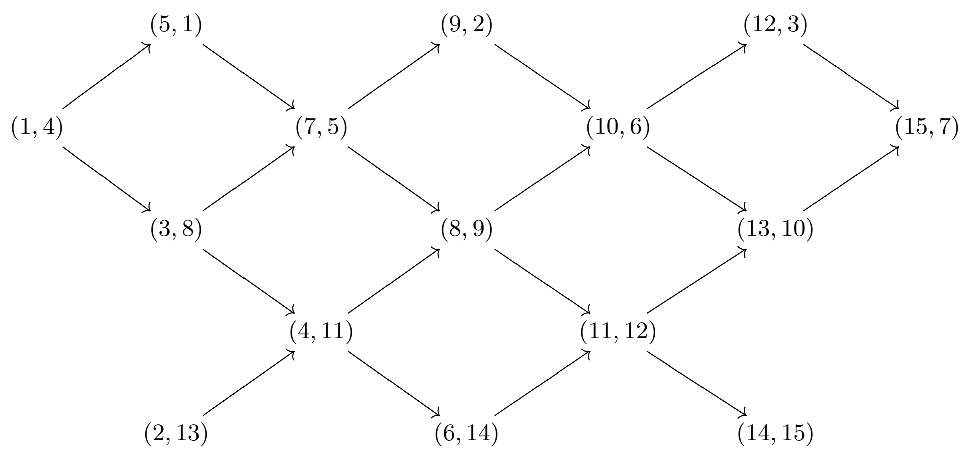
4.2. Let $\mathcal{M}_\Omega$ be the set of pairs (f, g) of functions $f\colon R^+ \to \{1, 2, \dots, \nu\}$ and $g\colon R^+ \to \mathbf{N}$ such that f is a bijection belonging to E_Ω. (E_Ω has been defined in 2.5.)

To such a pair $(f, g) \in \mathcal{M}_\Omega$, we can associate a monomial $\mathcal{E}(f, g)$ in $\mathbf{U}^+$ by

$$\mathcal{E}(f, g) = E_{i_1}^{(c_1)} E_{i_2}^{(c_2)} \cdots E_{i_\nu}^{(c_\nu)}$$

where $i_j = \rho_\Omega(f^{-1}(j))$ and $c_j = g(f^{-1}(j))$ for $j = 1, 2, \dots, \nu$.

Example 4.3. Let Ω be the quiver $\Omega : 1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 5$ of type A_5. The Auslander-Reiten quiver Γ_Ω of Ω is given in Example 2.2. Consider the pair (f, g) of functions $f\colon R^+ \to \{1, 2, \dots, \nu\}$ and $g\colon R^+ \to \mathbf{N}$ such that each value $(f(\alpha), g(\alpha))$ is written below at the position of the positive root α in the quiver Γ_Ω:



We have that $(f, g) \in \mathcal{M}_\Omega$ and

$$\mathcal{E}(f, g) = E_4^{(4)} E_1^{(13)} E_3^{(8)} E_2^{(11)} E_5^{(1)} E_1^{(14)} E_4^{(5)} E_3^{(9)} E_5^{(2)} E_4^{(6)} E_2^{(12)} E_5^{(3)} E_3^{(10)} E_1^{(15)} E_4^{(7)}.$$

Lemma 4.4. *Let (f_1, g) and (f_2, g) be two elements of $\mathcal{M}_\Omega$. Then $\mathcal{E}(f_1, g) = \mathcal{E}(f_2, g)$.*

Proof. Fix a reduced expression $\mathbf{i}$ of w_0 adapted to the quiver Ω. Because f_1 and f_2 both belong to E_Ω, there exist two reduced expressions $\mathbf{i}'$ and $\mathbf{i}''$ of w_0 both belonging to the commutation class $[\mathbf{i}]$ such that $f_1 = \pi_{\mathbf{i}'}$ and $f_2 = \pi_{\mathbf{i}''}$. This follows from Theorem 2.5. So there is a sequence $\mathbf{i}' = \mathbf{i}_0, \mathbf{i}_1, \dots, \mathbf{i}_p = \mathbf{i}''$ of reduced expressions of w_0 such that $\mathbf{i}_{q+1}$ is related to $\mathbf{i}_q$ by a short braid relation for $q = 0, 1, \dots, (p-1)$. To prove the lemma, it is then enough to prove it with the hypothesis that $f_1 = \pi_{\mathbf{i}'}$, $f_2 = \pi_{\mathbf{i}''}$ and $\mathbf{i}''$ is related to $\mathbf{i}'$ by a short braid relation.

If $\mathbf{i}' = (i'_1, i'_2, \ldots, i'_\nu)$ and $\mathbf{i}'' = (i''_1, i''_2, \ldots, i''_\nu)$, then there exists $1 \leq m < \nu$ such that

$$i''_j = \begin{cases} i'_j, & \text{if } j \neq m, (m+1); \\ i'_{m+1}, & \text{if } j = m; \\ i'_m, & \text{if } j = m+1; \end{cases}$$

and $a_{i'_m i'_{m+1}} = 0$. We easily get that

$$\alpha^{(j)}(\mathbf{i}'') = \begin{cases} \alpha^{(j)}(\mathbf{i}'), & \text{if } j \neq m, m+1; \\ \alpha^{(m+1)}(\mathbf{i}'), & \text{if } j = m; \\ \alpha^{(m)}(\mathbf{i}'), & \text{if } j = m+1. \end{cases}$$

If we write $c''_j = g(f_2^{-1}(j))$ and $c'_j = g(f_1^{-1}(j))$ for $j = 1, 2, \ldots, \nu$, then we also get that

$$c''_j = \begin{cases} c'_j, & \text{if } j \neq m, (m+1); \\ c'_{m+1}, & \text{if } j = m; \\ c'_m, & \text{if } j = (m+1). \end{cases}$$

Because $a_{i'_m i'_{m+1}} = 0$, we get that

$$\begin{aligned} \mathcal{E}(f_2, g) &= E_{i''_1}^{(c''_1)} E_{i''_2}^{(c''_2)} \cdots E_{i''_\nu}^{(c''_\nu)} \\ &= E_{i'_1}^{(c'_1)} E_{i'_2}^{(c'_2)} \cdots E_{i'_{(m-1)}}^{(c'_{(m-1)})} E_{i'_{(m+1)}}^{(c'_{(m+1)})} E_{i'_m}^{(c'_m)} E_{i'_{(m+2)}}^{(c'_{(m+2)})} \cdots E_{i'_\nu}^{(c'_\nu)} \\ &= E_{i'_1}^{(c'_1)} E_{i'_2}^{(c'_2)} \cdots E_{i'_{(m-1)}}^{(c'_{(m-1)})} E_{i'_m}^{(c'_m)} E_{i'_{(m+1)}}^{(c'_{(m+1)})} E_{i'_{(m+2)}}^{(c'_{(m+2)})} \cdots E_{i'_\nu}^{(c'_\nu)} \\ &= \mathcal{E}(f_1, g). \end{aligned}$$

□

4.5. For a quiver Ω and a function $g\colon R^+ \to \mathbf{N}$, we define $\mathcal{E}(\Omega, g)$ to be $\mathcal{E}(f, g)$ where f is any element of E_Ω. By the previous lemma, $\mathcal{E}(\Omega, g)$ is well defined. Note also that if $\lambda \in \mathcal{C}'_\Omega$ and $g : R^+ \to \mathbf{N}$ is defined by $g(\alpha) = \lambda([e_\alpha])$ for all $\alpha \in R^+$, then the monomials $\mathcal{E}(\Omega, g)$ are the ones considered by Lusztig in section 16 of [11].

Theorem 4.6. *Let Ω, Ω' be two quivers with the same underlying graph Δ and let $g\colon R^+ \to \mathbf{N}$ be a function such that $g(\alpha + \beta) = g(\alpha) + g(\beta)$ whenever α, β and $\alpha + \beta \in R^+$. Then*

$$\mathcal{E}(\Omega, g) = \mathcal{E}(\Omega', g).$$

Proof. Let $\mathbf{i} = (i_1, i_2, \ldots, i_\nu)$ be a reduced expression of w_0. Denote by $\mathcal{E}(\mathbf{i}, g)$ the monomial

$$\mathcal{E}(\mathbf{i}, g) = E_{i_1}^{(c_1)} E_{i_2}^{(c_2)} \cdots E_{i_\nu}^{(c_\nu)} \quad \text{where } g(\alpha^{(j)}(\mathbf{i})) = c_j \text{ for } j = 1, 2, \ldots, \nu.$$

We will first prove that $\mathcal{E}(\mathbf{i}, g) = \mathcal{E}(\mathbf{j}, g)$ for any reduced expressions $\mathbf{i}$, $\mathbf{j}$ of w_0, As we have indicated in 1.3, there is a sequence $\mathbf{i} = \mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_p = \mathbf{j}$ of reduced expressions of w_0 such that $\mathbf{i}_{q+1}$ is related to $\mathbf{i}_q$ by either a short braid relation or by a long braid relation. So it is enough to prove that $\mathcal{E}(\mathbf{i}, g) = \mathcal{E}(\mathbf{j}, g)$ whenever $\mathbf{j}$ is related to $\mathbf{i}$ by a short braid relation or by a long braid relation. Write $\mathbf{i} = (i_1, i_2, \ldots, i_\nu)$, $\mathbf{j} = (j_1, j_2, \ldots, j_\nu)$, $c_k = g(\alpha^{(k)}(\mathbf{i}))$ and $c'_k = g(\alpha^{(k)}(\mathbf{j}))$.

If $\mathbf{j}$ is related to $\mathbf{i}$ by a short braid relation, then there exists an integer m, $1 \le m \le (\nu - 1)$ such that

$$j_k = \begin{cases} i_k, & \text{if } k \neq m, (m+1); \\ i_{m+1}, & \text{if } k = m; \\ i_m, & \text{if } k = (m+1); \end{cases}$$

with $a_{i_m i_{m+1}} = 0$. We also have

$$\alpha^{(k)}(\mathbf{j}) = \begin{cases} \alpha^{(k)}(\mathbf{i}), & \text{if } k \neq m, (m+1); \\ \alpha^{(m+1)}(\mathbf{i}), & \text{if } k = m; \\ \alpha^{(m)}(\mathbf{i}), & \text{if } k = (m+1); \end{cases}$$

and

$$c'_k = \begin{cases} c_k, & \text{if } k \neq m, (m+1); \\ c_{(m+1)}, & \text{if } k = m; \\ c_m, & \text{if } k = (m+1). \end{cases}$$

Thus

$$\begin{aligned} \mathcal{E}(\mathbf{j}, g) &= E_{j_1}^{(c'_1)} E_{j_2}^{(c'_2)} \cdots E_{j_\nu}^{(c'_\nu)} \\ &= E_{i_1}^{(c_1)} E_{i_2}^{(c_2)} \cdots E_{i_{(m-1)}}^{(c_{(m-1)})} E_{i_{(m+1)}}^{(c_{(m+1)})} E_{i_m}^{(c_m)} E_{i_{(m+2)}}^{(c_{(m+2)})} \cdots E_{i_\nu}^{(c_\nu)} \\ &= E_{i_1}^{(c_1)} E_{i_2}^{(c_2)} \cdots E_{i_{(m-1)}}^{(c_{(m-1)})} E_{i_m}^{(c_m)} E_{i_{(m+1)}}^{(c_{(m+1)})} E_{i_{(m+2)}}^{(c_{(m+2)})} \cdots E_{i_\nu}^{(c_\nu)} \\ &= \mathcal{E}(\mathbf{i}, g), \end{aligned}$$

because $a_{i_m i_{m+1}} = 0$ and $E_{i_m} E_{i_{(m+1)}} = E_{i_{(m+1)}} E_{i_m}$.

If $\mathbf{j}$ is related to $\mathbf{i}$ by a long braid relation, then there exists an integer m, $1 \le m \le (\nu - 2)$ such that

$$j_k = \begin{cases} i_k, & \text{if } k \neq m, (m+1), (m+2); \\ i_{m+1}, & \text{if } k = m, (m+2); \\ i_m, & \text{if } k = (m+1); \end{cases}$$

with $i_m = i_{(m+2)}$ and $a_{i_m i_{m+1}} = -1$. We also have that

$$\alpha^{(k)}(\mathbf{j}) = \begin{cases} \alpha^{(k)}(\mathbf{i}), & \text{if } k \neq m, (m+2); \\ \alpha^{(m+2)}(\mathbf{i}), & \textit{if } k = m; \\ \alpha^{(m)}(\mathbf{i}), & \text{if } k = (m+2); \end{cases}$$

and

$$c'_k = \begin{cases} c_k, & \text{if } k \neq m, (m+2); \\ c_{(m+2)}, & \text{if } k = m; \\ c_m, & \text{if } k = (m+2). \end{cases}$$

Because we have that $\alpha^{(m+1)}(\mathbf{i}) = \alpha^{(m)}(\mathbf{i}) + \alpha^{(m+2)}(\mathbf{i})$, we get that $c_{(m+1)} = g(\alpha^{(m+1)}(\mathbf{i})) = g(\alpha^{(m)}(\mathbf{i})) + g(\alpha^{(m+2)}(\mathbf{i})) = c_m + c_{(m+2)}$.

Note also that we have $E_i^{(b)} E_{i'}^{(b+c)} E_i^{(c)} = E_{i'}^{(c)} E_i^{(b+c)} E_{i'}^{(b)}$ where $a_{ii'} = -1$ and $b, c \in \mathbf{N}$. See for example Proposition 2.3 or Example 3.4 in [9].

Thus

$$\begin{aligned}\mathcal{E}(\mathbf{j},g) &= E_{j_1}^{(c'_1)}E_{j_2}^{(c'_2)}\cdots E_{j_\nu}^{(c'_\nu)}\\ &= E_{i_1}^{(c_1)}E_{i_2}^{(c_2)}\cdots E_{i_{(m-1)}}^{(c_{(m-1)})}E_{i_{(m+1)}}^{(c_{(m+2)})}E_{i_m}^{(c_{(m+1)})}E_{i_{(m+1)}}^{(c_m)}E_{i_{(m+3)}}^{(c_{(m+3)})}\cdots E_{i_\nu}^{(c_\nu)}\\ &= E_{i_1}^{(c_1)}E_{i_2}^{(c_2)}\cdots E_{i_{(m-1)}}^{(c_{(m-1)})}E_{i_m}^{(c_m)}E_{i_{(m+1)}}^{(c_{(m+1)})}E_{i_m}^{(c_{(m+2)})}E_{i_{(m+3)}}^{(c_{(m+3)})}\cdots E_{i_\nu}^{(c_\nu)}\\ &= \mathcal{E}(\mathbf{i},g).\end{aligned}$$

If $\mathbf{i}$ is adapted to the quiver Ω, then we easily get that $\mathcal{E}(\mathbf{i},g)=\mathcal{E}(\pi_{\mathbf{i}},g)=\mathcal{E}(\Omega,g)$. By choosing $\mathbf{i}$ adapted to Ω and $\mathbf{j}$ adapted to Ω', then we have that $\mathcal{E}(\Omega,g)=\mathcal{E}(\mathbf{i},g)=\mathcal{E}(\mathbf{j},g)=\mathcal{E}(\Omega',g)$. □

Proposition 4.7. *Let Ω be a quiver and $\lambda\in\mathcal{C}'_\Omega$ be such that $\lambda(r_{\alpha'})=0$ for all $\alpha'\in R^+\setminus\mathcal{P}(\Omega)$. Define the function $g:R^+\to\mathbf{N}$ by $g(\alpha)=\lambda([e_\alpha])$ for all $\alpha\in R^+$. Then $g(\alpha+\beta)=g(\alpha)+g(\beta)$ whenever α, β and $\alpha+\beta$ belong to R^+.*

Proof. By Theorem 3.8 (b) and because $x_\alpha=r_\alpha$ and $\lambda(r_\alpha)=0$ if $\alpha\in R^+\setminus\mathcal{P}(\Omega)$, we have that $\lambda=\sum_{\alpha\in R^+}\epsilon_\alpha\ \lambda(x_\alpha)\ \lambda_\alpha=\sum_{\alpha\in\mathcal{P}(\Omega)}\lambda([x_\alpha])\ \lambda_\alpha$. So $\lambda=\sum_{i=1}^n\lambda([e_{\alpha_i}])\lambda_{\dim(\mathbf{P}(i))}$. By Theorem 3.8 (c), we get easily that, whenever α, β and $\alpha+\beta$ belong to R^+, we have that $\lambda_{\dim(\mathbf{P}(i))}([e_\alpha])+\lambda_{\dim(\mathbf{P}(i))}([e_\beta])=\lambda_{\dim(\mathbf{P}(i))}([e_{(\alpha+\beta)}])$. From this, we get that $g(\alpha+\beta)=g(\alpha)+g(\beta)$ whenever α, β and $\alpha+\beta$ belong to R^+. □

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References

1. M. Auslander, *Relations for Grothendieck groups of artin algebras.*, Proc. Amer. Math. Soc. **91** (1984), 336–340. MR **85e:**16038
2. M. Auslander, I. Reiten and S.O. Smalø, Representation theory of artin algebras, Cambridge Studies in Advanced Mathematics **36**, Cambridge Univ. Press, Cambridge, 1995. MR **98e:**16011; MR **96c:**16015
3. R. Bédard, *On commutation classes of reduced words in Weyl groups*, European J. Combin. **20** (1999), 483–505. CMP 99:16
4. I.N. Bernstein, I.M. Gelfand and V.A. Ponomarev, *Coxeter functors and Gabriel's theorem*, Russian Math. Surveys **28** (1973), 17–32.
5. R.W. Carter and R.J. Marsh, *Regions of linearity, Lusztig cones and canonical basis elements for the quantized enveloping algebra of type A_4*, forthcoming paper 1998.
6. P. Gabriel, *Unzerlegbare darstellungen.* I, Manuscripta Math. **6** (1972), 71–103. MR **48:**11212
7. P. Gabriel, *Auslander-Reiten sequences and representation-finite algebras*, Lecture Notes in Math. **831**, Springer-Verlag, Berlin-Heidelberg-New York, 1980, pp. 1–71. MR **82i:**16030
8. M. Kashiwara, *On crystal bases of the Q-analogue of the universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516. MR **93b:**17045
9. G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498. MR **90m:**17023
10. G. Lusztig, *Canonical bases arising from quantized enveloping algebras.* II, Common trends in mathematics and quantum field theories (T. Eguchi et. als., eds), Progr. Theoret. Phys. Suppl., 102, 1990, 175–201. MR **93g:**17019
11. G. Lusztig, *Tight monomials in quantized enveloping algebras*, Quantum deformations of algebras and their representations (A. Joseph and S. Shnider, eds.), Israel Math. Conf. Proc **7** 1993, pp. 117–132. MR **95i:**17016

12. R.J. Marsh, *More tight monomials in quantized enveloping algebras*, J. Algebra **204** (1998), 711–732. MR **99j:**17023
13. R.J. Marsh, *The Lusztig cones of a quantized enveloping algebra of type A*, preprint, 1998.
14. R.J. Marsh, *Rectangle diagrams for the Lusztig cones of quantized enveloping algebras of type A*, preprint, 1998.
15. C.M. Ringel, *Tame algebras* (*On algorithms for solving vector space problems.* II), Lecture Notes in Math, **831**, Springer-Verlag, Berlin-Heidelberg-New York, 1980, pp. 137–287. MR **82j:**16056
16. C.M. Ringel, Tame algebras and integral quadratic forms, Lecture Notes in Math. **1099**, Springer-Verlag, Berlin-Heidelberg-New York, 1984. MR **87f:**16027
17. J. Tits, *Le problème des mots dans les groupes de Coxeter*, Symposia Mathematica (INDAM, Rome 1967/68), Academic Press, London 1969, vol. 1, pp. 175–185. MR **40:**7339

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