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ON THE GENERIC DEGREES OF CYCLOTOMIC ALGEBRAS

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Abstract. We determine the generic degrees of cyclotomic Hecke algebras attached to exceptional finite complex reflection groups. The results are used to introduce the notion of spetsial reflection group, which in a certain sense is a generalization of the finite Weyl group. In particular, to spetsial W there is attached a set of unipotent degrees which in the case of a Weyl group is just the set of degrees of unipotent characters of finite reductive groups with Weyl group W, and in general enjoys many of their combinatorial properties.

1. Introduction

Cyclotomic algebras are certain deformations of the group algebras of finite complex reflection groups, defined analogously to Iwahori-Hecke algebras for Weyl groups. These cyclotomic algebras seem to play an important role in the understanding of the representation theory of finite groups of Lie type (see for example [3]). At the moment, the properties of cyclotomic algebras are not yet fully understood. This paper is devoted to the study of certain numerical invariants attached to them, the generic degrees.

It is conjectured (and known in all but finitely many cases [16]) that the cyclotomic algebra $\mathcal{H} = \mathcal{H}(W, \mathbf{u})$ of an irreducible finite complex reflection group W is symmetric over the ground ring $A = \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$, such that the corresponding trace form t vanishes on all basis elements T_w for $w \in W$, $w \neq 1$ reduced. Since t is a trace form, it can be expressed as a sum $t = \sum_\phi 1/c_\phi \, \phi$ of irreducible characters of $\mathcal{H}$ with non-vanishing coefficients. The c_ϕ appearing as coefficients are called the Schur elements of $\mathcal{H}$ (with respect to t). In this paper we compute the Schur elements with respect to any trace form as above for the exceptional complex reflection groups. The method is an extension of [13] where the 2-dimensional case was considered.

In the final section we use the results obtained so far to introduce the so-called spetsial reflection groups (as announced in [14]). This is a subclass of all finite complex reflection groups, which includes in particular all those which can already be defined over the real numbers. It can be defined by a variety of equivalent characterizations all related to the Schur elements (see Prop.8.1). At present there is no conceptual understanding of the equivalence of these properties, even in the case

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of finite Coxeter groups (where they are always satisfied). It would be very interesting to find such an explanation, since this property is a necessary and sufficient condition for the existence of unipotent degrees attached to a complex reflection group W, i.e., for W behaving like the Weyl group of some as yet mysterious object generalizing the concept of an algebraic group see (Section 8.3).

2. Cyclotomic Hecke algebras

We recall the definition and some properties of cyclotomic Hecke algebras, some facts about character values, and some basic properties of symmetric algebras.

2A. The cyclotomic algebra of a complex reflection group. Let W be a finite irreducible complex reflection group on the vector space V. The ring of invariants of W in the symmetric algebra $S(V)$ of V is a polynomial ring, generated by homogeneous invariants of degrees $d_1, \ldots, d_n$, with $n = \dim(V)$. The Poincaré polynomial P_W of W is given by $P_W := \prod_{j=1}^{n}(x^{d_j} - 1)/(x-1)$. For an irreducible character $\phi \in \mathrm{Irr}(W)$ the *fake degree* is defined as

$$R_\phi := (x-1)^n P_W \frac{1}{|W|} \sum_{w \in W} \frac{\det_V(w)\phi(w)}{\det_V(x-w)} \in \mathbb{Z}[x]\,, \tag{2.1}$$

where $\det_V$ denotes the determinant on V. The *b-value of* $\phi \in \mathrm{Irr}(W)$ is the order of zero at $x = 0$ of the fake degree R_ϕ.

Let $\mathcal{D}$ be the diagram associated to W in [5]. This defines a presentation of W on a set of generators S with order relations $s^{d_s} = 1$ for $s \in S$, together with certain homogeneous relations, the so-called braid relations. The *braid group* $B = B(W)$ *associated to* W is by definition the group generated by a set $\{\mathbf{s} \mid s \in S\}$ in bijection with S, subject to the braid relations of $\mathcal{D}$. Let $\mathbf{u} = (u_{s,j} \mid s \in S,\, 0 \le j \le d_s - 1)$ be transcendentals over $\mathbb{Z}$, such that $u_{s,j} = u_{t,j}$ whenever s and t are conjugate in W. The generic cyclotomic Hecke algebra $\mathcal{H}(W, \mathbf{u})$ of W with parameter set $\mathbf{u}$ is defined to be the quotient

$$\mathcal{H}(W, \mathbf{u}) := \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]B/I, \qquad \text{with } I = \Big(\prod_{j=0}^{d_s-1} (\mathbf{s} - u_{s,j}) \mid s \in S\Big) \tag{2.2}$$

of the group algebra of B over $A := \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ by the ideal I generated by certain deformed order relations. We will write $T_{\mathbf{w}}$ for the image in $\mathcal{H}(W, \mathbf{u})$ of an element $\mathbf{w} = \mathbf{s}_1 \ldots \mathbf{s}_k \in B$. Any ring homomorphism $f : \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}] \to R$ endows R with an A-module structure, and we write

$$\mathcal{H}_R(W, \mathbf{u}) := \mathcal{H}(W, \mathbf{u}) \otimes_A R$$

for the corresponding specialization. Note that such a homomorphism f is uniquely determined by the images $f(u_{s,j})$, $s \in S$, $0 \le j \le d_s - 1$. A specialization of $\mathcal{H}$ will be called *admissible* if the specialization

$$u_{s,j} \mapsto \exp(2\pi i j/d_s) \qquad \text{for } s \in S,\ 0 \le j \le d_s - 1, \tag{2.3}$$

to the group algebra of the complex reflection group W factors through it. One particularly important example is the 1-*parameter specialization* $\mathcal{H}(W, x)$ of $\mathcal{H}(W, \mathbf{u})$ induced by the map

$$f_x : u_{s,j} \mapsto \begin{cases} x & j = 0, \\ \exp(2\pi i j/d_s) & j > 0, \end{cases} \tag{2.4}$$

where x is an indeterminate. This is the analogue of the usual 1-parameter Iwahori-Hecke algebra for real W.

Henceforth, we will make the following assumption:

Assumption 2.5 . *The cyclotomic algebra $\mathcal{H}(W, \mathbf{u})$ is free over $A = \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ of rank $|W|$.*

This assumption is known to hold for all infinite families of irreducible reflection groups and for some of the remaining 34 exceptional groups (see the references in [15]). We conjecture it to be true in all cases [5, Sect. 4].

2B. Character values. Let k be the character field of the reflection representation of W. So k is a finite cyclotomic extension of $\mathbb{Q}$ and we write $\mu(k)$ for the group of roots of unity in k. For $s \in S$, $0 \leq j \leq d_s - 1$, let $v_{s,j}$ be such that $v_{s,j}^{|\mu(k)|} = \exp(-2\pi ij/d_s)u_{s,j}$ and set $K_W := k(\mathbf{v})$. We extend the specialization (2.3) to $\mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$ by

$$v_{s,j} \mapsto 1 \qquad \text{for } s \in S,\ 0 \leq j \leq d_s - 1\,. \tag{2.6}$$

By [15, Thm. 5.2] it is known that K_W is a splitting field for $\mathcal{H}$. In particular, the values of all irreducible characters on an A-basis of $\mathcal{H}$ are contained in K_W. Furthermore, it follows from Assumption 2.5 and Tits' deformation theorem that $\mathcal{H}_{K_W}$ is isomorphic to the group algebra $K_W W$, and the specialization (2.6) induces a bijection

$$\mathrm{Irr}(W) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}_{K_W}), \qquad \phi \mapsto \phi_{\mathbf{v}}\,, \tag{2.7}$$

between $\mathrm{Irr}(\mathcal{H}_{K_W})$ and $\mathrm{Irr}(W)$ which furthermore carries over to any admissible specialization of $\mathcal{H}_{k[\mathbf{v},\mathbf{v}^{-1}]}$.

In [5] we defined a certain central element $\boldsymbol{\pi} \in Z(B)$ which maps to the identity under the canonical epimorphism $B \to W$. Clearly, its image $T_{\boldsymbol{\pi}}$ in $\mathcal{H}$ is also central. Thus, it acts as a scalar in any (absolutely) irreducible representation of $\mathcal{H}_{K_W}$. Extending an idea of Springer, Broué and Michel have observed that this allows evaluation of irreducible characters on roots of $T_{\boldsymbol{\pi}}$ without knowing the corresponding representation explicitly. Let us describe this method.

Let S' be a system of representatives of the generators in S up to conjugation in W. Write $\boldsymbol{\pi} = \mathbf{s}_1 \ldots \mathbf{s}_l$ in B. For $s \in S'$ let $N_s = |\{j \mid s_j \sim s\}|$ denote the number of factors in the decomposition of $\boldsymbol{\pi}$ conjugate to $\mathbf{s}$. (This does not depend on the chosen expression for $\boldsymbol{\pi}$, as is seen by evaluating linear characters of B.) For an irreducible character ϕ of W let $m_{s,j}^{\phi}$ denote the multiplicity of the eigenvalue $\exp(2\pi ij/d_s)$ of s in a representation affording ϕ.

Proposition 2.8 (Broué-Michel [6])**.** *Let $\mathbf{w} \in B$ such that $\mathbf{w}^d = \boldsymbol{\pi}$ for some $d \geq 1$. Then the value of the irreducible character $\phi_{\mathbf{v}} \in \mathrm{Irr}(\mathcal{H}_{K_W})$ on $T_{\mathbf{w}}^l$ is given by*

$$\phi_{\mathbf{v}}(T_{\mathbf{w}}^l) = \phi(w^l) \prod_{s \in S'} \prod_{j=0}^{d_s-1} v_{s,j}^{l|\mu(k)|m_{s,j}^{\phi} N_s / d\phi(1)}.$$

It can be shown by a general argument that $m_{s,j}^{\phi} N_s/\phi(1)$ is always an integer (see [6, 4.17]). As a consequence of [15, Thm. 5.2], which relies on a case-by-case analysis, the exponent $|\mu(k)| m_{s,j}^{\phi} N_s / d\phi(1)$ in Proposition 2.8 has to be integral unless $\phi(w) = 0$.

2C. Symmetrizing forms. By what we saw in 2A the cyclotomic algebra $\mathcal{H}$ is isomorphic to the group algebra of W over the splitting field K_W. Since $K_W W$ is a symmetric algebra, the same is true for $\mathcal{H}_{K_W}$. Thus there exists a symmetric form $\langle\ ,\ \rangle : \mathcal{H} \otimes \mathcal{H} \to K_W$ on $\mathcal{H}$. Moreover, this can be normalized such that the associated trace form $t_{\mathbf{u}} : \mathcal{H}(W, \mathbf{u}) \to K_W$ defined by $t_{\mathbf{u}}(h) := \langle 1, h\rangle$ under (2.3) specializes to the canonical trace form on the group algebra of W. Over the splitting field K_W of $\mathcal{H}(W, \mathbf{u})$, we may write $t_{\mathbf{u}}$ as a sum over the irreducible characters of $\mathcal{H}_{K_W}$ with non-vanishing coefficients:

$$t_{\mathbf{u}} = \sum_{\phi \in \mathrm{Irr}(W)} \frac{1}{c_\phi}\, \phi_{\mathbf{v}}\, .$$

The c_ϕ are called *Schur elements* of $\mathcal{H}(W, \mathbf{u})$ (with respect to $t_{\mathbf{u}}$). A basis C of $\mathcal{H}$ with $1 \in C$ is called *quasi-symmetric* (with respect to $t_{\mathbf{u}}$) if $t_{\mathbf{u}}(T) = \delta_{T,1}$ for $T \in C$. Thus for any quasi-symmetric basis, the (inverses of the) Schur elements are uniquely determined by the linear system

$$\sum_{\phi \in \mathrm{Irr}(W)} c_\phi^{-1}\, \phi_{\mathbf{v}}(T) = \delta_{T,1} \qquad (\text{for } T \in C)\, . \tag{2.9}$$

Here, already the equations on a subset of C whose image under specialization to W covers all conjugacy classes yield a system of maximal rank.

It is well-known that for Coxeter groups W the Iwahori-Hecke algebra $\mathcal{H}(W)$ carries a canonical symmetrizing form endowing it with a structure of symmetric algebra over A. This form is characterized by the fact that the standard basis elements form a quasi-symmetric basis (see [8, Prop. 8.1.1]). The Schur elements with respect to this form are explicitly known in all cases.

It was shown in [16] for all but finitely many irreducible complex reflection groups W that there exists a symmetrizing form $t_{\mathbf{u}}$ on $\mathcal{H}(W, \mathbf{u})$ making it a symmetric algebra over A (i.e., the form has Gram matrix invertible over A). Moreover, there exists a quasi-symmetric basis with respect to $t_{\mathbf{u}}$ consisting of monomials in the generators $T_{\mathbf{s}}$, $s \in S$. Furthermore, restriction of $t_{\mathbf{u}}$ to parabolic subalgebras gives the corresponding symmetrizing form there. The Schur elements of this symmetrizing form were determined explicitly in [7].

On the other hand, in [13] we dealt with the 2-dimensional primitive groups. It is the purpose of the present paper to analyze the remaining exceptional cases. More precisely, we will determine explicitly the Schur elements c_ϕ with respect to any symmetrizing form which behaves nicely under restriction to parabolic subalgebras and which vanishes on some suitable monomials in the generators.

2D. Symmetries and action on $\mathrm{Irr}(W)$**.** It will be convenient to introduce and study the action of several groups of automorphisms on cyclotomic algebras and associated objects. Let $\tilde{k} := k(\exp(2\pi i/|\mu(k)|^2))$, and $\tilde{K}_W := \tilde{k}(\mathbf{v})$ the composite with K_W. Then $\tilde{K}_W$ is a Galois extension of $\mathbb{Q}(\mathbf{u})$. Its Galois group $G := \mathrm{Gal}(\tilde{K}_W/\mathbb{Q}(\mathbf{u}))$ acts naturally on $\mathcal{H}_{\tilde{K}_W}$, hence on $\mathrm{Irr}(\mathcal{H})$. Via the bijection (2.7) we thus obtain a G-action on $\mathrm{Irr}(W)$ where $g(\phi)$ is defined by the condition $g(\phi)_{\mathbf{v}} = g(\phi_{\mathbf{v}})$.

On the other hand, the factor group $\mathrm{Gal}(\tilde{k}(\mathbf{u})/\mathbb{Q}(\mathbf{u}))$ of G acts naturally on $\mathrm{Irr}(W)$. Now restriction yields an isomorphism between the subgroup $G_0 := \mathrm{Gal}(\tilde{K}_W/\mathbb{Q}(\mathbf{v}))$ of G and $\mathrm{Gal}(\tilde{k}/\mathbb{Q})$. The definition of the bijection (2.7) via the specialization (2.6) shows that the natural action of $\mathrm{Gal}(k/\mathbb{Q})$ on $\mathrm{Irr}(W)$ and its action induced by the identification with G_0 do coincide. If $\mathcal{H}$ is symmetric over

A, the linear independence of the irreducible characters together with (2.9) shows that the corresponding Schur elements satisfy

Lemma 2.10. *If $\mathcal{H}$ is symmetric with the corresponding Schur elements c_ϕ ($\phi \in \mathrm{Irr}(W)$) and $G := \mathrm{Gal}(\tilde{K}_W/\mathbb{Q}(\mathbf{u}))$, then we have*

$$c_{g(\phi)} = g(c_\phi) \qquad \text{for all } \phi \in \mathrm{Irr}(W),\ g \in G. \tag{2.11}$$

Next observe that for $s \in S' = S/W$ the symmetric group $\mathfrak{S}_{d_s}$ acts (from the left) on the set of indeterminates $\{u_{s,j} \mid 0 \le j \le d_s - 1\}$ by

$$\sigma(u_{s,j}) := u_{s,\sigma(j)}\,.$$

Let $\mathfrak{S} = \mathfrak{S}(W) := \prod_{s\in S'} \mathfrak{S}_{d_s}$, the *symmetry group of* W. By (2.2) $\mathcal{H}(W,\mathbf{u})$ is already defined over the ring of invariants $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]^{\mathfrak{S}}$ under this symmetry group. By trivial action on the constants $\tilde{k}$ the $\mathfrak{S}$-action on $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$ extends to an action on $\tilde{k}[\mathbf{u},\mathbf{u}^{-1}]$. Setting

$$\sigma(v_{s,j}) := \exp(2\pi i(\sigma(j)-j)/d_s|\mu(k)|)v_{s,\sigma(j)}$$

further extends to an action on $\tilde{k}[\mathbf{v},\mathbf{v}^{-1}]$. This induces an $\mathfrak{S}$-action on $\mathcal{H}$ and then also on $\mathrm{Irr}(\mathcal{H})$ by

$$\sigma(\phi)(T) := \sigma(\phi(\sigma^{-1}(T))) \qquad \text{for } \phi \in \mathrm{Irr}(\mathcal{H}),\ T \in \mathcal{H}_{K_W},\ \sigma \in \mathfrak{S}.$$

Via (2.7) this defines an $\mathfrak{S}$-action on $\mathrm{Irr}(W)$.

Now assume that $\mathcal{H}$ has a quasi-symmetric $\mathfrak{S}$-invariant basis, for example, one consisting of images of group elements in the braid group under the natural epimorphism $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]B \to \mathcal{H}$, i.e., of monomials in the generators $T_\mathbf{s}$, $s \in S$. Then $\mathcal{H}$ is symmetric already over $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]^{\mathfrak{S}}$ and (2.9) shows

Lemma 2.12 . *If $\mathcal{H}$ is symmetric with respect to an $\mathfrak{S}$-invariant quasi-symmetric basis, with corresponding Schur elements c_ϕ ($\phi \in \mathrm{Irr}(W)$), then we have*

$$c_{\sigma(\phi)} = \sigma(c_\phi) \qquad \text{for all } \phi \in \mathrm{Irr}(W),\ \sigma \in \mathfrak{S}. \tag{2.13}$$

This observation on symmetries together with (2.11) will be used later on in order to reduce the number of Schur elements to be printed explicitly.

By (2.9) the Schur elements are rational functions in $\mathbf{v}$. But, moreover, we have

Lemma 2.14. *Assume that $\mathcal{H}$ has a quasi-symmetric basis C consisting of monomials in the $T_\mathbf{s}$. Then for any $s \in S$ the Schur elements are homogeneous of degree 0 in $\{v_{s,j} \mid 0 \le j \le d_s - 1\}$.*

Proof. Indeed, let $(\alpha_s \mid s \in S)$ be invertible elements $\alpha_s \in k^\times$ such that $\alpha_s = \alpha_t$ whenever $s \sim t$. Then the elements $\{\alpha_s T_\mathbf{s} \mid s \in S\}$ of $\mathcal{H}(W,\mathbf{u})$ satisfy the relations of the cyclotomic algebra $\mathcal{H}(W,\mathbf{u}')$ with $\mathbf{u}' = (\alpha_s u_{s,j} \mid s \in S,\ 0 \le j \le d_s - 1)$. Any monomial in the $T_\mathbf{s}$ is a non-zero scalar multiple of a monomial in the $\alpha_s T_\mathbf{s}$. But if C is quasi-symmetric, then any basis C' containing 1 and consisting of non-zero scalar multiples of the elements of C is quasi-symmetric with respect to the same symmetrizing form. Thus the Schur elements of $\mathcal{H}$ remain invariant by replacing the $v_{s,j}$ with non-zero scalar multiples (which only depend on s) as claimed. $\square$

3. Generic degrees for G_{24}, G_{27}, G_{29}

Here and in the next section we compute the Schur elements for those exceptional non-real reflection groups of dimension $n \geq 3$, which are generated by n involutive reflections. In the classification of Shephard and Todd, these are the groups G_i, $i \in \{24, 27, 29, 33, 34\}$. In these cases, all reflections are conjugate, so the cyclotomic algebra has just two parameters (u_1, u_2). Since the Schur elements are homogeneous in the parameters, they can be recovered from the Schur elements for the (admissible) specialization with parameters $(x, -1)$. So it suffices to consider this latter case.

It turns out that in all cases the Schur elements (with the above specialization) divide the Poincaré polynomial $P(W)$ of W, considered as elements of the ring $k[x]$. We call $\delta_\phi := P(W)/c_\phi$ the *generic degree of* ϕ. Clearly, it is enough to know the generic degrees in order to recover the Schur elements.

We denote the irreducible characters $\phi \in \mathrm{Irr}(W)$ as $\phi_{d,b}$, where $d = \phi(1)$ is the degree and b is the b-invariant, that is, the order of zero at $x = 0$ of the fake degree R_ϕ (see 2A). In most cases, this distinguishes the characters unambiguously. In the few remaining cases, we give more detailed information below.

In the tables, we only list one character from each pair of complex conjugate ones, since the generic degree of the other one can be obtained by applying complex conjugation, according to (2.11). Moreover, if ϵ denotes the sign character of W, we only list one of ϕ, $\epsilon \otimes \phi$, since, as it turns out,

$$\delta_\phi(x) = x^N \, \delta_{\epsilon\otimes\phi}(x^{-1}) \tag{3.1}$$

in all cases, where N is the number of reflections in W. Combining this with (2.11) it can be rephrased to say that δ_ϕ is semi-palindromic in the sense of [15, 6B].

3A. The complex reflection group G_{24}. The 3-dimensional primitive complex reflection group $W = G_{24}$, abstractly isomorphic to $\mathrm{L}_2(7) \times 2$, has twelve irreducible characters. It is generated by three conjugate reflections s_1, s_2, s_3 of order 2. It has two classes of maximal parabolic subgroups, one of type B_2 and one of type A_2, hence both are Coxeter groups.

Let $\mathbf{c} := \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3$ be a Coxeter element in the braid group $B(W)$ of W. Then the central element $\boldsymbol{\pi} \in B(W)$ is given by $\boldsymbol{\pi} = \mathbf{c}^{14}$ (see [5, Table 4]).

Theorem 3.2. *Let t be a symmetrizing form on $\mathcal{H}(G_{24}; (x, -1))$ restricting to the canonical symmetrizing form on the maximal parabolic subalgebras and such that there exists a quasi-symmetric basis containing $T_{\mathbf{c}}^l$ $(1 \leq l \leq 7)$. Then the generic degrees with respect to t are given in Table* 3.3.

Proof. Assume that t is a symmetrizing form on $\mathcal{H}(W)$ which restricts to the canonical symmetrizing forms of the Iwahori-Hecke algebras of these parabolic subgroups. Since the generic degrees of finite Coxeter groups are known, restriction of characters to these subgroups yields conditions on the generic degrees of $\mathcal{H}(W)$ with respect to t. More precisely, since five of the conjugacy classes of W have representatives in one of these parabolic subgroups, we obtain five equations.

Since $\boldsymbol{\pi}$ is a power of $\mathbf{c}$, Proposition 2.8 allows us to compute the values of all irreducible characters on $T_{\mathbf{c}}^l$. Plugging these into (2.9) gives linear equations for the unknown generic degrees, which together with the equations from restriction to parabolic subalgebras yield a system of full rank. □

Table 3.3. Generic degrees for G_{24}

ϕ	δ_ϕ	$\bar\phi$	$\epsilon\otimes\phi$	$\epsilon\otimes\bar\phi$
$\phi_{1,0}$	1		$\phi_{1,21}$	
$\phi_{3,1}$	$\frac{\sqrt{-7}}{14}x\Phi_3\Phi_4\Phi_6\Phi_7'\Phi_{14}''$	$\phi_{3,3}$	$\phi_{3,8}$	$\phi_{3,10}$
$\phi_{6,2}$	$\frac{1}{2}x\Phi_2^2\Phi_3\Phi_6\Phi_{14}$		$\phi_{6,9}$	
$\phi_{7,3}$	$x^3\Phi_7\Phi_{14}$		$\phi_{7,6}$	
$\phi_{8,4}$	$\frac{1}{2}x^4\Phi_2^3\Phi_4\Phi_6\Phi_{14}$		$\phi_{8,5}$	

Here, as in the subsequent tables, Φ_n denotes the n-th cyclotomic polynomial over $\mathbb{Q}$. Moreover, with $\mu=(-1+\sqrt{-7})/2=\zeta_7+\zeta_7^2+\zeta_7^4$, $\overline{\mu}=(-1-\sqrt{-7})/2$, where $\zeta_7:=\exp(2\pi i/7)$, we let $\Phi_7'=x^3-\mu x^2+\overline{\mu}x-1$, $\Phi_{14}'=x^3+\mu x^2+\overline{\mu}x+1$.

3B. The complex reflection group G_{27}. The 3-dimensional primitive complex reflection group $W=G_{27}$, abstractly isomorphic to $3.A_7\times 2$, has 34 irreducible characters. It is generated by three conjugate reflections s_1,s_2,s_3 of order 2. It has four conjugacy classes of maximal parabolic subgroups, of types H_2,B_2 and A_2 (two classes). All of these are Coxeter groups, hence their Iwahori-Hecke algebras have a canonical symmetrizing form. Restriction to these subgroups gives eight equations for the generic degrees. The Coxeter element $\mathbf{c}:=\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3$ satisfies $\mathbf{c}^{30}=\boldsymbol{\pi}$ [5, Table 4]. Its fifth power $\mathbf{c}^5$ is central, thus more generally we may compute the value of any irreducible character on products $T_\mathbf{w}T_\mathbf{c}^{5l}$ for all $T_\mathbf{w}$ lying in proper parabolic subalgebras of $\mathcal{H}$. This yields enough independent equations, and as above we obtain:

Theorem 3.4. *Let t be a symmetrizing form on $\mathcal{H}(G_{27};(x,-1))$ such that there exists a quasi-symmetric basis containing $T_\mathbf{w}T_\mathbf{c}^{5l}$ $(0\le l\le 5)$ for $\mathbf{w}$ running through representatives of minimal length of the conjugacy classes of the maximal parabolic subgroups of W. Then the generic degrees with respect to t are given in Table* 3.5.

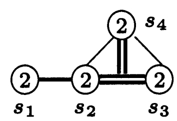
Table 3.5. Generic degrees for G_{27}

ϕ	δ_ϕ	$\bar\phi$	$\epsilon\otimes\phi$	$\epsilon\otimes\bar\phi$
$\phi_{1,0}$	1		$\phi_{1,45}$	
$\phi_{3,1,25}$	$-\frac{\sqrt{-15}\zeta_3^2}{30}x\Phi_3''^3\Phi_4\Phi_5'\Phi_6'^3\Phi_{10}'\Phi_{12}\Phi_{15}'\Phi_{15}^{ba}\Phi_{30}'\Phi_{30}^{ab}$	$\phi_{3,5,29}$	$\phi_{3,16,40}$	$\phi_{3,20,44}$
$\phi_{3,5,23}$	$-\frac{\sqrt{-15}\zeta_3}{30}x\Phi_3'^3\Phi_4\Phi_5''\Phi_6''^3\Phi_{10}''\Phi_{12}\Phi_{15}''\Phi_{15}^{aa}\Phi_{30}'\Phi_{30}^{ba}$	$\phi_{3,7,25}$	$\phi_{3,20,38}$	$\phi_{3,22,40}$
$\phi_{5,6}'$	$\frac{1}{2}x^3\Phi_4\Phi_5\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{30}$		$\phi_{5,15}''$	
$\phi_{5,6}''$	$\frac{1}{2}x^3\Phi_4\Phi_5\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{30}$		$\phi_{5,15}'$	
$\phi_{6,2}$	$\frac{1-\zeta_3^2}{6}x\Phi_2^2\Phi_3'^3\Phi_6^2\Phi_6''\Phi_{10}\Phi_{12}'\Phi_{15}''\Phi_{30}$	$\phi_{6,4}$	$\phi_{6,17}$	$\phi_{6,19}$
$\phi_{8,6,36}$	$\frac{(3-\lambda)}{10}x^6\Phi_2^3\Phi_4\Phi_5'\Phi_6^3\Phi_{10}\Phi_{12}\Phi_{15}^B\Phi_{30}$		$\phi_{8,9,39}$	
$\phi_{8,9,33}$	$\frac{(2+\lambda)}{10}x^6\Phi_2^3\Phi_4\Phi_5''\Phi_6^3\Phi_{10}\Phi_{12}\Phi_{15}^A\Phi_{30}$		$\phi_{8,12,36}$	
$\phi_{9,4}$	$\frac{1}{3}x^4\Phi_3^3\Phi_6^3\Phi_{12}\Phi_{15}\Phi_{30}$	$\phi_{9,8}$	$\phi_{9,13}$	$\phi_{9,11}$
$\phi_{9,6}$	$\frac{1}{3}x^4\Phi_3^3\Phi_6^3\Phi_{12}\Phi_{15}\Phi_{30}$		$\phi_{9,9}$	
$\phi_{10,3}$	$\frac{1}{2}x^3\Phi_2^2\Phi_5\Phi_6^2\Phi_{10}\Phi_{15}\Phi_{30}$		$\phi_{10,12}$	
$\phi_{15,5}$	$\frac{1-\zeta_3^2}{3}x^5\Phi_3''^3\Phi_5\Phi_6'^3\Phi_{10}\Phi_{12}''\Phi_{15}\Phi_{30}$	$\phi_{15,7}$	$\phi_{15,8}$	$\phi_{15,10}$

Here, $\Phi_3'=x-\zeta_3$, $\Phi_6'=x+\zeta_3^2$, $\Phi_{12}'=x^2+\zeta_3^2$, $\Phi_5''=x^2+\lambda x+1$, $\Phi_{10}'=x^2-\lambda x+1$, $\Phi_{15}'=x^4+\zeta_3^2x^3+\zeta_3x^2+x+\zeta_3^2$, $\Phi_{15}^A=x^4-\lambda x^3+\lambda x^2-\lambda x+1$, $\Phi_{15}^{aa}=x^2+\lambda\zeta_3^2x+\zeta_3$, $\Phi_{15}^{ab}=x^2+\lambda\zeta_3x+\zeta_3^2$, $\Phi_{30}''=\Phi_{15}'(-x)$, $\Phi_{30}^{aa}=\Phi_{15}^{aa}(-x)$, with $\lambda=(1+\sqrt{5})/2=1+\zeta_5+\zeta_5^4$.

Here, in order to distinguish the characters of degree 3 and of degree 8 among themselves, we have given as a third index the degree in x of the fake degree R_ϕ. We denote by $\phi'_{5,6}$, $\phi'_{5,15}$ the characters of degree 5 which occur in the permutation character of W on the parabolic subgroup of type A_2, and by $\phi''_{5,6}$, $\phi''_{5,15}$ the other two. (It turns out that the two characters $\phi_{5,6}$, respectively $\phi_{5,15}$ have identical generic degrees.)

3C. The complex reflection group G_{29}. The 4-dimensional primitive complex reflection group $W = G_{29}$ has 37 irreducible characters. It is generated by four conjugate reflections $s_1, \dots, s_4$ of order 2, satisfying the relations implied by the Coxeter diagram



together with $(s_2s_3s_4)^2 = (s_4s_2s_3)^2$. It has five conjugacy classes of maximal parabolic subgroups, of types B_3, A_3 (2 classes), A_2+A_1, $G(4,4,3)$. While the first four of these are Coxeter groups, the last one is an imprimitive non-real reflection group.

In order to use induction we first determine generic degrees for $G(4,4,3)$. Now all maximal parabolic subgroups of $G(4,4,3)$ are Coxeter groups, and restriction of characters to these yields five equations. Furthermore, evaluation of characters on the first five powers of the Coxeter element gives five additional equations, thus determining the generic degrees with respect to any quasi-symmetric basis containing these elements. We give the results in Table 3.7.

Let us return to $W = G_{29}$. Restriction to the maximal parabolic subgroups gives 16 equations. The Coxeter element $\mathbf{c} := \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4$ in the braid group $B(W)$ of W satisfies $\mathbf{c}^{20} = \boldsymbol{\pi}$. The equations given by evaluating the irreducible characters of $\mathcal{H}$ on $T_{\mathbf{c}}^l$, $1 \le l \le 18$, are linearly independent from the equations obtained by restriction. The character tables of all maximal parabolic subalgebras of $\mathcal{H}(G_{29}; (x,-1))$ are known, respectively, and can easily be computed for the non-Coxeter group $G(4,4,3)$. Since $\mathbf{c}^5$ is central in $B(W)$, the values of irreducible characters on elements $T_{\mathbf{w}}T_{\mathbf{c}}^5$ with $T_{\mathbf{w}}$ in some parabolic subalgebra of $\mathcal{H}$ are known. The corresponding equations determine the generic degrees uniquely.

Theorem 3.6. *Let t be a symmetrizing form on $\mathcal{H}(G_{29}; (x,-1))$ restricting to the canonical symmetrizing form on the maximal parabolic subalgebras of Coxeter type (respectively to the form determined above for $G(4,4,3)$) and such that there exists a quasi-symmetric basis containing $T_{\mathbf{c}}^l$ $(1 \le l \le 18)$ and elements $T_{\mathbf{w}}T_{\mathbf{c}}^5$ for $\mathbf{w}$ running through representatives of minimal length of the conjugacy classes of parabolic subgroups of W. Then the generic degrees with respect to t are given in Table* 3.8.

All characters of W are distinguished by their degree and their b-invariant, except for four characters of degree 6 with $b = 10$ and for two pairs of characters of degree 15. We denote by $\phi''_{15,4}$ the character of degree 15 occurring in the tensor product of the reflection character $\phi_{4,1}$ with $\phi_{4,3}$, and by $\phi''_{15,12}$ its tensor product with the sign character $\phi_{1,40}$. We denote by $\phi'_{6,10}$ the real character of type $(6,10)$ occurring in the permutation character of W on the parabolic subgroup of type A_3, and by $\phi''_{6,10}$ the other real character of degree 6 with $b = 10$ (these two have identical generic degrees). By $\phi'''_{6,10}$ we denote the character of degree 6 appearing in the tensor product of the reflection character with itself, and by $\phi''''_{6,10}$ its complex conjugate.

Table 3.7. Generic degrees for $G(4,4,3)$

ϕ	δ_ϕ	$\bar\phi$	$\epsilon\otimes\phi$	$\epsilon\otimes\bar\phi$
$\phi_{1,0}$	1		$\phi_{1,12}$	
$\phi_{2,4}$	$x^4\Phi_8$			
$\phi_{3,1}$	$\frac{1+i}{4}x\Phi_3{\Phi'_4}^2\Phi''_8$	$\phi_{3,2}$	$\phi_{3,5}$	$\phi_{3,6}$
$\phi_{3,2}$	$\frac{1}{2}x\Phi_3\Phi_8$		$\phi_{3,6}$	
$\phi_{6,3}$	$x^3\Phi_3\Phi_8$			

Table 3.8. Generic degrees for G_{29}

ϕ	δ_ϕ	$\bar\phi$	$\epsilon\otimes\phi$	$\epsilon\otimes\bar\phi$
$\phi_{1,0}$	1		$\phi_{1,40}$	
$\phi_{4,1}$	$\frac{1+i}{4}x\Phi_2^2{\Phi''_4}^2\Phi_6\Phi'_8\Phi_{10}\Phi^a_{12}\Phi''_{20}$	$\phi_{4,3}$	$\phi_{4,21}$	$\phi_{4,23}$
$\phi_{4,4}$	$\frac{1}{2}x\Phi_4^3\Phi_{12}\Phi_{20}$		$\phi_{4,24}$	
$\phi_{5,8}$	$\frac{1}{2}x^4\Phi_5\Phi_8\Phi_{10}\Phi_{12}\Phi_{20}$		$\phi_{5,16}$	
$\phi'_{6,10}$	$\frac{1}{5}x^6\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{12}\Phi_{20}$		$\phi''_{6,10}$	
$\phi'''_{6,10}$	$-\frac{1}{20}x^6\Phi_3{\Phi'_4}^4\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{12}\Phi''_{20}$	$\phi''''_{6,10}$		
$\phi_{6,12}$	$\frac{1}{4}x^6\Phi_3\Phi_4^2\Phi_6\Phi_8\Phi_{10}\Phi_{12}\Phi_{20}$			
$\phi_{10,2}$	$x^2\Phi_5\Phi_8\Phi_{10}\Phi_{20}$		$\phi_{10,18}$	
$\phi_{10,6}$	$\frac{1}{2}x^4\Phi_4^2\Phi_5\Phi_{10}\Phi_{12}\Phi_{20}$	$\phi_{10,14}$		
$\phi'_{15,4}$	$x^4\Phi_3\Phi_5\Phi_6\Phi_{10}\Phi_{12}\Phi_{20}$		$\phi'_{15,12}$	
$\phi''_{15,4}$	$\frac{1}{2}x^4\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{20}$		$\phi''_{15,12}$	
$\phi_{16,3}$	$\frac{1}{2}x^3\Phi_4^4\Phi_8\Phi_{12}\Phi_{20}$	$\phi_{16,5}$	$\phi_{16,15}$	$\phi_{16,13}$
$\phi_{20,5}$	$\frac{1-i}{4}x^5\Phi_2^2{\Phi''_4}^2\Phi_5\Phi_6\Phi''_8\Phi_{10}\Phi^a_{12}\Phi_{20}$	$\phi_{20,7}$	$\phi_{20,9}$	$\phi_{20,11}$
$\phi_{20,6}$	$\frac{1}{2}x^5\Phi_4^3\Phi_5\Phi_{10}\Phi_{12}\Phi_{20}$		$\phi_{20,10}$	
$\phi_{24,6}$	$\frac{1}{20}x^6\Phi_2^4\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{12}\Phi_{20}$			
$\phi_{24,7}$	$\frac{1}{4}x^6\Phi_2^2\Phi_3\Phi_4^3\Phi_6\Phi_{10}\Phi_{12}\Phi_{20}$	$\phi_{24,9}$		
$\phi_{30,8}$	$\frac{1}{4}x^6\Phi_3\Phi_4^2\Phi_5\Phi_6\Phi_8\Phi_{12}\Phi_{20}$			

Here $\Phi'_4=x-i$, $\Phi'_8=x^2-i$, $\Phi^a_{12}=x^2-ix-1$, $\Phi''_{20}=x^4-ix^3-x^2+ix+1$.

4. Generic degrees for G_{33} and G_{34}

Here we conclude the determination of Schur elements for exceptional non-real reflection groups of dimension $n\geq 3$ generated by n involutive reflections by considering the cases G_{33} and G_{34} in the Shephard-Todd classification.

4A. Some imprimitive groups. We first compute the generic degrees for some imprimitive complex reflection groups which occur as parabolic subgroups in the primitive groups G_{33} and G_{34}. More precisely, these are the groups $G(3,3,3)$, $G(3,3,4)$ and $G(3,3,5)$. Again, the results can easily be obtained by the methods presented so far, and we give the lists in Tables 4.1–4.3.

TABLE 4.1. Generic degrees for $G(3,3,3)$

ϕ	δ_ϕ	$\bar{\phi}$	$\epsilon\otimes\phi$	$\epsilon\otimes\bar{\phi}$
$\phi_{1,0}$	1		$\phi_{1,9}$	
$\phi_{2,3}$	$x^3\Phi_2\Phi_6$			
$\phi_{3,1}$	$\frac{1-\zeta_3^2}{3}x{\Phi'_3}^3\Phi''_6$	$\phi_{3,2}$	$\phi_{3,4}$	$\phi_{3,5}$

TABLE 4.2. Generic degrees for $G(3,3,4)$.

ϕ	δ_ϕ	$\bar{\phi}$	$\epsilon\otimes\phi$	$\epsilon\otimes\bar{\phi}$
$\phi_{1,0}$	1		$\phi_{1,18}$	
$\phi_{2,6}$	$\frac{1}{3}x^4\Phi_4\Phi_6\Phi_9$			
$\phi_{3,3}$	$x^3\Phi_9$		$\phi_{3,9}$	
$\phi_{4,1}$	$\frac{1-\zeta_3^2}{3}x\Phi_2\Phi_4\Phi'_6\Phi''_9$	$\phi_{4,2}$	$\phi_{4,10}$	$\phi_{4,11}$
$\phi_{6,2}$	$x^2\Phi_4\Phi_9$		$\phi_{6,8}$	
$\phi'_{6,5}$	$-\frac{\zeta_3^2}{3}x^4{\Phi'_3}^3\Phi_4\Phi_6\Phi''_9$	$\phi''_{6,5}$		
$\phi_{8,4}$	$\frac{1}{3}x^4\Phi_2^2\Phi_4\Phi_9$	$\phi_{8,5}$		
$\phi_{12,3}$	$x^3\Phi_2\Phi_4\Phi_9$		$\phi_{12,6}$	

Here, we have denoted by $\phi'_{6,5}$ the character of degree 6 with b-value 5 appearing in the tensor square of the reflection character $\phi_{4,1}$, by $\phi''_{6,5}$ its complex conjugate.

TABLE 4.3. Generic degrees for $G(3,3,5)$

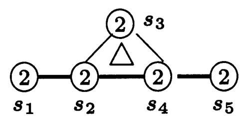
ϕ	δ_ϕ	$\bar{\phi}$	$\epsilon\otimes\phi$	$\epsilon\otimes\bar{\phi}$
$\phi_{1,0}$	1		$\phi_{1,30}$	
$\phi_{4,3}$	$x^3\Phi_2\Phi_4\Phi_6\Phi_{12}$		$\phi_{4,18}$	
$\phi_{5,1}$	$\frac{1-\zeta_3^2}{3}x\Phi_5\Phi'_6\Phi'_9\Phi''_{12}$	$\phi_{5,2}$	$\phi_{5,19}$	$\phi_{5,20}$
$\phi_{5,6}$	$\frac{1}{3}x^4\Phi_5\Phi_6\Phi_9\Phi_{12}$		$\phi_{5,12}$	
$\phi_{6,9}$	$x^9\Phi_4\Phi_9\Phi_{12}$			
$\phi_{10,2}$	$\frac{1-\zeta_3^2}{3}x^2\Phi_4\Phi_5\Phi'_6\Phi''_9\Phi_{12}$	$\phi_{10,3}$	$\phi_{10,14}$	$\phi_{10,15}$
$\phi_{10,5}$	$-\frac{\zeta_3^2}{3}x^4\Phi_4\Phi_5{\Phi'_6}^2\Phi_9\Phi''_{12}$	$\phi_{10,6}$	$\phi_{10,11}$	$\phi_{10,12}$
$\phi_{10,7}$	$\frac{1}{3}x^6\Phi_4\Phi_5\Phi_6\Phi_9\Phi_{12}$	$\phi_{10,8}$		
$\phi_{15,4}$	$\frac{1}{3}x^4{\Phi''_3}^3\Phi_5{\Phi'_6}^2\Phi''_9\Phi_{12}$	$\phi_{15,5}$	$\phi_{15,10}$	$\phi_{15,11}$
$\phi_{20,3}$	$x^3\Phi_2\Phi_4\Phi_5\Phi_6^2\Phi_{12}$		$\phi_{20,12}$	
$\phi_{20,5}$	$\frac{1-\zeta_3}{3}x^5\Phi_2\Phi_4\Phi_5\Phi_6\Phi''_6\Phi'_9\Phi_{12}$	$\phi_{20,6}$	$\phi_{20,8}$	$\phi_{20,9}$
$\phi_{30,4}$	$x^4\Phi_4\Phi_5\Phi_9\Phi_{12}$		$\phi_{30,10}$	
$\phi'_{30,7}$	$\frac{1}{3}x^6{\Phi'_3}^3\Phi_4\Phi_5{\Phi''_6}^2\Phi'_9\Phi_{12}$	$\phi''_{30,7}$		
$\phi_{40,6}$	$\frac{1}{3}x^6\Phi_2^2\Phi_4\Phi_5\Phi_9\Phi_{12}$			

Here, $\phi''_{30,7}$ denotes the character of degree 30 occurring in the fourth tensor power of the reflection character $\phi_{5,1}$, and $\phi'_{30,7}$ its complex conjugate.

In all tables of this section the notation for cyclotomic polynomials is as follows. Let $\zeta_3 := \exp(2\pi i/3)$ be a third root of unity. Then for any j multiple of 3 we have a factorization $\Phi_j = \Phi'_j\Phi''_j$ over $\mathbb{Q}(\zeta_3)$ and we choose notation such that

$\Phi'_j(\exp(2\pi i/j)) = 0$. More specifically, this means that $\Phi'_3 = x - \zeta_3$, $\Phi'_6 = x + \zeta_3^2$, $\Phi'_9 = x^3 - \zeta_3$, $\Phi'_{12} = x^2 + \zeta_3^2$, $\Phi'_{18} = x^3 + \zeta_3^2$, ...

4B. The complex reflection group G_{33}. The 5-dimensional primitive complex reflection group $W = G_{33}$, abstractly isomorphic to $2 \times O_5(3)$, has 40 irreducible characters. It is generated by five conjugate reflections $s_1, \dots, s_5$ of order 2 satisfying the relations implied by the Coxeter diagram



together with $(s_2 s_3 s_4)^2 = (s_3 s_4 s_2)^2 = (s_4 s_2 s_3)^2$. It has four conjugacy classes of maximal parabolic subgroups, of types D_4, A_4, $A_3 + A_1$, $G(3,3,4)$. The first three are Coxeter groups, and the last one was considered in 4A. Restriction to these subgroups gives 22 equations. The Coxeter element $\mathbf{c} := \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\mathbf{s}_5$ satisfies $\mathbf{c}^{18} = \boldsymbol{\pi}$. The equations given by evaluating the irreducible characters of $\mathcal{H}$ on $T_{\mathbf{c}}^m$, $1 \le m \le 17$, are linearly independent from the equations obtained by restriction. Furthermore, the defining relations of $B(W)$ (i.e., the braid relations of W) show that $\mathbf{w}^5 = \mathbf{c}^9$ for the element $\mathbf{w} := \mathbf{c}\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4 \in B(W)$ of length 9, hence it is a 10-th root of $\boldsymbol{\pi}$. Evaluating characters on $T_{\mathbf{w}}$ gives a further independent equation, to yield a system of full rank.

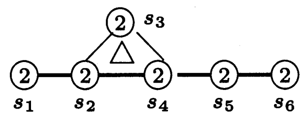
Theorem 4.4. *Let t be a symmetrizing form on $\mathcal{H}(G_{33};(x,-1))$ restricting to the canonical symmetrizing form on the maximal parabolic subalgebras of Coxeter type (respectively to the form determined above for $G(3,3,4)$) and such that there exists a quasi-symmetric basis containing $T_{\mathbf{c}}^l$ $(1 \le l \le 17)$ and $T_{\mathbf{w}}$. Then the generic degrees with respect to t are given in Table* 4.5.

TABLE 4.5. Generic degrees for G_{33}

ϕ	δ_ϕ	$\bar\phi$	$\epsilon \otimes \phi$	$\epsilon \otimes \bar\phi$
$\phi_{1,0}$	1		$\phi_{1,45}$	
$\phi_{5,1}$	$\frac{1-\zeta_3^2}{3} x\Phi_5\Phi'_9\Phi_{10}\Phi'_{12}\Phi''_{18}$	$\phi_{5,3}$	$\phi_{5,28}$	$\phi_{5,30}$
$\phi_{6,5}$	$\frac{1}{2}x^3\Phi_4^2\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$		$\phi_{6,20}$	
$\phi'_{10,8}$	$\frac{\zeta_3^2}{6}x^4\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_9\Phi_{10}\Phi_{12}\Phi'_{18}$	$\phi''_{10,8}$	$\phi'_{10,17}$	$\phi''_{10,17}$
$\phi_{15,2}$	$x^2\Phi_5\Phi_9\Phi_{10}\Phi_{18}$		$\phi_{15,23}$	
$\phi_{15,9}$	$x^9\Phi_5\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$		$\phi_{15,12}$	
$\phi_{20,6}$	$\frac{1}{3}x^4\Phi_4^2\Phi_5\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$		$\phi_{20,15}$	
$\phi_{24,4}$	$\frac{1}{2}x^3\Phi_2^4\Phi_6^2\Phi_9\Phi_{10}\Phi_{18}$		$\phi_{24,19}$	
$\phi_{30,3}$	$\frac{1}{2}x^3\Phi_4^2\Phi_5\Phi_9\Phi_{12}\Phi_{18}$		$\phi_{30,18}$	
$\phi_{30,4}$	$-\frac{\zeta_3^2}{6}x^4{\Phi'_3}^3\Phi_4^2\Phi_5\Phi''_9\Phi_{10}\Phi_{12}\Phi_{18}$	$\phi_{30,6}$	$\phi_{30,13}$	$\phi_{30,15}$
$\phi'_{40,5}$	$\frac{1-\zeta_3}{6}x^4\Phi_2^4\Phi_5\Phi_6^2\Phi''_6\Phi''_9\Phi_{10}\Phi'_{12}\Phi_{18}$	$\phi''_{40,5}$	$\phi'_{40,14}$	$\phi''_{40,14}$
$\phi_{45,7}$	$\frac{1-\zeta_3^2}{3}x^7{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_9\Phi_{10}\Phi''_{12}\Phi_{18}$	$\phi_{45,9}$	$\phi_{45,10}$	$\phi_{45,12}$
$\phi_{60,7}$	$x^7\Phi_4^2\Phi_5\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$		$\phi_{60,10}$	
$\phi_{64,8}$	$\frac{1}{2}x^8\Phi_2^5\Phi_4^2\Phi_6^3\Phi_{10}\Phi_{12}\Phi_{18}$		$\phi_{64,9}$	
$\phi_{81,6}$	$x^6\Phi_3^3\Phi_6^3\Phi_9\Phi_{12}\Phi_{18}$		$\phi_{81,11}$	

All irreducible characters of W are determined by their degree and their b-value, except for two pairs of characters of degree 10 and two pairs of characters of degree 40. We denote by $\phi'_{10,8}$ the character of degree 10 whose fake degree is of the form $x^{28}+x^{26}+$ lower powers of x, and by $\phi'_{40,5}$ the one of degree 40 with fake degree $x^{31}+x^{29}+$ lower powers of x. The notation for the other characters is now fixed by the information given in Table 4.5.

4C. The complex reflection group G_{34}. The 6-dimensional primitive complex reflection group $W = G_{34}$, abstractly isomorphic to $6.U_4(2).2$, has 169 irreducible characters. It is generated by a set $S = \{s_1, \ldots, s_6\}$ of six conjugate reflections of order 2, satisfying the relations implied by the Coxeter diagram



together with $(s_2s_3s_4)^2 = (s_3s_4s_2)^2 = (s_4s_2s_3)^2$. It has eight conjugacy classes of maximal parabolic subgroups of types G_{33}, $G(3,3,5)$, D_5, A_5 (2 classes), $G(3,3,4)+A_1$, A_4+A_1, A_3+A_2. All of these only involve real groups, one of the imprimitive groups considered above, or G_{33}. Thus for all maximal parabolic subgroups we have a well defined symmetrizing form. Restriction to these subgroups gives 71 equations.

As usual we denote by $\mathbf{s}_j$ the canonical preimage in $B(W)$ of the generator $s_j \in S$. The Coxeter element $\mathbf{c} := \mathbf{s}_1 \ldots \mathbf{s}_6$ satisfies $\mathbf{c}^{42} = \boldsymbol{\pi}$ in $B(W)$ [5, Table 4]. Its seventh power is central in $B(W)$.

Theorem 4.6. *The generic degrees of the cyclotomic Hecke algebra $\mathcal{H}(G_{34};(x,-1))$ with respect to any quasi-symmetric basis containing only $T_{\mathbf{c}}^m$, $1 \le m \le 41$, and elements of the form $T_{\mathbf{v}}T_{\mathbf{c}}^{7l}$ $(1 \le l \le 5)$ for $\mathbf{v}$ running through representatives of minimal length of the conjugacy classes of the maximal parabolic subgroups of W are given in Table* 4.7.

Proof. The character values of $\mathcal{H}$ on the powers of $T_{\mathbf{c}}$ can be evaluated with Proposition 2.8. Moreover, character values on a basis of any proper parabolic subalgebra are known by the preceding section, respectively by [3]. Since $T_{\mathbf{c}}^{7l}$ is central in $\mathcal{H}$, the values on any product $T_{\mathbf{v}}T_{\mathbf{c}}^{5l}$ can be determined, once those on $T_{\mathbf{v}}$ are known. Thus the values of all irreducible characters on all the elements occurring in the statement can be computed. Using the computer algebra system GAP it was checked that this system has maximal rank, and thus determines the generic degrees uniquely. □

All irreducible characters of W are determined by their degree and their b-value, except for seven pairs and three triples of characters. The notation for the characters of degree 105 and 840 is fixed by Table 4.7, since for each ambiguous character, its complex conjugate is determined by its degree and b-value. In the three triples, we choose $\phi'_{70,9}$, $\phi'_{70,45}$, $\phi'_{560,18}$ to be the real characters. We denote by $\phi''_{20,33}$, $\phi'''_{70,9}$ the characters of degree 20, respectively 70 occurring in the tensor product of the reflection character $\phi_{6,1}$ with $\phi_{15,14}$, and by $\phi'_{20,33}$, $\phi''_{70,9}$ their complex conjugates. The tensor product of $\phi''_{70,9}$ with the sign character is $\phi''_{70,45}$, its complex conjugate $\phi'''_{70,45}$. We write $\phi'_{120,21}$ for the character of type $(120,21)$ occurring in the permutation character on the parabolic subgroup of type A_5. Finally, we call $\phi'_{280,12}$, $\phi''_{280,30}$, $\phi'''_{560,18}$ the characters of the respective types which occur in $\phi_{6,1} \otimes \phi_{336,17}$, and $\phi'_{540,21}$ the one occurring in $\phi_{6,1} \otimes \phi_{105,20}$. This fixes the notation for all elements of $\mathrm{Irr}(W)$.

TABLE 4.7. Generic degrees for G_{34}

ϕ	δ_ϕ	$\bar\phi$	$\epsilon\otimes\phi$	$\epsilon\otimes\bar\phi$
$\phi_{1,0}$	1		$\phi_{1,126}$	
$\phi_{6,1}$	$-\frac{\sqrt{-3}}{3}\zeta_3 x{\Phi'_3}^3{\Phi''_6}^3\Phi_8\Phi'_{12}\Phi_{24}\Phi''_{15}\Phi'_{21}\Phi'_{30}\Phi''_{42}$	$\phi_{6,5}$	$\phi_{6,85}$	$\phi_{6,89}$
$\phi_{15,14}$	$\frac{\zeta_3}{6}x^4{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_8\Phi'_9\Phi_{10}{\Phi'_{12}}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi''_{42}$	$\phi_{15,16}$	$\phi_{15,56}$	$\phi_{15,58}$
$\phi'_{20,33}$	$\frac{1}{42}x^{15}\Phi_4^2\Phi_5{\Phi'_6}^6\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi''_{42}$	$\phi''_{20,33}$		
$\phi_{21,2}$	$-\frac{\sqrt{-3}}{3}\zeta_3 x^2{\Phi''_3}^3{\Phi'_6}^3\Phi_7\Phi''_{12}\Phi_{14}\Phi'_{15}\Phi_{21}\Phi'_{24}\Phi''_{30}\Phi_{42}$	$\phi_{21,4}$	$\phi_{21,68}$	$\phi_{21,70}$
$\phi_{21,6}$	$\frac12 x^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{21,60}$	
$\phi_{35,6}$	$\frac12 x^3\Phi_5\Phi_7\Phi_8\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{42}$		$\phi_{35,60}$	
$\phi_{35,18}$	$\frac13 x^{10}\Phi_5\Phi_7\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{35,36}$	
$\phi_{56,3}$	$\frac12 x^3\Phi_2^4\Phi_6^4\Phi_7\Phi_{10}\Phi_{14}\Phi_{18}\Phi_{30}\Phi_{21}\Phi_{42}$		$\phi_{56,57}$	
$\phi_{56,9}$	$\frac13 x^5\Phi_4^2\Phi_7\Phi_8\Phi_9\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{56,45}$	
$\phi'_{70,9}$	$x^9\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi'_{70,45}$	
$\phi''_{70,9}$	$-\frac{\zeta_3^2}{3}x^5\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'_{12}}^2\Phi_{14}\Phi'_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi''_{30}\Phi_{42}$	$\phi'''_{70,9}$	$\phi''_{70,45}$	$\phi'''_{70,45}$
$\phi_{84,13}$	$-\frac{\zeta_3^2}{6}x^7{\Phi''_3}^3\Phi_4^2{\Phi'_6}^3\Phi_7\Phi_8\Phi'_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi''_{30}\Phi_{42}$	$\phi_{84,17}$	$\phi_{84,37}$	$\phi_{84,41}$
$\phi_{90,6}$	$\frac13 x^4\Phi_3^3\Phi_5\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{90,48}$	
$\phi_{105,4}$	$-\frac{\zeta_3^2}{6}x^4{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi'_9\Phi_{10}{\Phi''_{12}}^2\Phi_{15}\Phi'_{18}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi'_{105,8}$	$\phi_{105,46}$	$\phi_{105,50}$
$\phi''_{105,8}$	$\frac13 x^6{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_7\Phi''_9\Phi_{10}{\Phi''_{12}}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{105,10}$	$\phi_{105,38}$	$\phi_{105,40}$
$\phi_{105,20}$	$-\frac16 x^{13}{\Phi'_3}^3\Phi_5{\Phi''_6}^6\Phi_7\Phi_8\Phi''_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi'_{30}\Phi_{42}$	$\phi_{105,22}$	$\phi_{105,26}$	$\phi_{105,28}$
$\phi_{120,5}$	$-\frac{\sqrt{-3}}{6}\zeta_3 x^4\Phi_2^4{\Phi'_3}^3\Phi_5\Phi_6^4\Phi''_6\Phi_{10}\Phi'_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi'_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	$\phi_{120,7}$	$\phi_{120,47}$	$\phi_{120,49}$
$\phi'_{120,21}$	$\frac17 x^{15}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi''_{120,21}$	
$\phi_{126,5}$	$\frac13 x^5\Phi_3^3\Phi_6^3\Phi_7\Phi_8\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{126,7}$	$\phi_{126,41}$	$\phi_{126,43}$
$\phi_{140,12}$	$\frac12 x^9\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{140,30}$	
$\phi_{140,21}$	$\frac12 x^{18}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$			
$\phi_{189,18}$	$\frac16 x^{13}\Phi_3^3\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{189,24}$	
$\phi_{210,8}$	$\frac13 x^6{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi'_9\Phi_{10}{\Phi''_{12}}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	$\phi_{210,10}$	$\phi_{210,38}$	$\phi_{210,40}$
$\phi_{210,12}$	$x^{12}\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{210,30}$	
$\phi_{210,13}$	$-\frac{\zeta_3}{3}x^{11}{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi''_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{210,17}$	$\phi_{210,25}$	$\phi_{210,29}$
$\phi'_{280,12}$	$-\frac{\zeta_3^2}{3}x^{10}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	$\phi''_{280,12}$	$\phi'_{280,30}$	$\phi''_{280,30}$
$\phi_{315,6}$	$\frac13 x^6\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{315,36}$	
$\phi_{315,10}$	$\frac13 x^{10}\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{315,14}$	$\phi_{315,28}$	$\phi_{315,32}$
$\phi_{315,18}$	$\frac16 x^{13}\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{315,24}$	
$\phi_{336,8}$	$-\frac{\sqrt{-3}}{6}\zeta_3 x^7\Phi_2^4{\Phi'_3}^3\Phi_6^4\Phi''_6\Phi_7\Phi_8\Phi_{10}\Phi'_{12}\Phi_{14}\Phi''_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{336,10}$	$\phi_{336,32}$	$\phi_{336,34}$
$\phi_{336,17}$	$-\frac{\zeta_3^2}{6}x^{13}\Phi_2^4{\Phi'_3}^3\Phi_6^4{\Phi''_6}^2\Phi_7\Phi_8\Phi'_9\Phi_{10}{\Phi'_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	$\phi_{336,19}$	$\phi_{336,23}$	$\phi_{336,25}$
$\phi_{384,8}$	$-\frac{\sqrt{-3}}{6}\zeta_3 x^8\Phi_2^5{\Phi''_3}^3\Phi_4^2\Phi_6^5\Phi'_6\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi'_{15}\Phi_{18}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{384,10}$	$\phi_{384,32}$	$\phi_{384,34}$
$\phi_{384,11}$	$-\frac{\sqrt{-3}}{6}\zeta_3 x^8\Phi_2^5{\Phi''_3}^3\Phi_4^2\Phi_6^5\Phi'_6\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi'_{15}\Phi_{18}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{384,13}$	$\phi_{384,29}$	$\phi_{384,31}$
$\phi_{420,7}$	$-\frac{\zeta_3^2}{6}x^7{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi''_9\Phi_{12}^2\Phi_{14}\Phi'_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{420,11}$	$\phi_{420,31}$	$\phi_{420,35}$
$\phi_{420,12}$	$\frac12 x^9\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{420,30}$	
$\phi_{420,14}$	$\frac{\zeta_3^2}{6}x^{13}{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^6\Phi_7\Phi_8\Phi'_9{\Phi'_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{420,16}$	$\phi_{420,20}$	$\phi_{420,22}$
$\phi_{420,21}$	$\frac12 x^{18}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$			
$\phi_{504,9}$	$\frac13 x^7\Phi_3^3\Phi_4^2\Phi_6^3\Phi_7\Phi_8\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{504,33}$	
$\phi_{504,15}$	$\frac16 x^{13}\Phi_2^4\Phi_3^3\Phi_6^5\Phi_7\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{504,21}$	
$\phi_{540,17}$	$\frac16 x^{15}\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{540,19}$		
$\phi'_{540,21}$	$-\frac{1}{42}x^{15}{\Phi''_3}^6\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi''_{540,21}$		
$\phi_{560,9}$	$\frac12 x^9\Phi_2^4\Phi_5\Phi_6^4\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{560,27}$	
$\phi'_{560,18}$	$\frac12 x^{18}\Phi_2^4\Phi_5\Phi_6^4\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$			
$\phi''_{560,18}$	$\frac16 x^{15}\Phi_2^4\Phi_5\Phi_6^4{\Phi'_6}^2\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi'''_{560,18}$		
$\phi_{630,11}$	$-\frac{\sqrt{-3}}{3}\zeta_3 x^{11}{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{630,13}$	$\phi_{630,23}$	$\phi_{630,25}$
$\phi_{630,14}$	$\frac{\sqrt{-3}}{3}\zeta_3^2 x^{14}{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi'_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{630,16}$	$\phi_{630,20}$	$\phi_{630,22}$
$\phi_{630,15}$	$\frac13 x^{11}\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{630,27}$	
$\phi_{720,16}$	$\frac16 x^{15}\Phi_2^4\Phi_3^3\Phi_5\Phi_6^5\Phi_8\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{720,20}$		
$\phi_{729,10}$	$\frac13 x^{10}\Phi_3^6\Phi_6^6\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{729,14}$	$\phi_{729,28}$	$\phi_{729,26}$
$\phi_{729,12}$	$\frac13 x^{10}\Phi_3^6\Phi_6^6\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{729,24}$	
$\phi_{756,14}$	$-\frac{\zeta_3^2}{6}x^{13}{\Phi'_3}^6\Phi_4^2{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'_{12}}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{756,16}$	$\phi_{756,20}$	$\phi_{756,22}$
$\phi_{840,11}$	$\frac13 x^{11}{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi''_9\Phi_{10}{\Phi''_{12}}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi'_{840,13}$	$\phi''_{840,23}$	$\phi_{840,25}$
$\phi''_{840,13}$	$\frac16 x^{13}\Phi_2^4{\Phi''_3}^3\Phi_5\Phi_6^4{\Phi'_6}^2\Phi_7\Phi'_9\Phi_{10}{\Phi''_{12}}^2\Phi_{14}\Phi'_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{840,17}$	$\phi_{840,19}$	$\phi'_{840,23}$
$\phi_{896,12}$	$\frac12 x^{12}\Phi_2^5\Phi_4^2\Phi_6^5\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{896,24}$	
$\phi_{896,15}$	$\frac12 x^{12}\Phi_2^5\Phi_4^2\Phi_6^5\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$		$\phi_{896,21}$	
$\phi_{945,14}$	$\frac16 x^{13}{\Phi'_3}^6\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi''_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{945,16}$	$\phi_{945,20}$	$\phi_{945,22}$
$\phi_{1260,17}$	$\frac16 x^{15}\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	$\phi_{1260,19}$		
$\phi_{1280,15}$	$\frac{1}{42}x^{15}\Phi_2^6\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$			
$\phi_{1280,18}$	$\frac16 x^{15}\Phi_2^6\Phi_4^2\Phi_5\Phi_6^4\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$			

5. Schur elements for G_{31}

The 4-dimensional primitive complex reflection group $W = G_{31}$ has a single class of reflections. It differs from all other exceptional reflection groups of dimension n at least three in that it cannot be generated by n of its reflections. It has 59 irreducible characters and 3 conjugacy classes of maximal parabolic subgroups, of types $G(4,2,3)$, A_3, $A_2 + A_1$. In order to be able to use induction on parabolic subalgebras, we first have to compute the Schur elements of the imprimitive group $G(4,2,3)$.

5A. The imprimitive group $G(4,2,3)$. There exists a unique symmetrizing form on the cyclotomic Hecke algebra of the imprimitive group $G(4,2,3)$ restricting to the canonical symmetrizing form on the maximal parabolic subalgebras (of Coxeter type) and vanishing on $T_{\mathbf{c}}^l$ $(1 \le l \le 5)$. The corresponding Schur elements can then be computed from (2.9). It turns out that they agree with the values conjectured in [12, Satz 5.13].

They can also be described in the following way. The group $G(4,2,3)$ occurs as the relative Weyl group of the Φ_4-Sylow torus in finite reductive groups of type D_6. Conjecturally, this should imply that the Schur elements of $\mathcal{H}(G(4,2,3);(x,-1))$ are related to the degree polynomials of the unipotent characters of ${}^2D_6$ in the principal Φ_4-block (see [3]). It turns out that this is in fact true. For this, let $U_4({}^2D_6)$ denote the set of unipotent characters of ${}^2D_6$ whose degree polynomial is not divisible by Φ_4, and write $\deg(\gamma) \in \mathbb{Q}[x]$ for the degree polynomial of $\gamma \in U_4({}^2D_6)$.

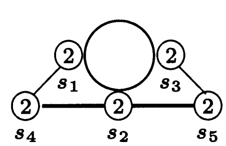
Theorem 5.1. *There is a bijection*

$$\mathrm{Irr}(G(4,2,3)) \longrightarrow U_4({}^2D_6), \qquad \phi \mapsto \gamma_\phi,$$

between the irreducible characters of the cyclotomic algebra $\mathcal{H}(G(4,2,3);(x,-1))$ and $U_4({}^2D_6)$ (see Table 5.3) such that the Schur element c_ϕ of $\phi \in \mathrm{Irr}(G(4,2,3))$ is given by

$$c_\phi(x) = \frac{(x+1)(x^2-1)(x^4-1)(x^5+1)(x^6-1)}{(x-1)^3 \deg(\gamma_\phi)(i\sqrt{x})}.$$

5B. The complex reflection group G_{31}. We return to the primitive reflection group $W = G_{31}$. Twenty of its conjugacy classes have non-empty intersection with proper parabolic subgroups, so by restriction we obtain 20 independent equations for the Schur elements. Let $\{s_1, \dots, s_5\}$ denote the set of standard generators of W, with the relations implied by the Coxeter diagram



together with $s_1s_2s_3 = s_2s_3s_1 = s_3s_1s_2$. The Coxeter element $\mathbf{c} := \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\mathbf{s}_5$ of the braid group $B(W)$ satisfies $\mathbf{c}^{24} = \boldsymbol{\pi}$ (see [5]). The equations given by evaluating the irreducible characters of $\mathcal{H}$ on $T_{\mathbf{c}}^m$, $1 \le m \le 23$, are linearly independent from the equations obtained by restriction.

TABLE 5.2. A bijection $\mathrm{Irr}(G(4,2,3)) \longleftrightarrow U_4({}^2D_6)$

ϕ	γ_ϕ	$\bar\phi$	$\epsilon\otimes\phi$	$\epsilon\otimes\bar\phi$
$\phi_{1,0}$	$(06\vert-)$		$\phi_{1,15}$	
$\phi_{1,3}$	$(016\vert 1)$		$\phi_{1,12}$	
$\phi_{2,4}$	$(0126\vert 12)$		$\phi_{2,7}$	
$\phi_{3,2,6}$	$(24\vert-)$		$\phi_{3,9}$	
$\phi_{3,5,9}$	$(124\vert 1)$		$\phi_{3,6,10}$	
$\phi_{3,1}$	$(012\vert 5)$	$\phi_{3,3}$	$\phi_{3,8}$	$\phi_{3,6,14}$
$\phi_{3,2,10}$	$(013\vert 4)$	$\phi_{3,4}$	$\phi_{3,7}$	$\phi_{3,5,13}$
$\phi_{6,3}$	$(023\vert 3)$		$\phi_{6,4}$	

Here, we denote the unipotent characters of ${}^2D_6$ by symbols of rank 6 as in [10].

TABLE 5.3. A bijection $\mathrm{Irr}(G_{31}) \longleftrightarrow U_4(E_8)$

ϕ	γ_ϕ	$\bar\phi$	$\epsilon\otimes\phi$	$\epsilon\otimes\bar\phi$
$\phi_{1,0}$	$\phi_{1,0}$		$\phi_{1,60}$	
$\phi_{4,1}$	$D_4,\phi_{1,0}$	$\phi_{4,7}$	$\phi_{4,37}$	$\phi_{4,31}$
$\phi_{5,4}$	$\phi_{35,2}$		$\phi_{5,40}$	
$\phi_{5,12}$	$\phi_{525,12}$		$\phi_{5,24}$	
$\phi_{6,14}$	$\phi_{70,32}$	$\phi_{6,18}$		
$\phi_{9,8}$	$\phi_{567,6}$		$\phi_{9,28}$	
$\phi_{10,2}$	$\phi_{50,8}$	$\phi_{10,6}$	$\phi_{10,30}$	$\phi_{10,26}$
$\phi_{10,12}$	$\phi_{1050,10}$		$\phi_{10,24}$	
$\phi'_{15,8}$	$\phi_{1575,10}$		$\phi'_{15,20}$	
$\phi''_{15,8}$	$\phi_{175,12}$		$\phi''_{15,20}$	
$\phi_{16,16}$	$D_4,\phi_{16,5}$			
$\phi_{20,3}$	$D_4,\phi'_{1,12}$	$\phi_{20,5}$	$\phi_{20,23}$	$\phi_{20,21}$
$\phi_{20,7}$	$\phi_{2800,13}$	$\phi'_{20,13}$	$\phi_{20,19}$	$\phi''_{20,13}$
$\phi_{20,14}$	$\phi_{2100,20}$			
$\phi_{24,6}$	$E_8^{\mathrm{II}}[1]$	$\phi_{24,14}$		
$\phi_{30,4}$	$\phi_{350,14}$		$\phi_{30,16}$	
$\phi'_{30,10}$	$\phi_{1134,20}$	$\phi''_{30,10}$		
$\phi_{36,5}$	$\phi_{1296,13}$	$\phi_{36,7}$	$\phi_{36,17}$	$\phi_{36,15}$
$\phi_{36,10}$	$\phi_{420,20}$			
$\phi_{40,6}$	$D_4,\phi'_{8,3}$		$\phi_{40,18}$	
$\phi_{40,7}$	$\phi_{3360,13}$	$\phi_{40,9}$	$\phi_{40,15}$	$\phi_{40,13}$
$\phi_{40,10}$	$\phi_{2688,20}$	$\phi_{40,14}$		
$\phi'_{45,8}$	$\phi_{2835,14}$		$\phi'_{45,12}$	
$\phi''_{45,8}$	$\phi_{6072,14}$		$\phi''_{45,12}$	
$\phi_{64,9}$	$E_8[i]$	$\phi_{64,11}$		

Furthermore, let $\mathbf{w} = \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\mathbf{s}_5\mathbf{s}_1 = \mathbf{c}\mathbf{s}_1$. From the defining relations of $B(W)$ it follows that $\mathbf{w}^5 = \mathbf{c}^6$, hence $\mathbf{w}$ is a 20-th root of $\boldsymbol{\pi}$ in $B(W)$. (In fact, the image of $\mathbf{w}$ in W is a Coxeter element in the maximal rank reflection subgroup $G_{29} < G_{31}$.) Evaluating characters on powers of $T_{\mathbf{w}}$ give 14 independent equations. The remaining two equations are obtained by evaluating the irreducible characters on products of central elements by elements in proper parabolic subalgebras.

The group G_{31} occurs as the relative Weyl group of the Φ_4-Sylow torus in finite reductive groups of type E_8. Conjecturally, this should imply that the Schur elements of $\mathcal{H}(W,x)$ are related to the degree polynomials of the unipotent characters of E_8 in the principal Φ_4-block. It turns out that this is in fact true. For this, let $U_4(E_8)$ denote the set of unipotent characters of E_8 whose degree polynomial is not divisible by Φ_4, and write $\deg(\gamma) \in \mathbb{Q}[x]$ for the degree polynomial of $\gamma \in U_4(E_8)$. We follow the naming convention in [10] for the unipotent characters of E_8.

Theorem 5.4. *There is a bijection*

$$\operatorname{Irr}(G_{31}) \longrightarrow U_4(E_8), \qquad \phi \mapsto \gamma_\phi,$$

between the irreducible characters of the cyclotomic Hecke algebra $\mathcal{H}(G_{31};(x,-1))$ and $U_4(E_8)$ (given by Table 5.3). This bijection is such that the Schur element c_ϕ of $\phi \in \operatorname{Irr}(G_{31})$ with respect to any quasi-symmetric basis containing $T_{\mathbf{c}}^m$ $(1 \le m \le 23)$ and $T_{\mathbf{w}}^m$ $(1 \le m \le 16)$ and elements of the form $T_{\mathbf{v}}T_{\mathbf{c}}^{6l}$ $(0 \le l \le 3)$ with $\mathbf{v}$ contained in a proper parabolic subgroups of $B(W)$ is given by

$$c_\phi(x) = \frac{(x+1)(x^4-1)(x^6-1)(x^7+1)(x^9+1)(x^{10}-1)(x^{12}-1)(x^{15}+1)}{(x-1)^4 \deg(\gamma_\phi)(i\sqrt{x})}\,.$$

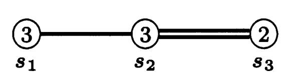
Except for five pairs the irreducible characters of G_{31} are determined by their degree and their b-invariant. We denote by $\phi'_{15,8}$, $\phi'_{15,20}$, $\phi''_{45,8}$ the irreducible characters of the respective types occurring in the tensor product $\phi_{4,1} \otimes \phi_{20,7}$, and we let $\phi'_{30,10}$ be the character of degree 30 with fake degree $x^{50}+x^{46}+$ lower powers of x. The remaining notation is now fixed by the information in Table 5.3.

6. Schur elements for G_{25}, G_{26} and G_{32}

The three exceptional reflection groups G_{25}, G_{26} and G_{32} do contain reflections of order 3. Thus their Schur elements cannot be recovered from a 1-parameter specialization.

6A. The complex reflection group G_{26}. The 3-dimensional primitive complex reflection group $W = G_{26}$ is the direct product of G_{25} with a central element of order 2. It can be shown (see [3, Bem. 4.14(b)]) that the generic cyclotomic algebra for G_{25} is a subalgebra of index 2 of a suitable specialization of the cyclotomic algebra for G_{26}. This allows us to study G_{25} using results from G_{26}. So we will first consider the latter group.

The complex reflection group $W = G_{26}$ is generated by three reflections s_1, s_2, s_3 of orders 3,3,2 respectively, which satisfy the defining relations indicated by this Coxeter type diagram:



We denote by $T_{\mathbf{s}_1}, T_{\mathbf{s}_2}, T_{\mathbf{s}_3}$ the corresponding generators of the generic cyclotomic algebra $\mathcal{H} = \mathcal{H}(W,(\mathbf{x},\mathbf{y}))$ of type G_{26}, hence

$$(T_{\mathbf{s}_j}-y_1)(T_{\mathbf{s}_j}-y_2)(T_{\mathbf{s}_j}-y_3)=0 \quad \text{for } j=1,2, \quad (T_{\mathbf{s}_3}-x_1)(T_{\mathbf{s}_3}-x_2)=0,$$

with the five parameters $\mathbf{x} = (x_1,x_2), \mathbf{y} = (y_1,y_2,y_3)$. The group W has three conjugacy classes of maximal parabolic subgroups of types $G(3,1,2)$, $G(3,1,1)+A_1$ and G_4. Fifteen of the 48 conjugacy classes of G_{26} have a representative in one of these parabolic subgroups. The Coxeter element $\mathbf{c} := \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3$ of the braid group $B(W)$ satisfies $\mathbf{c}^{18} = \boldsymbol{\pi}$ (see [5]). Its third power $\mathbf{c}^3$ is central. Since the irreducible representations of all proper parabolic subalgebras are explicitly known by [13, 3A] and [3, 5B], we may evaluate the irreducible characters of $\mathcal{H}$ on all elements of the form $T_{\mathbf{c}}^l$ or $T_{\mathbf{w}}T_{\mathbf{c}}^{3l}$ where $T_{\mathbf{w}}$ lies in some maximal parabolic subalgebra. This gives enough linear independent equations for the determination of Schur elements.

For $1 \le j \le 3$ let $\mathbf{z} := (z_1, z_2, z_3)$ with $z_j := \sqrt{-x_1x_2y_1y_2y_3/y_j}$ in an algebraic closure of $\mathbb{Q}(\mathbf{x}, \mathbf{y})$. We consider the specialization

$$x_j \mapsto (-1)^{j-1}, \qquad y_j \mapsto \zeta_3^{j-1}, \qquad z_j \mapsto (-\zeta_3^2)^j,$$

of the generic cyclotomic algebra $\mathcal{H}$ to the group algebra of W. By [15, Table 8.2] (or [3, 6A]) the field $\mathbb{Q}(\zeta_3)[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ is a splitting field for $\mathcal{H}(W)$, so by Tits' deformation theorem this identifies $\mathrm{Irr}(\mathcal{H}(W))$ with $\mathrm{Irr}(W)$. According to 2D the symmetry group $\mathfrak{S} = \mathfrak{S}_2 \times \mathfrak{S}_3$ of W acts on $\mathrm{Irr}(W)$ by permuting the first two and the last three parameters among themselves. In order to minimize the number of Schur elements to be printed, we record in Table 6.1 the action of $\mathfrak{S}$ on $\mathrm{Irr}(W)$ induced by the (left) action on $\{x_1, x_2, y_1, y_2, y_3\}$. We let $\tau_1 := (x_1, x_2)$, $\tau_2 := (y_1, y_2)$ and $\sigma := (y_1, y_2, y_3)$.

TABLE 6.1. $\mathfrak{S}_2 \times \mathfrak{S}_3$-action on $\mathrm{Irr}(G_{26})$

ι	σ	σ^2	τ_2	$\sigma\tau_2$	$\tau_2\sigma$	τ_1	$\tau_1\sigma$	$\tau_1\sigma^2$	$\tau_1\tau_2$	$\tau_1\sigma\tau_2$	$\tau_1\tau_2\sigma$
$\phi_{1,0}$	$\phi_{1,12}$	$\phi_{1,24}$	$\phi_{1,12}$	$\phi_{1,24}$	$\phi_{1,0}$	$\phi_{1,9}$	$\phi_{1,21}$	$\phi_{1,33}$	$\phi_{1,21}$	$\phi_{1,33}$	$\phi_{1,9}$
$\phi_{2,3}$	$\phi_{2,15}$	$\phi_{2,9}$	$\phi_{2,3}$	$\phi_{2,15}$	$\phi_{2,9}$	$\phi_{2,12}$	$\phi_{2,24}$	$\phi_{2,18}$	$\phi_{2,12}$	$\phi_{2,24}$	$\phi_{2,18}$
$\phi_{3,1}$	$\phi'_{3,13}$	$\phi''_{3,13}$	$\phi''_{3,5}$	$\phi_{3,17}$	$\phi'_{3,5}$	$\phi_{3,4}$	$\phi'_{3,16}$	$\phi''_{3,16}$	$\phi''_{3,8}$	$\phi_{3,20}$	$\phi'_{3,8}$
$\phi_{3,6}$						$\phi_{3,15}$					
$\phi_{6,2}$	$\phi'_{6,8}$	$\phi''_{6,8}$	$\phi''_{6,4}$	$\phi_{6,10}$	$\phi'_{6,4}$	$\phi_{6,5}$	$\phi'_{6,11}$	$\phi''_{6,11}$	$\phi''_{6,7}$	$\phi_{6,13}$	$\phi'_{6,7}$
$\phi_{8,3}$	$\phi''_{8,6}$	$\phi''_{8,9}$	$\phi''_{8,6}$	$\phi''_{8,9}$	$\phi_{8,3}$						
$\phi'_{8,6}$	$\phi'_{8,9}$	$\phi_{8,12}$	$\phi'_{8,9}$	$\phi_{8,12}$	$\phi'_{8,6}$						
$\phi_{9,5}$						$\phi_{9,8}$					
$\phi_{9,7}$						$\phi_{9,10}$					

We denote the elements of $\mathrm{Irr}(W)$ by $\phi_{d,b}$, where $d = \phi_{d,b}(1)$ and b is the b-value of $\phi_{d,b}$. This distinguishes the characters except for ten pairs. The eight pairs with $d = 3$ or $d = 6$ have the property that the complex conjugate of any character in a pair is already determined by d, b. We will see in Theorem 6.2 that the generic degrees for the relevant characters are $\mathbb{Q}(\mathbf{x}, \mathbf{y})$-rational. Thus interchanging y_2 with y_3 induces complex conjugation on $\mathrm{Irr}(W)$. So the complex conjugates can be read of from Table 6.1 and thus also the notation for the first eight ambiguous pairs is fixed. In each of the two pairs of characters of degree 8, although the b-values agree, the fake degrees themselves are different for the members of a pair. We choose notation such that the fake degrees of $\phi'_{8,6}$ and $\phi'_{8,9}$ are monic as polynomials in x. This fixes the notation for the irreducible characters of W.

Theorem 6.2. *The Schur elements of the cyclotomic algebra $\mathcal{H}(G_{26}; (\mathbf{x}, \mathbf{y}))$ with respect to any quasi-symmetric basis containing $T_{\mathbf{c}}^l$ $(1 \le l \le 7)$ and $T_{\mathbf{w}}T_{\mathbf{c}}^{3l}$ $(0 \le l \le 5)$ for $\mathbf{w}$ running through representatives of minimal length of the conjugacy classes* *of the maximal parabolic subgroups of W are as given below:*

$\phi_{1,0}$:$(x_2-x_1)(y_1-y_2)(y_1-y_3)(x_1y_1+x_2y_2)(x_1y_1+x_2y_3)(x_1y_1^2-x_2y_2^2)$
$(x_1y_1^2-x_2y_3^2)(x_1y_1^3+x_2y_2^2y_3)(x_1y_1^3+x_2y_2y_3^2)(x_1^2y_1^4-x_1x_2y_1^2y_2y_3+x_2^2y_2^2y_3^2)$
$(y_1^2+y_2y_3)(y_1^2-y_1y_2+y_2^2)(y_1^2-y_1y_3+y_3^2)/x_2^9y_2^{12}y_3^{12}$,

$\phi_{2,3}$:$(x_1-x_2)(y_1-y_3)(y_2-y_3)(x_1y_1-x_2y_3)(x_1y_2-x_2y_3)(x_1y_1+x_2y_3)$
$(x_1y_2+x_2y_3)(x_1y_1+x_2y_2)(x_1y_2+x_2y_1)(x_1^2y_1^2y_2^2-x_1x_2y_1y_2y_3^2+x_2^2y_3^4)$
$(y_1y_2+y_3^2)(y_1^2-y_1y_2+y_2^2)/x_2^9y_1^2y_2^2y_3^{12}$,

$\phi_{3,1}$:$(x_1-x_2)(y_1-y_2)(y_1^2-y_3^2)(y_2-y_3)(y_1y_2+y_3^2)(y_1^2+y_2y_3)(y_1^2-y_1y_3+y_3^2)$
$(x_1y_1+x_2y_3)(x_1y_2+x_2y_1)(x_1y_1^2-x_2y_2^2)(y_1^2y_2x_1+y_3^3x_2)/x_1x_2^4y_1y_2^4y_3^{12}$,

$\phi_{3,6}$:$(x_2^3-x_1^3)(x_1y_1+x_2y_2)(x_1y_1+x_2y_3)(x_1y_2+x_2y_1)(x_1y_2+x_2y_3)(x_1y_3+x_2y_1)$
$(x_1y_3+x_2y_2)(y_1y_2+y_3^2)(y_1y_3+y_2^2)(y_2y_3+y_1^2)/x_2^9y_1^4y_2^4y_3^4$,

$\phi_{6,2}$:$(x_2-x_1)(y_1-y_2)(y_1-y_3)(y_2-y_3)(y_1^3+y_3^3)(y_1y_3+y_2^2)(x_1y_2-x_2y_3)$
$(x_1y_3+x_2y_1)(x_1y_2+x_2y_3)(x_1y_1^3+x_2y_2^2y_3)/x_1x_2^4y_1^2y_2^4y_3^8$,

$\phi_{8,3}$:$2(y_1-y_2)(y_1-y_3)(x_1y_2+x_2y_3)(x_2y_2+x_1y_3)(z_3y_1+x_2y_3^2)(x_2y_1y_2-z_3y_3)$
$(z_3y_1+x_2y_2^2)(x_2y_1y_3-z_3y_2)(x_1y_1^2-z_3y_1-x_2y_2y_3)$
$(x_2y_1^2-z_3y_1-x_1y_2y_3)/x_1^3x_2^5y_1^2y_2^7y_3^7$,

$\phi_{9,5}$:$(\zeta_3^2-1)(y_1^2-y_1y_2+y_2^2)(y_1^2-y_1y_3+y_3^2)(y_2^2-y_2y_3+y_3^2)(x_2y_1^2+\zeta_3^2x_1y_2y_3)$,
$(x_2y_2^2+\zeta_3^2x_1y_1y_3)(x_2y_3^2+\zeta_3^2x_1y_1y_2)(x_1-x_2)(x_1-\zeta_3x_2)/x_1x_2^4y_1^4y_2^4y_3^4$.

The Schur element of $\phi'_{8,6}$ is the image of the one for $\phi_{8,3}$ under the non-trivial automorphism of $k(\mathbf{x},\mathbf{y},z_3)/k(\mathbf{x},\mathbf{y})$, the Schur element of $\phi_{9,7}$ is the complex conjugate of the one for $\phi_{9,5}$.

6B. The complex reflection group G_{25}. In order to treat the 3-dimensional primitive complex reflection group $W_1 := G_{25}$ we consider the specialization

$$x_1 \mapsto 1, \quad x_2 \mapsto -1$$

of the generic cyclotomic algebra for G_{26}. By [3, Bem. 4.14] its generic cyclotomic algebra $\mathcal{H}_1 := \mathcal{H}(G_{25};\mathbf{y})$ is a subalgebra of index 2 of the specialization $\mathcal{H}' := \mathcal{H}(G_{26};((1,-1),\mathbf{y}))$. It was shown in [3, Satz 4.7] that $\mathcal{H}_1$ satisfies Assumption 2.5. Furthermore, by [15, Table 8.2] (or [3, Bem. 6.5]) the field $\mathbb{Q}(\zeta_3)(\mathbf{y})$ is a splitting field for $\mathcal{H}_1$. So by Tits' deformation theorem the specialization $y_j \mapsto \zeta_3^j$ induces a bijection between $\mathrm{Irr}(\mathcal{H}_1)$ and $\mathrm{Irr}(W_1)$. On the other hand, W is isomorphic to the direct product of W_1 with the cyclic group of order 2. Since W_1 has no normal subgroup of order 2, this allows us to identify $\mathrm{Irr}(W_1)$ with the subset of elements of $\mathrm{Irr}(W)$ having this direct factor in the kernel. It turns out that these are precisely the characters appearing in the first six columns of Table 6.1.

Having thus given a labeling for the irreducible characters of $\mathcal{H}(W_1)$ by a subset of $\mathrm{Irr}(W)$, we may state the following:

Theorem 6.3. *The generic degree d_ϕ of an irreducible character ϕ of the cyclotomic Hecke algebra $\mathcal{H}_1 = \mathcal{H}(G_{25};\mathbf{y})$ with respect to the restriction of a symmetric form on $\mathcal{H}' = \mathcal{H}(G_{26};\mathbf{y},1,-1)$ as in Theorem 6.2 is given by twice the corresponding generic degree for $\mathcal{H}'$.*

Proof. This follows from the above observations by an easy descent argument given for example in [12, Lemma 5.11]. □

These Schur elements were already computed in [3, Bem. 5.15], but without giving the labeling by irreducible characters of G_{25}, so we include them once more for the convenience of the reader:

$$
\begin{aligned}
\phi_{1,0} :&(y_1 - y_2)^2(y_1 - y_3)^2(y_1^2 + y_2^2)(y_1^2 + y_3^2)(y_1^3 - y_2^2y_3)(y_1^3 - y_2y_3^2)(y_1^2 + y_2y_3)\\
&(y_1^4 + y_1^2y_2y_3 + y_2^2y_3^2)(y_1^2 - y_1y_2 + y_2^2)(y_1^2 - y_1y_3 + y_3^2)/y_2^{12}y_3^{12}\\
\phi_{2,3} :&(y_1 - y_2)^2(y_1 - y_3)^2(y_2 - y_3)^2(y_1 + y_3)(y_2 + y_3)(y_1^2y_2^2 + y_1y_2y_3^2 + y_3^4)\\
&(y_1y_2 + y_3^2)(y_1^2 - y_1y_2 + y_2^2)/y_1^2y_2^2y_3^{12},\\
\phi_{3,1} :&- (y_1 - y_2)^2(y_1 - y_3)^2(y_2 - y_3)(y_1y_2 + y_3^2)(y_1^2 + y_2y_3)(y_1^3 + y_3^3)(y_1^2 + y_2^2)\\
&(y_1^2y_2 - y_3^3)/y_1y_2^4y_3^{12},\\
\phi_{3,6} :&- (y_1 - y_2)^2(y_1 - y_3)^2(y_2 - y_3)^2(y_1y_2 + y_3^2)(y_1y_3 + y_2^2)(y_1^2 + y_2y_3)/y_1^4y_2^4y_3^4,\\
\phi_{6,2} :&(y_1-y_2)(y_1-y_3)^2(y_2-y_3)^2(y_1^3 + y_3^3)(y_1y_3 + y_2^2)(y_2 + y_3)(y_1^3 - y_2^2y_3)/y_1^2y_2^4y_3^8,\\
\phi_{8,3} :&(y_1-y_3)(y_1-y_2)(y_2-y_3)^2(y_2^3-y_1^2y_3)(y_1^2y_2-y_3^3)(y_2^2y_3^2 + y_1^2y_2y_3 + y_1^4)/y_1^2y_2^6y_3^6,\\
\phi_{9,5} :&(\zeta_3 - \zeta_3^2)(y_1^2 - y_1y_2 + y_2^2)(y_1^2 - y_1y_3 + y_3^2)(y_2^2 - y_2y_3 + y_3^2)(y_1^2 - \zeta_3^2y_2y_3)\\
&(y_2^2 - \zeta_3^2y_1y_3)(y_3^2 - \zeta_3^2y_1y_2)/y_1^4y_2^4y_3^4.
\end{aligned}
$$

The Schur element of $\phi_{9,7}$ is the complex conjugate of the one for $\phi_{9,5}$.

6C. The complex reflection group G_{32}. The 4-dimensional primitive complex reflection group $W = G_{32}$ is generated by four reflections $s_1, \dots, s_4$ of order 3, which satisfy the braid relations of the A_4-diagram. It has two conjugacy classes of maximal parabolic subgroups of types G_{25} and $G(3,1,1) + G_4$. Out of the 102 conjugacy classes of G_{32}, 32 have a representative in one of these parabolic subgroups. For both of these, a symmetrizing form and the corresponding Schur elements are known by virtue of the previous section and by [3], respectively. Let $T_{\mathbf{s}_1}, \dots, T_{\mathbf{s}_4}$ be the generators of the generic cyclotomic algebra $\mathcal{H} = \mathcal{H}(W, \mathbf{y})$ of type G_{32} satisfying

$$(T_{\mathbf{s}_j} - y_1)(T_{\mathbf{s}_j} - y_2)(T_{\mathbf{s}_j} - y_3) = 0 \quad \text{for } j = 1, \dots, 4,$$

with the parameters $\mathbf{y} = (y_1, y_2, y_3)$.

We consider the specialization $y_j \mapsto \zeta_3^{j-1}$ of $\mathcal{H}$ to the group algebra of W. By [15, Table 8.2] (or [3, 6A]) the field $K_W := \mathbb{Q}(\zeta_3)[\sqrt[6]{y_j} \mid 1 \le j \le 3]$ is a splitting field for $\mathcal{H}(W; \mathbf{y})$. By 2D the symmetry group $\mathfrak{S} = \mathfrak{S}_3$ of W in its action on the parameters induces an action on $\mathrm{Irr}(W)$. As in the case of G_{26} we first give a table showing the action of $\mathfrak{S}$ on $\mathrm{Irr}(W)$ induced by the (left) action on $\{y_1, y_2, y_3\}$.

The three characters $\phi_{81,10}$, $\phi_{81,12}$, $\phi_{81,14}$ are invariant under the $\mathfrak{S}_3$-action.

We denote the elements of $\mathrm{Irr}(W)$ by $\phi_{d,b}$, where $d = \phi_{d,b}(1)$ and b is the b-value of $\phi_{d,b}$. This distinguishes the characters except for six pairs. We denote by $\phi''_{30,12}$ and $\phi''_{60,15}$ the only two rational characters in these pairs. The other characters in these pairs have the property that their complex conjugate is already determined by its d, b. Interchanging y_2 with y_3 induces complex conjugation on $\mathrm{Irr}(W)$. So the complex conjugates can be read of from Table 6.4 and thus the notation for the ambiguous pairs is fixed.

TABLE 6.4. $\mathfrak{S}_3$-action on $\mathrm{Irr}(G_{32})$

(1)	$(1,2,3)$	$(1,3,2)$	$(1,2)$	$(1,3)$	$(2,3)$
$\phi_{1,0}$	$\phi_{1,40}$	$\phi_{1,80}$	$\phi_{1,40}$	$\phi_{1,80}$	$\phi_{1,0}$
$\phi_{4,1}$	$\phi_{4,41}$	$\phi_{4,51}$	$\phi_{4,21}$	$\phi_{4,61}$	$\phi_{4,11}$
$\phi_{5,4}$	$\phi_{5,44}$	$\phi_{5,36}$	$\phi_{5,12}$	$\phi_{5,52}$	$\phi_{5,20}$
$\phi_{6,8}$	$\phi_{6,48}$	$\phi_{6,28}$	$\phi_{6,8}$	$\phi_{6,48}$	$\phi_{6,28}$
$\phi_{10,2}$	$\phi_{10,30}$	$\phi_{10,34}$	$\phi_{10,14}$	$\phi_{10,42}$	$\phi_{10,10}$
$\phi_{15,6}$	$\phi_{15,22}$	$\phi_{15,38}$	$\phi_{15,22}$	$\phi_{15,38}$	$\phi_{15,6}$
$\phi_{15,8}$	$\phi_{15,24}$	$\phi_{15,16}$	$\phi_{15,8}$	$\phi_{15,24}$	$\phi_{15,16}$
$\phi_{20,3}$	$\phi_{20,25}$	$\phi'_{20,29}$	$\phi_{20,13}$	$\phi_{20,35}$	$\phi'_{20,9}$
$\phi_{20,5}$	$\phi_{20,33}$	$\phi_{20,19}$	$\phi_{20,5}$	$\phi_{20,33}$	$\phi_{20,19}$
$\phi_{20,7}$	$\phi''_{20,29}$	$\phi_{20,21}$	$\phi''_{20,9}$	$\phi_{20,31}$	$\phi_{20,17}$
$\phi_{20,12}$	$\phi_{20,16}$	$\phi_{20,20}$	$\phi_{20,16}$	$\phi_{20,20}$	$\phi_{20,12}$
$\phi_{24,6}$	$\phi_{24,16}$	$\phi_{24,26}$	$\phi_{24,16}$	$\phi_{24,26}$	$\phi_{24,6}$
$\phi_{30,4}$	$\phi''_{30,20}$	$\phi_{30,24}$	$\phi'_{30,12}$	$\phi_{30,28}$	$\phi_{30,8}$
$\phi''_{30,12}$	$\phi_{30,16}$	$\phi'_{30,20}$	$\phi_{30,16}$	$\phi'_{30,20}$	$\phi''_{30,12}$
$\phi_{36,5}$	$\phi_{36,15}$	$\phi_{36,25}$	$\phi_{36,15}$	$\phi_{36,25}$	$\phi_{36,5}$
$\phi_{36,7}$	$\phi_{36,17}$	$\phi_{36,27}$	$\phi_{36,17}$	$\phi_{36,27}$	$\phi_{36,7}$
$\phi_{40,8}$	$\phi_{40,18}$	$\phi_{40,22}$	$\phi_{40,14}$	$\phi_{40,24}$	$\phi_{40,10}$
$\phi_{45,6}$	$\phi_{45,22}$	$\phi_{45,14}$	$\phi_{45,6}$	$\phi_{45,22}$	$\phi_{45,14}$
$\phi_{45,10}$	$\phi_{45,26}$	$\phi_{45,18}$	$\phi_{45,10}$	$\phi_{45,26}$	$\phi_{45,18}$
$\phi_{60,7}$	$\phi_{60,17}$	$\phi'_{60,15}$	$\phi_{60,9}$	$\phi_{60,19}$	$\phi'_{60,11}$
$\phi''_{60,11}$	$\phi''_{60,15}$	$\phi_{60,13}$	$\phi''_{60,11}$	$\phi''_{60,15}$	$\phi_{60,13}$
$\phi_{60,12}$	$\phi_{60,16}$	$\phi_{60,20}$	$\phi_{60,16}$	$\phi_{60,20}$	$\phi_{60,12}$
$\phi_{64,8}$	$\phi_{64,18}$	$\phi_{64,13}$	$\phi_{64,8}$	$\phi_{64,18}$	$\phi_{64,13}$
$\phi_{64,11}$	$\phi_{64,21}$	$\phi_{64,16}$	$\phi_{64,11}$	$\phi_{64,21}$	$\phi_{64,16}$
$\phi_{80,9}$	$\phi_{80,13}$	$\phi_{80,17}$	$\phi_{80,13}$	$\phi_{80,17}$	$\phi_{80,9}$

We denote by $\mathbf{c} := \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4$ the Coxeter element of the braid group $B(W)$ of W. It satisfies $\mathbf{c}^{30} = \boldsymbol{\pi}$ and its fifth power is central in $B(W)$. Let $\mathbf{w} := \mathbf{s}_1^2\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4 = \mathbf{s}_1\mathbf{c}$. By direct computation with the defining relations of $B(W)$ we find that $\mathbf{w}^4 = \mathbf{c}^5$, hence $\mathbf{w}$ is a 24-th root of $\boldsymbol{\pi}$ in the braid group.

Theorem 6.5. *The Schur elements of the cyclotomic Hecke algebra $\mathcal{H}(G_{32}; \mathbf{y})$ with respect to any quasi-symmetric basis only containing $T_{\mathbf{c}}^l$ $(1 \le l \le 29)$, $T_{\mathbf{w}}^l$ $(1 \le l \le 23)$ and elements of the form $T_{\mathbf{v}}T_{\mathbf{c}}^{5l}$ $(1 \le l \le 5)$ for $\mathbf{v}$ running through representatives of minimal length of the conjugacy classes of the maximal parabolic subgroups of $B(W)$ are given in the table below.*

Proof. The character values of $\mathcal{H}$ on the powers of $T_{\mathbf{c}}$ and of $T_{\mathbf{w}}$ can be evaluated with Proposition 2.8. Moreover, character values on all elements of proper parabolic subalgebras are known by the preceding section, respectively by [3]. Since $T_{\mathbf{c}}^{5l}$ is central in $\mathcal{H}$, the values on any product $T_{\mathbf{v}}T_{\mathbf{c}}^{5l}$ can be determined, once those on $T_{\mathbf{v}}$ are known. Thus the values of all irreducible characters on all the elements occurring in the statement can be computed. Using the computer algebra system GAP it was checked that this system has maximal rank, and thus determines the generic degrees uniquely. ∎

$$\begin{aligned}
\phi_{1,0}:&(y^2z^2+yzx^2+x^4)(y^2-yx+x^2)(z^2-zx+x^2)(y^2z-x^3)(yz^2-x^3)\\
&(y^4z^4+y^3z^3x^2+y^2z^2x^4+yzx^6+x^8)(y^4-y^3x+y^2x^2-yx^3+x^4)\\
&(z^4-z^3x+z^2x^2-zx^3+x^4)(y^3z+x^4)(yz^3+x^4)(y^4z^2-y^2zx^3+x^6)\\
&(y^2z^4-yz^2x^3+x^6)(y^2z^3-x^5)(y^3z^2-x^5)(y^2z^2+x^4)(y^3z^3+x^6)(y^2+x^2)\\
&(z^2+x^2)(yz+x^2)(y-x)^2(z-x)^2/y^{40}z^{40},\\
\phi_{4,1}:&(x^8-x^7z+z^3x^5-x^4z^4+z^5x^3-z^7x+z^8)(y^4-y^3x+y^2x^2-yx^3+x^4)\\
&(z^2y^2+zyx^2+x^4)(x^3+z^3)(y^2-yx+x^2)(z^4y+x^5)(yx^3+z^4)(z^3y-x^4)\\
&(z^2y-x^3)(zy^2-x^3)(yx^2-z^3)(z^2y+x^3)(x^2+z^2)(yz+x^2)(xy+z^2)\\
&(x-z)^2(x-y)(y-z)/xy^{15}z^{40},\\
\phi_{5,4}:&(y^2x^2+yxz^2+z^4)(z^4-z^2x^2+x^4)(x^4y+z^5)(y^2x^3-z^5)(z^2-zx+x^2)^2\\
&(y^2-yx+x^2)(yx^2-z^3)(z^2y+x^3)(y^2x+z^3)(y^2+x^2)(zy+x^2)(yx+z^2)\\
&(z-x)^2(y-x)^2(z-y)^2(z+y)(z+x)^2/x^2y^6z^{40},\\
\phi_{6,8}:&(x^4y^4+x^3y^3z^2+x^2y^2z^4+xyz^6+z^8)(x^4-x^3y+x^2y^2-xy^3+y^4)\\
&(x^3y^3+z^6)(z^2-zy+y^2)(z^2-zx+x^2)(z^3-x^2y)(xy^2-z^3)(x^2y+z^3)\\
&(xy^2+z^3)(zx+y^2)(zy+x^2)(xy+z^2)(z-y)^2(x-y)^2(z-x)^2(z+x)\\
&(z+y)/x^4y^4z^{40},\\
\phi_{10,2}:&-(x^4y^2-x^2yz^3+z^6)(y^2z^2+yzx^2+x^4)(x^2-xz+z^2)(x^2-xy+y^2)\\
&(y^3z+x^4)(x^2y-z^3)(yz^2-x^3)(yz^2+x^3)(x^2+z^2)(xy+z^2)\\
&(xz+y^2)(yz+x^2)(x-z)^3(y-z)^2(x-y)(x+z)^2(y+z)/x^2y^{12}z^{30},\\
\phi_{15,6}:&(y^2z^2-yzx^2+x^4)(y^2z^2+x^4)(x^2-xz+z^2)(x^2-xy+y^2)(x^3y+z^4)\\
&(x^3z+y^4)(x^3-y^2z)(yz^2-x^3)(xy+z^2)(xz+y^2)(yz+x^2)^2(x-z)^2\\
&(y-z)^2(x-y)^2(x+y)(x+z)/x^2y^{20}z^{20},\\
\phi_{15,8}:&(x^2y^2+z^4)(z^2-zy+y^2)(z^2-zx+x^2)(zx^2-y^3)(x^3-y^2z)(zx+y^2)\\
&(zy+x^2)(xy+z^2)^2(z-y)^2(x-y)^2(z-x)^2(z+x)^2(z+y)^2/x^6y^6z^{20},\\
\phi_{20,3}:&(y^4z^2-y^2zx^3+x^6)(x^2y^2+xyz^2+z^4)(z^2-zx+x^2)(z^3y-x^4)(zx^2-y^3)\\
&(yz^2-x^3)(xy^2+z^3)(z^2+x^2)(zx+y^2)(zy+x^2)(xy+z^2)(z-y)(x-y)^2\\
&(z-x)^2(z+x)^2/x^3y^{12}z^{25},\\
\phi_{20,5}:&(y^2x^2+xyz^2+z^4)(y^2x^2-xyz^2+z^4)(y^2-yx+x^2)(z^2-zx+x^2)\\
&(y^2-yz+z^2)(x^3-y^2z)(x^2y-z^3)(xy^2-z^3)(zx^2-y^3)(yz^2+x^3)\\
&(z^2x+y^3)(yx+z^2)^2(z-x)^2(y-z)^2/x^5y^5z^{30},\\
\phi_{20,7}:&-(y^2z^2+yzx^2+x^4)(x^3+z^3)^2(y^2-yz+z^2)(x^3z+y^4)(x^3y-z^4)\\
&(xy^2-z^3)(xy^2+z^3)(xy+z^2)(xz+y^2)(x-y)^2(x-z)^2(y-z)^2/x^5y^6z^{25},\\
\phi_{20,12}:&2\,(z^2y^2+yzx^2+x^4)(z^2-zy+y^2)^2(z^2x+y^3)(xy^2+z^3)(zy+x^2)^2(y-z)^2\\
&(x-y)^3(z-x)^3(x+z)^2(x+y)^2/x^6y^{12}z^{12},\\
\phi_{24,6}:&(y^4z^4+y^3z^3x^2+y^2z^2x^4+yzx^6+x^8)(x^3+z^3)(x^3+y^3)(x^2y-z^3)(y^3-zx^2)\\
&(x^2+z^2)(x^2+y^2)(xy+z^2)(xz+y^2)(x-y)^2(x-z)^2(y-z)^2/x^4y^{15}z^{15},
\end{aligned}$$

$$\begin{aligned}
\phi_{30,4}:&(x^5-z^2y^3)(x^3+y^3)(y^3+z^3)(x^3+z^3)(yz^4+x^5)(xz^2+y^3)(x^4-z^4)\\
&(xz+y^2)(zy+x^2)(x-y)^2(y-z)^2(x-z)^2/x^4y^{12}z^{20},\\
\phi'_{30,12}:&(y^3+z^3)^2(x^2-xy+y^2)(z^2-zx+x^2)(zx^2-y^3)(x^2y-z^3)(zx+y^2)\\
&(xy+z^2)(x-y)^3(x-z)^3(x+z)^2(x+y)^2/x^6y^{12}z^{12},\\
\phi_{36,5}:&(\zeta_3^2-1)(\zeta_3z^3+x^2y)(\zeta_3y^3+x^2z)(xy-\zeta_3z^2)(xz-\zeta_3y^2)(x^2-\zeta_3^2yz)(x-y)\\
&(\zeta_3^2y^4+xy^3+\zeta_3x^2y^2+\zeta_3^2x^3y+x^4)(\zeta_3^2z^4+xz^3+\zeta_3x^2z^2+\zeta_3^2x^3z+x^4)\\
&(x^6+y^3z^3)(x^2-xz+z^2)(y^2-yz+z^2)(x^2-xy+y^2)(x-z)/x^4y^{15}z^{15},\\
\phi_{40,8}:&(x^2z^2+xzy^2+y^4)(y^2x^3-z^5)(x^3+y^3)(z^2-zy+y^2)(zy^2-x^3)(x^2+y^2)\\
&(x^2+z^2)(xy+z^2)(zy+x^2)(x-y)^2(z-y)(x-z)^3(x+z)/x^5y^{12}z^{15},\\
\phi_{45,6}:&(1-\zeta_3^2)(\zeta_3x^3+y^2z)(\zeta_3y^3+zx^2)(-\zeta_3^2y^2+zx)(-\zeta_3^2x^2+zy)(\zeta_3xy-z^2)\\
&(\zeta_3xy+z^2)(\zeta_3^2xy+z^2)(\zeta_3^2y^2+z^2)(\zeta_3^2x^2+z^2)(\zeta_3y+z)^2(\zeta_3^2y+z)\\
&(x^2-xy+y^2)(\zeta_3x+z)^2(\zeta_3^2x+z)(xy+z^2)(z-y)(z-x)/x^6y^6z^{20},\\
\phi_{60,7}:&(x^3+y^3)(z^2-zy+y^2)(yx^4+z^5)(y^2x-z^3)(z^2y-x^3)(x^2+y^2)(xz+y^2)\\
&(x^2+yz)(x-y)^2(y^2-z^2)^2(x-z)^2/x^6y^9z^{15},\\
\phi''_{60,11}:&(y^2x^2-xyz^2+z^4)(y^2-yx+x^2)(yz^3-x^4)(z^3x-y^4)(yz+x^2)(yx+z^2)^2\\
&(zx+y^2)(x^2-y^2)^2(x-z)^2(y-z)^2/x^9y^9z^{12},\\
\phi_{60,12}:&2\,(y^2z^2-yzx^2+x^4)(y^3+z^3)^2(xy^2-z^3)(z^2x-y^3)(x^2+z^2)(x^2+y^2)\\
&(yz+x^2)^2(x-z)^3(x-y)^3/x^6y^{12}z^{12},\\
\phi_{64,8}:&2\,(x^2y^2-rxyz+xyz^2-rz^3+z^4)(x^2y^2+xyz^2+z^4)(zx^2+ry^2)(rx^2+zy^2)\\
&(y^4-ry^3+rxy^2-x^2y^2+rx^2y-rx^3+x^4)(xr-z^2)(ry-z^2)(y^3-z^2x)\\
&(x^3-yz^2)(y-z)(x-y)^2(x-z)/rx^8y^8z^{15},\\
\phi_{80,9}:&2\,(y^2z^2+yzx^2+x^4)(y^4-y^2z^2+z^4)(y^2z^2+x^4)(xy^2-z^3)(z^2x-y^3)\\
&(x^2+z^2)(x^2+y^2)(x-z)^2(y-z)^2(x-y)^2(x+y)(x+z)/x^6y^{12}z^{12},\\
\phi_{81,10}:&3\,(sx^2+syz+s^2x+yzx+x^3)(sz^2+sxy+s^2z+yzx+z^3)\\
&(sy^2+sxz+s^2y+yzx+y^3)(xy+sz)(xz+sy)(yz+sx)(sz+x^2)\\
&(sz+y^2)(sx+z^2)(sy+x^2)(sx+y^2)(sy+z^2)(xy+z^2)(xz+y^2)\\
&(x^2+yz)(s+z)(s+y)(s+x)/x^{12}y^{12}z^{12}.
\end{aligned}$$

In the table, we do not print the Schur elements for $\phi_{36,7}$, $\phi_{45,10}$, since they are the complex conjugates of those for $\phi_{36,5}$, $\phi_{45,6}$, respectively. In order to avoid indices, we write (x,y,z) instead of (y_1,y_2,y_3). Finally, r,s denote elements of K_W with $r^2=xy$ and $s^3=xyz$.

7. d-Howlett-Lehrer-Lusztig theories for unipotent blocks

It was first noted by Springer that non-real reflection groups naturally appear inside Weyl groups as what is now called relative Weyl groups, that is, normalizers modulo centralizers of subspaces in the natural reflection representation. By the standard dictionary between Weyl groups and finite reductive groups these relative Weyl groups can also be seen as normalizer modulo centralizer of (so-called d-split) Levi subgroups in groups of Lie type. The importance of this construction was revealed in the work of Broué, Michel and the author on the ℓ-blocks of characters of finite groups of Lie type. There it was shown that the unipotent characters inside an ℓ-block are parametrized by the irreducible characters of a relative Weyl group, which in general is a non-real reflection group.

A possible conceptual interpretation of this result was subsequently proposed by Broué and the author [3]. Namely, it was conjectured there that the cyclotomic Hecke algebras attached to relative Weyl groups govern the decomposition of Lusztig-induced d-cuspidal characters. If this does in fact hold, it implies that the degrees of the constituents of those Lusztig induced characters can be expressed in terms of the Schur elements of the corresponding cyclotomic algebras.

This latter fact was already verified in [12, Folg. 3.16 and 6.11] for all relative Weyl groups which are imprimitive in their natural reflection representation, in [13, Prop. 5.2] for the 2-dimensional ones, and in [3, Folg. 5.16] for G_{25}. With the Schur elements computed above, we can check the remaining cases and thus obtain (we refer to [3] or [13] for the notation):

Proposition 7.1. *Let G be a group of Lie type, $\mathcal{E}(G,(L,\lambda))$ a Φ_d-block of unipotent characters. Then the degrees of the unipotent characters $\gamma \in \mathcal{E}(G,(L,\lambda))$ satisfy conjecture (d-HV6) in* [3] *with respect to the values of parameters given in* [3, 2B and Table 8.1].

Proof. By the references cited above, it only remains to consider the cases

$$(G,d) \in \{(E_7,3),(E_7,6),(E_8,4),(E_8,3),(E_8,6)\},$$

with relative Weyl groups $G_{26}, G_{26}, G_{31}, G_{32}, G_{32}$ respectively. The Schur elements for the cyclotomic Hecke algebras of these latter groups have all been determined above, and the result follows by comparing the specialized Schur elements with the unipotent character degrees. (This was done explicitly in Theorem 5.4 for the case $(E_8,4)$.) □

8. Spetsial reflection groups

In this last section we observe some general properties of the Schur elements of complex reflection groups, using the explicit results derived here, respectively in [7] and [13]. These will bring to light an important property shared by certain finite complex reflection groups which seems to lie at the heart of the existence of so-called unipotent degrees (see for example [12] and Section 8.3). This result was announced in the ICM-report [14].

8A. A property of some complex reflection groups. Let W be a finite irreducible complex reflection group defined over k, $\mathbb{Z}_k$ the ring of integers of k, $\mathcal{H}(W,x)$ the 1-parameter cyclotomic algebra (2.4) over $\mathbb{Z}[x,x^{-1}]$, $k(y)$ with $y^{|\mu(k)|} = x$ a splitting field for $\mathcal{H}(W,x)$ (see 2A).

We now associate a certain rational function c_ϕ in y to each irreducible character of W. If W is a real reflection group, c_ϕ denotes the Schur element of ϕ with respect to the canonical symmetrizing form on the Iwahori-Hecke algebra of W (see [8, 9.4]). Similarly, if $W = G(m,p,n)$, c_ϕ denotes the Schur element of ϕ with respect to the symmetrizing form on $\mathcal{H}(W,x)$ constructed by Bremke and the author. According to [7, Cor. 1.5] and [12, Satz 5.13], it is given as follows. The irreducible characters of W are indexed by so-called m-symbols $\mathcal{S}$. For each symbol $\mathcal{S}$ let $D_\mathcal{S}$ be the rational function defined in [12, (5.12)]. We let $c_\phi = 1/D_\mathcal{S}(x;w_0,\dots,w_{m-1})$ if ϕ is indexed by $\mathcal{S}$, where the w_j are given by

$$w_j = \begin{cases} x^{\frac{1}{p}-1} & \text{if } j = 0, \\ x^{\frac{1}{p}}\zeta_m^j & \text{if } t|j,\ j \neq 0, \\ \zeta_m^j & \text{otherwise,} \end{cases}$$

with $\zeta_m := \exp(2\pi i/m)$. If W is a 2-dimensional exceptional reflection group, we let c_ϕ be the Schur element corresponding to ϕ computed in [13]. Finally, if W is a non-real exceptional reflection group in dimension at least 3, then c_ϕ is the 1-parameter specialization of the Schur element computed in the previous sections.

Thus, at least conjecturally, the c_ϕ are the Schur elements with respect to a nice (natural) symmetric form on the 1-parameter cyclotomic algebra $\mathcal{H}(W, x)$ of W.

For $\phi \in \mathrm{Irr}(W)$ we now call $\delta_\phi := P(W)/c_\phi$ the *generic degree of* ϕ. It is a rational function in y. We write $a(\phi)$ for $|\mu(k)|^{-1}$ times the order of zero of δ_ϕ at $y = 0$, and $b(\phi)$ for the order of zero of the fake degree R_ϕ at $x = 0$. Following the notation of Lusztig [9] for the real case, a character ϕ with $a(\phi) = b(\phi)$ is called *special*.

The explicit formulae for the generic degrees now allow us to observe the following:

Proposition 8.1. *Let W be a finite irreducible complex reflection group. Then the following are equivalent:*

(i) *for all $\phi \in \mathrm{Irr}(W)$ there exists a special $\psi \in \mathrm{Irr}(W)$ with $a(\phi) = a(\psi)$;*
(ii) *$a(\phi) \leq b(\phi)$ for all $\phi \in \mathrm{Irr}(W)$;*
(iii) *(rationality) $\delta_\phi \in k(x)$ for all $\phi \in \mathrm{Irr}(W)$;*
(iv) *(integrality) $\delta_\phi \in k[y]$ for all $\phi \in \mathrm{Irr}(W)$;*
(v) *$\delta_1 = 1$ (i.e. $c_1 = P(W)$);*
(vi) *(representability) the k-subspaces $\langle \delta_\phi \mid \phi \rangle$ and $\langle R_\phi \mid \phi \rangle$ of $k(y)$ coincide.*

Proof. First note that the fake degrees R_ϕ lie in $\mathbb{Z}[x]$, so statement (vi) implies (iii) and (iv). For the other implications, the proof is case by case. If W is a real reflection group, then it is known that statements (i)–(vi) hold. For example, they can be checked from the explicitly known values of the c_ϕ. In the case of Weyl groups, it is possible to give general proofs for some of the statements by using the fact that the generic degrees specialize to the degrees of actual characters of groups of Lie type with this Weyl group. Properties (i) and (v) for all real types but H_4 are contained in [9, 12.d) and e)], property (ii) in [9, (2.1)], (iii) and (iv) in [2, Th. 2.6], and (vi) in [10, (4.24)]. The case of H_4 is treated in [1].

For the monomial groups $G(m, 1, n)$, (i) and (ii) hold by [12, Bem. 2.24], (iii) by definition of the c_ϕ in this case (see [12, (2.21)]), (iv) by [12, Folg. 3.17], (v) by inspection and (vi) by [12, Satz 4.17]. For the imprimitive groups $G(m, m, n)$, (i) and (ii) are shown in [12, Lemma 5.16], (iii) follows from [12, (5.14)], (iv) from [12, Folg. 6.12], (v) again by inspection and (vi) from [12, Satz 6.26].

Now assume that $W = G(m, p, n)$ with $n \geq 2$, $p \neq 1, m$. We claim that none of the statements (i)–(vi) holds for W. For this we compute the generic degrees of the trivial character and of the reflection character. The trivial character is parametrized by the m-tuple of partitions $(n, -, \ldots, -)$. The formula in [12] yields

$$\delta_1 = \frac{x^{nt} - 1}{x^n - 1} \prod_{k=1}^{n} \frac{x^{(k-1)p+1} - 1}{x^{(k-1)m+t} - 1} \prod_{k=1}^{n-1} \frac{x^{mk} - 1}{x^{pk} - 1},$$

where $t := m/p$. For $p \neq 1, m$ the factor $x^{(n-1)m+t} - 1$ in the denominator does not cancel, hence δ_1 is not integral and (iv), (v) and (vi) fail. The reflection character ρ is parametrized by the multi-partition $(n-1, 1, -, \dots, -)$. Here, for $p \neq m$ we find

$$\delta_\rho = \frac{x^{2-\frac{1}{p}}(x^n-1)(x^{m(n-1)+t}-1)(x^{p(n-1)}-1)(x^{n-2+\frac{1}{p}}-\zeta)(x^{\frac{1}{p}}-\zeta)}{(x^{n-1}-1)(x^{p(n-1)+1}-1)(x-1)(x^{n-1+\frac{1}{p}}-\zeta)(x^{\frac{p-1}{p}}-\zeta^{-1})(x-\zeta^p)}\,\delta_1\,,$$

with $\zeta = \exp(2\pi i/m)$. Thus $a(\rho) = 2 - \frac{1}{p}$ is not integral, and (i), (iii) cannot hold. Moreover, by definition the reflection character occurs in the first symmetric power of the natural representation of W, so $b(\rho) = 1 < a(\rho)$, and (ii) is violated.

The generic degrees of the 2-dimensional primitive reflection groups were computed in [13]. From these, it can be checked that (i)–(vi) hold for the groups G_4, G_6, G_8, G_{14}.

For the remaining 2-dimensional groups, the explicit formulae in [13] show that δ_1 is not integral, so (iv), (v) and (vi) fail to hold. Also, in all cases there exists a non-rational degree, so (iii) is violated. For all groups except G_{12} and G_{22} there even exists some ϕ with non-integral $a(\phi)$, which contradicts (i). For G_{12}, the reflection character ρ has $a(\rho) = 2$, but there is no ψ with $b(\psi) = 2$. Thus (i) and (ii) do not hold. The group G_{22} has a 3-dimensional character ϕ with $a(\phi) = 1$, but $a(\rho) = 3$, so there is no special character ψ with $a(\psi) = 1$, again contradicting (i) and (ii). It can be checked that $a(\rho) > b(\rho)$ for G_7, G_{11}, G_{13}, G_{15} and G_{19}. For the remaining groups, condition (ii) is violated for some other irreducible character.

It remains to consider the exceptional, non-real groups of dimension at least 3. The results of the previous sections show that for G_{31}, none of the statements hold, while the other groups satisfy (i)–(vi). $\square$

A reflection group all of whose irreducible components satisfy the above (very special) equivalent conditions will be called *spetsial*. From the proof of Proposition 8.1 we see that the irreducible spetsial groups are

$$\mathfrak{S}_n, \quad G(m,1,n), \quad G(m,m,n), \quad G_i \text{ with } i \in \{4,6,8,14,23,\dots,30,32,\dots,37\}.$$

With this list it is straightforward to verify that spetsiality is in a certain sense a local property:

Proposition 8.2. *A reflection group is spetsial if and only if all of its* 2*-dimensional parabolic subgroups are spetsial.*

Since taking parabolic subgroups is transitive, we obtain the immediate consequence that all parabolic subgroups of a spetsial reflection group are spetsial.

8B. Special characters of spetsial groups. The explicit formulae for the generic degrees obtained in the previous sections or given in the above-mentioned references allow us to compile the list of all special characters of irreducible spetsial reflection groups W. For real W, the special characters were listed in [9] and [1]. For $W = G(m,1,n)$ the special characters are described in [12, Bem. 2.24] in terms of the associated symbols, for $W = G(m,m,n)$ in [12, Lemma 5.16]. For the exceptional spetsial reflection groups, the special characters can be read off from the explicit data in [13] and in the previous sections. Thus we obtain the following lists:

G_4: $\phi_{1,0}$, $\phi_{2,1}$, $\phi_{3,2}$, $\phi_{1,4}$.
G_6: $\phi_{1,0}$, $\phi_{2,1}$, $\phi_{3,4}$, $\phi_{2,5,13}$, $\phi_{1,10}$.
G_8: $\phi_{1,0}$, $\phi_{2,1}$, $\phi_{3,2}$, $\phi_{4,3}$, $\phi_{1,6}$.

G_{14}: $\phi_{1,0}$, $\phi_{2,1}$, $\phi_{4,5}$, $\phi_{3,6}$, $\phi_{2,9}$, $\phi_{1,20}$.
G_{24}: $\phi_{1,0}$, $\phi_{3,1}$, $\phi_{7,3}$, $\phi_{8,4}$, $\phi_{7,6}$, $\phi_{3,8}$, $\phi_{1,21}$.
G_{25}: $\phi_{1,0}$, $\phi_{3,1}$, $\phi_{6,2}$, $\phi''_{6,4}$, $\phi_{8,6}$, $\phi'_{6,8}$, $\phi_{1,12}$.
G_{26}: $\phi_{1,0}$, $\phi_{3,1}$, $\phi_{6,2}$, $\phi_{8,3}$, $\phi'_{6,4}$, $\phi_{6,5}$, $\phi''_{8,6}$, $\phi''_{6,11}$, $\phi''_{3,16}$, $\phi_{1,21}$.
G_{27}: $\phi_{1,0}$, $\phi_{3,1}$, $\phi_{10,3}$, $\phi_{9,4}$, $\phi_{15,5}$, $\phi_{8,6}$, $\phi_{15,8}$, $\phi_{9,9}$, $\phi_{10,12}$, $\phi_{3,16}$, $\phi_{1,45}$.
G_{29}: $\phi_{1,0}$, $\phi_{4,1}$, $\phi_{10,2}$, $\phi_{16,3}$, $\phi'_{15,4}$, $\phi''_{15,4}$, $\phi_{20,5}$, $\phi_{24,6}$, $\phi_{20,9}$, $\phi''_{15,12}$, $\phi'_{15,12}$, $\phi_{16,13}$, $\phi_{10,18}$, $\phi_{4,21}$, $\phi_{1,40}$.
G_{32}: $\phi_{1,0}$, $\phi_{4,1}$, $\phi_{10,2}$, $\phi_{20,3}$, $\phi_{30,4}$, $\phi_{20,5}$, $\phi_{45,6}$, $\phi_{64,8}$, $\phi_{60,9}$, $\phi_{81,10}$, $\phi_{30,12}$, $\phi_{36,15}$, $\phi_{30,20}$, $\phi_{20,25}$, $\phi_{10,30}$, $\phi_{1,40}$.
G_{33}: $\phi_{1,0}$, $\phi_{5,1}$, $\phi_{15,2}$, $\phi_{30,3}$, $\phi_{30,4}$, $\phi_{81,6}$, $\phi_{60,7}$, $\phi_{45,7}$, $\phi_{64,8}$, $\phi_{15,9}$, $\phi_{45,10}$, $\phi_{81,11}$, $\phi_{60,10}$, $\phi_{15,12}$, $\phi_{30,13}$, $\phi_{30,18}$, $\phi_{15,23}$, $\phi_{5,28}$, $\phi_{1,45}$.
G_{34}: $\phi_{1,0}$, $\phi_{6,1}$, $\phi_{21,2}$, $\phi_{56,3}$, $\phi_{105,4}$, $\phi_{126,5}$, $\phi_{315,6}$, $\phi_{420,7}$, $\phi_{384,8}$, $\phi'_{70,9}$, $\phi_{560,9}$, $\phi_{315,10}$, $\phi_{729,10}$, $\phi_{630,11}$, $\phi_{840,11}$, $\phi_{896,12}$, $\phi_{210,12}$, $\phi'_{840,13}$, $\phi_{630,14}$, $\phi_{1280,15}$, $\phi'_{560,18}$, $\phi_{630,20}$, $\phi_{840,19}$, $\phi_{210,30}$, $\phi_{896,21}$, $\phi''_{840,23}$, $\phi_{630,23}$, $\phi_{729,24}$, $\phi_{315,28}$, $\phi_{560,27}$, $\phi'_{70,45}$, $\phi_{384,29}$, $\phi_{420,31}$, $\phi_{315,36}$, $\phi_{126,41}$, $\phi_{105,46}$, $\phi_{56,57}$, $\phi_{21,68}$, $\phi_{6,85}$, $\phi_{1,126}$.

Recall that an n-dimensional irreducible finite complex reflection group W is called *well-generated* if it can be generated by n of its reflections. The proof of Proposition 8.1, together with results from [15], immediately yields the following characterization:

Corollary 8.3. *Let W be irreducible. Then the following are equivalent:*
(i) *W is well-generated;*
(ii) *the reflection character ρ of W is special;*
(iii) *the generic degree δ_ρ is rational.*

Proof. We proceed case by case. The groups $G(m,p,n)$, $n \geq 2$, $p \neq 1, m$, are not well-generated. In the proof of Proposition 8.1 we saw that $a(\rho) = 2 - 1/p$, so neither is ρ special nor δ_ρ rational. For the remaining non-well-generated groups G_i, $i \in \{7, 11, 12, 13, 15, 19, 22, 31\}$ the assertion has either been verified in the course of proving Proposition 8.1, or it is a consequence of the explicit formulae for δ_ρ in Theorem 5.4 or in [13].

Now assume that W is well-generated. Moreover, if W is spetsial, then δ_ρ is rational by Proposition 8.1. The a-value can be checked from the lists of Schur elements determined here or in [12]. All non-spetsial well-generated groups are 2-dimensional, and for them the result follows from the formulae in loc. cit. $\square$

8C. Unipotent degrees and spetses. We end by giving some motivation for the study of spetsial reflection groups. In the course of his classification of the irreducible characters of finite reductive groups $G(q) = G^F$ Lusztig [10] determined the important class of unipotent characters and showed that they can be indexed by a set which only depends on the Weyl group W of the associated algebraic group G, together with the action of the Frobenius endomorphism F on it. Moreover, the degrees of these unipotent characters can naturally be written as polynomials with rational coefficients in the size q of the underlying field. We call this the (multi-) set of unipotent degrees attached to (W, F). Many properties of unipotent characters are already reflected by combinatorial properties of the unipotent degrees, for example, the distribution into Harish-Chandra series, and the distribution into families (which restricted to principal series characters are just the Kazhdan-Lusztig cells).

In the course of this classification Lusztig observed that similar sets can formally also be attached to those finite real reflection groups W which are not Weyl groups

[11]. We showed in [12] that in fact such sets of unipotent degrees exist for all imprimitive spetsial complex reflection groups, and in [4] corresponding sets are constructed for the primitive spetsial groups (using the results of the present paper).

More precisely, we obtain sets $\mathcal{E}(W)$ together with a degree map

$$\mathrm{Deg} : \mathcal{E}(W) \to \frac{1}{|W|}\mathcal{O}_K[x],$$

where $\mathcal{O}_K$ denotes the ring of integers of the character field K of W (the field of definition of the reflection representation of W). The Galois group $\mathrm{Gal}(K/\mathbb{Q})$ acts on $\mathcal{E}(W)$ such that Deg is equivariant. The set $\mathcal{E}(W)$ falls into Φ-Harish-Chandra series for each cyclotomic polynomial Φ over K, and into families. The Φ-Harish-Chandra series are completely described by the Schur elements of cyclotomic Hecke algebras attached to certain relative Weyl groups. For $\Phi = x - 1$ and W a Weyl group, these are just the ordinary Harish-Chandra series of unipotent characters. Each family contains a unique special character, and the degrees in the family are connected to the corresponding fake degrees by a Fourier transform matrix. To each element of $\mathcal{E}(W)$ is attached a root of unity (called Frobenius eigenvalue), such that for each family the diagonal matrix of Frobenius eigenvalues together with the Fourier matrix give a representation of $\mathrm{SL}_2(\mathbb{Z})$. For details and references we refer to [12] and [14]. It turns out that all (equivalent) conditions in Proposition 8.1 are necessary for the existence of such sets of unipotent degrees, so that we can state:

Theorem 8.4. *A finite complex reflection group has unipotent degrees if and only if it is spetsial.*

Thus in a certain sense spetsial complex reflection groups behave just as if they were the Weyl groups of an algebraic group. It is tempting to speculate about an underlying algebraic structure, baptized 'spets' in [14], giving rise to the unipotent degrees attached to complex reflection groups. We don't yet know what these spetses should be, but a lot of intriguing evidence for their existence has been collected.

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