REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 4, Pages 370–397 (September 11, 2000)
S 1088-4165(00)00093-5

ON THE REPRESENTATION THEORY OF IWAHORI–HECKE ALGEBRAS OF EXTENDED FINITE WEYL GROUPS

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ABSTRACT. We apply Lusztig's theory of cells and asymptotic algebras to the Iwahori–Hecke algebra of a finite Weyl group extended by a group of graph automorphisms. This yields general results about splitting fields (extending earlier results by Digne–Michel) and decomposition matrices (generalizing earlier results by the author). Our main application is to establish an explicit formula for the number of simple modules in type D_n (except in characteristic 2), using the known results about type B_n due to Dipper, James, and Murphy and Ariki and Mathas.

1. INTRODUCTION

1.1. We consider a finite Weyl group W_1 and let S_1 be the set of simple reflections of W_1. In addition, we assume that we are given a group homomorphism $\pi\colon \Omega \to \mathrm{Aut}(W_1, S_1)$ where Ω is a finite group and $\mathrm{Aut}(W_1, S_1)$ is the group of all automorphisms of W_1 which leaves the set S_1 invariant. (This is slightly more general than the set-up in [27] where it is assumed that π is injective.) We form the semidirect product $W = W_1 \rtimes \Omega$ so that, in W, we have the identity $\omega w_1 \omega^{-1} = \pi(\omega)(w_1)$ for $w_1 \in W_1$ and $\omega \in \Omega$. The group W is called an *extended Weyl group*. We have a length function $l\colon W \to \mathbb{N}_0$ given by the formula $l(w_1\omega) := l_1(w_1)$ where $w_1 \in W_1$, $\omega \in \Omega$, and $l_1\colon W_1 \to \mathbb{N}_0$ is the usual length function on W_1. In particular, this means that all elements of Ω have length 0.

1.2. We now define the corresponding extended Iwahori–Hecke algebra. Let F be a finite extension field of $\mathbb{Q}$ and R be the ring of algebraic integers in F or the localization of that ring of integers in some prime ideal. (The relevance of this assumption will become more transparent in Definition 3.3 below.) Let $A = R[v, v^{-1}]$ be the ring of Laurent polynomials in an indeterminate v and H be the generic Iwahori–Hecke algebra associated with (W_1, S_1, Ω). By definition, H is a free A-module with basis $\{T_w \mid w \in W\}$ and multiplication given by $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$ and $T_s^2 = uT_1 + (u-1)T_s$ for $s \in S_1$, where we have set $u := v^2$. In particular, we have

$$T_{w_1} T_\omega = T_{w_1\omega} \quad \text{and} \quad T_\omega T_{w_1} = T_{\omega w_1} \quad \text{for } w_1 \in W_1 \text{ and } \omega \in \Omega.$$

The unit element of H is T_1. (For more details about the construction of Iwahori–Hecke algebras corresponding to extended finite Weyl groups, even in a slightly more general context, see [3, §10.11].) Let H_1 be the A-subspace of H spanned by

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Received by the editors January 19, 2000 and, in revised form, August 7, 2000.
2000 *Mathematics Subject Classification.* Primary 20C08; Secondary 20C20.

all T_{w_1} with $w_1 \in W_1$. Then H_1 is nothing but the (ordinary) generic Iwahori–Hecke algebra associated with (W_1, S_1). If $\theta \colon A \to k$ is any ring homomorphism into a field k, we consider k as an A-module and obtain a corresponding specialized algebra $H_k := k \otimes_A H$.

The aim of this paper is to develop some basic aspects of the representation theory of extended Iwahori–Hecke algebras and their specializations in a systematic way, using Lusztig's asymptotic algebra and the related constructions in [25, 26, 27]. Our motivation comes from applications to the following situations, in which extended finite Weyl groups and their Iwahori–Hecke algebras arise naturally.

Example 1.3. Let G be a connected reductive group defined over a finite field $\mathbb{F}_q$. Let W_1 be the Weyl group of G with a set of simple reflections S_1 and consider a graph automorphism $\sigma \in \mathrm{Aut}(W_1, S_1)$. If σ is induced by the Frobenius map corresponding to the $\mathbb{F}_q$-rational structure of G, then the character theory of H_K (where K is the field of fractions of A) plays a role in the study of $\mathcal{L}$-functions of the Deligne–Lusztig varieties of G; see [23] and [5]. In a different direction, assume that σ is induced by a graph automorphism $\tilde{\sigma} \colon G \to G$. In this setting, H_K occurs as the endomorphism algebra of an induced representation for the finite group $G(\mathbb{F}_q) \rtimes \langle \tilde{\sigma} \rangle$. For applications to certain problems arising in inverse Galois theory, see [28, 29].

Example 1.4. Assume that V is a finite dimensional Euclidean vector space and that we have an embedding $W_1 \subset \mathrm{GL}(V)$, where the elements of S_1 are represented by reflections. Let $W \subset \mathrm{GL}(V)$ be a subgroup such that W_1 is a normal subgroup of W. Then, by a result due to Howlett, W_1 has a natural complement Ω in W which leaves S_1 invariant. This situation arises in the study of induced cuspidal k-representations of a reductive group over the finite field $\mathbb{F}_q$, where k is a field whose characteristic is either 0 or a prime which does not divide q. From this point of view, it is of considerable interest to understand the representations of the algebra H_k, which occurs as the endomorphism algebra of the induced reresentation; see [3, Chap. 10] for ordinary representations and [15] for modular representations. For example, in order to determine [15, Table 5.4], it is required to know the number of isomorphism classes of simple H_k-modules where H_k is non-semisimple.

Example 1.5. Assume that (W_1, S_1) is of type D_n and that Ω is the group generated by a single graph automorphism of order 2. Then W can be identified with a Weyl group of type B_n. (This case will be studied in detail in Section 6.) Now the modular representation theory of Iwahori–Hecke algebras of type B_n has been studied extensively; see the work of Dipper, James, and Murphy [8], Ariki [1] and Ariki and Mathas [2]. On the other hand, the representation theory of Iwahori–Hecke algebras of type D_n can be developed to some extent along similar lines as that of type B_n (see [32]) but there are some particularly intricate problems in the case where n is even. It is therefore desirable to develop tools which allow us to obtain information on type D_n from known results on type B_n.

In Section 2, we introduce the basic notation and the results from Lusztig's papers [25, 26, 27] that we shall need here. A first demonstration of the power of the asymptotic methods is given towards the end of that section, where we consider the question of splitting fields for the extended Iwahori–Hecke algebras in the semisimple case. This puts the results of Digne and Michel (which are concerned with the case where Ω is cyclic) into a general framework.

In Section 3, we use the structure as symmetric algebra to obtain general results about splitting fields and semisimplicity for extended Iwahori–Hecke algebras under specialization.

In Section 4, we apply Clifford theory to study relations between the irreducible representations of extended and non-extended Iwahori–Hecke algebras. In combination with the results in Section 3 this leads to general semisimplicity criteria for extended asymptotic algebras and Iwahori–Hecke algebras; see Theorem 3.2, Corollary 4.7, and (4.8).

The main purpose of Section 5 is to extend the results of [14, 18] to the extended Iwahori–Hecke algebra. In particular, this shows that decomposition matrices for extended Iwahori–Hecke algebras again have a unitriangular shape. In terms of Lusztig's a-function, we obtain a basic result relating the triangular shapes for the extended and the non-extended algebra; see Theorem 5.5 and its corollary. This result allows us to prove in Theorem 5.8 a basic formula on the number of simple modules for specialized algebras. See Examples 5.9 and 5.10.

Finally, Section 6 contains the applications to type D_n. The main result is Theorem 6.3 which yields an explicit formula for the number of simple modules. We point out that our proof requires, in an essential way, the results of Ariki and Mathas [2] about type B_n and the methods developed in Section 4 (involving the extended asymptotic algebra). These methods yield a classification of the simple modules in type B_n which may be different from that of Dipper, James, and Murphy [8]. Towards the end of Section 6, we consider the problem of relating the two classifications. A solution to that problem (which remains open) would lead to a natural parametrization of the simple modules for type D_n (in the modular case).

2. The extended asymptotic algebra

In this section, we briefly recall the basic results about Lusztig's asymptotic version of the extended generic Iwahori–Hecke algebra H as defined in (1.2). Originally, the asymptotic algebra was only defined for the case $\Omega = \{1\}$ in [25]. But, as remarked in [26, 27], the theory trivially extends to the general case. We indicate at some places exactly how this extension works; see, for example, the formulas in (2.1) and (2.2).

2.1. As in [27, 3.1], for any $y, w \in W$, we define a polynomial $P_{y,w} \in \mathbb{Z}[u]$ as follows. Writing $y = y_1\omega$ and $w = w_1\omega'$ with $y_1, w_1 \in W_1$ and $\omega, \omega' \in \Omega$, we have $P_{y,w} = 0$ if $\omega \neq \omega'$ and $P_{y,w} = P_{y_1,w_1}$ if $\omega = \omega'$, where $P_{y_1,w_1} \in \mathbb{Z}[u]$ is the *Kazhdan–Lusztig polynomial* corresponding to $y_1, w_1 \in W_1$. (Note that $P_{y_1,w_1} = 0$ unless $y_1 \leq w_1$ where $\leq$ denotes the Bruhat–Chevalley order on W_1; see [22].) Then we have a new basis $\{C_w \mid w \in W\}$ of H, where

$$C_w := \sum_{y \in W} (-1)^{l(w)-l(y)} v^{l(w)-2l(y)} P_{y,w}(v^{-2})\, T_y.$$

For any $x, y \in W$, we write $C_x\, C_y = \sum_{z \in W} h_{xyz}\, C_z$ with $h_{x,y,z} \in A$. Given $z \in W$, we denote by $a(z)$ the smallest integer $i \geq 0$ such that $v^i h_{x,y,z} \in R[v]$ for all $x, y \in W$. This yields a function $a\colon W \to \mathbb{N}_0$. A deep fact about that function is the following result due to Lusztig [24, Theorem 5.4]:

(a) If $h_{x,y,z} \neq 0$, then $a(z) \geq a(x)$ and $a(z) \geq a(y)$.

Furthermore, for any $w_1 \in W_1$ and $\omega \in \Omega$, we have

(b) $C_{w_1\omega} = C_{w_1} C_\omega$ and $C_\omega = T_\omega$,

(c) $a(w_1\omega) = a(w_1)$.

The latter two properties are immediate consequences of the definition.

2.2. For $x, y, z \in W$, we denote by $\gamma_{x,y,z} \in R$ the constant term of $(-v)^{a(z)} h_{x,y,z^{-1}}$. Following [27, 3.1i], the constants $\{\gamma_{x,y,z}\}$ can be used to construct an *asymptotic algebra*, as follows. Let J be the free R-module with basis $\{t_w \mid w \in W\}$ and multiplication defined by $t_x t_y = \sum_{z \in W} \gamma_{x,y,z}\, t_{z^{-1}}$. Then J is an associative algebra with unit element $\sum_{d \in \mathcal{D}} t_d$, where $\mathcal{D}$ is a certain set of involutions in W. We have

$$\mathcal{D} = \{d \in W \mid a(d) = l(d) - 2 \deg P_{1,d}\}.$$

(For proofs in the case where $\Omega = \{1\}$ see [25].) Let J_A be the A-algebra obtained from J by extending scalars from R to A. Then, by [27, 3.2a], the map $\phi\colon H \to J_A$ defined by

$$\phi(C_w) = \sum_{\substack{d \in \mathcal{D},\, z \in W \\ a(d) = a(z)}} h_{w,d,z}\, t_z$$

is a homomorphism of A-algebras which preserves the unit elements. The formula in [26, 1.3c] shows that the determinant of ϕ is a polynomial in $R[v]$ with constant term 1. A deep fact about the constants $\gamma_{x,y,z}$ is the following result due to Lusztig [24, Theorem 6.1] (see also [26, Theorem 1.8]):

(a) For any $x, y, z \in W$, we have $\gamma_{x,y,z} = \gamma_{y,z,x}$.

On the other hand, the following properties are immediate consequences of the definition. For any $w_1 \in W_1$ and $\omega \in \Omega$, we have

(b) $\mathcal{D} \subseteq W_1$ and $t_{w_1\omega} = t_{w_1} t_\omega$,

(c) $\phi(C_{w_1\omega}) = \phi(C_{w_1}) t_\omega$.

Let J_1 be the A-submodule of J spanned by all elements t_{w_1} for $w_1 \in W_1$. Then J_1 is nothing but the (ordinary) asymptotic algebra associated with H_1 and $\phi\colon H \to J_A$ restricts to the homomorphism $\phi_1\colon H_1 \to (J_1)_A$ defined with respect to (W_1, S_1) and H_1.

Now we turn to the application of the above constructions to the representation theory of H and its specializations.

2.3. Let $\theta\colon A \to k$ be any homomorphim into a field k. By extension of scalars, we obtain corresponding algebras $H_k = k \otimes_A H$, $J_k = k \otimes_A J_A$, and an induced homomorphism $\phi_k\colon H_k \to J_k$. We write again C_w and t_w for $1 \otimes C_w$ and $1 \otimes t_w$, respectively. Then any J_k-module E can also be regarded as an H_k-module via ϕ_k; we denote that H_k-module by E^*. The assignment $E \mapsto E^*$ induces a group homomorphism $\phi^*\colon R_0(J_k) \to R_0(H_k)$. (For any finite dimensional algebra T over a field, $R_0(T)$ denotes the Grothendieck group of the category of finite dimensional T-modules; the class of a T-module V in $R_0(R)$ will be denoted by $[V]$.) We have the following basic result:

Theorem 2.4 (Lusztig [26, Lemma 1.9])**.** *Let M be a simple H_k-module and E be a simple J_k-module. Then we define corresponding integers $a_M \geq 0$ and $a_E \geq 0$ as follows:*

$$a_M := \max\{i \geq 0 \mid C_w M \neq 0 \text{ for some } w \in W \text{ with } a(w) = i\},$$
$$a_E := a(w), \text{ where } w \in W \text{ is such that } t_w E \neq 0.$$

(Note that a_E is well-defined; see the remarks below.) With these definitions, the following holds. For any simple H_k-module M, there exists a J_k-module $\tilde{M}_J$ and a surjective H_k-module homomorphism $p\colon \tilde{M}_J^* \to M$ such that the following two conditions are satisfied:

(a) *We have* $a_E = a_M$ *for each simple* J_k*-module* E *which occurs as a composition factor of* $\tilde{M}_J$*.*
(b) *We have* $a_{M'} < a_M$ *for each simple* H_k*-module* M' *which occurs as a composition factor of* $\ker(p)$*.*

Thus, in $R_0(H_k)$*, we have* $\phi_k^*([\tilde{M}_J]) = [M]+$ *sum of terms* $[M']$ *where* M' *are simple* H_k*-modules with* $a_{M'} < a_M$*. Consequently, the homomorphism* $\phi^*\colon R_0(H_k) \to R_0(J_k)$ *is surjective.*

We briefly sketch the main ingredients of the proof. For any $i \geq 0$, we introduce the following subspaces, following [26, §1]:

$$J_k^i = \text{subspace of } J_k \text{ generated by all } t_w \text{ with } a(w) = i,$$
$$H_k^{\geq i} = \text{subspace of } H_k \text{ generated by all } C_w \text{ with } a(w) \geq i.$$

As a consequence of the deep results (2.1a) and (2.2a), the subspaces J_k^i and $H_k^{\geq i}$ are in fact two-sided ideals. Moreover, we have $J_k = \bigoplus_i J_k^i$; see [26, 1.3d]. In particular, this shows that a_E is well-defined. Let $H_k^i := H_k^{\geq i}/H_k^{\geq i+1}$; this is an (H_k, H_k)-bimodule in a natural way. There is also a natural left action of J_k on H_k^i which we denote by $j\colon f \mapsto j \circ f$; we have

$$hf = \phi_k(h) \circ f \qquad \text{for all } h \in H_k \text{ and } f \in H_k^i.$$

Now we construct $\tilde{M}_J$ as follows. Let $\tilde{M} := H_k^a \otimes_{H_k} M$, where $a = a_M$ and where we regard H_k^a as a right H_k-module and M as a left H_k-module. Then $\tilde{M}$ is naturally a left H_k-module. Let $\tilde{M}_J$ be the J_k-module whose underlying vector space is $\tilde{M}$ and J_k acts via $j\colon (f \otimes m) \mapsto (j \circ f) \otimes m$. Then we have $\tilde{M} = \tilde{M}_J^*$ and (b) follows by the argument in (b) of the proof of [26, Cor. 3.6] (see also [14, 2.7(2)]). The map p is defined by $p(f \otimes m) = \dot{f}m$, where $\dot{f} \in H_k^{\geq a}$ is a representative of $f \in H_k^a$. Then (a) and the last assertion are proved by the same arguments as those in the proof of [26, Lemma 1.9].

Corollary 2.5. *The kernel of* $\phi_k\colon H_k \to J_k$ *is contained in the Jacobson radical of* H_k*. In particular,* ϕ_k *is an isomorphism if* H_k *is semisimple.*

Proof. As already mentioned in [14, Remark 2.9], this immediately follows from the fact that ϕ_k^* is surjective. □

Example 2.6. Consider the specialization homomorphism $\theta\colon A \to F$, $v \mapsto 1$. Then, in H_F, we have $T_s^2 = T_1$ for all $s \in S$, and so H_F is naturally isomorphic to the group algebra $F[W]$. Thus, we have an F-algebra homomorphism $\phi_F\colon F[W] \to J_F$. Note that J_F is just obtained from J by extending scalars from R to F.

Since F has characteristic 0, Maschke's Theorem shows that $F[W]$ is semisimple and so, by Corollary 2.5, ϕ_F is an isomorphism. In particular, this means that J_F is semisimple. This can also be proved using the fact that J is a *based ring*; see [27, 1.2a and 3.1j]. If F is also a splitting field for W, we can conclude that J_F is a split semisimple algebra.

Theorem 2.7. *Assume that* F *is a splitting field for* W*. Let* K *be the field of fractions of* A*. Then* H_K *is a split semisimple algebra. Moreover, if every simple* J_F*-module can be realized over* R*, then every simple* H_K*-module can be realized over* A*.*

Proof. By Example 2.6, J_F is split semisimple. Since K is an extension field of F, it follows that J_K is also split semisimple. We have already remarked in (2.2) that the determinant of ϕ is non-zero and so $\phi_K\colon H_K \to J_K$ is an isomorphism. Consequently, H_K must be split semisimple, too. To prove the last assertion note that, since J_F is already split semisimple, every simple J_K-module can be realized over F. The assertion then follows from the fact that ϕ_K is defined over A. □

Remark 2.8. Every simple J_F-module can be realized over R if R is a principal ideal domain. (This follows by a general argument which is explained in [19, Satz 12.2].) This is the case, for example, when R is the localization of the ring of integers of F in some prime ideal. In general, if R is not a principal ideal domain, a realization of the simple J_F-modules over R can always be achieved by passing to a suitable finite extension of F. (This follows from a general number theoretic argument; see [19, Satz 12.5(b)].)

Example 2.9. Assume that Ω is the group generated by a single graph automorphism $\sigma\colon W_1 \to W_1$. Furthermore, we assume that σ is *ordinary* in the sense of [23, 3.1], i.e., whenever $s \neq s'$ in S_1 are in the same σ-orbit, the product ss' has order 2 or 3. This is the case which arises naturally in the representation theory of finite groups of Lie type; see [23] and [5, Chap. II].

Assume that σ has order $d \geq 1$ and let $\zeta_d \in \mathbb{C}$ be a primitive d-th root of unity. We claim that

(a) $\mathbb{Q}(\zeta_d)$ is a splitting field for W.

To prove this we may assume, by a standard reduction technique (see the proof of [23, Prop. 3.2]) that (W_1, S_1) is irreducible. Using the classification of irreducible finite Weyl groups, we then see that $d \in \{1, 2, 3\}$. Now we use Clifford theory for the complex irreducible characters of W with respect to W_1. Let χ be an irreducible character of W. We must show that χ can be realized over $\mathbb{Q}(\zeta_d)$. Now, since Ω is cyclic of prime order, there exists an irreducible character ψ of W_1 such that χ is obtained by either inducing or by extending ψ from W_1 to W (see [20, 6.20]). Since $\mathbb{Q}$ is a splitting field for every finite Weyl group (see [16, Theorem 6.3.8]), we know that ψ can be realized over $\mathbb{Q}$. Thus, if χ is obtained by inducing ψ, then χ certainly is realized over $\mathbb{Q}$. On the other hand, if ψ can be extended to χ, then Lusztig [23, Prop. 3.2] has shown that ψ can be extended to an irreducible character $\tilde{\psi}$ of W which can be realized over $\mathbb{Q}$. But then there exists a linear character η of W with W_1 in its kernel such that χ is obtained from $\tilde{\psi}$ by tensoring with η (see [20, 6.17]). The character η can be regarded as a character of Ω and, hence, is realized over $\mathbb{Q}(\zeta_d)$. Thus, (a) is proved.

Finally, if $d \leq 2$, then we have $F = \mathbb{Q}$ and $R = \mathbb{Z}$; if $d = 3$, then $\mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$ and R is a principal ideal domain (see [34, Theorem 4.17]). So, in all cases, the simple J_F-modules can be realized over R (see Remark 2.8). Hence, assuming that F is a splitting field for W and using Theorem 2.7, we conclude that

(b) H_K is split semisimple and every simple H_K-module can be realized over A.

More precise results about the possible character values of H_K have been obtained by Digne and Michel in [5, Théorème II.3.3].

Example 2.10. We keep the setting of the previous example but drop the assumption that σ is ordinary. This case arises if σ is the unique non-trivial graph authorphism of order 2 for (W_1, S_1) of type B_2, G_2 or F_4. By checking the complex character tables of W (see [17, §7]), one sees that not all character values are rational. In fact, all the values of all irreducible characters of W lie in $\mathbb{Q}(\sqrt{2})$ (for type B_2 or F_4) or in $\mathbb{Q}(\sqrt{3})$ (for type G_2). The constructions in [17, Example 7.6] show that these fields are splitting in type B_2 and G_2. As far as type F_4 is concerned, the explicit computations required to obtain [17, Table 5] show that $\mathbb{Q}(\sqrt{2})$ is a splitting field.

We can summarize the above results as follows.

Theorem 2.11. *Assume that Ω is the group generated by an arbitrary graph automorphism $\sigma\colon W_1 \to W_1$. Let $d \geq 1$ be the order of σ and $\zeta_d \in \mathbb{C}$ be a primitive d-th root of unity. For any $s, t \in S_1$, denote by m_{st} the order of $st \in W_1$. Then*

$$F := \mathbb{Q}(\zeta_d, \cos(\pi/m_{s\sigma(s)}) \mid s \in S_1)$$

is a splitting field for W. (Note that $F = \mathbb{Q}$ if σ is the identity.)

Proof. By a standard reduction technique (see the proof of [23, Prop. 3.2]), we may assume that (W_1, S_1) is irreducible. If σ is ordinary, we have already seen in Example 2.9 that $\mathbb{Q}(\zeta_d)$ is a splitting field. So it remains to consider the case where σ is not ordinary. Then σ has order 2 and (W_1, S_1) is of type B_2, G_2 or F_4. In these cases, there exist a generator $s \in S_1$ such that $m_{s\sigma(s)}$ equals 4 (in type B_2, F_4) or 6 (in type G_2). Thus, F (as defined above) contains $\sqrt{2}$ (in type B_2, F_4) or $\sqrt{3}$ (in type G_2). Hence F is a splitting field for W by the remarks in Example 2.10. □

3. Splitting fields and semisimplicity

We will now use the methods in Section 2 to obtain results about splitting fields for specialized algebras. Recall that R is the ring of algebraic integers in a number field F or the localization of that ring in some prime ideal. Furthermore, $K = F(v)$ is the field of fractions of $A = R[v, v^{-1}]$. We begin with the asymptotic algebra J.

3.1. Assume that F is a splitting field for W. Then we have seen in Example 2.6 that J_F is a split semisimple algebra. Since R is integrally closed in F, we have $\text{Trace}(t_w, E) \in R$ for every simple J_F-module E and every $w \in W$ (see [16, Prop. 7.3.8]). Moreover, since J is a based ring by [27, 3.1j], there exist non-zero elements $f_E \in F$ such that

$$f_E \dim_F E = \sum_{w \in W} \text{Trace}(t_w, E)\,\text{Trace}(t_{w^{-1}}, E);$$

we have in fact $f_E \in R$ (see [26, 1.3]). Now consider the linear map $\tau_0\colon J \to R$ defined by $\tau(t_w) = 1$ if $w \in \mathcal{D}$ and $\tau(t_w) = 0$ otherwise. Then, by [27, 1.1c], τ_0 is a trace function and the Gram matrix of the bilinear form $J \times J \to R$, $(j, j') \mapsto \tau_0(jj')$, is invertible over R. Thus, J is a symmetric algebra. The bases $\{t_w\}$ and $\{t_{w^{-1}}\}$ are dual bases of J with respect to the above bilinear form. Hence, using the notation of [16, §7.2], we see that the elements f_E are the *Schur elements* of J_F with respect to the canonical extension of τ_0 to J_F.

Theorem 3.2. *Assume that F is a splitting field for W. Let $\mathfrak{p} \subset R$ be a prime ideal and k be the field of fractions of $R/\mathfrak{p}$. Then $J_k = k \otimes_R J$ is split semisimple if and only if $f_E \notin \mathfrak{p}$ for all simple J_F-modules E.*

Proof. First assume that J_k is split semisimple. Then we necessarily have $f_E \notin \mathfrak{p}$ for all simple J_F-modules E, by a general semisimplicity criterion for symmetric algebras (see [16, Theorem 7.4.7]).

To prove the converse, first note that if $\mathfrak{p} = \{0\}$, then $F = k$ and the desired assertions are already contained in (3.1). Assume now that $\mathfrak{p} \neq \{0\}$ and, thus, k is a finite field. Then there exists a finite Galois extension $k' \supseteq k$ such that $J_{k'}$ is split; moreover, J_k is semisimple if and only if $J_{k'}$ is semisimple (see [4, Cor. 7.14]). We can extend the canonical map $R \mapsto k$ to a ring homomorphism $R' \to k'$ where R' is the ring of algebraic integers in a finite extension $F' \supseteq F$. Since J_F is split semisimple by Example 2.6, the same holds for $J_{F'}$; moreover, the scalar extension from F to F' defines a bijection between the isomorphism classes of simple J_F-modules and simple $J_{F'}$-modules. Clearly, if E is a simple J_F-module, $f_E \in R$ is also the Schur element of the simple $J_{F'}$-module $F' \otimes_F E$. Hence, if $f_E \notin \mathfrak{p}$ for all E, the above mentioned semisimplicity criterion now implies that $J_{k'}$ is split semisimple. Consequently, J_k is also semisimple. It remains to show that J_k is split, i.e., that every simple $J_{F'}$-module M' can be realized over k. Since k' is still a finite field, it is enough to show that $\text{Trace}(t_w, M') \in k$ for all $w \in W$ (see [9, Theorem I.19.3]). But, since J_F and $J_{k'}$ are both split semisimple, we can apply Tits' Deformation Theorem (see [4, §68A]). This shows that, for each M' there exists a simple $J_{F'}$-module E' (unique up to isomorphism) such that, for any $w \in W$, we have $\text{Trace}(t_w, E') \mapsto \text{Trace}(t_w, M')$ under the canonical map $R' \mapsto k'$. (Note that $\text{Trace}(t_w, E') \in R'$ since R' is integrally closed in F'.) Hence it is enough to show that $\text{Trace}(t_w, E') \in F$ for all $w \in W$. But this follows from the fact (already mentioned earlier in the proof), that every simple $J_{F'}$-module is obtained by scalar extension from a simple J_F-module. □

Now we turn to the algebra H and its specializations. The basic result Theorem 2.4 is valid for any ring homomorphism from A into a field. For further applications, it will be convenient to introduce the following notation.

Definition 3.3. Let k be a field. We say that a ring homomorphism $\theta\colon A \to k$ is an *admissible specialization* if the following conditions are satisfied.

(a) F is a splitting field for W. This implies that J_F and H_K are split semisimple algebras; see Example 2.6 and Theorem 2.7.
(b) R is the localization of the ring of algebraic integers of F in a prime ideal $\mathfrak{p}$, say, and k is the residue field of R. (We allow the case that $\mathfrak{p} = \{0\}$ and $R = F = k$.) In particular, R is a discrete valuation ring and so every simple H_K-module can be realized over A and every simple J_F-module can be realized over R; see Theorem 2.7 and Remark 2.8.
(c) There exists an invertible element $\zeta \in R$ such that θ is the composition of the specialization $A \to R$, $v \mapsto \zeta$, followed by the canonical map $R \mapsto k = R/\mathfrak{p}$.

It also follows that $\mathfrak{p}$ is either 0 or contains a rational prime number $\ell > 0$ and so k is either equal to F or a finite field of characteristic ℓ. Note that, for any invertible element $\zeta \in R$, the specialization map

$$\theta\colon A \to F, \qquad v \mapsto \zeta,$$

is an admissible specialization. See also [13, §3] for examples and a discussion of the connections with the modular representation theory of finite reductive groups.

Note that J_k only depends on $\mathfrak{p}$ but not on the specialization $v \mapsto \zeta$, since J_A is already defined over R. More precicely, we have a canonical isomorphism $J_k = k \otimes_A J_A = k \otimes_R J$.

3.4. Define an A-linear map $\tau\colon H \to A$ by $\tau(T_w) = 0$ if $1 \neq w \in W$ and $\tau(T_1) = 1$. Then we have $\tau(T_w T_{w'}) = u^{l(w)}$ if $w' = w^{-1}$ and $\tau(T_w T_{w'}) = 0$ otherwise. (For details of the proof, see [3, Prop. 10.9.1].) It follows that τ is a symmetrizing trace for H and, hence, H is a symmetric algebra. The basis dual to $\{T_w \mid w \in W\}$ is given by $\{u^{-l(w)} T_{w^{-1}} \mid w \in W\}$. We extend τ canonically to a trace function on H_K. Then, for any simple H_K-module V, we also have a corresponding *Schur element* $c_V \in K$ given by

$$c_V \, \dim_K V = \sum_{w \in W} u^{-l(w)} \operatorname{Trace}(T_w, V) \operatorname{Trace}(T_{w^{-1}}, V).$$

By [16, Prop. 7.3.9], we have in fact $c_V \in A$, since A is integrally closed in K. Moreover, we have $c_V \neq 0$, since H_K is semisimple; see [16, Theorem 7.2.6].

Theorem 3.5. *Let $\theta\colon A \to k$ be an admissible specialization and assume that $\theta(f_E) \neq 0$ for all simple J_F-modules E. Then the algebra H_k is split. Furthermore, H_k is semisimple if and only if $\theta(c_V) \neq 0$ for all simple H_K-modules V.*

Proof. By Theorem 2.7, H_K is split semisimple. Let us first show that H_k is split under the above assumptions on θ. For this purpose, we argue as follows. (Compare with the proof of Theorem 3.2.) Since k is a perfect field, there exists a finite Galois extension $k' \supseteq k$ such that $H_{k'}$ is split (see [4, Cor. 7.14]). Then we can extend the canonical map $R \mapsto k$ to a ring homomorphism $R' \to k'$ where R' is the ring of algebraic integers in a finite extension $F' \supseteq F$. Let $A' = R'[v, v^{-1}]$ and $K' = F'(v)$ be the field of fractions of A'. We extend θ to a ring homomorphism $\theta'\colon A' \to k'$.

Now we fix a simple $H_{k'}$-module M'. We must show that M' can be realized over k. First we show that

$$(*) \qquad \operatorname{Trace}(T_w, M') \in k \qquad \text{for all } w \in W.$$

To see this, we consider the group homomorphism $\phi_{k'}^*\colon R_0(J_{k'}) \to R_0(H_{k'})$ of (2.3). By Theorem 2.4, $\phi_{k'}^*$ is surjective. On the level of characters, this means that every irreducible character of $H_{k'}$ is an integral linear combination of the irreducible characters of $J_{k'}$. Thus, it remains to show that if E' is any simple $J_{k'}$-module, then we have $\operatorname{Trace}(t_w, E') \in k$ for all $w \in W$. This is seen as follows. By Theorem 3.2, J_k is split semisimple. Hence the same holds for $J_{k'}$ and, moreover, the scalar extension from k to k' defines a bijection between the isomorphism classes of simple J_k-modules and simple $J_{k'}$-modules. Thus, $(*)$ follows.

Now, if k is a finite field, then k' is still finite and $(*)$ implies that M' can be realized over k (see [9, Theorem I.19.3]). So we can now assume that $k = F$. To proceed, we note that Theorem 2.4 actually shows more than just the surjectivity of $\phi_{k'}^*$. Namely, there exists a $J_{k'}$-module E' and a surjective $H_{k'}$-module homomorphism $p\colon (E')^* \to M'$ such that M' does not occur as a composition factor of $\ker(p)$. We may certainly assume that E' is simple. Then (as already mentioned), E' is obtained by scalar extension from a simple J_k-module E. Hence $(E')^*$ is obtained by scalar extension from E^* and so we have a natural action of the Galois group of k'/k on $(E')^*$. Now, since F has characteristic 0, the composition factors of any $H_{F'}$-module are uniquely determined by the character of that module. Therefore, using $(*)$ and the fact that M' has multiplicity 1 in $(E')^*$, the submodule $\ker(p) \subset (E')^*$ is seen to be invariant under the action of the Galois group. So, by [19, Hilfssatz 13.2], that submodule is obtained by scalar extension from a submodule $U \subset E^*$. Thus, we have $M' \cong (E')^*/\ker(p) \cong k' \otimes_k (E^*/U)$, as required.

Finally, since H_k is split, we can now apply the general semisimplicity criterion for symmetric algebras which we already used in the proof of Theorem 3.2. It yields that H_k is semisimple if and only if $\theta(c_V) \neq 0$ for all simple H_K-modules V. □

In order to be able to apply the above results in a concrete example, we therefore need to solve the following two problems:

(A) Determine a splitting field F for W.

(B) Find the Schur elements f_E of J_F, where F is a splitting field.

In the case where Ω is cyclic and generated by a graph automorphism of (W_1, S_1), the answer to (A) is given in Theorem 2.11. In general, it is always enough to take a field F containing all m-th roots of unity, where m is the exponent of W; see [20, Theorem 10.3]. In the case where $\Omega = \{1\}$, a complete answer to (B) is given by the formula [27, 3.4e] (see also (4.5b) below) and the tables in [23, Chap. 4]. A solution to this problem in general will be given in the following section; see Corollary 4.7.

4. Clifford theory

We assume from now on that F is a splitting field for W, so that J_F and H_K are split semisimple algebras; see Example 2.6 and Theorem 2.7. The Clifford theory for characters of finite groups yields information about the induction and restriction of characters between W and its normal subgroup W_1. In this section, we apply a generalization of that theory (see [4, §11C]) to obtain similar results for the algebras H, H_1, and their specializations. We begin with the following discussion concerning simple modules over K.

4.1. Consider the isomorphism $\phi_F \colon F[W] \to J_F$ of Example 2.6. By extension of scalars from F to K, we obtain an isomorphism of K-algebras $\phi'_F \colon K[W] \to J_K$. Now consider the composition $\Psi_K := (\phi'_F)^{-1} \circ \phi_K \colon H_K \to K[W]$, where ϕ_K is obtained from ϕ in (2.2) by extending scalars from A to K. We already remarked above that the determinant of ϕ_K is non-zero. Hence, ϕ_K is an isomorphism which is defined over A. Since $(\phi'_F)^{-1}$ is an isomorphism which is defined over F, we conclude that Ψ_K is an isomorphism such that

(a) $$\Psi_K(T_w) = \sum_{z \in W} \xi_{z,w}(v)\, z \in K[W] \qquad \text{with } \xi_{z,w}(v) \in F[v, v^{-1}]$$

for all $w, z \in W$, where the coefficients $\xi_{z,w}(v)$ satisfy the condition that

(b) $$\xi_{w,z}(1) = 1 \text{ if } w = z \text{ and } \xi_{w,z}(1) = 0 \text{ otherwise.}$$

Now let E be a $K[W]$-module. Via composition with Ψ_K, we can also regard E as an H_K-module, which we denote by E_v. Since Ψ_K is an isomorphism, the correspondence $E \mapsto E_v$ defines a bijection between the isomorphism classes of simple modules for $K[W]$ and for H_K, respectively. By construction, we have $\mathrm{Trace}(T_w, E_v) \in F[v, v^{-1}]$ for all $w \in W$. Furthermore, (b) implies that

(c) $$\mathrm{Trace}(w, E) = \mathrm{Trace}(T_w, E_v)|_{v=1} \qquad \text{for all } w \in W.$$

Thus, the correspondence $E \mapsto E_v$ is entirely determined by the specialization $v \mapsto 1$. Alternatively, that correspondence can be established using Tits' Deformation

Theorem; see [4, §68A]. Applying the above constructions to W_1 and H_1, we obtain a corresponding K-algebra isomorphism $\Psi_{1,K}\colon H_{1,K} \to K[W_1]$, which is just the restriction of Ψ_K from H_K to $H_{1,K}$.

Lemma 4.2. *Let V be a simple H_K-module and V_1 be a simple $H_{1,K}$-module. Then the multiplicity of V_1 in the restriction of V to $H_{1,K}$ equals the multiplicity of E_1 in the restriction of E to W_1, where E is a simple $K[W]$-module such that $V \cong E_v$ and E_1 is a simple $K[W_1]$-module such that $V_1 \cong (E_1)_v$. Furthermore, if the above multiplicity is non-zero, then $\dim_K V_1$ divides $\dim_K V$ and the quotient $(\dim_K V)/(\dim_K V_1)$ divides $|\Omega|$.*

Proof. The first statement follows immediately from the fact that $\Psi_{1,K}$ is the restriction of Ψ_K from H_K to $H_{1,K}$. This also reduces the proof of the second statement to E_1 and E. The required assertion in this case is contained in [20, Lemma 6.8 and Cor. 11.29]. □

The above result is just a special case of a more general compatibility relation which we will consider next.

4.3. Let $\theta\colon A \to k$ be any ring homomorphism into a field k. For $\omega \in \Omega$, we set

$$H_{k,\omega} := \langle T_{w_1\omega} \mid w_1 \in W_1\rangle_k \subseteq H_k.$$

Then $H_{1,k} = k \otimes_A H_1$ is the Iwahori–Hecke algebra associated with (W_1, S_1) and we have $H_\omega = H_{1,k}T_\omega$ for all $\omega \in \Omega$. Since $H_k = \bigoplus_{\omega\in\Omega} H_{k,\omega}$ and $H_{k,\omega} \cdot H_{k,\omega'} = H_{k,\omega\omega'}$ for all $\omega, \omega' \in \Omega$, we see that the family of subspaces $\{H_{k,\omega}\}$ forms an Ω-*graded Clifford system* in H_k, in the sense of [4, Def. 11.12].

For any $\omega \in W$ and $w_1 \in W_1$, we have $T_\omega^{-1} T_{w_1} T_\omega = T_{\omega^{-1}w_1\omega}$. Thus, conjugation by a fixed T_ω defines a k-algebra automorphism of $H_{1,k}$. Given any $H_{1,k}$-module M_1, we can define a new $H_{1,k}$-module structure on M_1 by composing the original action with the above automorphism. We denote that new $H_{1,k}$-module by ${}^\omega M_1$. Thus, we have ${}^\omega M_1 = M_1$ as k-vector spaces, but $h_1 \in H_{1,k}$ acts on ${}^\omega M_1$ in the same way as $T_\omega^{-1} h_1 T_\omega$ acts on M_1. Now, by Clifford's Theorem (see [4, Prop. 11.16]), we have:

(a) Let M be a simple H_k-module and let M_1 be a simple submodule of the restriction of M to $H_{1,k}$. Then that restriction is the direct sum of simple $H_{1,k}$-modules which are all of the form ${}^\omega M_1$ for various $\omega \in \Omega$.

Similarly, if we define $J_{\omega,k} = \langle t_{w_1\omega} \mid w_1 \in W_1\rangle_k \subseteq J_k$ for any $\omega \in \Omega$, then the subspaces $\{J_{\omega,k}\}$ form an Ω-graded Clifford system in J_k and a statement analogous to (a) also holds for the restriction of a simple J_k-module to $J_{1,k}$.

We use the symbol Res_1 to denote the restriction of modules from H_k to $H_{1,k}$ (resp., from J_k to $J_{1,k}$). Now consider the homomorphisms $\phi_k\colon H_k \to J_k$ and $(\phi_1)_k\colon H_{1,k} \to J_{1,k}$. We have already remarked in (2.2) that $(\phi_1)_k$ is the restriction of ϕ_k to $H_{1,k}$. This yields the following compatibility result:

(b) For any J_k-module E, we have $\mathrm{Res}_1(E)^* = \mathrm{Res}_1(E^*)$.

Here, E^* is regarded as an H_k-module via ϕ_k and $\mathrm{Res}_1(E)^*$ is regarded as an $H_{1,k}$-module via $(\phi_1)_k$; see (2.3).

Lemma 4.4. *In the above set-up, assume that M is a simple H_k-module and let M_1 be a simple submodule of the restriction of M of $H_{1,k}$. Then all modules ${}^\omega M_1$ $(\omega \in \Omega)$ have the same a-invariant and this is equal to the a-invariant of M.*

Proof. First we show that ${}^{\omega}M_1$ has the same a-invariant as M_1 for all $\omega \in \Omega$. Let $w_1 \in W_1$. Using the formulas in (2.1b) and (1.2), we have

$$T_\omega^{-1} C_{w_1} T_\omega = T_{\omega^{-1}} C_{w_1} T_\omega = C_{\omega^{-1} w_1 \omega}.$$

Thus, we see that $C_{w_1}({}^{\omega}M_1) \neq 0$ if and only if $C_{\omega^{-1}w_1\omega} M_1 \neq 0$. Since $a(\omega^{-1}w_1\omega) = a(w_1)$ by (2.1c), we conclude that the a-invariants of ${}^{\omega}M_1$ and M_1 are the same.

Now we can show that $a_{M_1} = a_M$. Indeed, since M_1 is a submodule of M, it is clear that $a_{M_1} \leq a_M$. In order to prove the reverse inequality, let $w \in W$ be such that $a(w) = a_M$ and $C_w M \neq 0$. We write $w = w_1\omega'$ with $w_1 \in W_1$, $\omega' \in \Omega$. Then we have $C_w = C_{w_1} T_{\omega'}$; see (2.1b). Now $T_{\omega'}$ is an invertible element in H_k, so the condition that $C_w M \neq 0$ implies that $C_{w_1} M \neq 0$. Hence, using Clifford's Theorem as in (4.3a), we see that C_{w_1} does not act as 0 on some simple direct summand of $\mathrm{Res}_1(M)$. Thus, there exists some $\omega \in \Omega$ such that $C_{w_1}({}^{\omega}M_1) \neq 0$. Consequently, $a(w_1)$ is less than or equal to the a-invariant of ${}^{\omega}M_1$. We have seen before that the latter a-invariant equals that of M_1. Thus, using (2.1c), we have $a_M = a(w) = a(w_1) \leq a_{M_1}$, as desired. □

As far as the simple H_K-modules are concerned, we can even obtain a more precise result involving the Schur elements. This is based on the following remarks.

4.5. Let V be a simple H_K-module. Since ϕ_K is an isomorphism, there exists a simple J_K-module E such that V is isomorphic to E^*. Then the question arises in which way the a-invariants and the Schur elements of E and V are related. As far as the a-invariants are concerned, the answer is that we have

(a) $\qquad a_E = a_V = \min\{i \geq 0 \mid v^{i-l(w)}\,\mathrm{Trace}(T_w, V) \in R[v] \text{ for all } w \in W\}.$

First note that, since A is integrally closed in K, we have $\mathrm{Trace}(T_w, V) \in A$ for every $w \in W$ (see [16, Prop. 7.3.8]). Now, if $w \in W$ is such that $a(w) = a_V$ and $C_w V \neq 0$, then, since $V \cong E^*$, we also have $\phi(C_w)E \neq 0$. The defining formula for ϕ shows that there exists some $z \in W$ and $d \in \mathcal{D}$ such that $a(d) = a(z)$, $h_{w,d,z} \neq 0$ and $t_z E \neq 0$. Then we have $a_E = a(z) = a(d)$. But, since $h_{w,d,z} \neq 0$, we must have $a(d) \geq a(w)$ by (2.1a) and so $a_E \leq a_V$. The reverse inequality and the identity relating a_V with the character values of V are contained in [27, Prop. 3.3 and 3.4a].

The above identity shows that $v^{a_V - l(w)}\mathrm{Trace}(T_w, V)$ is a polynomial in $R[v]$ for all $w \in W$. Moreover, by [27, 3.4b], the constant term of that polynomial in fact equals $(-1)^{a_V}\,\mathrm{Trace}(t_w, E)$. As shown in [27, 3.4e], this implies that

(b) $\qquad c_V = u^{-a_V} f_E + R\text{-linear combination of higher powers of } v,$

where $0 \neq f_E \in R$ is the Schur element of E; see (3.1).

Proposition 4.6. *Let V be a simple H_K-module and $0 \neq c_V \in A$ be the corresponding Schur element. Then we have*

$$c_V \dim_K V = |\Omega|\, c_{V_1} \dim_K V_1$$

for any simple $H_{1,K}$-module V_1 which occurs in the restriction of V to $H_{1,K}$. Consequently, we have $a_V = a_{V_1}$ and

$$f_E \dim_K V = |\Omega|\, f_{E_1} \dim_K V_1,$$

where E is a simple J_K-module such that $V \cong E^*$ and E_1 is a simple $J_{1,K}$-module such that $V_1 \cong E_1^*$.

Proof. We use the following interpretation of the Schur elements (see [16, Theorem 7.2.1]). Assume that $\dim_K V = m$ and let $\rho\colon H_K \to M_m(K)$ be the matrix representation afforded by V with respect to some basis of V. Then we have

$$m\, c_V \,\mathrm{id}_m = \sum_{w \in W} u^{-l(w)} \rho(T_w)\, \rho(T_{w^{-1}}),$$

where id_m denotes the $m \times m$-identity matrix. Now, writing each $w \in W$ in the form $w = w_1\omega$ (with $w_1 \in W_1$, $\omega \in \Omega$) and using the relations in (1.2), we find that

$$\begin{aligned} m\, c_V \,\mathrm{id}_m &= \sum_{w_1 \in W_1, \omega \in \Omega} u^{-l(w_1)} \rho(T_{w_1} T_\omega) \rho(T_{\omega^{-1}} T_{w_1^{-1}}) \\ &= \sum_{w_1 \in W_1} u^{-l(w_1)} \rho(T_{w_1}) \Big(\sum_{\omega \in \Omega} \rho(T_\omega)\rho(T_{\omega^{-1}}) \Big) \rho(T_{w_1^{-1}}) \\ &= |\Omega| \sum_{w_1 \in W_1} u^{-l(w_1)} \rho(T_{w_1}) \rho(T_{w_1^{-1}}) \end{aligned}$$

where the last equality holds since $T_{\omega^{-1}} = T_\omega^{-1}$ for all $\omega \in \Omega$. Now, by (4.3a), we may assume that the restriction of ρ to $H_{1,K}$ is the matrix direct sum of irreducible representations of $H_{1,K}$ affording the simple direct summands of the restriction of V. Denote these direct summands by $V_1, \dots, V_d$, let $m_1, \dots, m_d$ be their dimensions and $\rho_1, \dots, \rho_d$ the corresponding matrix representations. Then we have

$$m_i\, c_{V_i} \,\mathrm{id}_{m_i} = \sum_{w_1 \in W_1} u^{-l(w_1)} \rho_i(T_{w_1}) \rho_i(T_{w_1^{-1}}) \qquad \text{for } 1 \leq i \leq d.$$

So we conclude that $m\, c_V \,\mathrm{id}_m$ is $|\Omega|$ times a block diagonal matrix, where each diagonal block has the form $m_i\, c_{V_i} \,\mathrm{id}_{m_i}$. Thus, we have $m\, c_V = |\Omega| m_i c_{V_i}$ for $1 \leq i \leq d$, as desired. The assertions about the a-invariants and the Schur elements then follow from the formula in (4.5b). ∎

In order to state the following result about the Schur elements of J_F, we recall the notion of "bad primes". A prime number p is "bad" for (W_1, S_1) if p divides a Schur element of $J_{1,F}$. (Note that $\mathbb{Q}$ is a splitting field for W_1 (see Theorem 2.11) and so all Schur elements of $J_{1,F}$ are rational integers by (3.1).) Thus, p is bad if p is bad for some irreducible component of (W_1, S_1). Using the tables in [23, Chap. 4], we see that the conditions for the various irreducible types are as follows:

$$\begin{aligned} A_n &: \quad \text{none}, \\ B_n, C_n, D_n &: \quad p = 2, \\ G_2, F_4, E_6, E_7 &: \quad p \in \{2, 3\}, \\ E_8 &: \quad p \in \{2, 3, 5\}. \end{aligned}$$

Corollary 4.7. *Recall that F is assumed to be a splitting field for W. Then we have $f_E \in \mathbb{Z}$ for all simple J_F-modules E. Furthermore, the only primes dividing f_E are the "bad primes" for (W_1, S_1) and the prime divisors of the order of Ω.*

Proof. By Proposition 4.6 and Lemma 4.2 we have $f_E = m\, f_{E_1}$, where m is an integer dividing the order of Ω. It remains to use the fact that $\mathbb{Q}$ is a splitting field for W_1 (see Theorem 2.11) and so $f_{E_1} \in \mathbb{Z}$ by (3.1). ∎

4.8. The results in Proposition 4.6 and Lemma 4.2 show that the Schur elements of H_K are, up to integer factors which divide the order of $|\Omega|$, equal to the Schur elements of $H_{1,K}$. Now the latter are explicitly known for all types of (W_1, S_1);

see, for example, [16]. These explicit results show that each c_V is a product of an integral power of u, various cyclotomic polynomials in u, and an integer which is only divisible by bad primes and the prime divisors of the order of $|\Omega|$.

Now assume that $\theta\colon A \to k$ is an admissible specialization such that the characteristic of k is either 0 or a prime which is neither a bad prime nor a prime divisor of $|\Omega|$. Then $\theta(c_V)$ can only be zero for some $V \in \operatorname{Irr}(H_K)$ if $\theta(u)$ is a root of unity in k. Hence, by Theorem 3.5, the specialized algebra H_k if semisimple unless $\theta(u)$ is a root unity.

5. Decomposition numbers

In order to study modular representations of H, we will now place ourselves in the following standard setting.

5.1. We assume that F is a splitting field for W. Then H_K and J_F are split semisimple algebras. Now let $\theta\colon A \to k$ be an admissible specialization as in Definition 3.3. We assume that the characteristic of k is either 0 or a prime which is not bad for (W_1, S_1) and which does not divide the order of Ω. Then, by Corollary 4.7, the Schur elements of J_F remain non-zero in k and so, by Theorem 3.2 and Theorem 3.5, J_k is split semisimple and H_k is a split algebra. The same statements also hold for the algebras $J_{1,F}$, $J_{1,k}$, $H_{1,K}$ and $H_{1,k}$ (see also [14]). Then, by [16, Theorem 7.4.3], we have well-defined *decomposition maps*

$$d_\theta\colon R_0(H_K) \to R_0(H_k) \qquad \text{and} \qquad d_\theta^1\colon R_0(H_{1,K}) \to R_0(H_{1,k}).$$

Since all simple H_K-modules can be realized over A, the map d_θ is given as follows. Let V be a simple H_K-module. The condition that V can be realized over A means that there exists an H-module $\hat{V}$ which is finitely generated and free as an A-module such that $V \cong K \otimes_A \hat{V}$. Then $d_\theta([V]) = [k \otimes_A \hat{V}]$. The map d_θ^1 is determined similarly. We shall write

$$d_\theta([V]) = \sum_{M \in \operatorname{Irr}(H_k)} (V : M)\,[M] \qquad \text{for all } V \in \operatorname{Irr}(H_K),$$

$$d_\theta^1([V_1]) = \sum_{M_1 \in \operatorname{Irr}(H_{1,k})} (V_1 : M_1)\,[M_1] \qquad \text{for all } V_1 \in \operatorname{Irr}(H_{1,K}),$$

where $(V : M) \in \mathbb{N}_0$ and $(V_1 : M_1) \in \mathbb{N}_0$ are called the *decomposition numbers*. (For any finite dimensional algebra T over a field, we denote by $\operatorname{Irr}(T)$ the set of simple T-modules, up to isomorphism.) As we have seen in (4.8), the algebras H_k and $H_{1,k}$ are semisimple unless $\theta(u)$ is a root of unity. Note that, if this is the case, the decomposition matrices are the identity matrices by Tits' Deformation Theorem [4, §68A]. Thus, we will be mainly interested in the case where $\theta(u)$ is a root of unity. We define e by

$$e = \min\{i \geq 2 \mid 1 + \theta(u) + \theta(u)^2 + \cdots + \theta(u)^{i-1} = 0\}.$$

(If no integer $i \geq 2$ satisfying the above condition exists, we set $e = \infty$.) Note that, if $\theta(u) = 1$ and k has characteristic $\ell \geq 0$, then $e = \infty$ (for $\ell = 0$) or $e = \ell$ (for $\ell > 0$); in all other cases, e is the order of $\theta(u)$ in the multiplicative group of k.

Lemma 5.2. *The restriction of modules from H_K to $H_{1,K}$ (resp., from H_k to $H_{1,k}$) induces maps on the level of Grothendieck groups and we have a commutative* *diagram*

$$\begin{array}{ccc} R_0(H_K) & \xrightarrow{\mathrm{Res}_1} & R_0(H_{1,K}) \\ \Big\downarrow{\scriptstyle d_\theta} & & \Big\downarrow{\scriptstyle d_\theta^1} \\ R_0(H_k) & \xrightarrow{\mathrm{Res}_1} & R_0(H_{1,k}) \end{array}$$

Moreover, for any $V_1 \in \mathrm{Irr}(H_{1,K})$ *and* $M_1 \in \mathrm{Irr}(H_{1,K})$*, we have*

$$(*) \qquad (V_1 : M_1) = ({}^{\omega}V_1 : {}^{\omega}M_1) \qquad \text{for all } \omega \in \Omega.$$

Proof. The commutativity of the diagram is readily established using the characterization of decomposition maps in [13, §2]. To prove $(*)$, it suffices to note that, for any $\omega \in \Omega$, the map $T_{w_1} \mapsto T_{\omega^{-1}w_1\omega}$ defines algebra automorphisms of $H_{1,K}$ and of $H_{1,k}$. The compatibility with the decomposition map is again established using [13, §2]. □

The following result extends [14, Theorem 3.3] and [18, Cor. 4.3].

Theorem 5.3. *Let* $\theta\colon A \to k$ *be an admissible specialization satisfying the conditions in* (5.1)*. We consider the following subset of* $\mathrm{Irr}(H_K)$*:*

$$\mathcal{B} := \{V \in \mathrm{Irr}(H_K) \mid (V : M) \neq 0 \text{ and } a_V = a_M \text{ for some } M \in \mathrm{Irr}(H_k)\}.$$

Then there exists a unique bijection $\mathcal{B} \leftrightarrow \mathrm{Irr}(H_k)$, $V \leftrightarrow \overline{V}$*, such that the following two conditions hold:*

(a) *For all* $V \in \mathcal{B}$*, we have* $(V : \overline{V}) = 1$ *and* $a_V = a_{\overline{V}}$.
(b) *If* $V \in \mathrm{Irr}(H_K)$ *and* $M \in \mathrm{Irr}(H_k)$ *are such that* $(V : M) \neq 0$*, then we have* $a_M \leq a_V$*, with equality only for* $V \in \mathcal{B}$ *and* $M = \overline{V}$.

In particular, the matrix of all decomposition numbers $(V : \overline{V}')$ $(V, V' \in \mathcal{B})$ *is square unitriangular, if we order the simple modules according to increasing* a*-invariants.*

Proof. In the case where $\Omega = \{1\}$, this has been proved in [14, Theorem 3.3] and [18, Cor. 4.3]. The same proofs apply here again, based on the observation that J_k is split semisimple and that d_θ can be interpreted in terms of Lusztig's homomorphism $\phi_k\colon H_k \to J_k$. All the required properties of that homomorphism also hold in the case where $\Omega \neq \{1\}$ (see [25], [26], [27] and the remarks in Section 2). □

5.4. Note that, while the a-invariants of the simple H_K-modules are known via the formulas in (4.5), Proposition 4.6 and the tables in [23, Chap. 4], there does not seem to be an efficient way of determining directly the a-invariants of the simple H_k-modules. However, once the numbers $(V : M)$ are known (for some labelling of the simple H_k-modules) the a-invariant of $M \in \mathrm{Irr}(H_k)$ is determined by

$$a_M = \min\{a_V \mid (V : M) \neq 0\};$$

see also [14, Remark 3.4]. Moreover, Theorem 5.3 shows that for a given $M \in \mathrm{Irr}(H_k)$, there exists a unique $V \in \mathrm{Irr}(H_K)$ such that $(V : M) \neq 0$ and $a_V = a_M$. Thus, the decomposition matrix uniquely determines $\mathcal{B}$ and the bijection $\mathcal{B} \leftrightarrow \mathrm{Irr}(H_k)$.

Theorem 5.3 applies, in particular, to H_1 (the case where $\Omega = \{1\}$). Denote by $\mathcal{B}_1$ the corresponding subset of $\mathrm{Irr}(H_{1,K})$. The following result shows that $\mathcal{B}$ and $\mathcal{B}_1$ determine each other.

Theorem 5.5. *Under the assumptions of Theorem 5.3, the following hold:*

(a) *$\mathcal{B}_1$ is the set of all $V_1 \in \mathrm{Irr}(H_{1,K})$ such that $V_1 \subseteq \mathrm{Res}_1(V)$ for some $V \in \mathcal{B}$.*
(b) *$\mathcal{B}$ is the set of all $V \in \mathrm{Irr}(H_K)$ such that $V_1 \subseteq \mathrm{Res}_1(V)$ for some $V_1 \in \mathcal{B}_1$.*

In particular, if $V_1 \in \mathcal{B}_1$, then all conjugates of V_1 lie in $\mathcal{B}_1$; furthermore, if $V \in \mathcal{B}$ then all simple submodules of $\mathrm{Res}_1(V)$ lie in $\mathcal{B}_1$.

Proof. Let $V \in \mathrm{Irr}(H_K)$ and $V_1 \in \mathrm{Irr}(H_{1,K})$ be such that V_1 occurs in $\mathrm{Res}_1(V)$. We must show that $V \in \mathcal{B}$ if and only if $V_1 \in \mathcal{B}_1$.

First assume that $V \in \mathcal{B}$. Then we have $a_V = a_{\overline{V}}$ and $d_\theta([V]) = [\overline{V}]+$ lower terms, where the expression "lower terms" stands for a sum of simple H_k-modules whose a-invariants are strictly less than that of V. The commutative diagram in Lemma 5.2 and the statements in Lemma 4.4 yield that

$$d_\theta^1 \circ \mathrm{Res}_1([V]) = \mathrm{Res}_1 \circ d_\theta([V]) = \mathrm{Res}_1([\overline{V}]) + \text{lower terms}.$$

Now let $M_1 \in \mathrm{Irr}(H_{1,K})$ be a simple submodule of $\mathrm{Res}_1(\overline{V})$. Then, by Lemma 4.4, the a-invariant of M_1 equals $a_V = a_{\overline{V}}$. On the other hand, by (4.3), we can write $[\mathrm{Res}_1(V)]$ as a sum of terms $[{}^\omega V_1]$, for various $\omega \in \Omega$. It follows that there exists some $\omega \in \Omega$ such that

$$d_\theta^1([{}^\omega V_1]) = [M_1] + \text{sum of further terms } [M_1'] \text{ with } M_1' \in \mathrm{Irr}(H_{1,k}).$$

Now, by Lemma 4.4, ${}^\omega V_1$ has the same a-invariant as V, and this equals a_{M_1}. Thus, we have ${}^\omega V_1 \in \mathcal{B}_1$. But then the compatibility relation $(*)$ in Lemma 5.2 implies also that $V_1 \in \mathcal{B}_1$, as desired.

Now assume that $V \notin \mathcal{B}$. Using (4.3) and once more Lemma 5.2, we have that

$$\mathrm{Res}_1 \circ d_\theta([V]) = d_\theta^1([\mathrm{Res}_1(V)]) = d_\theta^1([V_1]) + \text{sum of terms } d_\theta^1([{}^\omega V_1]),$$

for various $\omega \in \Omega$. Now, the fact that V is not in $\mathcal{B}$ means that $d_\theta([V])$ is a sum of terms $[M]$, where $M \in \mathrm{Irr}(H_k)$ are such that $a_M < a_V$. Using Lemma 4.4, it follows that the left-hand side of the above identity is a sum of terms $[M_1]$, where $M_1 \in \mathrm{Irr}(H_{1,k})$ are such that $a_{M_1} < a_V$. Consequently, a similar statement also holds for $d_\theta^1([V_1])$. This means that $V_1 \notin \mathcal{B}_1$. □

Corollary 5.6. *Let $V \in \mathcal{B}$ and write $\mathrm{Res}_1(V) = V_1 \oplus \cdots \oplus V_m$ with $V_i \in \mathrm{Irr}(H_{1,K})$. Then $V_i \in \mathcal{B}_1$ for all i and we have*

$$\mathrm{Res}_1(\overline{V}) = \overline{V}_1 \oplus \cdots \oplus \overline{V}_m,$$

where $\overline{V}$ and $\overline{V}_1, \ldots, \overline{V}_m$ are defined as in Theorem 5.3.

Proof. By Theorem 5.5, we have $V_i \in \mathcal{B}_1$ for all i. So, as in the above proof, we can write $d_\theta^1([V_i]) = [\overline{V}_i]+$ lower terms. It follows that

(a) $$d_\theta^1([\mathrm{Res}_1(V)]) = [\overline{V}_1] + \cdots + [\overline{V}_m] + \text{lower terms}.$$

On the other hand, the fact that $V \in \mathcal{B}$ implies that $d_\theta([V]) = [\overline{V}]+$ lower terms. So, using Lemma 4.4, we can also write

(b) $$\mathrm{Res}_1(d_\theta([V])) = [\mathrm{Res}_1(\overline{V})] + \text{lower terms}.$$

Comparing (a) and (b) using the compatibility in Lemma 5.2, we conclude that $[\mathrm{Res}_1(\overline{V})] = [\overline{V}_1] + \cdots + [\overline{V}_m]$. Thus, the composition factors of $\mathrm{Res}_1(\overline{V})$ are determined. On the other hand, by Clifford's Theorem (see (4.3a)), we know that $\mathrm{Res}_1(\overline{V})$ is a direct sum of simple modules. □

Example 5.7. Assume that the order of Ω is a prime number p. Then we have the following relations between the simple modules for H_k and for $H_{1,k}$, respectively.

Let $V \in \mathcal{B}$. Then, via Lemma 4.2 and [20, 6.20], there are two cases:

(1) $V_1 := \text{Res}_1(V)$ is simple and $V_1 \in \mathcal{B}_1$. Then we also have $\overline{V}_1 = \text{Res}_1(\overline{V})$. Moreover, there are precisely p non-isomorphic simple H_k-modules whose restriction to $H_{1,k}$ is $\overline{V}_1$.

(2) $\text{Res}_1(V)$ is the direct sum of p pairwise non-isomorphic simple $H_{1,K}$-modules $V_1, \dots, V_p \in \mathcal{B}_1$. Then we also have $\text{Res}_1(\overline{V}) = \overline{V}_1 \oplus \cdots \oplus \overline{V}_p$ and $\overline{V}_i \not\cong \overline{V}_j$ for $i \neq j$.

This follows immediately from Corollary 5.6. Thus, knowing the decomposition pattern for the restriction of the simple H_K-modules, we see that a classification of the simple H_k-modules determines a classification of the simple $H_{1,k}$-modules and vice versa.

Let us consider in more detail the case where $p = 2$. We introduce the following notation. Let $\mathcal{B}^I$ (resp., $\mathcal{B}^{II}$) be the set of all $V \in \mathcal{B}$ such that $\text{Res}_1(V)$ is simple (resp., splits into a direct sum of two simple modules). Then we have $\mathcal{B} = \mathcal{B}^I \cup \mathcal{B}^{II}$ and we obtain a corresponding decomposition of $\mathcal{B}_1 = \mathcal{B}_1^I \cup \mathcal{B}_1^{II}$, where $\mathcal{B}_1^I$ (resp., $\mathcal{B}_1^{II}$) is the set of all $V_1 \in \mathcal{B}_1$ such that V_1 occurs in $\text{Res}_1(V)$ for some $V \in \mathcal{B}^I$ (resp., for some $V \in \mathcal{B}^{II}$). From the above discussion we see that

$$|\mathcal{B}^I| = 2\,|\mathcal{B}_1^I| \qquad \text{and} \qquad 2\,|\mathcal{B}^{II}| = |\mathcal{B}_1^{II}|.$$

Using the bijection in Theorem 5.3, we also obtain corresponding decompositions

$$\text{Irr}(H_k) = \text{Irr}(H_k)^I \cup \text{Irr}(H_k)^{II} \quad \text{and} \quad \text{Irr}(H_{1,k}) = \text{Irr}(H_{1,k})^I \cup \text{Irr}(H_{1,k})^{II},$$

where $\text{Irr}(H_k)^I \leftrightarrow \mathcal{B}^I$, $\text{Irr}(H_k)^{II} \leftrightarrow \mathcal{B}^{II}$, $\text{Irr}(H_{1,k})^I \leftrightarrow \mathcal{B}_1^I$ and $\text{Irr}(H_{1,k})^{II} \leftrightarrow \mathcal{B}_1^{II}$. With this notation, we have the following relations between the decomposition numbers of H_k and $H_{1,k}$. Let $V \in \mathcal{B}$ and $V_1 \in \mathcal{B}_1$ be such that $V_1 \subseteq \text{Res}_1(V)$. Let $M \in \text{Irr}(H_k)$ and $M_1 \in \text{Irr}(H_{1,k})$ be such that $M_1 \subseteq \text{Res}_1(M)$. Then we have

$$\begin{aligned}
(V_1 : M_1) &= (V : M) + (V : M') && \text{if } V \in \mathcal{B}^I \text{ and } M \in \text{Irr}(H_k)^I,\\
(V_1 : M_1) &= (V : M) && \text{if } V \in \mathcal{B}^I \text{ and } M \in \text{Irr}(H_k)^{II},\\
(V_1 : M_1) &= \frac{1}{2}((V : M) + (V : M')) && \text{if } V \in \mathcal{B}^{II} \text{ and } M \in \text{Irr}(H_k)^I,
\end{aligned}$$

where M' is the second simple H_k-module such that $\text{Res}_1(M') = \text{Res}_1(M)$ in the case where $M \in \text{Irr}(H_k)^I$. Thus, the remaining problem is to describe the decomposition numbers $(V_1 : M_1)$ where $V \in \mathcal{B}_1^{II}$ and $M \in \text{Irr}(H_k)^{II}$.

Finally, we will establish a formula relating the cardinalities of the sets $\text{Irr}(H_k)$ and $\text{Irr}(H_{1,k})$ (in the case where Ω is cyclic of order 2). This will be based on an argument involving the *character table* of $H_{1,K}$, which is defined in [16, 8.2.9]. In the following discussion, we identify $\text{Irr}(H_K)$ and $\text{Irr}(H_{1,K})$ with the sets of the corresponding characters. Let $\mathcal{R}$ be a set of representatives of minimal length in the conjugacy classes of W_1. Then the character table of $H_{1,K}$ is the matrix of all character values $\chi(T_w)$ for $\chi \in \text{Irr}(H_{1,K})$ and $w \in \mathcal{R}$. We decompose $\mathcal{R}$ as a disjoint union of subsets $\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_1'$, where $\mathcal{R}_0$ consists of all $w \in \mathcal{R}$ such that $\sigma w \sigma$ is conjugate to w in W_1, $\mathcal{R}_1 \cup \mathcal{R}_1' = \mathcal{R} \setminus \mathcal{R}_0$, and $\mathcal{R}_1' = \{\sigma w \sigma \mid w \in \mathcal{R}_1\}$. We define the following matrix of character values:

$$\Delta(H_1, \sigma) := \left[\chi(T_w) - {}^{\sigma}\chi(T_w)\right]_{\chi, w},$$

where χ runs over all irreducible characters of $H_{1,K}$ such that $\chi \neq {}^{\sigma}\chi$ and w runs over all elements in $\mathcal{R}_1$. Now consider the specialization map $\theta\colon A \to k$. We denote this map simply by $a \mapsto \bar{a}$. Since all character values at basis elements T_w lie in A, we can apply θ to all entries of the above matrix; we denote the specialized matrix, with coefficients in k, by $\bar{\Delta}(H_1, \sigma)$. Now we can state

Theorem 5.8. *Recall that we assume that Ω is cyclic of order 2 and that the characteristic of k is not 2. Then, with the above notation, we have $|\mathcal{B}^{II}| = \operatorname{rank} \bar{\Delta}(H_1, \sigma)$ and*

$$2\,|\operatorname{Irr}(H_{1,k})| = |\operatorname{Irr}(H_k)| + 3 \operatorname{rank} \bar{\Delta}(H_1, \sigma).$$

Proof. First note that $|\operatorname{Irr}(H_{1,k})| = |\mathcal{B}_1|$, by Theorem 5.3. Next, by the discussion in Example 5.7, we have

$$|\mathcal{B}_1| = \frac{1}{2}\,|\mathcal{B}^{I}| + 2\,|\mathcal{B}^{II}| = \frac{1}{2}(|\mathcal{B}| + 3\,|\mathcal{B}^{II}|) = \frac{1}{2}(|\operatorname{Irr}(H_k)| + 3\,|\mathcal{B}^{II}|).$$

Thus, we must show that

$$(*) \qquad |\mathcal{B}^{II}| = \operatorname{rank} \bar{\Delta}(H_1, \sigma).$$

In order to prove $(*)$, let us write $\operatorname{Irr}(H_K) = \{\chi_1, \chi_1', \ldots, \chi_r, \chi_r', \psi_1, \ldots, \psi_s\}$ for $r, s \geq 1$, where the notation is such that χ_i and χ_i' have the same restriction to $H_{1,K}$ and the restriction of ψ_j to $H_{1,K}$ is of the form $\psi_j^+ + \psi_j^-$ with $\psi_j^{\pm} \in \operatorname{Irr}(H_{1,K})$. Now note that we have the following relations among the character values:

$$\begin{array}{lllllll} \chi_i(T_w) & = & \chi_i'(T_w) & \text{and} & \psi_j^+(T_w) & = & \psi_j^-(T_w) \qquad \text{for } w \in \mathcal{R}_0, \\ \chi_i(T_w) & = & \chi_i(T_{\sigma w \sigma}) & \text{and} & \psi_j^+(T_w) & = & \psi_j^-(T_{\sigma w \sigma}) \qquad \text{for } w \in \mathcal{R}_1. \end{array}$$

(For proofs in the case where (W_1, S_1) is of type D_n, see [16, 10.4.6]; the same proofs work in general.) These relations allow us to partition the whole character table of $H_{1,K}$ into blocks. We define

$$X_0 := (\chi_i(T_w))_{\substack{1\leqslant i\leqslant r\\ w\in\mathcal{R}_0}}, \qquad Y_0 := (\psi_j(T_w))_{\substack{1\leqslant j\leqslant s\\ w\in\mathcal{R}_0}},$$
$$X_1 := (\chi_i(T_w))_{\substack{1\leqslant i\leqslant r\\ w\in\mathcal{R}_1}}, \qquad Y_1^{\pm} := (\psi_j^{\pm}(T_w))_{\substack{1\leqslant j\leqslant s\\ w\in\mathcal{R}_1}}.$$

Then the character table of $H_{1,K}$ is of the following form:

$$\mathfrak{X} := \begin{bmatrix} X_0 & X_1 & X_1 \\ \frac{1}{2}Y_0 & Y_1^+ & Y_1^- \\ \frac{1}{2}Y_0 & Y_1^- & Y_1^+ \end{bmatrix}.$$

Applying the specialization map $\theta\colon A \to k$, we obtain a new matrix with coefficients in k (note that the characteristic of k is not 2)

$$\bar{\mathfrak{X}} = \begin{bmatrix} \bar{X}_0 & \bar{X}_1 & \bar{X}_1 \\ \frac{1}{2}\bar{Y}_0 & \bar{Y}_1^+ & \bar{Y}_1^- \\ \frac{1}{2}\bar{Y}_0 & \bar{Y}_1^- & \bar{Y}_1^+ \end{bmatrix}.$$

Now, we know from the discussion in Example 5.7 that there are subsets $I \subseteq \{1, \ldots, r\}$ and $J \subseteq \{1, \ldots, s\}$ such that

$$\mathcal{B}_1 = \{\chi_i \mid i \in I\} \cup \{\psi_j^+ \mid j \in J\} \cup \{\psi_j^- \mid j \in J\}.$$

We denote by $X_0[I]$ and $X_1[I]$ the submatrices of X_0 and X_1, respectively, with rows in I. A similar notation will be used for submatrices of Y_0 and $Y_1^{\pm}$. Recalling the definition of $\Delta(H_1,\sigma)$, we see that $(*)$ is equivalent to

$$(*') \qquad |J| = \text{rank } (\bar{Y}_1^+ - \bar{Y}_1^-).$$

Since the decomposition matrix of $H_{1,k}$ has a triangular shape with 1 along the diagonal by Theorem 5.3, we can apply [16, 7.5.7 and 8.2.9] and deduce that $|\text{Irr}(H_{1,k})| = \text{rank } \bar{\mathfrak{X}}$. More precisely, since the triangular shape is given by the subset $\mathcal{B}_1 \subseteq \text{Irr}(H_{1,K})$, we have that rank $\bar{\mathfrak{X}} = \text{rank } \bar{\mathfrak{X}}(I,J)$, where

$$\bar{\mathfrak{X}}(I,J) := \begin{bmatrix} \bar{X}_0[I] & \bar{X}_1[I] & \bar{X}_1[I] \\ \frac{1}{2}\bar{Y}_0[J] & \bar{Y}_1^+[J] & \bar{Y}_1^-[J] \\ \frac{1}{2}\bar{Y}_0[J] & \bar{Y}_1^-[J] & \bar{Y}_1^+[J] \end{bmatrix}.$$

In the following discussion, we will consider the matrix $\bar{\mathfrak{X}}(I,J')$ where J' is any subset of $\{1,\ldots,s\}$. First note that we have

$$(**) \qquad \text{rank } \bar{\mathfrak{X}}(I,J') = \text{rank } \bar{\mathfrak{X}}(I,J) = \text{rank } \bar{\mathfrak{X}} \qquad \text{if } J' \supseteq J.$$

Now, after some elementary row and column operations, we see that $\bar{\mathfrak{X}}(I,J')$ has the same rank as the following block diagonal matrix:

$$\bar{\mathfrak{Y}}(I,J') := \left[\begin{array}{cc|c} \bar{X}_0[I] & \bar{X}_1[I] & 0 \\ \bar{Y}_0[J'] & \bar{Y}_1^+[J'] + \bar{Y}_1^-[J'] & 0 \\ \hline 0 & 0 & \bar{Y}_1^+[J'] - \bar{Y}_1^-[J'] \end{array}\right].$$

By $(**)$, the rows of the above matrix are linearly independent for $J' = J$. Hence the rows of each of the two diagonal blocks are linearly independent. In particular, this shows that

$$|J| = \text{rank } (\bar{Y}_1^+[J] - \bar{Y}_1^-[J]).$$

Furthermore, if we replace J by any subset J' of $\{1,\ldots,s\}$ with $J' \supseteq J$, the rank does not change by $(**)$. This must hold for each of the two diagonal blocks in $\bar{\mathfrak{Y}}(I,J')$ and so we conclude that

$$\text{rank } (\bar{Y}_1^+[J'] - \bar{Y}_1^-[J']) = \text{rank } (\bar{Y}_1^+[J] - \bar{Y}_1^-[J]) = |J| \quad \text{for all } J' \supseteq J.$$

Consequently, $(*')$ follows by taking $J' = \{1,\ldots,s\}$. □

Example 5.9. Let $w_0 \in W_1$ be the unique element of maximal length. Then w_0 has order 2 and conjugation with w_0 defines an automorphism $\sigma_0 \in \text{Aut}(W_1,S_1)$. Let $\Omega = \langle \sigma_0 \rangle$. (Note that σ_0 may be the identity.) Then, with the above notation, we clearly have $\mathcal{R} = \mathcal{R}_0$ and so $\Delta(H_1,\sigma_0)$ is the empty matrix. Thus, by Theorem 5.8, we have

$$|\text{Irr}(H_k)| = 2\,|\text{Irr}(H_{1,k})|.$$

This can also be seen as follows. Define $\sigma_0' := w_0\sigma_0 \in W$. Then σ_0' has order 2 and it is readily checked that σ_0' commutes with all elements of W_1. Thus, W is the direct product of W_1 and $\langle \sigma_0' \rangle$. Using Lemma 4.2, we conclude that

$$\text{Res}_1(V) \in \text{Irr}(H_{1,K}) \qquad \text{for all } V \in \text{Irr}(H_K).$$

Moreover, for each $V_1 \in \mathrm{Irr}(H_{1,K})$ there exist precisely two simple H_K-modules $V, V' \in \mathrm{Irr}(H_K)$ such that $\mathrm{Res}_1(V) = \mathrm{Res}_1(V') = V_1$. Thus, only case (1) in Example 5.7 occurs.

The cases in which w_0 is not central arise when (W_1, S_1) is of type A_{n-1}, E_6 or D_n with n odd. If (W_1, S_1) is of type A_{n-1} ($n \geq 1$), then it is known by [7] that the simple $H_{1,k}$-modules have a natural labelling by the e-regular partitions of n, with e defined as in (5.1). (See [14, 3.5] for the identification of the set $\mathcal{B}_1$ in this case.) Recall that a partition, written in exponential form $(1^{n_1}, 2^{n_2}, 3^{n_3}, \dots)$, is called e-regular if $n_i < e$ for all i. Hence we conclude that the number of simple H_k-modules is $2p_e(n)$, where $p_e(n)$ is the number of e-regular partitions of n. A generating function for $p_e(n)$ is given by

$$1 + \sum_{n \geq 1} p_e(n) X^n = \prod_{i \geq 1} \frac{1 - X^{ei}}{1 - X^i};$$

see the proof of [15, Lemma 2.6].

If (W_1, S_1) is of type E_6, the cardinalities of the sets $\mathrm{Irr}(H_{1,k})$ are known from [12]; they are given by $8, 13, 19, 23, 20, 24, 24, 24$ for $e = 2, 3, 4, 5, 6, 8, 9, 12$, respectively, and 25 otherwise. Finally, type D_n will be considered in detail in Section 6. Thus, whenever Ω is generated by a non-trivial graph automorphism which is given by conjugation with w_0, we know explicitly the number of simple H_k-modules (with the assumptions on k as specified in (5.1)).

Example 5.10. Assume that (W_1, S_1) and σ are as in Example 2.10.

If (W_1, S_1) is of type B_2 or G_2 with generators $S_1 = \{s, t\}$, then the character tables in [16, Table 8.1] show that all irreducible characters of $H_{1,K}$ are invariant under σ except for the two linear characters ε_s and ε_t given by $\varepsilon_s(T_s) = u$, $\varepsilon_s(T_t) = -1$ and $\varepsilon_t(T_s) = -1$, $\varepsilon_t(T_t) = u$. Hence, in these cases, the matrix $\Delta(H_1, \sigma)$ only has one row and one column. Taking $\mathcal{R}_1 = \{s\}$, we obtain

$$\Delta(H_1, \sigma) = \left[\ \varepsilon_s(T_s) - \varepsilon_t(T_s) \ \right] = \left[\ u + 1 \ \right].$$

Thus, we see that the rank of $\bar{\Delta}(H_1, \sigma)$ is 0 or 1 according to whether $\theta(u) = -1$ or $\theta(u) \neq -1$. The cardinalities of the sets $\mathrm{Irr}(H_{1,k})$ are easily computed using the character tables in [16, Table 8.1]. (By [16, 7.5.7], the required cardinality is the rank of the specialized character table.) In type B_2 they are given by $2, 4$ for $e = 2, 4$, respectively, and 5 otherwise. In type G_2 they are given by $3, 5, 5$ for $e = 2, 3, 6$, respectively, and 6 otherwise.

Finally, assume that (W_1, S_1) is of type F_4 with generators $S_1 = \{a, b, c, d\}$ such that $\sigma(a) = d$ and $\sigma(b) = c$. From the known character table of $H_{1,K}$ (see [16, Chap. 11]), we extract the matrix $\Delta(H_1, \sigma)$. The result is given in Table 1. (For the labelling of the irreducible characters, see also [3, p. 413].) Then it can be computed that the rank of $\bar{\Delta}(H_1, \sigma)$ is $1, 3, 6, 5$ for $e = 2, 3, 4, 6$, respectively, and 7 otherwise, where e is the smallest integer $i \geq 2$ such that $1 + \theta(u) + \theta(u)^2 + \cdots + \theta(u)^{i-1} = 0$. The cardinalities of the sets $\mathrm{Irr}(H_{1,k})$ are known from [11]; they are given by $8, 15, 19, 20, 24, 24$ for $e = 2, 3, 4, 6, 8, 12$, respectively, and 25 otherwise.

6. Modular representations in type D_n

Throughout this section, we let (W_1, S_1) be a Weyl group of type D_n ($n \geq 2$) and consider the case where Ω is generated by a graph automorphism of order 2. Then it turns out that W is a Weyl group of type B_n. Using the techniques developed

TABLE 1. The matrix $\Delta(W_1, \sigma)$ in type F_4

$\psi^+ - \psi^-$	w_1	w_2	w_3	w_4
1_3-1_2	$-u^2+1$	u^6-u^4	$-u-1$	$-u^6-u^3$
2_3-2_1	$-2u^2-u$	$-u^8-2u^7$	$-u-1$	u^9-3u^6
2_2-2_4	$u+2$	$2u^3+u^2$	$-u-1$	$-3u^3+1$
4_4-4_3	$-2u^2+2$	u^6-u^4	$-2u-2$	$-u^6+3u^5+3u^4-u^3$
8_3-8_1	$-3u^2-u+1$	$2u^6+u^5$	$-2u-2$	$u^9-3u^6+3u^5+3u^4$
8_2-8_4	$-u^2+u+3$	$-u^5-2u^4$	$-2u-2$	$3u^5+3u^4-3u^3+1$
9_3-9_2	$-3u^2+3$	0	$-3u-3$	$-6u^6+3u^5+3u^4-6u^3$

$\psi^+ - \psi^-$	w_5	w_6	w_7
1_3-1_2	$-u^2-u$	$-u^2-u$	$-u^3-u^2$
2_3-2_1	u^3	u^3	$-u^5-u^3$
2_2-2_4	1	1	$-u^2-1$
4_4-4_3	u^3+1	$-u^2-u$	0
8_3-8_1	u^3-u^2-u	u^3	$-u^5+u^2$
8_2-8_4	$-u^2-u+1$	1	u^3-1
9_3-9_2	u^3-u^2-u+1	0	u^3+u^2

$w_1 = ba$
$w_2 = bcbcdcbacd$
$w_3 = a$
$w_4 = bcbcdcbcd$
$w_5 = adc$
$w_6 = dcb$
$w_7 = bcbac$

in the previous sections, we shall obtain results about modular representations of the Iwahori–Hecke algebra associated with (W_1, S_1) from known results about Iwahori–Hecke algebras of type B_n.

6.1. We assume that W_1 is generated by $S_1 = \{s_1', s_1, s_2, \dots, s_{n-1}\}$, where the defining relations are given as follows:

$$s_1's_1 = s_1s_1', \qquad s_1's_2s_1' = s_2s_1's_2, \qquad s_1's_i = s_is_1' \text{ for } i \geq 3,$$
$$s_is_j = s_js_i \text{ if } |i-j| \geq 2, \qquad s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \text{ for } i \geq 1.$$

We consider the graph automorphism $\sigma\colon W_1 \to W_1$ defined by $\sigma(s_1') = s_1$, $\sigma(s_1) = s_1'$, and $\sigma(s_i) = s_i$ for all $i \geq 2$. Then, since $\sigma s_1 \sigma = s_1'$, we find that W has a presentation with generators $S = \{\sigma, s_1, \dots, s_{n-1}\}$ and defining relations

$$\sigma s_1 \sigma s_1 = s_1 \sigma s_1 \sigma, \qquad \sigma s_i = s_i \sigma \text{ for } i \geq 2,$$
$$s_is_j = s_js_i \text{ if } |i-j| \geq 2, \qquad s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \text{ for } i \geq 1.$$

Thus, we see that (W, S) is a Weyl group of type B_n. Let F be any finite extension field of $\mathbb{Q}$. The corresponding extended Iwahori–Hecke algebra H is an Iwahori–Hecke algebra of type B_n where the generators satisfy the following relations:

$$T_\sigma^2 = T_1 \quad \text{and} \quad T_{s_i}^2 = uT_1 + (u-1)T_{s_i} \quad \text{for } 1 \leq i \leq n-1.$$

By Example 2.9 and Corollary 4.7, we have:

(a) $K = F(v)$ is a splitting field for H_K and all simple H_K-modules can be realized over $A = R[v, v^{-1}]$.
(b) f_E is a power of 2 for all simple J_F-modules E.

Finally, let H_1 be the subalgebra of H generated by T_{w_1} ($w_1 \in W_1$). Then H_1 is the generic Iwahori–Hecke algebra of (W_1, S_1) and K is also a splitting field for H_1.

We consider an admissible specialization $\theta\colon A \to k$ as in Definition 3.3, where the characteristic of k is not equal to 2. Then all the assumptions in (5.1) are satisfied. The difficulty in finding the decomposition numbers depends strongly on whether e (defined as in (5.1)) is even or odd. Note that we have that

(c) e is even if and only if $-1 \in k$ is a power of $\theta(u)$.

Indeed, let $f \geq 1$ be the multiplicative order of $\theta(u)$. Since -1 has order 2 and the multiplicative group of k is cyclic, it follows that $\theta(u)^i = -1$ for some $i \geq 1$ if and only if f is even. Now, if $\theta(u) \neq 1$, then $e = f$. On the other hand, suppose that $\theta(u) = 1$ and let ℓ be the characteristic of k. Then we have $e = \infty$ if $\ell = 0$ and $e = \ell$ if $\ell > 0$.

6.2. The simple H_K-modules are naturally parametrized by the set Λ consisting of all pairs of partitions (λ, μ) such that $|\lambda| + |\mu| = n$. Denote by $V^{(\lambda,\mu)}$ the simple H_K-module labelled by (λ, μ). Thus, we have

$$\operatorname{Irr}(H_K) = \{V^{(\lambda,\mu)} \mid (\lambda, \mu) \in \Lambda\}.$$

A classification of the simple $H_{1,K}$-modules is obtained as follows. If $\lambda \neq \mu$, then $V^{(\lambda,\mu)}$ and $V^{(\mu,\lambda)}$ have the same restriction to $H_{1,K}$ and this restriction is a simple $H_{1,K}$-module which will be denoted by $V^{[\lambda,\mu]}$. If $\lambda = \mu$, then the restriction of $V^{(\lambda,\lambda)}$ splits into a direct sum of two non-isomorphic simple $H_{1,K}$-modules which are denoted by $V^{[\lambda,+]}$ and $V^{[\lambda,-]}$. Every simple $H_{1,K}$-module arises exactly once in this way. For more details see, for example, [16, §10.4]. The a-invariant of a simple module of H_K or of $H_{1,K}$ labelled by (λ, μ) is given by

$$\begin{aligned} a(\lambda,\mu) = -\frac{1}{6}m(m-1)(2m-1) + \sum_{i=1}^{m}(i-1)(\lambda_i+\mu_i) \\ + \sum_{i,j=1}^{m} \min\{\lambda_i + m - i, \mu_j + m - j\}, \end{aligned}$$

where we assume that $m \geq 1$ is chosen such that λ and μ have m parts $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0$ and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m \geq 0$, respectively. (The above formula is obtained by rewriting that in [23, 4.6.3], which is in terms of symbols.)

Our first main result determines the number of simple $H_{1,k}$-modules in terms of the number of simple H_k-modules.

Theorem 6.3. *Set $d = e/2$ if e is an even integer and $d = e$ otherwise. Furthermore, recall that the characteristic of k is not equal to 2. Then the number of simple $H_{1,k}$-modules is given by*

$$\frac{1}{2}\Big(|\operatorname{Irr}(H_k)| + 3\,p_d(n/2)\Big),$$

where $p_d(n/2)$ denotes the number of d-regular partitions of $n/2$ if n is even (see the definition in Example 5.9) and where this number is interpreted as 0 if n is odd.

Proof. Using Theorem 5.8, we see that it is enough to prove that

$$\operatorname{rank} \bar{\Delta}(H_1, \sigma) = p_d(n/2).$$

Now, if n is odd, then $\Delta(H_1, \sigma)$ is empty and so both sides of the above identity are 0. So it remains to consider the case where n is even. By [16, Theorem 10.4.7], the rows and columns of $\Delta(H_1, \sigma)$ have a natural labelling by the partitions of $n/2$.

Furthermore, there is a choice for the class representatives in $\mathcal{R}_1$ such that the entry of $\Delta(H_1, \sigma)$ in the row labelled by $\lambda \vdash n/2$ and the column labelled by $\mu \vdash n/2$ is given by

$$(u+1)^{l(\mu)} \chi_\lambda^\mu(u^2),$$

where $l(\mu)$ denotes the number of parts of μ and $(\chi_\mu^\lambda(u^2))_{\lambda\mu}$ is the character table of the generic Iwahori–Hecke algebra with parameter u^2 associated with the symmetric group $\mathfrak{S}_{n/2}$. Now, if $\theta(u) = -1$, the above matrix specializes to 0. On the other hand, we have $e = 2$ and $d = 1$ in this case and there are no 1-regular partitions. Thus, the desired formula holds if $\theta(u) = -1$.

Now assume that $\theta(u) \neq -1$. Then note that d is the smallest $i \geq 2$ such that $1 + \theta(u^2) + \theta(u^2)^2 + \cdots + \theta(u^2)^{i-1} = 0$. Furthermore, $\bar{\Delta}(H_1, \sigma)$ is the product of a diagonal matrix (with non-zero diagonal entries) and the specialized character table $(\bar{\chi}_\lambda^\mu(\bar{u}^2))_{\lambda\mu}$. By a similar argument as in the proof of Theorem 5.8, the rank of the specialized character table is the number of simple modules for the specialized algebra. Thus, using the known results for the Iwahori–Hecke algebra associated with $\mathfrak{S}_{n/2}$ (see Example 5.9), we obtain again the desired formula. ☐

Remark 6.4. (a) Using [2, Theorem A], we see that the number of simple modules in type B_n only depends on e. Hence the above result establishes a similar statement for type D_n (assuming that the characteristic of k is not 2). Furthermore, [2, Theorem C] actually provides a generating function for the number of simple modules in type B_n. Combining this with the above formula we also obtain a generating function for the number of simple modules in type D_n.

(b) Recall that we assume that the characteristic of k is not 2. In the case where the characteristic of k is 2, the above formula need no longer be true. This can be seen, for example, in the case $n = 4$. The following table contains the cardinalities of $\mathrm{Irr}(H_k)$ and $\mathrm{Irr}(H_{1,k})$:

$n = 4$	$\mathrm{char}(k) = 0$		$\mathrm{char}(k) = 2$	
	$e = 2$	$e = 3$	$e = 2$	$e = 3$
$\lvert\mathrm{Irr}(H_k)\rvert$	6	16	2	8
$\lvert\mathrm{Irr}(H_{1,k})\rvert$	3	11	2	10

Thus, in characteristic 2, the cardinalities of $\mathrm{Irr}(H_k)$ and $\mathrm{Irr}(H_{1,k})$ are not related by the formula in Theorem 6.3.

(c) The above result shows that the cardinality of $\mathcal{B}^{II}$ is given by $p_d(n/2)$. It may be conjectured that $\mathcal{B}^{II}$ consists precisely of all $(\lambda, \mu) \in \Lambda$ such that $\lambda = \mu$ and λ is d-regular. This is certainly true in the case where $e = 2$, since then $d = 1$ and there are no 1-regular partitions of $n/2$. Hence, in this case, we have $\mathcal{B} = \mathcal{B}^I$. Consequently, the decomposition numbers of the simple modules in $\mathcal{B}_1$ are completely determined for $e = 2$ by the decomposition numbers of H_k.

The remainder of this section will deal with the problem of describing the set $\mathcal{B} \subseteq \mathrm{Irr}(H_K)$ explicitly. For this purpose, we first recall some facts about the Dipper–James–Murphy construction of the simple H_k-modules.

6.5. For any $(\lambda, \mu) \in \Lambda$, Dipper, James, and Murphy [8] have constructed a so-called *Specht module* $S^{(\lambda,\mu)}$ which is an H-module, finitely generated and free over A, such that $K \otimes_A S^{(\lambda,\mu)} \cong V^{(\lambda,\mu)}$. By extension of scalars from A to k, we obtain a corresponding Specht module $S_k^{(\lambda,\mu)}$ for H_k. Furthermore, each Specht module for H_k has a natural H_k-invariant bilinear form and $D^{(\lambda,\mu)} := S_k^{(\lambda,\mu)}/\text{rad}\, S_k^{(\lambda,\mu)}$ is either zero or a simple module. Let Λ_0 be the set of all pairs of partitions (λ,μ) such that $D^{(\lambda,\mu)} \neq \{0\}$. Then we have

$$\text{Irr}(H_k) = \{D^{(\lambda,\mu)} \mid (\lambda,\mu) \in \Lambda_0\}.$$

For any $(\lambda,\mu) \in \Lambda_0$ and $(\sigma,\tau) \in \Lambda$, the following hold (see [8, §6]):

(a) We have $(V^{(\lambda,\mu)} : D^{(\lambda,\mu)}) = 1$.

(b) We have $(V^{(\sigma,\tau)} : D^{(\lambda,\mu)}) = 0$ unless $(\sigma,\tau) \trianglelefteq (\lambda,\mu)$.

Here, $\trianglelefteq$ denotes the dominance order on Λ; see [8, p. 508]. (Note that the results in [8] are formulated in terms of "dual" Specht modules $\tilde{S}^{(\lambda,\mu)}$; the passage from Specht modules to their duals is provided by the map $(\lambda,\mu) \mapsto (\mu^t,\lambda^t)$, where λ^t and μ^t denote the conjugate partitions.) Thus, we have a similar statement as in Theorem 5.3. The following result shows that, using the a-invariants, we obtain a canonical bijection between Λ_0 and $\mathcal{B}$.

Lemma 6.6. *For any $(\lambda,\mu) \in \Lambda_0$, there exists a unique simple H_K-module, denoted ${}^{a}V^{(\lambda,\mu)}$, such that the following two conditions are satisfied:*

(a) *We have $({}^{a}V^{(\lambda,\mu)} : D^{(\lambda,\mu)}) \neq 0$.*

(b) *For any $V \in \text{Irr}(H_K)$ with $(V : D^{(\lambda,\mu)}) \neq 0$, the a-invariant of V is bigger than or equal to the a-invariant of ${}^{a}V^{(\lambda,\mu)}$.*

Thus, we have a canonical bijection $\Lambda_0 \leftrightarrow \mathcal{B}$, $(\lambda,\mu) \leftrightarrow {}^{a}V^{(\lambda,\mu)}$, and we have in fact $({}^{a}V^{(\lambda,\mu)} : D^{(\lambda,\mu)}) = 1$.

Proof. Fix $(\lambda,\mu) \in \Lambda_0$ and let $\mathcal{M} := \{V \in \text{Irr}(H_K) \mid (V : D^{(\lambda,\mu)}) \neq 0\}$. As remarked in (5.4), the function $V \mapsto a_V$ takes its minimum at exactly one element of $\mathcal{M}$, which we denote by ${}^{a}V^{(\lambda,\mu)}$. Thus, in the notation of Theorem 5.3, we have

$$\overline{{}^{a}V^{(\lambda,\mu)}} = D^{(\lambda,\mu)}.$$

Moreover, we have $({}^{a}V^{(\lambda,\mu)} : D^{(\lambda,\mu)}) = 1$ by Theorem 5.3(a). □

Thus, it is possible to obtain a description of the set $\mathcal{B}$ via a description of the set Λ_0 and the bijection $\Lambda_0 \leftrightarrow \mathcal{B}$. Using the deep results of Ariki [1] and Ariki and Mathas [2], it is known that a pair of partitions (λ,μ) belongs to Λ_0 if and only if (λ,μ) is a Kleshchev bipartition (see the definition in [2]).[1] We may therefore concentrate on the bijection $\Lambda_0 \leftrightarrow \mathcal{B}$. The following examples show that this bijection is not the "identity", in general.

Example 6.7. Assume that (W,S) is of type B_2, B_3 or B_4. We consider the case where $F = k$ and $\theta(u)$ is a primitive e-th root of unity, with $e \in \{2,4\}$. Using the methods described in [11, 12], we obtain the corresponding decomposition matrices; they are printed in Table 2 (where . stands for 0). In all cases, the ordering of Λ is chosen such that if $(\sigma,\tau) \trianglelefteq (\lambda,\mu)$, then (λ,μ) comes before (σ,τ). Then the

[1] It is conjectured by Dipper, James, and Murphy [8] that $(\lambda,\mu) \in \Lambda_0$ if and only if (λ,μ) is $(1,e)$-restricted in the sense of [8, Def. 8.12]. When $e = 2$ this holds by Mathas [30]; as Ariki informed me, the general case is still open. Ariki and the referee independently pointed out that it might also be interesting to compare this with the labelling of the simple modules given by Jimbo et al. [21] and Foda et. al. [10]. For small values of n, that labelling is consistent with the examples in Table 2. Ariki found that this does not seem to be the case for larger values of n.

Table 2. Decomposition numbers for type B_2, B_3, B_4, where u is specialized to a primitive e-th root of unity in $\mathbb{C}$

B_2	a_V	$e=2$		
$(2,\varnothing)$	0	$*$	1	.
$(11,\varnothing)$	2		1	.
$(1,1)$	1		1	1
$(\varnothing,2)$	0	$*$	.	1
$(\varnothing,11)$	2		.	1

B_3	a_V	$e=2$				
$(3,\varnothing)$	0	$*$	1	.	.	.
$(21,\varnothing)$	2		.	1	.	.
$(111,\varnothing)$	6		1	.	.	.
$(2,1)$	1	$*$	.	1	1	.
$(11,1)$	3		.	1	1	.
$(1,2)$	1	$*$	1	.	.	1
$(1,11)$	3		1	.	.	1
$(\varnothing,3)$	0	$*$	.	.	1	.
$(\varnothing,21)$	2		.	.	.	1
$(\varnothing,111)$	6		.	.	1	.

B_4	a_V	$e=2$							$e=4$															
$(4,\varnothing)$	0	$*$	1	.	.	.	.	.	$*$	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$(31,\varnothing)$	2		1	1	.	.	.	.	$*$	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.
$(22,\varnothing)$	3		.	1	.	.	.	.	$*$	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
$(211,\varnothing)$	6		1	1	.	.	.	.		.	1	.	1	.	.	.	.	.	.	.	.	.	.	.
$(1111,\varnothing)$	12		1	.	.	.	.	.		.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
$(3,1)$	1	$*$	1	1	1	.	.	.	$*$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
$(21,1)$	3	$*$	.	.	.	1	.	.	$*$	1	1	.	1	.	1	.	.	.	.	.	.	.	.	.
$(111,1)$	7		1	1	1	.	.	.	$*$	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
$(2,2)$	2		1	1	1	.	1	.	$*$	1	.	.	.	.	1	.	1	.	.	.	.	.	.	.
$(11,2)$	3		1	1	1	.	1	.	$*$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.
$(2,11)$	3		1	1	1	.	1	.	$*$	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.
$(11,11)$	6		1	1	1	.	1	.		.	.	.	1	.	1	.	.	.	.	1	.	.	.	.
$(1,3)$	1	$*$	1	.	1	.	1	.	$*$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.
$(1,21)$	3	$*$	.	.	.	.	.	1	$*$	.	.	.	.	.	1	.	1	.	.	1	.	1	.	.
$(1,111)$	7		1	.	1	.	1	.	$*$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
$(\varnothing,4)$	0	$*$	.	.	1	.	.	.	$*$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
$(\varnothing,31)$	2		.	.	1	.	1	.	$*$	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.
$(\varnothing,22)$	3		.	.	.	.	1	.	$*$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
$(\varnothing,211)$	6		.	.	1	.	1	.		.	.	.	.	.	.	.	.	.	.	1	.	1	.	.
$(\varnothing,1111)$	12		.	.	1	.	.	.		.	.	.	.	.	.	.	.	.	.	1	.	.	.	.

two conditions (a) and (b) in (6.5) determine Λ_0 uniquely from the decomposition matrices. The sets $\mathcal{B}$ are determined by the method in (5.4); they are marked with a star in the tables. From these tables, one obtains the canonical bijections $\Lambda_0 \leftrightarrow \mathcal{B}$ using the conditions in Lemma 6.6; see Table 3. (We do not know how to describe in general the sets $\mathcal{B}$ and the bijections $\Lambda_0 \leftrightarrow \mathcal{B}$ in a purely combinatorial way.)

In any case, we see that the labelling of $\mathrm{Irr}(H_k)$ provided by the asymptotic algebra is not the same as that given by Dipper, James, and Murphy! Note that, for example in type B_2, the restriction of $D^{(2,-)}$ to $H_{1,k}$ remains simple and the restriction of $D^{(1,1)}$ splits into a sum of two simple $H_{1,k}$-modules, one of which is the restriction of $D^{(2,-)}$. Thus, it seems to be difficult to obtain a classification of

TABLE 3. The canonical bijections $\Lambda_0 \leftrightarrow \mathcal{B}$ for type B_2, B_3 and B_4

Λ_0	$\leftrightarrow$	$\mathcal{B}$
for $n=2$, $e=2$		
$(2,\varnothing)$	$\leftrightarrow$	$(2,\varnothing)$
$(1,1)$	$\leftrightarrow$	$(\varnothing,2)$

Λ_0	$\leftrightarrow$	$\mathcal{B}$
for $n=3$, $e=2$		
$(3,\varnothing)$	$\leftrightarrow$	$(3,\varnothing)$
$(21,\varnothing)$	$\leftrightarrow$	$(2,1)$
$(2,1)$	$\leftrightarrow$	$(\varnothing,3)$
$(1,2)$	$\leftrightarrow$	$(1,2)$

Λ_0	$\leftrightarrow$	$\mathcal{B}$
for $n=4$, $e=2$		
$(4,\varnothing)$	$\leftrightarrow$	$(4,\varnothing)$
$(31,\varnothing)$	$\leftrightarrow$	$(3,1)$
$(3,1)$	$\leftrightarrow$	$(\varnothing,4)$
$(21,1)$	$\leftrightarrow$	$(21,1)$
$(2,2)$	$\leftrightarrow$	$(1,3)$
$(1,21)$	$\leftrightarrow$	$(1,21)$

Λ_0	$\leftrightarrow$	$\mathcal{B}$
for $n=4$, $e=4$		
$(4,\varnothing)$	$\leftrightarrow$	$(4,\varnothing)$
$(31,\varnothing)$	$\leftrightarrow$	$(31,\varnothing)$
$(22,\varnothing)$	$\leftrightarrow$	$(22,\varnothing)$
$(211,\varnothing)$	$\leftrightarrow$	$(21,1)$
$(3,1)$	$\leftrightarrow$	$(3,1)$
$(21,1)$	$\leftrightarrow$	$(2,2)$
$(111,1)$	$\leftrightarrow$	$(111,1)$
$(2,2)$	$\leftrightarrow$	$(\varnothing,4)$
$(11,2)$	$\leftrightarrow$	$(11,2)$
$(2,11)$	$\leftrightarrow$	$(2,11)$
$(11,11)$	$\leftrightarrow$	$(1,21)$
$(1,3)$	$\leftrightarrow$	$(1,3)$
$(1,21)$	$\leftrightarrow$	$(\varnothing,31)$
$(1,111)$	$\leftrightarrow$	$(1,111)$
$(\varnothing,22)$	$\leftrightarrow$	$(\varnothing,22)$

the simple $H_{1,k}$-modules using the modules $D^{(\lambda,\mu)}$. On the other hand, once $\mathcal{B}$ is known, the set $\mathcal{B}_1 \leftrightarrow \operatorname{Irr}(H_{1,k})$ is readily determined; see Theorem 5.5.

The above example was only concerned with cases where e is even. What happens if e is odd? In this case, we have $\theta(u)^i \neq -1$ for $0 \leq i \leq n-1$ and so we can apply the results of Dipper and James [7]. In particular, by [7, Theorem 5.8] (see also [8, §7]), the set Λ_0 consists precisely of all $(\lambda,\mu) \in \Lambda$ where both λ and μ are e-regular.

Proposition 6.8. *If e is odd, then the canonical bijection $\Lambda_0 \leftrightarrow \mathcal{B}$ of Lemma 6.6 is the identity, i.e., we have ${}^{a}V^{(\lambda,\mu)} = V^{(\lambda,\mu)}$ for all $(\lambda,\mu) \in \Lambda_0$.*

Proof. The results in [7] yield a description of the decomposition numbers in terms of decomposition numbers for Iwahori–Hecke algebras associated with the symmetric groups $\mathfrak{S}_r$ for $0 \leq r \leq n$. Let $H(\mathfrak{S}_r)$ be the generic Iwahori–Hecke algebra over A associated with $\mathfrak{S}_r$. Then the specialization θ also determines a decomposition map between the Grothendieck groups of $H_K(\mathfrak{S}_r)$ and $H_k(\mathfrak{S}_r)$. Dipper and James [6] have shown that the simple $H_K(\mathfrak{S}_r)$-modules have a natural parametrization by the partitions $\lambda \vdash r$ and the simple $H_k(\mathfrak{S}_r)$-modules have a natural parametrization by the e-regular partitions of r. Denote by $d_{\lambda'\lambda}$ the corresponding decomposition numbers, where $\lambda', \lambda \vdash r$ and λ is e-regular. With this notation, we have

$$(V^{(\lambda',\mu')} : D^{(\lambda,\mu)}) = \begin{cases} d_{\lambda'\lambda} \cdot d_{\mu'\mu} & \text{if } |\lambda'| = |\lambda| \text{ and } |\mu'| = |\mu|, \\ 0 & \text{otherwise,} \end{cases}$$

for any $(\lambda',\mu') \in \Lambda$ and $(\lambda,\mu) \in \Lambda_0$. Taking into account the conditions in Lemma 6.6, we must establish the following relation, for a fixed $(\lambda,\mu) \in \Lambda_0$:

$$d_{\lambda'\lambda} \neq 0 \quad \text{and} \quad d_{\mu'\mu} \neq 0 \quad \Rightarrow \quad a(\lambda,\mu) \leq a(\lambda',\mu').$$

To prove this, we first note that if λ', μ' are such that $d_{\lambda'\lambda} \neq 0$ and $d_{\mu'\mu} \neq 0$, then we have $\lambda' \trianglelefteq \lambda$ and $\mu' \trianglelefteq \mu$ by [6, Theorem 7.6]. Hence it is enough to prove that

$$(*) \qquad \kappa' \trianglelefteq \kappa \quad \text{and} \quad \nu' \trianglelefteq \nu \quad \Rightarrow \quad a(\kappa, \nu) \leq a(\kappa', \nu')$$

for any $(\kappa, \nu), (\kappa', \nu') \in \Lambda$ with $|\kappa| = |\kappa'|$ and $|\nu| = |\nu'|$. Now, since $a(\kappa, \nu)$ does not change if we interchange the roles of κ and ν, it is actually enough to prove $(*)$ in the case where $\kappa = \kappa'$. But this follows easily using the formula in (6.2). □

6.9. We can now draw the following conclusions about modular representations in type D_n. Assume that e is odd. First, Proposition 6.8 in combination with Example 5.7 shows that we have

$$\mathcal{B}_1 = \{V^{[\lambda,\mu]} \mid (\lambda, \mu) \in \Lambda_0, \lambda \neq \mu\} \cup \{V^{[\lambda,\pm]} \mid (\lambda, \lambda) \in \Lambda_0\},$$

where Λ_0 is the set of all $(\lambda, \mu) \in \Lambda$ such that both λ and μ are e-regular. In combination with Theorem 5.3, we obtain the following classification of the simple $H_{1,k}$-modules:

$$\operatorname{Irr}(H_{1,k}) = \{\overline{V}^{[\lambda,\mu]} \mid (\lambda, \mu) \in \Lambda_0, \lambda \neq \mu\} \cup \{\overline{V}^{[\lambda,\pm]} \mid (\lambda, \lambda) \in \Lambda_0\}.$$

Furthermore, Example 5.7 yields the following dimension formulas:

$$\begin{aligned} \dim_k \overline{V}^{[\lambda,\mu]} &= \dim_k \overline{V}^{[\lambda,\mu]} && \text{for } \lambda \neq \mu, \\ \dim_k \overline{V}^{[\lambda,\pm]} &= \frac{1}{2} \dim_k \overline{V}^{[\lambda,\lambda]} && \text{for } \lambda = \mu. \end{aligned}$$

Finally, the above formulas imply that *James' Conjecture* (as formulated in [13, §3]) holds for H_k if and only if it holds for $H_{1,k}$. We also remark that, in the case where both e and n are odd, Pallikaros [33] has shown that the decomposition matrix of $H_{1,k}$ is completely determined by the decomposition matrices of Iwahori–Hecke algebras associated with the symmetric groups $\mathfrak{S}_r$ for $0 \leq r \leq (n+1)/2$ (compare the similar result for H_k used in the proof of Proposition 6.8).

Acknowledgements. I am indebted to Gunter Malle for carefully reading an earlier version of this paper and spotting a number of misprints.

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