REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 5, Pages 129–146 (June 5, 2001)
S 1088-4165(01)00122-4

GENERIC CENTRAL EXTENSIONS AND PROJECTIVE REPRESENTATIONS OF FINITE GROUPS

RACHEL QUINLAN

Abstract. Any free presentation for the finite group G determines a central extension (R, F) for G having the projective lifting property for G over any field k. The irreducible representations of F which arise as lifts of irreducible projective representations of G are investigated by considering the structure of the group algebra kF, which is greatly influenced by the fact that the set of torsion elements of F is equal to its commutator subgroup and, in particular, is finite. A correspondence between projective equivalence classes of absolutely irreducible projective representations of G and F-orbits of absolutely irreducible characters of F' is established and employed in a discussion of realizability of projective representations over small fields.

1. Preliminaries

The complex irreducible projective representations of a finite group G may be described in terms of the complex irreducible ordinary representations of a covering group $\hat{G}$ for G, which takes the form of a central extension of the Schur Multiplier $M(G)$ of G by G. If we wish to discuss projective representations over non-algebraically closed fields however, no finite central extension for G can in general play the role of $\hat{G}$. However, given a free presentation for G we may construct a central extension F of a certain infinite abelian group R by G, which behaves as a covering group for G with respect to all fields. We will refer to groups such as F as *generic central extensions* for G. Their representation theory yields information on the irreducible projective representations of G over various fields.

Throughout this paper G will denote a finite group and k a field of characteristic zero. We begin with the requisite definitions.

Definition 1.1. A projective representation T of G over k (of degree n) is a map

$$T : G \longrightarrow GL(n, k)$$

satisfying the conditions

$$\begin{aligned} T(1_G) &= 1_{GL(n,k)}, \\ T(g)T(h) &= f(g, h)T(gh), \quad \forall g, h \in G, \text{ where } f(g, h) \in k^\times. \end{aligned}$$

The function $f : G \times G \longrightarrow k^\times$ is the *cocycle associated to* T. The projective representation T of G extends by k-linearity to a ring homomorphism of the twisted group ring $k^f G$ into $M_n(k)$, which is completely reducible by Maschke's theorem (since char $k = 0$) and which we also denote by T. We say that T is an

Received by the editors February 26, 2001 and, in revised form, March 23, 2001.
2000 *Mathematics Subject Classification.* Primary 20C25; Secondary 20C07.

irreducible projective representation of G if its k-linear extension is an irreducible representation of $k^f G$.

If T_1 and T_2 are projective k-representations of G of degree n, we say that T_1 and T_2 are *projectively equivalent* over k if there exist a matrix $A \in GL(n,k)$ and a function $\mu : G \longrightarrow k^\times$ for which

$$T_2(g) = \mu(g) A^{-1} T_1(g) A, \quad \forall g \in G.$$

From Definition 1.1 above it follows that the cocycle f associated to a projective k-representation of G satisfies the following properties:

(1) $f(x,1) = f(1,x) = 1,\ \forall x \in G,$
(2) $f(x,y)f(xy,z) = f(x,yz)f(y,z),\ \forall x,y,z \in G.$

Indeed the cocycles of G are precisely the functions from $G \times G$ into $k^\times$ satisfying (1) and (2) above, and they form a group, denoted by $Z^2(G,k^\times)$, under pointwise multiplication.

The projective representation T is projectively equivalent to an ordinary k-representation of G if and only if its cocycle f has the property

$$f(x,y) = \mu(x)\mu(y)\mu(xy)^{-1}, \forall x,y \in G,$$

for some function $\mu : G \longrightarrow k^\times$. In this case f is called a coboundary. Within the group of cocycles the coboundaries form a subgroup, denoted by $B^2(G,k^\times)$. If T_1 and T_2 are projectively equivalent k-representations of G, then their cocycles represent the same class in the quotient $Z^2(G,k^\times)/B^2(G,k^\times)$ which is denoted $H^2(G,k^\times)$. This abelian group may in general be infinite but it is finite when k is sufficiently large, for example, if k is algebraically closed.

We remark that unlike the usual definition of equivalence of ordinary representations to which it is (somewhat) analogous, this definition of projective equivalence depends on the field under consideration. It is possible that a pair of representations which are projectively inequivalent over a given field may become projectively equivalent over some of its extensions. For example, we may define rational projective representations of degree 2 of the cyclic group of order 2 by sending the generator either to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or to $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. These representations are projectively inequivalent over $\mathbb{Q}$ but not over $\mathbb{Q}\left(\sqrt{2}\right)$. Thus in order to discuss the projective equivalence of representations, we need to specify a field over which to work.

Now let $T : G \longrightarrow GL(n,k)$ be a projective k-representation of G. The subgroup of $GL(n,k)$ generated by $\{T(g), g \in G\}$ need not be a homomorphic image of G but it is an extension of some subgroup of $k^\times$ (which we identify with the centre of $GL(n,k)$) by such an image. This relates T to an ordinary representation of some (possibly infinite) group having a homomorphic image of G as quotient modulo a central subgroup. Suppose now that H is a group having G as image under a homomorphism θ with $A = \ker\theta \subseteq \mathcal{Z}(H)$. In this situation we will refer to (A,H,θ) (or sometimes (A,H) or just H) as a *central extension* for G. The central extension (A,H,θ) is said to have the *projective lifting property* for G with respect to the field k if whenever $T : G \longrightarrow GL(n,k)$ is a projective representation of G over k, there exists an ordinary representation $\tilde{T} : H \longrightarrow GL(n,k)$ for which the following diagram commutes:

$$\begin{tikzcd} H \arrow[rr, "\tilde{T}"] \arrow[d, "\theta"'] & & GL(n,k) \arrow[d, "\pi"] \\ G \arrow[r, "T"'] & GL(n,k) \arrow[r, "\pi"'] & PGL(n,k) \end{tikzcd}$$

Here π denotes the usual projection of $GL(n,k)$ on $PGL(n,k)$.

2. Generic central extensions and their group algebras

If $\langle\tilde{F}|\tilde{R}\rangle$ is a free presentation for G, we may define the groups

$$F := \tilde{F}/[\tilde{F},\tilde{R}] \quad \text{and} \quad R := \tilde{R}/[\tilde{F},\tilde{R}].$$

Then $R \subseteq \mathcal{Z}(F)$ and if $\tilde{\phi}$ denotes the surjection of $\tilde{F}$ on G with kernel $\tilde{R}$, then $\tilde{\phi}$ induces a surjection $\phi : F \longrightarrow G$ with kernel R. Thus (R, F, ϕ) is a central extension for G. Extensions of this type were introduced by Schur in his description of finite covering groups, and as outlined in the next lemma, which follows easily from the freeness of $\tilde{F}$, they have a particular universal property amongst all central extensions for G. For this reason we shall refer to them as *generic central extensions* for G.

Lemma 2.1. *Let (A, H, θ) be a central extension for G, and let (R, F, ϕ) be as above. Then there exists a homomorphism $\alpha : F \longrightarrow H$ for which the following diagram commutes:*

$$\begin{tikzcd} 1 \arrow[r] & R \arrow[r] \arrow[d, "\alpha|_R"] & F \arrow[r, "\phi"] \arrow[d, "\alpha"] & G \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\ 1 \arrow[r] & A \arrow[r] & H \arrow[r, "\theta"] & G \arrow[r] & 1 \end{tikzcd}$$

It is well known (see, for example, [10]) that if T is a projective k-representation of G, then T lifts to an ordinary representation of a central extension of $k^\times$ by G, in which the group operation is defined in terms of the cocycle in $Z^2(G, k^\times)$ associated to T. The following theorem is then an immediate consequence of Lemma 2.1.

Theorem 2.2. *Let (R, F, ϕ) be a generic central extension for G, and let $T : G \longrightarrow GL(n,k)$ be a projective representation of G over the field k. Then there exists an ordinary k-representation $\tilde{T}$ of F for which the following diagram commutes:*

$$\begin{tikzcd} F \arrow[rr, "\tilde{T}"] \arrow[d, "\phi"'] & & GL(n,k) \arrow[d, "\pi"] \\ G \arrow[r, "T"'] & GL(n,k) \arrow[r, "\pi"'] & PGL(n,k) \end{tikzcd}$$

In the context of Theorem 2.2, it is apparent that $\tilde{T}$ is an irreducible representation of F if and only if T is irreducible as a projective representation of G. Thus we obtain a connection between irreducible projective k-representations of G and simple images of the group ring kF under k-algebra homomorphisms sending R into $k^\times$. Our next aim is to establish some properties of generic central extensions and their group algebras which will facilitate the study of such representations.

Throughout the remainder of this paper we assume that (R, F, ϕ) is a generic central extension for the finite group G. The following properties of F were established by Schur (see [1], for example).

Theorem 2.3. *The set of torsion elements of F is precisely equal to its commutator subgroup F', and is determined up to isomorphism by G. The group F' is finite and is a central extension of $F' \cap R$, which is isomorphic to the Schur multiplier $M(G)$ of G, by G'.*

Theorem 2.3 has important implications for the investigation of the irreducible k-representations of F in view of the following lemma, for which a proof can be found in [5].

Lemma 2.4. *Let $\mathcal{G}$ be a group and let $\mathcal{F}$ be a field. The support of every central idempotent of the group algebra $\mathcal{F}\mathcal{G}$ generates a finite normal subgroup of G.*

It is now immediate that every central idempotent of kF belongs to kF', which by Maschke's theorem is a completely reducible ring. Of course F acts by conjugation on the set I of primitive central idempotents of kF'. If for each $f \in I$ we choose a transversal T_f for $C_F(f)$ in F, then the full set $\mathcal{I}$ of primitive central idempotents of kF is given by

$$\mathcal{I} = \left\{ \sum_{x \in T_f} f^x \right\}_{f \in I}. \tag{2.1}$$

The group algebra kF decomposes as the direct sum

$$kF = \bigoplus_{e \in \mathcal{I}} kFe.$$

3. Projective equivalence

If T is an irreducible projective representation of G of degree n, let $\tilde{T}$ be a lift of T to F. Then $\tilde{T}$ extends by k-linearity to an irreducible representation of kF, also denoted by $\tilde{T}$, which sends some $e_T \in \mathcal{I}$ to the identity matrix in $M_n(k)$ and annihilates all other elements of $\mathcal{I}$. We will say that T *belongs to* the component $\langle e_T \rangle$ of kF, or simply to the idempotent e_T. This notation is justified, as it is easily checked that e_T does not depend on the choice of lift $\tilde{T}$; since $\mathcal{I} \subset kF'$, this follows from the observation that different lifts of T to F have the same restriction to F'.

The following lemma is a consequence of a related fact: if $\tilde{T}_1$ and $\tilde{T}_2$ are lifts to F of projectively equivalent projective k-representations of G, then the restrictions $\tilde{T}_1|_{F'}$ and $\tilde{T}_2|_{F'}$ are (linearly) equivalent.

Lemma 3.1. *Let T_1 and T_2 be projectively equivalent projective representations of G over k. Then T_1 and T_2 belong to the same component of kF.*

There is certainly no hope of the converse of Lemma 3.1 being true in general, since for an arbitrary field k the group $H^2(G, k^\times)$ may be in infinite, in which case G has infinitely many equivalence classes of irreducible projective k-representations. Thus, in general, the connection between irreducible projective k-representations belonging to the same component of kF is weaker than projective equivalence. In the case where the field under consideration is algebraically closed, however, this connection is as strong as we could hope for.

Theorem 3.2. *Let $\bar{k}$ be an algebraically closed field of characteristic zero, and let T_1 and T_2 be irreducible projective $\bar{k}$-representations of G. Then T_1 and T_2 belong to the same component of $\bar{k}F$ if and only if they are projectively equivalent over $\bar{k}$.*

The proof of Theorem 3.2 will require a number of steps. First we describe how for a given irreducible projective k-representation T of G, the cocycle associated to T may be recovered (up to cohomology) from any lift of T to F. This involves the *transgression map* $\mathrm{tra} : \mathrm{Hom}(R, k^\times) \longrightarrow H^2(G, k^\times)$, which is defined as follows. Let $\eta \in \mathrm{Hom}(R, k^\times)$, and let μ be a section for G in F. Then define $\mathrm{tra}\,\eta$ to be the class in $H^2(G, k^\times)$ of the cocycle η' defined for $x, y \in G$ by

$$\eta'(x, y) = \eta(\mu(x), \mu(y), \mu(xy)^{-1}).$$

Lemma 3.3. *Let T be an irreducible projective representation of G over the field k, with cocycle $f \in Z^2(G, k^\times)$, belonging to the class $\bar{f} \in H^2(G, k^\times)$. Let $\tilde{T}$ be a lift of T to F and let $\eta = \tilde{T}|_R$, regarded as a homomorphism of R into $k^\times$. Then $\bar{f} = \mathrm{tra}\,\eta$.*

Proof. Let μ be a section for G in F and use it to define the cocycle $\eta' \in Z^2(G, k^\times)$ as above. For $g, h \in G$ we have

$$\begin{aligned} \eta'(g, h) &= \eta(\mu(g)\mu(h)\mu(gh)^{-1}) \\ &= \tilde{T}(\mu(g)\mu(h)\mu(gh)^{-1}) \\ &= \tilde{T}(\mu(g))\,\tilde{T}(\mu(h))\,\tilde{T}\left(\mu(gh)^{-1}\right) \\ f(g, h) &= T(g)T(h)T(gh)^{-1}. \end{aligned}$$

Define a map $\psi : G \longrightarrow k^\times$ by

$$\psi(g) = T(g)^{-1}\tilde{T}(\mu(g)) \ (= \tilde{T}(\mu(g))T(g)^{-1}).$$

Then

$$\begin{aligned} f^{-1}(g, h)\eta'(g, h) &= T(gh)T(h)^{-1}T(g)^{-1}\tilde{T}(\mu(g))\,\tilde{T}(\mu(h))\,\tilde{T}\left(\mu(gh)^{-1}\right) \\ &= \psi(g)\psi(h)\psi(gh)^{-1}. \end{aligned}$$

Thus $f^{-1}\eta'$ is a coboundary in $Z^2(G, k^\times)$ and f and η' belong to the same class in $H^2(G, k)$; $\bar{f} = \mathrm{tra}\,\eta$. ☐

Let $\eta \in \mathrm{Hom}(R, \bar{k}^\times)$, where as before the field $\bar{k}$ is algebraically closed. Then $\eta \in \ker(\mathrm{tra})$ if and only if the restriction of η to $F' \cap R$ is trivial, and in general $\mathrm{tra}\,\eta$ depends only on this restriction. These facts follow directly from the divisibility of $\bar{k}^\times$ and the exactness of the *Hochschild-Serre sequence* (see [10], for example):

$$\cdots \longrightarrow \mathrm{Hom}(F, k^\times) \xrightarrow{\mathrm{res}} \mathrm{Hom}(R, k^\times) \xrightarrow{\mathrm{tra}} H^2(G, k^\times) \longrightarrow \cdots$$

Now suppose T is an irreducible projective representation of G over k, whose cocycle belongs to the class α of $H^2(G, \bar{k}^\times) \cong M(G)$. Then if $\tilde{T}$ is a lift to F of T, $\tilde{T}|_{F' \cap R}$ (as a homomorphism into $\bar{k}^\times$) depends only on α; we denote this homomorphism by θ_α. Furthermore, since θ_α uniquely determines α by the transgression mapping, we obtain a bijective correspondence between $H^2(G, \bar{k}^\times)$ and $\mathrm{Hom}(F' \cap R, \bar{k}^\times)$, given by $\alpha \longleftrightarrow \theta_\alpha$.

Finally, for each $\alpha \in H^2(G, \bar{k}^\times)$ we let I_α denote the kernel of θ_α, a subgroup of $F' \cap R$.

We next state a theorem of Tappe (see [9]) which will be invoked in the proof of Theorem 3.2. An element x of G is said to be f-regular for a cocycle $f \in H^2(G, \bar{k}^\times)$

if $f(x,y) = f(y,x)$ whenever $y \in C_G(x)$. It is easily checked that if $x \in G$ is f-regular, then so is each of its conjugates in G, and that each is also f'-regular whenever f and f' represent the same element of $H^2(G,\bar{k}^\times)$. Thus we may define for $\alpha \in H^2(G,\bar{k}^\times)$ the notion of an α-regular conjugacy class of G.

Theorem 3.4. *Let $\alpha \in H^2(G,\bar{k}^\times)$, and let $f \in Z^2(G,\bar{k}^\times)$ be a representative of the class α. Then the number n_α of projective equivalence classes of irreducible projective f-representations of G over $\bar{k}$ is equal to the number of α-regular conjugacy classes of G contained in G'.*

We now denote by $\mathcal{S}_F$ the set of conjugacy classes of F contained in F', and by $\mathcal{S}_G$ the set of conjugacy classes of G contained in G'. We will say that $\mathcal{C} \in \mathcal{S}_F$ *lies over* $C \in \mathcal{S}_G$ if C is the image of $\mathcal{C}$ under ϕ. For each $C \in \mathcal{S}_G$, we define a subset Z_C of $F' \cap R$ by choosing $\mathcal{C} \in \mathcal{S}_F$ lying over C, then choosing $x \in C$ and a preimage X for x in $\mathcal{C}$, and setting

$$Z_C = \{Z \in F' \cap R : ZX \in \mathcal{C}\}.$$

It is routine to check that Z_C is a group and that it does not depend on the choice of $\mathcal{C}$ or on the choices of x or X. Suppose $Z \in Z_C$. Then, since ZX is conjugate to X in F, $Z = Y^{-1}XYX^{-1}$, for some $Y \in F$. Since $Z \in R, Y \in \phi^{-1}(C_G(x))$. On the other hand, it is clear that $Y^{-1}XYX^{-1} \in Z_C$ for any $Y \in \phi^{-1}(C_G(x))$, hence

$$Z_C = \{Y^{-1}XYX^{-1} : Y \in \phi^{-1}(C_G(x))\}.$$

Lemma 3.5. *Let $C \in \mathcal{S}_G$, and let $\alpha \in H^2(G,\bar{k}^\times)$. Then C is α-regular if and only if $Z_C \subseteq I_\alpha$.*

Proof. Let $f \in Z^2(G,\bar{k}^\times)$ be a cocycle representing α, and let $x \in C$. Then x (and hence C) is α-regular if and only if $T(x)T(y) = T(y)T(x)$ whenever $y \in C_G(x)$ and T is an irreducible projective f-representation of G over $\bar{k}$. Let $\tilde{T}$ be a lift to F of such a representation T, and choose $X \in \phi^{-1}(x)$. Then if $Y \in \phi^{-1}(y)$ for some $y \in C_G(x)$, we have $\tilde{T}(X) \in \bar{k}^\times T(x)$ and $\tilde{T}(Y) \in \bar{k}^\times T(y)$, whence

$$\tilde{T}(YXY^{-1}X^{-1}) = T(x)T(y)T(x)^{-1}T(y)^{-1}.$$

Thus x is α-regular if and only if $\tilde{T}(YXY^{-1}X^{-1}) = 1$ for all $X \in \phi^{-1}(x)$ and $Y \in \phi^{-1}(C_G(x))$, whenever $\tilde{T}$ is a lift to F of an irreducible projective α-representation of G. This completes the proof since such a $\tilde{T}$ restricts on $F' \cap R$ to θ_α, and $Z_C = \{Y^{-1}XYX^{-1} : Y \in \phi^{-1}(C_G(x))\}$. □

The proof of Theorem 3.2 is now a matter of counting.

Proof of Theorem 3.2. In view of Lemma 3.1 it is sufficient to show that the number of components of $\bar{k}F$ is equal to the number $\sum_{\alpha \in H_2(G,\bar{k}^\times)} n_\alpha$ of mutually (projectively) inequivalent irreducible projective $\bar{k}$-representations of G. For $\alpha \in H^2(G,\bar{k}^\times)$, n_α here denotes the number of such representations having cocycle representing α. From Theorem 3.4 we have

$$n_\alpha = |\{C \in \mathcal{S}_G : C \text{ is } \alpha\text{−regular}\}|.$$

Of course $C \in \mathcal{S}_G$ is α-regular if and only if $\mathcal{Z}_C \subseteq I_\alpha$, i.e., if and only if θ_α factors through Z_C. Since

$$\{\theta_\alpha\}_{\alpha \in H^2(G,\bar{k}^\times)} = \mathrm{Hom}(F' \cap R, \bar{k}^\times),$$

the number of elements of $H^2(G,\bar{k}^\times)$ with respect to which $C \in \mathcal{S}_G$ is regular is $|\mathrm{Hom}(F' \cap R/Z_C, k^\times)| = [F' \cap R : Z_C]$, since $\bar{k}$ is algebraically closed. Then counting the ordered pairs of the form (C,α) where $\alpha \in H^2(G,\bar{k}^\times)$ and $C \in \mathcal{S}_G$ is α-regular leads to the equality

$$\sum_{\alpha \in H^2(G,\bar{k}^\times)} n_\alpha = \sum_{C \in \mathcal{S}_G} [F' \cap R : Z_C]. \tag{3.1}$$

If $\mathcal{C} \in \mathcal{S}_F$, let $\hat{\mathcal{C}}$ denote the element $\sum_{x \in \mathcal{C}} x$ of $\bar{k}F$. Then $\{\hat{\mathcal{C}}\}_{\mathcal{C} \in \mathcal{S}_F}$ has the same cardinality as the set $\mathcal{I}$ of primitive central idempotents of $\bar{k}F$, since each is a basis for the same vector space over $\bar{k}$, namely $\mathcal{Z}(\bar{k}F) \cap \bar{k}F'$. Thus the number of components of $\bar{k}F$ is $|\mathcal{S}_F|$. Now let $\mathcal{C} \in \mathcal{S}_F$ lie over $C \in \mathcal{S}_G$. Then it is easily observed that the elements of $\mathcal{S}_F$ lying over C are precisely those of the form $r\mathcal{C}$, where $r \in F' \cap R$. Furthermore, if $r \in F' \cap R$, then $r\mathcal{C} = \mathcal{C}$ if and only if $r \in Z_C$. Thus the number of elements of $\mathcal{S}_F$ lying over $C \in \mathcal{S}_G$ is $[F' \cap R : Z_C]$ and

$$|\mathcal{S}_F| = \sum_{C \in \mathcal{S}_G} [F' \cap R : Z_C] = \sum_{\alpha \in H^2(G,\bar{k}^\times)} n_\alpha.$$

This completes the proof of Theorem 3.2. □

4. Realizability and projective splitting fields

We now employ the results established in Section 3 in a discussion of realizability of complex irreducible projective representations over subfields of $\mathbb{C}$. We begin with some standard definitions, each of which is a straightforward extension of a corresponding definition from the theory of ordinary representations.

Definition 4.1. Let $T : G \longrightarrow GL(n,\mathbb{C})$ be a complex projective representation of G, and let E be a subfield of $\mathbb{C}$. Then T is *projectively realizable* over E if there exists a matrix $A \in GL(n,\mathbb{C})$ and a function $\mu : G \longrightarrow \mathbb{C}^\times$ for which

$$\mu(g)A^{-1}T(g)A \in GL(n,E),\ \forall g \in G.$$

In this situation the projective representation T' of G defined on $g \in G$ by $T'(g) = \mu(g)A^{-1}T(g)A$ is projectively equivalent (over $\mathbb{C}$) to T and is called a *projective realization* of T over E.

T is said to be *linearly realizable* over E if the above can be accomplished with $\mu(g) = 1,\ \forall g \in G$.

Definition 4.2. Let $T : G \longrightarrow GL(n,k)$ be an irreducible projective representation of G over a field k. Then T is *absolutely irreducible* if it remains irreducible when regarded as a representation over any field extension of k.

This is the case if and only if the k-linear span in $M_n(k)$ of $\{T(g)\}_{g \in G}$ is $M_n(k)$. Thus T is absolutely irreducible if and only if every lift of T to a generic central extension F for G is an absolutely irreducible representation of F.

Definition 4.3. If k is a field with algebraic closure $\bar{k}$, then k is called a *projective splitting field* for G if every projective $\bar{k}$-representation of G is projectively realizable over k. If every ordinary $\bar{k}$-representation of G is realizable over k, k is a *splitting field* (or *ordinary* splitting field) for G.

It is known that every finite group has a cyclotomic projective splitting field; Reynolds [8] shows that if $|G| = n$, then $\mathbb{Q}(\xi_n)$ is a projective splitting field for G

(ξ_i will in general denote a root of unity of order i in $\mathbb{C}$). H. Opolka [4] provides an example which shows that if $l = \exp(G)$, then $\mathbb{Q}(\xi_l)$ need not be a projective splitting field for G, although it is of course an ordinary splitting field for G. In the same paper it is shown that if $m = \exp(G')\exp(M(G))$, then $\mathbb{Q}(\xi_m)$ is a projective splitting field for G. We will establish this result as a consequence of a more general one: if F is a generic central extension for G, then any ordinary splitting field for the finite group F' is a projective splitting field for G.

The general idea is as follows: if k is a subfield of $\mathbb{C}$ containing all F-invariant character values of F', then it follows from the relation (2.1) that the group algebras kF and $\mathbb{C}F$ have the same set $\mathcal{I}$ of primitive central idempotents. If for every $e \in \mathcal{I}$, F (or kF) has an absolutely irreducible k-representation $\tilde{T}_k$ belonging to e and arising as a lift of a projective representation T_k of G, then by Theorem 3.2 T_k is projectively equivalent to any complex (absolutely) irreducible representation of G belonging to the component $\langle e \rangle$ of $\mathbb{C}F$. In this case, since every irreducible projective $\mathbb{C}$-representation of G belongs to *some* $e \in \mathcal{I}$, we can conclude that k is a projective splitting field for G.

We will establish, after an investigation of the general structure of simple images of kF under maps sending R into $k^\times$, that the condition that k be an ordinary splitting field for F' is sufficient to guarantee the existence of absolutely irreducible projective k-representations of G belonging to every component of kF.

4.1. **Extending the Centre.** Throughout the following we assume that $k \subseteq \mathbb{C}$ is an ordinary splitting field for F'. This assumption on k is not required throughout the entire following discussion, and in many places relaxing it would lead to only minor complications.

Any irreducible k-representation of F which sends R into $k^\times$ is a lift to F of an irreducible projective k-representation of G. Such a representation extends by k-linearity to a mapping of the group algebra kF onto a simple subring of $M_n(k)$ for some n; in particular, onto a finite-dimensional simple k-algebra. The group algebra kF is of course not finite-dimensional over k, nor is it completely reducible. However, F is a centre-by-finite group and kF certainly has finite rank as a module over its central subring kR. In this section we show that kF embeds in a completely reducible algebra having finite dimension over a central subfield which is a purely transcendental extension of k. The simple components of this completely reducible ring are closely related to the images of kF under lifts of irreducible projective k-representations of G.

Let S be a torsion-free complement for $F' \cap R$ in R. Then $S \cong R/F' \cap R \cong RF'/F'$. Since RF'/F' has finite index in the free abelian group F/F', S is itself free abelian of rank equal to the free rank r of $\tilde{F}$. Then the central subring kS of kF is a ring of Laurent polynomials in r commuting variables. Thus kS is an integral domain and, furthermore, no element of kS can be a zerodivisor in kF; this follows from the fact that any transversal for S in F forms a basis for kF as a right module over kS. Then we can form from kF a ring of quotients $(kS)^{-1}kF$, in which every element of kS is invertible. We will denote this ring by KF, where K denotes the field of fractions of kS, which is a purely transcendental field extension of k of transcendence degree r.

Theorem 4.4. *KF is a completely reducible K-algebra.*

Proof. Let $\hat{G} = F/S$ and for $g \in \hat{G}$ choose $\mu(g) \in F$ for which $g = \mu(g)S$. Then the correspondence $g \longleftrightarrow \mu(g)$ leads to a K-algebra isomorphism of KF with the twisted group ring $K^\alpha \hat{G}$, where the cocycle $\alpha : \hat{G} \times \hat{G} \longrightarrow K^\times$ is defined for $g, h \in \hat{G}$ by

$$\alpha(g, h) = \mu(g)\mu(h)\mu(gh)^{-1}.$$

That KF is completely reducible now follows from Maschke's theorem as it applies to twisted group rings (see [6]). $\square$

Now KF is a direct sum of simple K-algebras:

$$KF = \bigoplus_{i=1}^{m} M_{n_i}(D_i),$$

where for $i = 1, \dots, m$, D_i is a finite dimensional K-division algebra.

The next theorem is of fundamental importance since it establishes (with Theorem 3.2) a bijective correspondence between the set of simple components of KF and the set of projective equivalence classes of irreducible complex projective representations of G.

Theorem 4.5. *KF and kF have the same set of primitive central idempotents.*

Proof. Let e be a primitive central idempotent of kF. Then e is central in KF, and we need to show that the two-sided ideal of KF generated by e is a simple ring. Certainly KFe is completely reducible since KF is, and thus $\mathcal{Z}(KFe)$ is a direct sum of fields. To show that $\mathcal{Z}(KFe)$ is in fact a field it suffices to show that $\mathcal{Z}(kFe)$ is a domain, since for every nonzero $\alpha \in KF$ we can find a nonzero $A \in kS \subseteq \mathcal{Z}(kF)$ for which $0 \neq A\alpha \in kF$.

Let f be a primitive central idempotent of kF' for which $ef = f$; $e \in kF'$ by Lemma 2.4. Then $e = \sum_{x \in T} f^x$ where $\mathbf{T}$ is a transversal in F for $F_1 = C_F(f)$. Let $s = [F : F_1]$; this index is of course finite since $\mathcal{Z}(F)$ has finite index in F. Then $kFe \cong M_s(kF_1 f)$ (see [5]) and we now need to show that $\mathcal{Z}(kF_1 f)$ contains no zerodivisors. The idempotent f is central in kF_1 by definition of F_1, and primitive in kF_1 since it is primitive in kF'. Let $B_1 = kF'f$, so $B_1 \cong M_n(k)$ for some n, since k is a splitting field for F'. Let $\mathcal{E}$ be a set of n^2 matrix units in B_1, and let Λ denote the centralizer of $\mathcal{E}$ in $kF_1 f$. Then $kF_1 f \cong M_n(\Lambda)$. Since $F' \trianglelefteq F_1$ and $f \in kF'$, the ring $kF_1 f$ is a crossed product over $kF'f$ by F_1/F' which is free abelian of rank r since it has finite index in F/F'. We will use this crossed product structure to show that Λ is also a crossed product over B_1, again by a free abelian group of rank r. Since B_1 is invariant under conjugation by elements of F_1, for each $t \in \mathbf{T}$ the set

$$\mathcal{E}^t = \{t^{-1}\varepsilon_{ij}t \,:\, \varepsilon_{ij} \in \mathcal{E}\}$$

is a system of matrix units in B_1. Then $\mathcal{E}$ and $\mathcal{E}^t$ are conjugate in B_1 since B_1 is a simple ring (see [2], Theorem 2.13). Thus for each $t \in \mathbf{T}$ we may choose an element $b(t)$ of $\mathcal{U}(B_1)$, determined by t up to multiplication by elements of $k^\times$, for which $c(t) = b(t)t$ centralizes $\mathcal{E}$.

Let $\mathcal{S} = \{c(t), t \in \mathbf{T}\}$. Then $\mathcal{S}$ is certainly right independent over k since $b(t) \in kF'$ for each t, and $\mathbf{T}$ is a transversal for F' in F. We now show that $\mathcal{S}$ generates Λ as a vector space over k—certainly $k[\mathcal{S}] \subseteq \Lambda$; on the other hand, suppose $\lambda \in \Lambda$. Then λ can be uniquely written in the form $\lambda = \sum_{t \in T} b_t t$, where

$b_t \in B_1$ for each t, and $b_t = 0$ for all but finitely many t. Since $\lambda \in \Lambda = C_{kF_1f}(\mathcal{E})$, we have for each $\varepsilon \in \mathcal{E}$

$$\varepsilon\lambda = \lambda\varepsilon \Longrightarrow \sum_{t\in T} \varepsilon b_t t = \sum_{t\in T} b_t t\varepsilon = \sum_{t\in T} b_t \varepsilon^{t^{-1}} t.$$

Then $\varepsilon b_t = b_t\varepsilon^{t^{-1}} = b_t t\varepsilon t^{-1}$, so $b_t t \in \Lambda$, $\forall t \in T$ and $\forall \varepsilon \in \mathcal{E}$. Thus for each $t \in T$ either $b_t = 0$ or $b_t t$ differs from $c(t) = b(t)t$ only by a scalar in $k^\times$. Thus $b_t t \in k^\times c(t)$ whenever $b_t \neq 0$, and $\lambda \in k[\mathcal{S}]$; hence $\Lambda = k[\mathcal{S}]$ and $\mathcal{S}$ is a basis for Λ as a vector space over k.

Now suppose $t_1, t_2 \in T$ and let $t \in \mathbf{T}$ represent the coset t_1t_2F'. Then

$$\begin{aligned} c(t_1)c(t_2) &= b(t_1)t_1b(t_2)t_2 = b(t_1)b(t_2)^{t_1^{-1}}t_1t_2 \\ &= b(t_1)b(t_2)^{t_1^{-1}}ct, \text{ for some } c \in F'. \end{aligned}$$

Then $c(t_1)c(t_2) \in k^\times c(t)$. Since $b'(t) := b(t_1)b(t_2)^{t_1^{-1}}c \in \mathcal{U}(B_1)$ and $c(t_1)c(t_2) = b'(t)t$ centralizes $\mathcal{E}$, it follows that $c(t_1)c(t_2)c(t)^{-1} \in C_{B_1}(\mathcal{E}) = k$. The correspondence $t \longleftrightarrow c(t)$ leads to an isomorphism between $\Lambda = k[\mathcal{S}]$ and a twisted group ring $k^\alpha(F_1/F')$ where the cocycle α is defined by $\alpha(t_1F', t_2F') = c(t_1)c(t_2)c(t)^{-1}$ where $t_1, t_2, t \in T$, and $tF' = t_1t_2F'$. Thus Λ is a twisted group ring of a free abelian group over k, and it is immediate that Λ is a domain, whence $\mathcal{Z}(kFe)$ is a domain and KFe is a simple ring. This completes the proof of Theorem 4.5. □

We remark that Theorem 4.5 remains true without restriction on the field k. We note also that it is evident from the proof of Theorem 4.5 not only that KFe is simple whenever e is a primitive central idempotent of kF, but also that kF_1f is simple whenever f is a primitive central idempotent of kF' and $F_1 = C_F(f)$. Indeed if F_0 is any subgroup of F which contains $\langle R, F'\rangle$ and centralizes f, then KF_0f is a simple component of the ring KF_0 which is completely reducible by virtue of being a twisted group ring of F_0/S over K. Furthermore, in this case if $kF'f \cong M_n(k)$, then kF_0f is a ring of $n \times n$ matrices over a domain, and KF_0f is a ring of $n \times n$ matrices over a K-division algebra. In particular, the simple rings KF_0f and $kF'f$ have the same matrix degree.

4.2. **Structure of the Simple Components of KF.** As in the proof of Theorem 4.5, let e be a primitive central idempotent of kF and let f be a primitive central idempotent of the completely reducible ring $kF'e$. If $F_1 = C_F(f)$ and $[F : F_1] = s$, we then have $KFe \cong M_s(KF_1f)$. From now on we denote the simple ring KF_1f by A_1, and we denote the centre of A_1 by Z. If $kF'f \cong M_n(k)$, the following series of results will show that A_1 is a ring of $n\times n$ matrices over a central Z-division algebra which can be described as a twisted group ring over Z of an abelian quotient of G. Let B denote the subalgebra of A_1 generated over Z by $F'f$. We begin by showing that B is central simple over Z. This makes use of the following lemma, of which a proof can be found in [7].

Lemma 4.6. *Let $\mathcal{A}$ be a finite dimensional algebra over a field $\mathcal{F}$. Let $\mathcal{B}$ and $\mathcal{C}$ be $\mathcal{F}$-subalgebras of $\mathcal{A}$ for which*

(i) *$\mathcal{B}$ is central simple over $\mathcal{F}$.*
(ii) *$\mathcal{C}$ centralizes $\mathcal{B}$.*
(iii) *$\mathcal{A} = \mathcal{B}\mathcal{C}$.*

Then $\mathcal{A} \cong \mathcal{B} \otimes_{\mathcal{F}} \mathcal{C}$.

Lemma 4.7. $B \cong M_n(Z)$.

Proof. Let S be a free abelian complement for $F' \cap R$ in R, such that K is the field of quotients of $k[S]$. Since $f \in kF'$ and the subgroup of F generated by S and F' is the direct product $S \times F'$, any k-basis for $kF'f$ remains independent over $k[S]$ and hence over K. Thus the subring of B generated over K by $F'f$ is isomorphic to the tensor product $K \otimes_k kF'f$. Then $K[F'f] \cong M_n(K)$, since $kF'f \cong M_n(k)$. We can now regard B as a K-algebra generated by the K-subalgebras Z and the central simple K-subalgebra $K[F'f]$ to apply Lemma 4.6 and conclude $B = Z \otimes_K K[F'f]$. Then $B \cong M_n(Z \otimes_K K) \cong M_n(Z)$. □

Since B is a simple Z-subalgebra of the central simple Z-algebra A_1, we have $A_1 = B \otimes_Z C$, where $C := C_{A_1}(B) = C_{A_1}(F')$ is again a central simple Z-algebra. In fact C is a division algebra, since the matrix degree of A_1 is the same as that of $kF'f$ and hence of B, by the remarks following the proof of Theorem 4.5.

Let $x \in F_1$. Then the map $\phi_x : kF'f \longrightarrow kF'f$ defined for $\alpha \in kF'f$ by $\phi_x(\alpha) = x^{-1}\alpha x$ is a central automorphism of the central simple k-algebra $kF'f$. This automorphism is inner by the Noether-Skolem theorem, so for each $x \in F_1$ there exists a unit β_x of $kF'f$ for which $\gamma_x := \beta_x x$ belongs to $C_{A_1}(B) = C$. Furthermore, since $\mathcal{Z}(kF'f) = k$, γ_x is determined by x up to multiplication by elements of $k^\times$. From now on we fix γ_x for each $x \in F_1$.

Next we show that $\{\gamma_x\}_{x\in F_1}$ generates the division algebra C as a vector space over K. Let $\alpha \in C$. After multiplying α by an element of $k[S]$, if necessary, we can assume $\alpha \in kF_1$. Let $\mathbf{T}$ be a transversal for F' in F_1. Then $\mathbf{T}$ is right independent over $kF'f$, so that α can be uniquely written in the form $\alpha = \sum_{t\in T} \alpha_t t$, where $\alpha_t \in kF'f$ and $\alpha_t = 0$ for all but finitely many $t \in \mathbf{T}$. Let $c \in F'$. Then since α centralizes F' we have

$$\begin{aligned} c\alpha = \sum_{t\in T} c\alpha_t t \quad &= \quad \alpha c = \sum_{t \in T} \alpha_t c^{t^{-1}} t \\ \implies c\alpha_t = \alpha_t t c t^{-1} \quad &\implies \quad c\alpha_t t = \alpha_t t c, \quad \forall t \in \mathbf{T},\ \forall c \in F'. \end{aligned}$$

Thus for each $t \in \mathbf{T}$, $\alpha_t t$ centralizes F', so either $\alpha_t = 0$ or $\alpha_t t \in k^\times \gamma_t$. Thus every element of C is a K-linear combination of elements of the set $\{\gamma_t\}_{t\in\mathbf{T}}$.

We now introduce some notation. We define

$$F'^{+} = \{x \in F_1 : xf \in B\}.$$

It is apparent that F'^{+} is a normal subgroup of F_1 containing F' and $\mathcal{Z}(F)$, and hence having finite index in F_1. Let $\mathcal{S}$ be a transversal for F'^{+} in F, and define $\mathcal{B} = \{\gamma_s\}_{s\in\mathcal{S}}$.

Lemma 4.8. *Let $x \in F_1$. Then $\gamma_x \in Z$ if and only if $x \in F'^{+}$.*

Proof. ($\Longleftarrow$): Suppose $xf \in B$. Then $\gamma_x = \beta_x x$ where $\beta_x \in B$. Thus $\gamma_x \in C_B(F') = \mathcal{Z}(B) = Z$.

($\Longrightarrow$): On the other hand, suppose $\gamma_x = \beta_x x \in Z$ where β_x is a unit of B. Then $xf = \beta_x^{-1}\gamma_x \in B$. □

Lemma 4.9. *Suppose $x, y \in F_1$. Then*

(i) $\gamma_x\gamma_y \in k^\times \gamma_{xy}$,
(ii) *If $x \in F'^{+}y$, then $\gamma_x \in Z^\times \gamma_y$.*

Proof. (i) $\gamma_x\gamma_y = x\beta_x y\beta_y = xy\beta_x^y\beta_y$, and $\gamma_{xy} = xy\beta_{xy}$. Thus

$$\begin{aligned}\gamma_{xy}^{-1}\gamma_x\gamma_y &= \beta_{xy}^{-1}\beta_x^y\beta_y \in C^\times \cap kF' = k^\times \\ \Longrightarrow & \quad \gamma_x\gamma_y \in k^\times\gamma_{xy}.\end{aligned}$$

(ii) This second statement is an immediate consequence of (i) above and Lemma 4.8. □

Lemma 4.10. *$\mathcal{B}$ is a basis for C over Z.*

Proof. Since $\mathcal{S}$ is right independent over kF'^+, any $\alpha \in C \cap kF_1$ can be written in a unique way in the form $\alpha = \sum_{s\in\mathcal{S}} \alpha_s s$, where $\alpha_s \in kF'^+$ for each $s \in \mathcal{S}$. Then as in the proof of Lemma 4.8 the requirement that α centralize C leads to the conclusion $\alpha_s s \in C$ for each $s \in \mathcal{S}$. Since $s = \beta_s^{-1}\gamma_s$, where β_s is a unit in kF', it follows that $\alpha_s\beta_s^{-1}\gamma_s \in C$, whence $\theta_s = \alpha_s\beta_s^{-1} \in C \cap kF'^+ = Z$, as γ_s is a unit in C. Thus every $\alpha \in C \cap kF_1$ has the form $\alpha = \sum_{s\in\mathcal{S}} \theta_s\gamma_s$ where $\theta_s \in Z$, $\forall s \in \mathcal{S}$. Thus $\mathcal{B}$ is a spanning set for C as a vector space over C, since every element of C is the product of an element of $C \cap kF_1$ and an element of K. That $\mathcal{B}$ is linearly independent over Z follows easily from the independence of $\mathcal{S}$ (and hence $\mathcal{B}$) over kF'^+. □

Thus C is a central simple Z-algebra of dimension $[F_1 : F'^+]$, and the order of the abelian group $\bar{F}_1 := F_1/F'^+$ is a square. In fact we can say more than this.

Theorem 4.11. *C is isomorphic to a twisted group ring of $\bar{F}_1$ over Z.*

Proof. If $s \in \mathcal{S}$, let $\bar{s}$ denote the element of F_1/F'^+ represented by s. Then the assignment $\bar{s} \longleftrightarrow \gamma_s$ establishes a bijective correspondence between the group F_1/F'^+ and the Z-basis B of C. Suppose for some $s_1, s_2 \in \mathcal{S}$ that $\bar{s}_1\bar{s}_2$ is represented by $s \in \mathcal{S}$. Then it is immediate from part (ii) of Lemma 4.9 that $\gamma_{s_1}\gamma_{s_2} \in Z^\times\gamma_s$, and $C \cong Z^f(\bar{F}_1)$; here the cocycle $f \in Z^2(\bar{F}_1, Z^\times)$ is defined for $s_1, s_2 \in \mathcal{S}$ by $f(\bar{s}_1, \bar{s}_2) = \gamma_s\gamma_{s_1}^{-1}\gamma_{s_2}^{-1}$ where $s \in \mathcal{S}$ represents $\bar{s}_1\bar{s}_2 \in \bar{F}_1$. □

From the fact that the finite abelian group $\bar{F}_1$ possesses central simple twisted group algebras over Z it follows that $\bar{F}_1$ is of symmetric type (i.e. the direct product of two isomorphic abelian groups) and that Z contains a root of unity of order equal to the exponent of $\bar{F}_1$ (see [10] for the details). The following result shows that such a root of unity must in fact belong not only to Z but to k.

Theorem 4.12. *Let $\{\bar{x}_1, \dots, \bar{x}_r\}$ be a basis for the free abelian group F'^+/F' and for $i = 1 \dots r$ let x_i be a representative for $\bar{x}_i$ in F'^+. Let $\Gamma = \{\gamma_{x_1}, \dots, \gamma_{x_r}\}$. Then Γ generates Z as a field over k, and Γ is algebraically independent over k.*

Proof. First we show $k(\Gamma)$ contains K. For this it is sufficient to show that $k(\Gamma)$ contains Rf. Let $a \in R$, and $a = x_1^{i_1} \dots x_r^{i_r} c$ for some integers $i_1, \dots, i_r$ and $c \in F'$. It follows that $\gamma_a \in k^\times(\gamma_{x_1})^{i_1} \dots (\gamma_{x_r})^{i_r}$ by Lemma 4.9, so $\gamma_a \in k(\Gamma)$. Then $af \in k(\Gamma)$ since $\gamma_a = a\beta_a$ where $\beta_a \in \mathcal{Z}(\mathcal{U}(kF'f)) = k^\times f$. Thus $K \subseteq k(\Gamma)$.

Now let $\alpha \in Z$. After multiplying α (if necessary) by a suitable element of kR we may assume $\alpha \in Z \cap kF$. Let $\mathcal{T} = \mathbf{T} \cap F'^+$, where $\mathbf{T}$ as before is a transversal for F' in F_1. We now show that $\{\gamma_t\}_{t\in\mathcal{T}}$ is a k-basis for $Z \cap kF$. Certainly $\mathbf{T}$ is right independent over kF' and so α can be written in the form $\alpha = \sum_{t\in\mathbf{T}} a_t t$, where $a_t \in kF'$, $a_t = 0$ for all but finitely many t. Let $x \in F_1$. Then

$$\begin{aligned}\alpha = \alpha^x &= \sum x^{-1}a_t t x \\ \Longrightarrow \sum a_t t &= \sum x^{-1}a_t t x t^{-1} t.\end{aligned}$$

Since $x^{-1}a_t txt^{-1} = x^{-1}a_t xx^{-1}txt^{-1} \in kF'$ for each $t \in \mathbf{T}$, it follows that $xa_t t = a_t tx,\ \forall x \in F_1,\ \forall t \in \mathbf{T}$. Thus for each $t \in \mathbf{T}$, $a_t t$ centralizes F_1 and thus belongs to Z. Then, in particular, $a_t t$ centralizes F' for each $t \in \mathbf{T}$, so $a_t t \in k^\times \gamma_t,\ \forall t \in \mathbf{T}$. Since by Lemma 4.8 $\gamma_t \in Z$ if and only if $t \in F'^+$, this forces $a_t = 0$ whenever $t \notin \mathcal{T}$. Thus $\alpha = \sum_{t\in\mathbf{T}} a_t t$ is a k-linear combination of elements of $\{\gamma_t\}_{t\in\mathcal{T}}$. That $\{\gamma_t\}_{t\in\mathcal{T}}$ is linearly independent over k is clear, since $\gamma_t \in \mathcal{U}(kF'f)t$ for each $t \in \mathcal{T}$, and $\mathcal{T}$ is independent over kF'. Finally, if $t \in \mathcal{T}$ then $tF' = x_i^{i_1} x_2^{i_2} \dots x_r^{i_r} F'$ for some $i_1, \dots, i_r$ and $\gamma_t \in k^\times(\gamma_{x_1})^{i_1} \dots (\gamma_{x_r})^{i_r}$ by Lemma 4.9. It follows that $\alpha \in k(\Gamma)$, and that $Z = k(\Gamma)$ since $K \subseteq k(\Gamma)$ also.

That Γ is algebraically independent over k is clear since $k(\Gamma)$ contains K as a subfield, and K has transcendence degree r over k. □

Theorem 4.12 shows, in particular, that Z is purely transcendental over k.

Our next aim is to describe the central Z-division algebra C as a tensor product of cyclic division algebras. If A and B are nonzero elements of a field $\mathcal{F}$ containing a root of unity ξ of order d, then the *symbol algebra* $\mathcal{A} = \left(\frac{a,b}{\xi,\mathcal{F}}\right)$ is a central simple $\mathcal{F}$-algebra of degree d, generated over $\mathcal{F}$ by elements a and b for which $a^d = A,\ b^d = B$, and $ab = \xi ba$. The symbol algebra $\mathcal{A}$ is isomorphic to the matrix ring $M_d(\mathcal{F})$ if and only if B is a norm for the cyclic field extension $\mathcal{F}(a)/\mathcal{F}$. The proof of Theorem 4.14 involves the following technicality.

Lemma 4.13. *Let $a_1, \dots, a_m, b$ be positive integers for which* $\gcd(a_1, \dots, a_m)$ *is relatively prime to b. Then there exist integers $t_2, \dots, t_m$ for which $a_1 + \sum_{i=2}^m t_i a_i$ is relatively prime to b.*

Proof. Let $p_{l+1}, \dots, p_t$ denote the prime divisors of $\gcd(a_1, b)$ and suppose $b = p_1^{i_1} \dots p_l^{i_l} p_{l+1}^{i_{l+1}} \dots p_t^{i_t}$ for distinct primes $p_1, \dots, p_t$ and positive integers $i_1, \dots, i_t$. Let $b' = p_1^{i_1} \dots p_l^{i_l}$. Then $b'|b$ and b' is relatively prime to a_1. Define

$$a = a_1 + b' \gcd(a_2, \dots, a_m).$$

Then a is relatively prime to b: for suppose p is a prime dividing b. Then p is one of $p_1, \dots, p_t$ and either p divides a_1 or p divides b' (not both). If $p|a_1$, then $p \not| \gcd(a_2, \dots, a_m)$ since $\gcd(a_2, \dots, a_m)$ is relatively prime to b. On the other hand ,if p divides b' it cannot divide a_1. Thus no prime divisor of b divides a, so $\gcd(a, b) = 1$. The result follows since $\gcd(a_2, \dots, a_m) = \sum_{i=2}^m t_i a_i$ for some integers $t_2, \dots, t_m$. □

We now return to the context and notation of Theorem 4.12 to state:

Theorem 4.14. *There exist elements $r_1, s_1, r_2, s_2, \dots, r_q, s_q, c_{p+1}, \dots, c_r$ of F_1 (where $p = 2q$) for which*

1. $F_1 = \langle r_1, s_1, \dots, r_q, s_q, c_{p+1}, \dots, c_r, F'\rangle$;
2. $F'^+ = \langle r_1^{d_1}, s_1^{d_1}, \dots, r_q^{d_q}, s_q^{d_q}, c_{p+1}, \dots, c_r, F'\rangle$, *where* $1 < d_q | d_{q-1} | \dots | d_1$ *and* $\bar{F}_1 = F_1/F'^+ \cong (C_{d_1} \times C_{d_1}) \times \dots \times (C_{d_q} \times C_{d_q})$;
3. $C = \left(\frac{R_1,S_1}{\xi_1,Z}\right) \otimes_Z \left(\frac{R_2,S_2}{\xi_2,Z}\right) \otimes_Z \dots \otimes_Z \left(\frac{R_q,S_q}{\xi_q,Z}\right)$, *where for* $i = 1, \dots, q$,
   (i) $R_i = (\gamma_{r_i})^{d_i},\ S_i = (\gamma_{s_i})^{d_i}$,
   (ii) *ξ_i is a root of unity of order d_i in k,*
   (iii) $\Gamma = \{R_1, S_1, \dots, R_q, S_q, \gamma_{c_{p+1}}, \dots, \gamma_{c_r}\}$ *is a transcendence basis for Z over k with $Z = k(\Gamma)$.*

Proof. The finite abelian group $\bar{F}_1$ has symmetric type by the remarks following Theorem 4.11; thus

$$\bar{F}_1 \cong (C_{d_1} \times C_{d_1}) \times \cdots \times (C_{d_q} \times C_{d_q}),$$

where $1 < d_q | d_{q-1} | \ldots | d_1 = \exp \bar{F}_1$. Setting $p = 2q$, we may choose elements $a_1, \ldots, a_p, c_{p+1}, \ldots, c_r$ of F_1 for which $\{a'_1, \ldots, c'_r\}$ is a basis for the free abelian group F_1/F' (a'_i and c'_j respectively denoting the cosets a_iF' and c_jF' for $i = 1, \ldots, p$ and $j = p+1, \ldots, r$), and for which the set

$$\{(a'_{2i-1})^{d_i}, (a'_{2i})^{d_i}, c'_j\}_{i=1,\ldots,q;\ j=p+1,\ldots,r}$$

is a basis for ${F'}^+/F'$.

By Lemma 4.10, the set $\mathcal{B} = \{\gamma_{a_1}^{i_1}, \ldots, \gamma_{a_p}^{i_p}\}_{1 \le i_j \le \lceil d_j/2 \rceil}$ is a Z-basis for C, and the correspondence

$$\bar{a}_1^{i_1} \ldots \bar{a}_p^{i_p} \longleftrightarrow (\gamma_{a_1})^{i_1} \ldots (\gamma_{a_p})^{i_p}$$

(where $\bar{a}_i = a_i{F'}^+$) establishes as in Theorem 4.11 an isomorphism of C with $Z^\alpha \bar{F}_1$, where the cocycle $\alpha : \bar{F}_1 \times \bar{F}_1 :\longrightarrow Z^\times$ is defined as in Theorem 4.11 We define an antisymmetric bilinear pairing ϕ on $\bar{F}_1$ by

$$\phi(\bar{a}, \bar{b}) = \frac{\alpha(\bar{a}, \bar{b})}{\alpha(\bar{b}, \bar{a})}, \text{ for } \bar{a}, \bar{b} \in \bar{F}_1.$$

That ϕ is bilinear follows easily from the cocycle law and the fact that $\bar{F}_1$ is abelian, and that ϕ is antisymmetric is clear. Furthermore, ϕ is nondegenerate; this follows from the fact that C is central simple over Z. Suppose for some $\bar{a} \in \bar{F}_1$ that $\phi(\bar{a}, \bar{b}) = 1$, $\forall\, \bar{b} \in \bar{F}_1$. If $\bar{a} = \bar{a}_1^{j_1} \ldots \bar{a}_p^{j_p}$, then the element $\gamma_{a_1}^{j_1} \ldots \gamma_{a_p}^{j_p}$ of $\mathcal{B}$ commutes with all other elements of $\mathcal{B}$ and is therefore central in C. Then $\gamma_{a_1}^{j_1} \ldots \gamma_{a_p}^{j_p} \in Z$ and it follows from Lemmas 4.9 and 4.8 that $a_1^{j_1} \ldots a_p^{j_p} \in {F'}^+$, so $\bar{a} = 1$ in $\bar{F}_1$.

Since $d_1 = \exp \bar{F}_1$, the values assumed by ϕ are the d_1th roots of unity in Z, and hence in k, since Z is purely transcendental over k. The element $\bar{a}_1$ of $\bar{F}_1$ has order d_1; for $i = 2, \ldots, p$ let $\zeta_i = \phi(\bar{a}_1, \bar{a}_i)$ in k. Then each ζ_i satisfies $\zeta_i^{d_1} = 1$ certainly; furthermore, the subgroup of $k^\times$ generated by $\zeta_2, \ldots, \zeta_p$ has order d_1, since if its order were a strict divisor d' of d_1, then the homomorphism $\phi(\bar{a}_1^{d'}, \_) : \bar{F}_1 \longrightarrow k^\times$ would be trivial, and $\gamma_{a_1}^{d'}$ would be central in C. Then there exists some $\bar{b} \in \bar{F}_1$ for which the order in $k^\times$ of $\phi(\bar{a}, \bar{b})$ is d_1, and in particular there exists a primitive d_1th root of unity ζ in k. For $i = 2, \ldots, p$ let $\xi_i = \phi(\bar{a}_1, \bar{a}_i) = \zeta^{j_i}$. Then $\gcd(j_2, \ldots, j_p)$ is relatively prime to d_1, and hence by Lemma 4.13 there exist integers $t_3, \ldots, t_p$ for which $j_2 + \sum_{i=3}^p t_i j_i$ is relatively prime to d_1. Then if $b = a_2 a_3^{t_3} \ldots a_p^{t_p}$ we find that $\phi(\bar{a}_1, \bar{b}) = \zeta^{j_2 + \sum_{i=3}^p r_i j_i} =: \zeta_i$, is a root of unity of order d_1 in $k^\times$. Now define $r_1 = a_1$, $s_1 = b$, and $\xi_1 = \phi(\bar{r_1}, \bar{s_1})$. Since $b = a_2 a_3^{r_3} \ldots a_p^{r_p}$, an expression in which a_2 appears with exponent 1, it is apparent that $F_1 = \langle r_1, s_1, a_3, \ldots, a_p, c_{p+1}, \ldots, c_r, F' \rangle$.

Now for $i = 3, \ldots, p$ we can define an element a_{i1} of F_1 by $a_{i1} = a_i r_1^{\alpha_1} s_1^{\alpha_2}$, where the exponents α_1 and α_2 are chosen to ensure that $\phi(\bar{r}_1, \bar{a}_{i1}) = \phi(\bar{s}_1, \bar{a}_{i1}) = 1$; there is no difficulty here since $\phi(\bar{r}_1, \bar{a}_i)$ is some power of $\xi_1 = \phi(\bar{r}_1, \bar{s}_1)$. It is clear that for $i = 3, \ldots, p$, a_{i1} has the same order as a_i modulo ${F'}^+$, and also that $F_1 = \langle r_1, s_1, a_{31}, \ldots, a_{p1}, c_{p+1}, \ldots, c_r, F' \rangle$. Now we may set $r_2 = a_{31}$ and repeat the application of Lemma 4.13 to define $s_2 = a_{41} \prod_{i=5}^p (a_{i1})^{l_i}$ for suitable $l_5, \ldots, l_p \in \mathbb{Z}$, so that $\xi_2 := \phi(\bar{r}_2, \bar{s}_2)$ is a root of unity of order d_2 in k.

For $i = 5, \ldots, p$ we may now replace a_{i1} by an element $a_{i2} = a_{i1} r_2^{\beta_1} s_2^{\beta_2}$ so that $\phi(\bar{a}_{i2}, \bar{r}_j) = \phi(\bar{a}_{i2}, \bar{s}_j) = 1$ for $j = 1, 2$, and

$$F_1 = \langle r_1, s_1, r_2, s_2, a_{52}, \ldots, a_{p2}, c_{p+1}, \ldots, c_r, F' \rangle.$$

Continuing in this manner we ultimately produce a basis

$$\{r'_1, s'_1, \ldots, r'_q, s'_q, c'_{p+1}, \ldots, c'_r\}$$

for F_1/F' for which

$$\{{r'_1}^{d_1}, {s'_1}^{d_1}, \ldots, {r'_q}^{d_q}, {s'_q}^{d_q}, c'_{p+1}, \ldots, c'_r\}$$

is a basis for ${F'}^+/F'$. For $i = 1, \ldots, q$, $\xi_i = \phi(\bar{r}_i, \bar{s}_i)$ is a root of unity of order d_i in k; also $\phi(\bar{r}_i, \bar{r}_j) = 1$, $\forall i, j$, $\phi(\bar{s}_i, \bar{s}_j) = 1$, $\forall i, j$, and $\phi(\bar{r}_i, \bar{s}_j) = 1$ for $j \neq i$.

By Lemmas 4.10 and 4.9, the set

$$\left\{ \prod_{i=1}^{q} \gamma_{r_i}^{j_i} \gamma_{s_i}^{k_i} \right\}_{1 \leq j_i, k_i \leq d_i}$$

is a Z-basis for C, and in particular C is generated by $\{\gamma_{r_1}, \gamma_{s_1}, \ldots, \gamma_{r_q}, \gamma_{s_q}\}$ as a Z-algebra. The subalgebra C_1 of C generated over Z by γ_{r_1} and γ_{s_1} has dimension d_1^2 and since $\gamma_{r_1}\gamma_{s_1} = \xi_1 \gamma_{s_1} \gamma_{r_1}$ where ξ_1 is a root of unity of order d_1 in Z, C_1 is a symbol algebra:

$$C_1 = Z(\gamma_{r_1}, \gamma_{s_1}) \cong \left(\frac{R_1, S_1}{\xi_1, Z} \right); \quad R_1 = (\gamma_{r_1})^{d_1}, \ S_1 = (\gamma_{s_1})^{d_1}.$$

If C'_1 is defined as the Z-subalgebra of C generated by $\gamma_{r_2}, \gamma_{s_2}, \ldots, \gamma_{r_q}, \gamma_{s_q}$, then C'_1 centralizes C_1 and since C_1 is central simple over Z, we can apply Lemma 4.6 to conclude that $C = C_1 \otimes_Z C'_1$. Continuing in this manner we obtain the required tensor product decomposition of the division algebra C:

$$C = \left(\frac{R_1, S_1}{\xi_1, Z} \right) \otimes_Z \left(\frac{R_2, S_2}{\xi_2, Z} \right) \otimes_Z \cdots \otimes_Z \left(\frac{R_q, S_q}{\xi_q, Z} \right),$$

where for $i = 1, \ldots, q$, $R_i = (\gamma_{r_i})^{d_i}$ and $S_i = (\gamma_{s_i})^{d_i}$.

The final statement of Theorem 4.14, that Γ is a transcendence basis for Z over k for which $Z = k(\Gamma)$, is an immediate consequence of Theorem 4.12. □

4.3. **Projective Realizability.** In this final section we show how the structure, as described above, of the simple components of the completely reducible ring KF is reproduced in the simple components of twisted group rings of G over k; i.e., in those simple k-algebras which arise as images of kF under (extensions of) lifts to F of irreducible projective k-representations of G.

We begin with some observations relating the rings kF_1f and $A_1 = KF_1f$ (where we retain all the notation of Section 4.2). We have shown that the set $\{\gamma_t\}_{t \in \mathbf{T}}$ is a k-basis for $C \cap kF_1$, and also of course generates kF_1f as a ring over $kF'f$. Moreover, $\{\gamma_t\}_{t \in \mathcal{T}}$ is a k-basis for $Z \cap kF_1$, as was shown in the proof of Theorem 4.12. It then follows from Lemma 4.9 that if $\Theta = \{\gamma_{r_1}, \gamma_{s_1}, \ldots, \gamma_{r_q}, \gamma_{s_q}\}$ and $\Gamma = \{R_1, S_1, \ldots, R_q, S_q, \gamma_{c_{p}+1}, \ldots, \gamma_{c_r}\}$ as in Theorem 4.14, then Γ generates $Z_1 := \mathcal{Z}(kF_1f) = Z \cap kF_1f$ as a k-algebra, and Θ generates $C \cap kF_1$ as a ring over $Z_1 = k(\Gamma)$. In particular, kF_1f is generated by Θ as a ring over the subring $Z_1\langle F' \rangle$ of kF_1f.

Now suppose that we wish to construct an irreducible complex representation $\tilde{T}$ of F which arises as a lift of an irreducible complex projective representation of G belonging to e. By Clifford's theorem such a $\tilde{T}$ will be induced from an irreducible complex representation $\tilde{T}_1$ of F_1 belonging to f. Since $\tilde{T}_1$ is absolutely irreducible, its linear extension to $\mathbb{C}F_1$ must map $\mathbb{C}F_1$ onto a simple ring A_1^T, which is isomorphic to a full ring of matrices over $\mathbb{C}$. If $B^T \subseteq A_1^T$ denotes the image of $\mathbb{C}F'$ under $\tilde{T}_1$, then $B^T \cong M_n(\mathbb{C})$, where n is the degree of the absolutely irreducible representation of F' corresponding to f. Also, $\tilde{T}_1$ must map Z_1 into $\mathbb{C}$, and since $\mathcal{Z}_1$ is generated over $\mathbb{C}$ by Γ, the restriction of $\tilde{T}_1$ to Z_1 is determined by choosing images $R_i^T, S_j^T, \gamma_{c_l}^T$ in $\mathbb{C}^\times$ for R_i, S_j, γ_{c_l} respectively, $1 \le i, j \le q,\ p+1 \le l \le r$. Since Γ is algebraically independent over $\mathbb{C}$, these choices may be made completely arbitrarily.

Now A_1^T is generated over B^T by $\Theta^T := \{\gamma_{r_1}^T, \gamma_{s_1}^T, \dots, \gamma_{r_q}^T, \gamma_{s_q}^T\}$ (where $\gamma_{r_i}^T$ and $\gamma_{r_i}^T$ denote respectively the images under $\tilde{T}_1$ of γ_{r_i} and γ_{s_i}) as $\mathbb{C}F_1 f$ is generated over $Z_1\langle F'\rangle$ by Θ. Furthermore, the subalgebra C^T of A_1^T generated over Z_1 by Θ^T centralizes B^T, and so $A_1^T \cong B^T \otimes_{\mathbb{C}} C^T$, by Lemma 4.6. Since $\tilde{T}_1$ restricts to an isomorphism of $\mathbb{C}$ with $Z^T := \mathcal{Z}(A_1^T)$, we have in C^T,

$$\gamma_{r_i}^T \gamma_{s_i}^T = \xi_i \gamma_{s_i}^T \gamma_{r_i}^T,\ i = 1, \dots, q,$$

ξ_i being a root of unity of order d_i in $Z^T \cong \mathbb{C}$. If for $i = 1, \dots, q,\ C_i^T$ denotes the subalgebra of C^T generated over Z^T by $\gamma_{r_i}^T$ and $\gamma_{s_i}^T$, then

$$C_i^T = \left(\frac{R_i^T, S_i^T}{\xi_i, Z^T}\right),\ \text{where } R_i^T = \tilde{T}_1(R_i) \in \mathbb{C}^\times,\ S_i^T = \tilde{T}_1(S_i) \in \mathbb{C}^\times.$$

In particular, $C_i^T \cong M_{d_i}(\mathbb{C})$ for $i = 1, \dots, q$, and

$$A_1^T \cong M_n(\mathbb{C}) \otimes_{\mathbb{C}} M_{d_1}(\mathbb{C}) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} M_{d_q}(\mathbb{C}) \cong M_{nd'}(\mathbb{C}),$$

where $d' = d_1 d_2 \dots d_q = \sqrt{[F_1 : F'^+]}$.

Thus $\tilde{T}_1$ is determined, at least up to the choice of a system of matrix units for $A_1^T \cong M_{nd'}(\mathbb{C})$ by the designation of images in $\mathbb{C}^\times$ for the elements of Γ. A complex irreducible projective representation T of G belonging to e may now be obtained by composing $\tilde{T} := \mathrm{Ind}_{F_1}^F \tilde{T}_1$ with a section for G in F. All such irreducible complex representations of G are projectively equivalent by Theorem 3.2, and have degree $d := n\sqrt{[F_1 : F'^+]}[F : F_1]$. In particular, the projective equivalence class of T does not depend on the choices of $R_1^T, S_1^T, \dots, R_q^T, S_q^T$ in $\mathbb{C}^\times$; this is of course due to the fact that all symbol algebras over $\mathbb{C}$ are split as the multiplicative group of $\mathbb{C}$ is divisible.

If the ground field under consideration is not algebraically closed, the choice of elements in $\tilde{T}(\Gamma)$ may be more influential. We now return to the case where $k \subseteq \mathbb{C}$ is an ordinary splitting field for the finite group F'. Then it is easily seen that kFe and $\mathbb{C}Fe$ determine the same subgroups F_1 and F'^+ of F, since this requires only that $f \in kF'$. Irreducible projective k-representations of G belonging to e of course again lift to irreducible ordinary k-representations of F, which are induced from irreducible k-representations of F_1 belonging to f.

If such a representation $\tilde{T}_{k1}$ of F_1 is to be *absolutely* irreducible, its linear extension to kF_1 must map kF_1 onto a full ring of matrices over k, and must, in particular, map $\mathcal{Z}(kF_1 f)$, which is generated over k by Γ, into k. The image $A_1^{T_k}$ of kF_1 under $\tilde{T}_{k1}$ (extended to kF_1) may be described, as above, as a tensor product of symbol algebras over k, as $kF'f \cong M_n(k)$ since k is a splitting field for F':

$$A_1^{T_k} \cong M_n(k) \otimes_k \left(\frac{R_1^{T_k}, S_1^{T_k}}{\xi_1, k} \right) \otimes_k \cdots \otimes_k \left(\frac{R_q^{T_k}, S_q^{T_k}}{\xi_q, k} \right),$$

where $R_1^{T_k}, S_1^{T_k}, \ldots, R_q^{T_k}, S_q^{T_k} \in k^\times \cong \mathcal{Z}(A_1^{T_k})$ denote respectively the images under $\tilde{T}_{k1}$ of $R_1, S_1, \ldots, R_q, S_q$, and ξ_i is a root of unity of order d_i in k. For $i = 1, \ldots, q$ the subalgebra C_i^k of $A_1^{T_k}$ generated over k by the images $\gamma_{r_i}^{T_k}$ and $\gamma_{s_i}^{T_k}$ of γ_{r_i} and γ_{s_i} respectively under $\tilde{T}_{k1}$ is again a symbol algebra:

$$C_i^k \cong \left(\frac{R_i^{T_k}, S_i^{T_k}}{\xi_i, k} \right).$$

In this case, however, the Schur index of C_i^k may depend on the choices of $R_i^{T_k}$ and $S_i^{T_k}$ in $k^\times$. Since Γ is algebraically independent over k, these choices may be made independently and completely arbitrarily; for example, to guarantee that $C_i \cong M_{d_i}(k)$ for $i = 1, \ldots, q$ we may choose $S_i^{T_k} \in (k^\times)^{d_i}$. In this case $A_1^{T_k} \cong M_{nd'}(k)$, and by choosing a system of matrix units in $A_1^{T_k}$ we may produce an absolutely irreducible k-representation $\tilde{T}_{k1} : F_1 \longrightarrow M_{nd'}(k)$ mapping $kF_1 f$ onto $M_{nd'}(k)$. Finally, if $T_k : G \longrightarrow GL(d, k)$ is defined as the composition of $\tilde{T}_k := \operatorname{Ind}_{F_1}^F \tilde{T}_{k1}$ with some section for G in F, then T_k is an absolutely irreducible projective k-representation of G belonging to e. If $T : G \longrightarrow GL(d, \mathbb{C})$ is any (complex) irreducible projective representation of G belonging to e, then by T is projectively equivalent (over $\mathbb{C}$) to T_k by Theorem 3.2, and so T is projectively realizable over k. We have proved the following result.

Theorem 4.15. *Let G be a finite group with generic central extension F, and let $k \subseteq \mathbb{C}$ be an ordinary splitting field for F'. Then k is a projective splitting field for G.*

We remark here that F' is isomorphic to $\hat{G}'$ where $\hat{G}$ is any Schur representation group for G; i.e., any finite central extension for G having the projective lifting property for G over $\mathbb{C}$ and having kernel isomorphic to $M(G)$.

Since $\exp(F')$ divides $\exp(G')\exp(M(G))$, the following is an immediate consequence of Theorem 4.15 and Brauer's theorem on splitting fields.

Corollary 4.16 ([4], Opolka, 1981)**.** *Let G be a finite group and let ξ be a root of unity of order $\exp(G')\exp(M(G))$ in $\mathbb{C}$. Then $\mathbb{Q}(\xi)$ is a projective splitting field for G.*

The following example of a group G with generic central extension F for which $\exp(F') < \exp(G')\exp(M(G))$ indicates that Theorem 4.15 is genuinely stronger than Corollary 4.16.

Example. Let $G = G_1 \times G_2$, where G_1 and G_2 are metacyclic groups of orders 81 and 18 respectively, having the following presentations:

$$G_1 = \langle x, y \mid x^9 = 1, y^9 = 1, y^{-1}xy = x^4 \rangle,$$
$$G_2 = \langle a, b \mid a^9 = 1,\, b^2 = 1,\, b^{-1}ab = a^8 \rangle.$$

Then $G_1' \cong C_3$, $G_2' \cong C_9$, and $G' \cong C_3 \times C_9$. Also $M(G_1) \cong C_3$ and $M(G_2)$ is trivial, whence $M(G) \cong C_3$ since $\gcd(|G_1/G_1'|, |G_2/G_2'|) = 1$. For descriptions of Schur multipliers of metacyclic groups and of direct products; see [3] and [1], respectively.

Let (R, F, ϕ) be a generic central extension for G. We will show that

$$\exp(F') = 9 < \exp(G')\exp(M(G)) = 27.$$

Choose preimages X, Y, A, B in F for the elements x, y, a, b respectively of G under ϕ, and let $c_1 = Y^{-1}XYX^{-1}$ and $c_2 = B^{-1}ABA^{-1}$ in F. Then $F'/F' \cap R \cong G'$ is generated by the elements $\bar{c}_1 = c_1(F' \cap R)$ and $\bar{c}_2 = c_2(F' \cap R)$ having orders 3 and 9 respectively; $F' \cap R \cong M(G)$ has order 3 and is central in F'.

Let $c \in F'$. Then $c = c_1^i c_2^j \alpha$ where $0 \leq i < 3$, $0 \leq j < 9$, and $\alpha \in F' \cap R$. Since G' is abelian and $c_1^3 \in F' \cap R$ we have $c^3 = c_2^{3j}\beta$ for some $\beta \in F' \cap R$. Thus $c^9 = c_2^{9j}$ since $\beta^3 = 1$. We now show that $c_2^9 = 1$.

Certainly c_2 commutes with A since $c_2 = B^{-1}ABA^{-1} \in RA^7$ and $R \subseteq \mathcal{Z}(F)$. Then $c_2 = A^{-1}B^{-1}AB$ and

$$c_2^2 = (A^{-1}B^{-1}AB)(B^{-1}ABA^{-1}) = A^{-2}B^{-1}A^2B.$$

Similarly, $c_2^n = A^{-n}B^{-1}A^nB$ for any positive integer n, and $c_2^9 = 1$ since $A^9 \in R \subseteq \mathcal{Z}(F)$. Thus $c^9 = 1$ for all $c \in F'$ and $\exp(F') = 9$. Then $\mathbb{Q}(\xi_9)$ is a projective splitting field for G by Theorem 4.15, although $9 < \exp(G')\exp(M(G))$.

Acknowledgment

I am grateful to my advisor, M. Shirvani, for many helpful discussions and suggestions, and for his careful reading of the material presented here. Thanks also to the referee for many helpful comments.

References

[1] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin-Heidelberg-New York, 1967. MR **37:**302

[2] N. Jacobson, *The theory of rings*, Amer. Math. Soc., New York, 1943. MR **5:**31f

[3] H.N. Ng, *Faithful irreducible projective representations of metabelian groups*, J. Algebra **38** (1976), 8–28. MR **55:**5732

[4] H. Opolka, *Projective representations of finite groups in cyclotomic fields*, Pacific J. Math. **94** (1981), no. 1, 207–210. MR **83a:**20017

[5] D.S. Passman, *The algebraic structure of group rings*, Wiley, New York, 1977. MR **81d:**16001; Reprint of 1977 original. MR **81j:**16001

[6] D.S. Passman, *Infinite crossed products*, Pure and Applied Mathematics, no. 135, Academic Press, Boston, 1989. MR **90g:**16002

[7] R.S. Pierce, *Associative algebras*, Graduate Texts in Math., 88, Springer-Verlag, New York, 1982. MR **84c:**16001

[8] W.F. Reynolds, *Projective representations of finite groups in cyclotomic fields*, Illinois J. Math **9** (1965), 191–198. MR **30:**4831

[9] J. Tappe, *Irreducible projective representations of finite groups*, Manuscripta Math. **22** (1977), 33–45. MR **57:**6172

[10] K. Yamazaki, *On projective representations and ring extensions of finite groups*, J. Fac. Sci. Univ. Tokyo Sect. 1 **10** (1964), 147–195. MR **31:**4842

Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, Canada, T6G 2G1

Current address: Department of Mathematics, University College Dublin, Dublin, Ireland

E-mail address: `rachel.quinlan@ucd.ie`