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GENERIC EXTENSIONS AND MULTIPLICATIVE BASES OF QUANTUM GROUPS AT $\mathbf{q} = 0$

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ABSTRACT. We show that the operation of taking generic extensions provides the set of isomorphism classes of representations of a quiver of Dynkin type with a monoid structure. Its monoid ring is isomorphic to the specialization at $q = 0$ of Ringel's Hall algebra. This provides the latter algebra with a multiplicatively closed basis. Using a crystal-type basis for a two-parameter quantum group, this multiplicative basis is related to Lusztig's canonical basis.

1. INTRODUCTION

The main motivation for this paper results from the analysis of C. M. Ringel's realization of quantized enveloping algebras as Hall algebras of representations of quivers. More precisely, for Q a quiver of Dynkin type, C. M. Ringel constructs an associative algebra structure on the free $\mathbf{Q}(v)$-vector space generated by the isomorphism classes of representations of Q, with structure constants given by counting exact sequences of representations of Q over finite fields. The resulting algebra is shown to be isomorphic to the positive part of the quantized enveloping algebra of the semisimple Lie algebra of the same type as Q.

Thus, in the analysis of these algebras, one is frequently led to the problem of whether a given module is an extension of two other modules. More generally, one tries to understand the set of extensions of two given modules together with the partial order given by degenerations. In this set, there are at least two distinguished extensions. The first one is the direct sum, which can be regarded as being trivial. The second one is much less known: it is the generic extension, which can be viewed as the 'most complicated' way of twisting the module structure on the direct sum. This special extension 'governs' all other extensions in a certain way. This is explained in section 2, after recalling the construction of the generic extension.

The first part of this paper (sections 2–5) originally grew out of an attempt to find reduction techniques for the calculation of the generic extension, say in terms of its decomposition into indecomposables. An algorithm for this calculation is given in section 3. This attempt leads to the following, rather surprising fact: the operation of taking the generic extension provides the set of all isomorphism classes with a monoid structure.

It is natural to ask whether this structure has some relation to the Hall algebra. We show in section 4 that there are in fact two such relations: the first relation is that the monoid ring of the monoid of generic extensions can be realized as the $q = 0$ specialization of Ringel's Hall algebra. The second relation starts with the

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quantized enveloping algebra itself, being a twisted form of the Hall algebra, and realizes the monoid ring via a simple form of Lusztig's 'J-ring' construction.

Once these two realizations are known, the natural basis of the monoid ring yields a basis of the 'degenerate' quantized enveloping algebra with a rather surprising property: it is closed under multiplication. All known bases for the quantized enveloping algebra itself are far from having this property. Thus, one can ask whether this basis has some relation to Lusztig's canonical basis, which is known to have several remarkable compatibility properties.

To attack this problem, it is neccessary to consider a framework where both the twisted and the untwisted forms of quantized enveloping algebras can be considered at the same time. Such a framework is found by using a generalization of Takeuchi's two-parameter quantization.

Thus, the second part of this paper (sections 6 and 7) develops the theory of canonical bases for these two-parameter quantized enveloping algebras. First, a realization as a two-parameter Hall algebra is given. Then, an analog of Lusztig's canonical basis is constructed. One of the main ingredients of the proofs given in section 6 is an induction technique which makes use of the properties of generic extensions developed in section 2. Since the construction of the Hall algebra model, as well as the construction of the canonical basis, follow closely the one-parameter case, they are presented in a very compact form. The efforts of section 6 are rewarded by the result of section 7, which shows that the multiplicative basis, coming from the monoid structure of generic extensions, is nothing else than the specialization of the two-parameter canonical basis to a degenerate case. This in turn can provide some further insight into the nature of this multiplicative basis, for example, by using Lusztig's realization of the canonical basis in terms of perverse sheaves.

Summing up, we proceed in the following way: starting from a problem in representation theory of quivers (the description of extensions), we develop a geometric tool for this problem (the generic extension). Its analysis leads to the theory of quantized enveloping algebras (the multiplicative basis for the degenerate quantized enveloping algebra). The analysis of this phenomenon leads to a general framework for studying the various twisted forms of these quantum deformations. The geometric methods simplify the proofs in the development of this set-up, and ultimately lead us back to the problem with which we started.

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2. Generic extensions

Let Q be a quiver of finite type, i.e. a disjoint union of oriented Dynkin diagrams. Let Q_0 (resp. Q_1) be its set of vertices (resp. arrows); let $t, h : Q_1 \to Q_0$ be the maps associating to an arrow α its tail and head, respectively. Thus the arrows in Q are of the form $t(\alpha) \xrightarrow{\alpha} h(\alpha)$.

Let mod kQ be the abelian category of finite dimensional k-representations of Q for k an algebraically closed field. For $M \in \text{mod}kQ$, we denote by $\underline{\dim}\, M \in \mathbf{N}^{Q_0}$ its dimension vector, i.e. $(\underline{\dim}\, M)_i = \dim_k M_i$ for all $i \in Q_0$.

The starting point for our construction is the following well-known lemma:

Lemma 2.1. *For k-representations M and N of Q, there exists a unique (up to isomorphism) extension G of M by N, (i.e. a representation G fitting into an exact sequence $0 \to N \to G \to M \to 0$), with minimal dimension of its endomorphism ring.*

We will see below that this lemma is equivalent to the fact that the 'variety of all extensions' admits a (Zariski-) dense orbit corresponding to the representation G. This motivates the following definition:

Definition 2.2. G is called the generic extension of M by N and is denoted by $G =: M * N$.

To recall the proof of the lemma, and to look at generic extensions in more detail we turn to the geometry of representations of quivers (as a general reference for this topic see [B1] or [B2]).

For $d \in \mathbf{N}^{Q_0}$, consider the affine space

$$R(d) = \prod_{\alpha \in Q_1} k^{d_{h(\alpha)} \times d_{t(\alpha)}}$$

parametrizing the k-representations of Q of dimension vector d. In the following, we always identify a point $M = (M_\alpha)_{\alpha \in Q_1}$ in $R(d)$ with the corresponding representation. The group

$$GL(d) = \prod_{i \in Q_0} GL_{d_i}(k)$$

acts on $R(d)$ via

$$(g_i)_i (M_\alpha)_\alpha = (g_{h(\alpha)} M_\alpha g_{t(\alpha)}^{-1})_\alpha.$$

So by definition, the orbits $\mathcal{O}_M$ of $GL(d)$ in $R(d)$ are in bijection with the isomorphism classes $[M]$ of k-representations of Q of dimension vector d. The stabilizer of a point M is precisely its group of kQ-automorphisms $\mathrm{Aut}_Q(M)$, which is open in the vector space of kQ-endomorphisms $\mathrm{End}_Q(M)$. So we see that

$$\dim \mathcal{O}_M = \dim GL(d) - \dim \mathrm{End}_Q(M).$$

We say that a representation M degenerates to another representation N, and write $M \leq N$, if $\mathcal{O}_N$ is contained in the closure of $\mathcal{O}_M$.

For subsets $\mathcal{M} \subset R(d')$ and $\mathcal{N} \subset R(d'')$ stable under $GL(d')$ and $GL(d'')$, respectively, we define $\mathcal{E}(\mathcal{M}, \mathcal{N}) \subset R(d)$ (where $d = d' + d''$) as the subset of all extensions of representations $M \in \mathcal{M}$ by representations $N \in \mathcal{N}$.

Lemma 2.3. *$\mathcal{E} := \mathcal{E}(\mathcal{M}, \mathcal{N})$ is a constructible subset of $R(d)$. Furthermore, if $\mathcal{M}$ and $\mathcal{N}$ are irreducible (resp. closed), then $\mathcal{E}$ is irreducible (resp. closed).*

Proof. Let $\mathcal{Z} = \mathcal{Z}(\mathcal{M}, \mathcal{N})$ be the subset of $R(d)$ consisting of representations of the form

$$\left(\begin{bmatrix} N_\alpha & \zeta_\alpha \\ 0 & M_\alpha \end{bmatrix}\right)_\alpha, \text{ where } (M_\alpha)_\alpha \in \mathcal{M},\ (N_\alpha)_\alpha \in \mathcal{N} \text{ and } \zeta_\alpha \in k^{d''_{h(\alpha)} \times d'_{t(\alpha)}}.$$

By definition, the image of the conjugation map

$$m : GL(d) \times \mathcal{Z} \to R(d), \quad (g, z) \mapsto gz$$

equals $\mathcal{E}$. Since $\mathcal{M}$ and $\mathcal{N}$ are finite unions of orbits, they are constructible subsets; so the same holds for $\mathcal{Z}$ and for $\mathcal{E}$ by definition.

Next, assume that $\mathcal{M}$ and $\mathcal{N}$ are irreducible subvarieties. The canonical projection $\pi : \mathcal{Z} \to \mathcal{M} \times \mathcal{N}$ is a trivial vector bundle over $\mathcal{M} \times \mathcal{N}$, so $\mathcal{Z}$ is irreducible. Thus, $\mathcal{E}$ is irreducible, being the image of the map m.

If $\mathcal{M}$ and $\mathcal{N}$ are closed subvarieties, then the same holds for $\mathcal{Z}$. But $\mathcal{Z}$ is stable under the action of the parabolic

$$P := \prod_{i \in Q_0} \begin{bmatrix} GL_{d''_i} & * \\ 0 & GL_{d'_i} \end{bmatrix} \subset GL(d),$$

so we see that $\mathcal{E}$, the $GL(d)$-saturation of $\mathcal{Z}$, is also a closed subvariety. □

It follows that the constructible set $\mathcal{E}(\mathcal{O}_M, \mathcal{O}_N)$ is irreducible for all representations M and N of Q, so it contains a dense orbit $\mathcal{O}_G$. By the calculation of the orbit dimension above, we see that G is in fact the generic extension of M by N. This proves Lemma 2.1.

We derive the following main result of this section:

Proposition 2.4. *Let M, N be representations of Q. The following properties of a representation X are equivalent:*

a) $M * N \leq X$,

b) *there exist degenerations M', N' of M, N, respectively, as well as an exact sequence $0 \to N' \to X \to M' \to 0$ in* $\mathrm{mod} kQ$.

Proof. We apply Lemma 2.3 to the closed irreducible subvarieties $\mathcal{M} = \overline{\mathcal{O}_M}$ and $\mathcal{N} = \overline{\mathcal{O}_N}$. So we see that $\mathcal{E}(\mathcal{M}, \mathcal{N})$ is a closed irreducible subvariety. Since $\mathcal{O}_M \times \mathcal{O}_N$ is open, thus dense in $\mathcal{M} \times \mathcal{N}$, we see that $\mathcal{Z}(\mathcal{O}_M, \mathcal{O}_N)$ is open, thus dense in $\mathcal{Z}(\mathcal{M}, \mathcal{N})$. We infer that $\mathcal{E}(\mathcal{M}, \mathcal{N})$ contains $\mathcal{E}(\mathcal{O}_M, \mathcal{O}_N)$ as a dense subset, which in turn contains $\mathcal{O}_{M*N}$ as a dense subset. Since $\mathcal{E}(\mathcal{M}, \mathcal{N})$ is closed, we get $\mathcal{E}(\mathcal{M}, \mathcal{N}) = \overline{\mathcal{O}_{M*N}}$, and we are done. □

Remarks. a) The existence of generic extensions holds more generally for the class of representation-finite algebras (see [B1]). But the preceding proposition is only valid for path algebras of Dynkin quivers, as the following example shows:

Let A be the path algebra of the quiver $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ subject to the relation $\beta\alpha = 0$. Denote by E_i (resp. P_i) the simple (resp. indecomposable projective) representation corresponding to the vertex i of Q. Let $M = E_1 = M'$, $N = P_2 \leq E_2 \oplus E_3 = N'$ and $X = E_3 \oplus P_1$. Then X is an extension of M' by N', but $M * N = P_2 \oplus E_1$ does not degenerate to X.

b) Proposition 2.4 gives a neccessary condition for a representation X to be an extension of representations M and N:

If there exists an exact sequence $0 \to N \to X \to M \to 0$, then $M * N \leq X \leq M \oplus N$.

The converse is not true, as can be seen from the following example for the quiver $Q = 1 \to 2 \to 3 \to 4$:

Denote by E_{ij} the indecomposable representation of Q supported on the vertices $i, i+1, \ldots, j$ for $1 \leq i \leq j \leq 4$. Set

$$M = E_{33} \oplus E_{12},\ N = E_{34} \oplus E_{24},\ X = E_{34} \oplus E_{14} \oplus E_{23}.$$

Then $M * N = E_{24} \oplus E_{14} \oplus E_{33} \leq X \leq M \oplus N$, but there is no exact sequence $0 \to N \to X \to M \to 0$.

3. The monoid of generic extensions

The above proposition can be interpreted as follows: the generic extension of two representations 'governs' the extensions of all their degenerations. Thus, it is desirable to have a method to calculate the generic extension explicitly. In this section, we will derive an algorithm for this calculation. The following statement serves as a key lemma.

Lemma 3.1. *The operation of taking the generic extension is associative, i.e.*

$$(M * N) * P = M * (N * P)$$

for all representations M, N, P *in* $\mathrm{mod}kQ$.

Proof. The representation $(M * N) * P$ fits into a diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & N & & \\
 & & & & & & \downarrow & & \\
0 & \to & P & \to & (M * N) * P & \to & M * N & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & M & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0. & &
\end{array}$$

Taking the pullback provides us with sequences:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \to & P & \to & Y & \to & N & \to & 0 \\
 & & & & \downarrow & & & & \\
 & & & & (M * N) * P & & & & \\
 & & & & \downarrow & & & & \\
 & & & & M & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0. & & & &
\end{array}$$

By definition, Y satisfies $N * P \leq Y$. We apply Proposition 2.4 and get

$$M * (N * P) \leq (M * N) * P.$$

Dually, by considering pushout sequences arising from the definition of $M*(N* P)$, we see that $(M * N) * P \leq M * (N * P)$. But since degenerating is a partial order, we are done. ∎

This lemma allows us to define the following structure on isoclasses of representations of Q:

Definition 3.2. Let $\mathcal{M} = \mathcal{M}(Q)$ be the monoid having as elements the isomorphism classes in $\mathrm{mod}kQ$, multiplication $*$ and unit element $1_{\mathcal{M}} = [0]$, the isoclass of the zero representation. $\mathcal{M}$ is called the monoid of generic extensions.

Remark. The associativity does not generalize to other representation-finite algebras, as the example at the end of the preceding section shows: with the same notations, we have $([E_1]*[E_2])*[E_3] = [E_3 \oplus P_1]$, but $[E_1]*([E_2]*[E_3]) = [P_2 \oplus E_1]$. Nevertheless, it is possible to define an analogue of the monoid of generic extensions for a representation-finite algebra A by taking the free monoid in the isoclasses in $\mathrm{mod}\, A$ and dividing by the relations $[M] \cdot [N] = [M * N]$. However, it is not clear how much of the structure of A is encoded in this object.

Now we analyze the structure of $\mathcal{M}$; our aim is a presentation by generators and relations.

Proposition 3.3. *$\mathcal{M}$ is generated by the isomorphism classes of simple representations E_i corresponding to the vertices i of Q.*

Proof. Since modkQ is representation-directed, there exists $X_1, \ldots, X_\nu$, an enumeration of the isoclasses of indecomposable representations of Q such that

$$\mathrm{Hom}(X_i, X_j) = 0 \text{ for } i > j \text{ and } \mathrm{Ext}^1(X_i, X_j) = 0 \text{ for } i \leq j.$$

Since an arbitrary representation M can be written as $M = \bigoplus_{i=1}^{\nu} X_i^{m_i}$, we get the following identity in the monoid $\mathcal{M}$:

$$[M] = [X_1]^{*m_1} * \ldots * [X_\nu]^{*m_\nu}.$$

So we are reduced to the problem of generating indecomposable representations X by simple ones in $\mathcal{M}$. But X fits into some exact sequence

$$0 \to M \to X \to E_i^s \to 0,$$

which is neccessarily generic since X has no proper self-extensions, which implies that the orbit $\mathcal{O}_X$ is dense in $R(\underline{\dim}\, X)$. All indecomposable direct summands of M have smaller dimension than X, so by induction on the dimension we are done. □

Lemma 3.4. *The following relations hold in $\mathcal{M}$:*

$$[E_i] * [E_j] = [E_j] * [E_i] \quad \text{if } i, j \text{ are not connected by an arrow,}$$

$$\left.\begin{array}{r} [E_i] * [E_j] * [E_i] = [E_i] * [E_i] * [E_j] \\ [E_j] * [E_i] * [E_j] = [E_i] * [E_j] * [E_j] \end{array}\right\} \quad \text{if there exists an arrow from } i \text{ to } j.$$

Proof. If i and j are not connected by an arrow, then $\mathrm{Ext}^1(E_i, E_j) = 0 = \mathrm{Ext}^1(E_j, E_i)$, so the only possible extension of these simples is $E_i \oplus E_j$.

If there is an arrow from i to j, then $\mathrm{Ext}^1(E_j, E_i) = 0$, and $\mathrm{Ext}^1(E_i, E_j)$ is one-dimensional. The only non-trivial extension of E_i by E_j is a two-dimensional indecomposable representation, which has no non-trivial extensions with E_i or E_j. The relations in $\mathcal{M}$ follow. □

At this point, an obvious question arises: do these relations suffice to present $\mathcal{M}$? In the following section we show that indeed they do.

But first we derive the above mentioned algorithm for computing generic extensions:

Suppose the representations M and N are given by the multiplicities of their indecomposable direct summands:

$$M = \bigoplus_{i=1}^{\nu} X_i^{m_i}, \quad N = \bigoplus_{i=1}^{\nu} X_i^{n_i}.$$

As in the proof of Proposition 3.3, $X_1, \ldots, X_\nu$ is a complete directed enumeration. Then, computing $[M] * [N] = [\bigoplus_{i=1}^{\nu} X_i^{x_i}]$ amounts to 'straightening' the word

$$[X_1]^{*m_1} * \ldots * [X_\nu]^{*m_\nu} * [X_1]^{*n_1} * \ldots * [X_\nu]^{*n_\nu}$$

to the form $[X_1]^{*x_1} * \ldots * [X_\nu]^{*x_\nu}$ in the monoid $\mathcal{M}$. So we are reduced to computing generic extensions between indecomposables. But by [B1], for indecomposables M and N such that $\mathrm{Ext}^1(M,N) \neq 0$ there exists an extension X of M by N without selfextensions, i.e. having an open orbit in the variety $R(d)$. Thus, $[M]*[N] = [X]$. With the aid of the Auslander-Reiten quiver of kQ, the decomposition of X into indecomposables can be computed explicitly. In particular, if $M = X_j$, $N = X_i$ with $i < j$, then all indecomposable direct summands X_k of $[M] * [N]$ satisfy $i \leq j \leq k$.

Thus, we see that successively replacing terms of the form $[X_j] * [X_i]$ (where $i < j$) by the element of $\mathcal{M}$ corresponding to the decomposition of $[X_j] * [X_i]$ into indecomposables reduces any word in $\mathcal{M}$ to the normal form given above.

4. Two realizations of $\mathbf{Q}\mathcal{M}$ in terms of quantum groups

Definition 4.1. Let U_q^+ be the $\mathbf{Q}[q]$-algebra with generators $E_i : i \in Q_0$ and relations

$$E_iE_j = E_jE_i \quad \text{if } i,\, j \text{ are not connected by an arrow,}$$

$$\left.\begin{aligned} E_i^2E_j - (q+1)E_iE_jE_i + qE_jE_i^2 = 0, \\ E_iE_j^2 - (q+1)E_jE_iE_j + qE_j^2E_i = 0 \end{aligned}\right\} \quad \text{if there exists an arrow from } i \text{ to } j.$$

This algebra is a twisted form of the positive part of the quantized enveloping algebra. For type A it is studied in [KT] and can be viewed as the specialization $U_{q,1}$ of Takeuchi's two-parameter quantization ([T]) of sl_n. We will come back to this two-parameter quantum group in section 6.

Let $H_q(Q)$ be the (generic, untwisted) Hall algebra associated by Ringel to the quiver Q (see [Ri1]). This is the $\mathbf{Q}[q]$-algebra with representatives $u_{[M]}$ of the isomorphism classes of representations of Q as a basis and multiplication

$$u_{[M]}u_{[N]} = \sum_{[X]} F_{M,N}^X(q)u_{[X]},$$

where $F_{M,N}^X(q)$ denotes the Hall polynomial counting subrepresentations of X isomorphic to N with quotient isomorphic to M over finite fields $\mathbf{F}_q$.

We can now describe our first realization of $\mathbf{Q}\mathcal{M}$:

Theorem 4.2. *The following algebras are isomorphic:*

a) *the monoid ring $\mathbf{Q}\mathcal{M}$ of $\mathcal{M}$,*
b) *the $\mathbf{Q}$-algebra with generators $i : i \in Q_0$ and relations*

$$\text{(I)} \quad ij = ji \qquad \textit{if } i \textit{ and } j \textit{ are not connected by an arrow,}$$

$$\text{(II)} \quad \left.\begin{aligned} iji = iij, \\ jij = ijj \end{aligned}\right\} \textit{ if there exists an arrow from } i \textit{ to } j,$$

c) *the specialization U_0^+ of U_q^+ to $q = 0$,*
d) *the specialization $H_0(Q)$ of $H_q(Q)$ to $q = 0$.*

Proof. By Ringel's Theorem ([Ri1]), the map $E_i \mapsto u_{[E_i]}$ induces an isomorphism $U_q^+ \xrightarrow{\sim} H_q(Q)$ of $\mathbf{Q}[q]$-algebras. Since specialization is compatible with presentations by generators and relations ([J], 5.20), U_0^+ is isomorphic to the $\mathbf{Q}$-algebra with generators $\{i : i \in Q_0\}$ and relations (I), (II) of the statement of the Theorem.

But by Lemma 3.4, this algebra has $\mathbf{Q}\mathcal{M}$ as a quotient. Thus, we get the following maps:

$$H_0(Q) \quad \overset{\sim}{\leftarrow} \quad U_0^+ \quad \simeq \quad \mathbf{Q}\langle i : i \in Q_0\rangle/((a),(b)) \quad \twoheadrightarrow \quad \mathbf{Q}\mathcal{M}.$$

The left- and rightmost algebras are $\mathbf{N}^{Q_0}$-graded by the dimension vector. Their homogeneous parts obviously have the same dimension, namely the number of isomorphism classes of representations of a fixed dimension vector. Thus, we see that they are isomorphic as $\mathbf{Q}$-algebras. □

The second realization of $\mathbf{Q}\mathcal{M}$ starts with the Drinfel'd-Jimbo quantum group and recovers the monoid ring by applying a simple form of the construction used by Lusztig in [L2]. We start with the general construction.

Lemma 4.3. *Let R be an associative algebra with 1 over the field $\mathbf{Q}(v)$. Suppose R has a basis $B = \{b_i : i \in I\}$ such that the structure constants of R with respect to B, defined by*

$$b_i b_j = \sum_k \alpha_{ij}^k b_k,$$

belong to $\mathbf{Q}[v, v^{-1}]$. Suppose further that there exists a function $f : I \times I \times I \to \mathbf{N}$ such that

$$\deg \alpha_{ij}^k \leq f(i,j,k) \text{ for all } i,j,k \in I,$$

where the degree of the zero Laurent polynomial is defined as $-\infty$.

If f satisfies the following conditions for all $i,j,k,m,p,q \in I$:

a) *$f(i,j,p) + f(p,k,m)$ does not depend on p,*
b) *$f(i,q,m) + f(j,k,q)$ does not depend on q,*
c) *the two terms above are equal,*

then the rule

$$b_i \tilde{\cdot} b_j := \sum_k (\text{coefficient of } v^{f(i,j,k)} \text{ in } \alpha_{ij}^k) b_k$$

defines an associative algebra structure with 1 on

$$J(R) = J(R, \{b_i\}_i, f) := \bigoplus_{i \in I} \mathbf{Q} b_i.$$

Proof. By definition, $v^{-f(i,j,k)}\alpha_{ij}^k$ belongs to $\mathbf{Q}[q^{-1}]$ for all $i,j,k \in I$, so the ring homomorphism given by evaluation at ∞ is well-defined. Obviously, $(v^{-f(i,j,k)}\alpha_{ij}^k)(\infty)$ equals the coefficient of $v^{f(i,j,k)}$ in α_{ij}^k. Using the above conditions for f and the associativity of R, the associativity of $J(R)$ follows by direct calculation. □

Remark. In [L2], Lusztig applies a similar construction to the Hecke algebra. But in that situation, the function $f(i,j,k)$ only depends on k.

We apply this lemma to the $\mathbf{Q}(v)$-algebra $\mathcal{U}_v^+$ with generators $E_i : i \in Q_0$ and relations

$$E_iE_j = E_jE_i \quad \text{if } i, j \text{ are not connected by an arrow,}$$

$$\left.\begin{array}{l} E_i^2E_j - (v+v^{-1})E_iE_jE_i + E_jE_i^2 = 0, \\ E_iE_j^2 - (v+v^{-1})E_jE_iE_j + E_j^2E_i = 0 \end{array}\right\} \text{if there is an arrow from } i \text{ to } j.$$

By C. M. Ringel's Theorem ([Ri1]), this algebra is isomorphic to a twisted version of the Hall algebra defined above in the following sense:

There exists a basis of $\mathcal{U}_v^+$, consisting of elements $E_{[M]}$ parametrized by the isoclasses $[M]$ of representations in modkQ, such that the multiplication in $\mathcal{U}_v^+$ is given on this basis by

$$E_{[M]} \cdot E_{[N]} = \sum_X \underbrace{v^{[M,M]+[N,N]+R(M,N)-[X,X]} F^X_{M,N}(v^2)}_{=:c^X_{M,N}} E_{[X]}.$$

(Here and in the sequel, we use the short notations $[M,N] = \dim \mathrm{Hom}(M,N)$, $[M,N]^1 = \dim \mathrm{Ext}^1(M,N)$ and $R(M,N) = [M,N] - [M,N]^1$.)

We define a function f by

$$f(M,N,X) = [X,X] - [M,M] - [N,N] - R(M,N),$$

if X is an extension of M by N, and $f(M,N,X) = -\infty$ otherwise.

Proposition 4.4. *With the notations above, we have $J(\mathcal{U}_v^+) \simeq \mathbf{Q}\mathcal{M}$.*

Proof. The conditions a), b) and c) for the function f are easily checked by a direct computation, using the biadditivity of $R(\_,\_)$ on exact sequences. We now have to verify the estimate $\deg c^X_{M,N} \leq f(M,N,X)$.

To do this, we use a formula of C. Riedtmann ([Rie]) for the Hall polynomial:

$$F^X_{M,N}(q) = q^{-[M,N]} \frac{a_X(q)}{a_M(q) a_N(q)} \# \mathrm{Ext}^1(M,N)_X(q),$$

where $a_M(q)$ denotes the number of automorphisms of M as a representation over $\mathbf{F}_q$, and $\#\mathrm{Ext}^1(M,N)_X(q)$ equals the number of $\mathbf{F}_q$-rational points of the constructible subset $\mathrm{Ext}^1(M,N)_X$ of extension classes in $\mathrm{Ext}^1(M,N)$ with middle term isomorphic to X. This last number is obviously bound by the number of elements of the $\mathbf{F}_q$-vector space $\mathrm{Ext}^1(M,N)$, i.e. by $q^{[M,N]^1}$. Combined with the fact that $a_M(q)$ is a polynomial in q of degree $[M,M]$, this implies

$$\begin{aligned} \deg F^X_{M,N}(q) &\leq -[M,N] + [X,X] - [M,M] - [N,N] + [M,N]^1 \\ &= [X,X] - [M,M] - [N,N] - R(M,N). \end{aligned}$$

The estimate for $\deg c^X_{M,N}$ follows by definition.

Thus, we can apply the previous lemma and form the ring $J(\mathcal{U}_v^+)$. The product of two elements $E_{[M]}$, $E_{[N]}$ in $J(\mathcal{U}_v^+)$ is given by

$$E_{[M]} \tilde{\cdot} E_{[N]} = \sum_{[X]} (\text{coefficient of } v^{f(M,N,X)} \text{ in } c^X_{M,N}) \cdot E_{[X]}.$$

Using the above, it remains first to decide for which X the equality

$$\deg F^X_{M,N}(q) = -[M,N] + [X,X] - [M,M] - [N,N] + [M,N]^1$$

holds, and second to determine the leading coefficient of $F^X_{M,N}(v^2)$ in these cases. The equality obviously holds if and only if $\#\mathrm{Ext}^1(M,N)_X(q)$ is a polynomial of degree $[M,N]^1$, which in turn is equivalent to the density of the constructible subset $\mathrm{Ext}^1(M,N)_X$ in the affine space $\mathrm{Ext}^1(M,N)$. But this holds exactly if $X = M * N$. In this case the leading coefficient of $\#\mathrm{Ext}^1(M,N)_X(q)$ is obviously equal to 1. Together with the fact that the polynomials $a_M(q)$ have leading coefficient 1, this implies the formula

$$E_{[M]} \tilde{\cdot} E_{[N]} = E_{[M*N]},$$

which proves the claimed isomorphism. ∎

5. The multiplicative basis of $H_0(Q)$

Using the isomorphism $\phi : \mathbf{Q}\mathcal{M} \xrightarrow{\sim} H_0(Q)$ constructed above, we can define a basis $\mathcal{A}$ of $H_0(Q)$ consisting of the elements

$$A_{[M]} := \phi([M])$$

for all isomorphism classes $[M]$. It is immediately clear, but nevertheless rather surprising, that this basis has the following property:

Corollary 5.1. $A_{[M]}A_{[N]} = A_{[M*N]}$ *for all representations* M, N.

Such a multiplicative basis consisting of homogeneous elements is obviously unique (up to scalars). This can be seen by induction on $\mathbf{N}^{Q_0}$ starting with the fact that E_i is the unique representation of dimension vector e_i.

In analogy with Lusztig's canonical basis ([L1]), we have the following property of the base change from the basis $\mathcal{A}$ to the natural basis of the Hall algebra at $q = 0$:

Lemma 5.2. *Writing* $A_{[M]} = \sum_{[N]} a_N^M u_{[N]}$, *we have*

a) $a_N^M \neq 0$ *only if* $M \leq N$,
b) $a_M^M = 1$.

Proof. First we reduce to the case of M being an exponent of an indecomposable:

Suppose the above statement holds for representations M and N satisfying $\mathrm{Ext}^1(M,N) = 0$ and $\mathrm{Hom}(N,M) = 0$. Then $M * N = M \oplus N$ and

$$\begin{aligned} A_{[M\oplus N]} = A_{[M]}A_{[N]} &= \sum_{M\leq M'} \sum_{N\leq N'} a_{M'}^M a_{N'}^N u_{[M']}u_{[N']} \\ &= \sum_{[X]} \Big(\sum_{\substack{M\leq M' \\ N\leq N'}} a_{M'}^M a_{N'}^N F_{M',N'}^X(0) \Big) u_{[X]}. \end{aligned}$$

So a representation X appears on the right-hand side with non-zero coefficient only if there exists an exact sequence

$$0 \to N' \to X \to M' \to 0,$$

where $M \leq M'$ and $N \leq N'$. But by Proposition 2.4, this is equivalent to $M * N \leq X$.

To compute the coefficient $a_{M\oplus N}^{M\oplus N}$, we have to study exact sequences of the form

$$0 \to N' \to M \oplus N \to M' \to 0.$$

Applying Proposition 2.4 twice we get

$$M \oplus N = M * N \leq M' * N \leq M' * N' \leq M \oplus N,$$

so $M' * N = M \oplus N$, yielding an exact sequence

$$0 \to N \to M \oplus N \to M' \to 0.$$

Since $\mathrm{Hom}(N,M) = 0$, we have $M' \simeq M$; analogously we get $N' \simeq N$. So we have

$$a_{M\oplus N}^{M\oplus N} = a_M^M a_N^N F_{M,N}^{M\oplus N}(0) = 1$$

since $F_{M,N}^{M\oplus N}(q)$ is easily seen to equal 1.

By the existence of complete directed enumerations of indecomposables, we are reduced to the case of an exponent M^a of an indecomposable representation M.

If M is simple, there is nothing to prove, since E_i^a is the unique representation of dimension vector ae_i.

Otherwise, let d be the dimension vector of M and choose an enumeration $1, \dots, n$ of Q_0 such that $i \to j$ implies $i < j$ (which is possible since Q contains no oriented cycles). Then by [Ri2], we have in $H_q(Q)$ and then also in $H_0(Q)$:

$$u_{[E_1^{d_1}]} \cdots u_{[E_n^{d_n}]} = \sum_{M' : \underline{\dim}\, M' = d} u_{[M']},$$

thus

$$A_{[M]} = A_{[E_1^{d_1}]} \cdots A_{[E_n^{d_n}]} = \sum_{M' : \underline{\dim}\, M' = d} u_{[M']}$$

in $H_0(Q)$. □

Corollary 5.3. *The following holds for the coefficients* a_N^M*:*

a) *If* $\mathrm{Hom}(N, M) = 0 = \mathrm{Ext}^1(M, N)$*, then*

$$a_X^{M \oplus M} = \sum_{\substack{M \leq M' \\ N \leq N'}} a_{M'}^M a_{N'}^N F_{M', N'}^X(0).$$

b) *If* $M = X^n$ *for an indecomposable* X*, then* $a_N^M = 1$ *for all representations* N *of the same dimension vector as* M*.*

Proof. This follows directly from the proof of the above lemma. □

The importance of this corollary lies in the following:

In the next section, we introduce a two-parameter quantization of the enveloping algebra which specializes to $\mathbf{Q}\mathcal{M}$. Then, we construct a canonical basis for this algebra in the sense of [L1]. We show that, specializing to $\mathbf{Q}\mathcal{M}$, the base change coefficients from this canonical basis to the basis of PBW-type satisfy the same recursive relations as the coefficients a_N^M, proving their equality.

6. The two-parameter quantization

Generalizing Takeushi's construction of a two-parameter quantization of $\mathcal{U}(\mathfrak{gl}_n)$, we associate to the quiver Q a two-parameter quantization of the positive part of the corresponding enveloping algebra.

Definition 6.1. $\mathcal{U}_{\alpha,\beta}^+$ is defined as the $\mathbf{Q}(\alpha, \beta)$-algebra with generators $e_i : i \in Q_0$ and relations

$$e_i e_j = e_j e_i$$

if i and j are not connected by an arrow,

$$e_i^2 e_j - (\alpha + \beta^{-1}) e_i e_j e_i + \alpha\beta^{-1} e_j e_i^2 = 0,$$

$$e_i e_j^2 - (\alpha + \beta^{-1}) e_j e_i e_j + \alpha\beta^{-1} e_j^2 e_i = 0$$

if there is an arrow from i to j.

$\mathcal{U}_{\alpha,\beta}^+$ carries a $\mathbf{Q}$-linear involution defined by

$$\overline{\alpha} = \beta^{-1},\ \overline{\beta} = \alpha^{-1},\ \overline{e_i} = e_i \text{ for all } i \in Q_0.$$

For $N \in \mathbf{N}$, define $[N] = \frac{\alpha^N - \beta^{-N}}{\alpha - \beta^{-1}}$. Define $[N]!$ and $\begin{bmatrix} M+N \\ N \end{bmatrix}$ as usual. All these elements belong to $\mathbf{Z}[\alpha, \beta, \beta^{-1}]$. We define divided powers in $\mathcal{U}^+_{\alpha,\beta}$ by $E_i^{(n)} := ([n]!)^{-1} E_i^n$.

Now we define a Hall algebra model for this quantization:

Let $H(Q)$ be the $\mathbf{Q}(\alpha, \beta)$-vector space with basis B consisting of elements $E_{[M]}$ indexed by the isoclasses $[M]$; define a multiplication on $H(Q)$ by

$$E_{[M]} \cdot E_{[N]} = \sum_{[X]} \underbrace{\beta^{[M,M]+[N,N]+R(M,N)-[X,X]} F^X_{M,N}(\alpha\beta)}_{=:c^X_{M,N}} E_{[X]}.$$

The following statements can be proved exactly in the same way as in the one-parameter case in ([Ri1], [Ri2]):

Lemma 6.2.

a) *The above multiplication defines a structure of an associative $\mathbf{Q}(\alpha, \beta)$-algebra with 1 on $H(Q)$.*

b) *If X is indecomposable, then $(E_{[X]})^n = [n]! E_{[X^n]}$.*

c) *If $[N, M] = 0 = [M, N]^1$, then $E_{[M]} E_{[N]} = E_{[M \oplus N]}$.*

d) *If Q_0 is indexed such that $i \to j$ implies $i < j$, then*

$$E^{(d_1)}_{[E_1]} \dots E^{(d_n)}_{[E_n]} = \sum_{[M]: \underline{\dim}\, M = d} \beta^{-[M,M]^1} E_{[M]}.$$

e) *The following relations hold in $H(Q)$:*

$$E_{[E_i]} E_{[E_j]} = E_{[E_j]} E_{[E_i]} \text{ if } i \text{ and } j \text{ are not connected by an arrow,}$$

$$E^2_{[E_i]} E_{[E_j]} - (\alpha + \beta^{-1}) E_{[E_i]} E_{[E_j]} E_{[E_i]} + \alpha\beta^{-1} E_{[E_j]} E^2_{[E_i]} = 0,$$
$$E_{[E_i]} E^2_{[E_j]} - (\alpha + \beta^{-1}) E_{[E_j]} E_{[E_i]} E_{[E_j]} + \alpha\beta^{-1} E^2_{[E_j]} E_{[E_i]} = 0$$

if there is an arrow from i to j.

Proposition 6.3. *The map $\eta : e_i \mapsto E_{[E_i]}$ extends to a $\mathbf{Q}(\alpha, \beta)$-algebra isomorphism*

$$\eta : \mathcal{U}^+_{\alpha,\beta} \xrightarrow{\sim} H(Q).$$

Proof. The map η extends to a map of $\mathbf{Q}(\alpha, \beta)$-algebras, since the defining relations of $\mathcal{U}^+_{\alpha,\beta}$ also hold for the elements $E_{[E_i]}$ by Lemma 6.2 e).

We show that η is surjective by showing that $H(Q)$ is generated by the $E_{[E_i]}$. Given an element $E_{[X]}$ of $H(Q)$, we write X as $\bigoplus_{k=1}^{\nu} X_k^{x_k}$ for $X_1, \dots, X_\nu$ a complete directed enumeration of the indecomposables. Using Lemma 6.2 b) and c), we have

$$E_{[X]} = \prod_{k=1}^{\nu} ([x_k]!)^{-1} \cdot E_{[X_1^{x_1}]} \dots E_{[X_\nu^{x_\nu}]}$$

in $H(Q)$. Thus, we only have to prove that elements $E_{[X]}$ for indecomposable X can be generated by the $E_{[E_i]}$. Since such a representation X fulfills $[X, X]^1 = 0$, we can apply Lemma 6.2 d) to get

$$E_{[X]} = E^{(d_1)}_{[E_1]} \dots E^{(d_n)}_{[E_n]} - \sum_{\substack{[M] \neq [X]: \\ \underline{\dim}\, M = d}} \beta^{-[M,M]^1} E_{[M]}.$$

But a module M appearing in this sum is a direct sum of indecomposables of dimension strictly smaller than the dimension of X. Thus, an induction on the dimension of X reduces to the case $\dim X = 1$, which already means $X = E_i$ for some $i \in Q_0$. Thus, $H(Q)$ is generated by the elements $E_{[E_i]}$.

The algebra $\mathcal{U}^+_{\alpha,\beta}$ is $\mathbf{N}^{Q_0}$-graded by defining the degree of the generator e_i to be the i-th coordinate function (note that the defining relations are then homogeneous), and $H(Q)$ is $\mathbf{N}^{Q_0}$-graded by the dimension vector. Obviously, the map η is compatible with the grading. We claim that the homogeneous parts $(\mathcal{U}^+_{\alpha,\beta})_d$, $H(Q)_d$ have the same dimension for all $d \in \mathbf{N}^{Q_0}$; from this it follows that η is an isomorhpism.

We consider the local ring $B = \mathbf{Q}[\alpha,\beta]_{(\alpha,\beta-1)}$ with residue field $\mathbf{Q}$ and field of fractions $\mathbf{Q}(\alpha,\beta)$. We define an auxiliary algebra $\mathcal{U}^+_B$ as the free B-algebra with generators e_i for $i \in Q_0$ and the same relations as for $\mathcal{U}^+_{\alpha,\beta}$. Again, we have a natural $\mathbf{N}^{Q_0}$-grading on $\mathcal{U}^+_B$. The specialization $\mathbf{Q} \otimes_B \mathcal{U}^+_B$ is isomorphic to $\mathbf{Q}\mathcal{M}$ by Theorem 4.2, and $\mathbf{Q}(\alpha,\beta) \otimes_B \mathcal{U}^+_B \simeq \mathcal{U}^+_{\alpha,\beta}$. Using Nakayama's lemma, we get the following estimate for all $d \in \mathbf{N}^{Q_0}$:

$$\begin{aligned}\dim_{\mathbf{Q}}(\mathbf{Q}\mathcal{M})_d &= \dim_{\mathbf{Q}}(\mathbf{Q} \otimes_B \mathcal{U}^+_B)_d \geq \dim_{\mathbf{Q}(\alpha,\beta)}(\mathbf{Q}(\alpha,\beta) \otimes_B \mathcal{U}^+_B)_d \\ &= \dim_{\mathbf{Q}(\alpha,\beta)}(\mathcal{U}^+_{\alpha,\beta})_d \geq \dim_{\mathbf{Q}(\alpha,\beta)} H(Q)_d.\end{aligned}$$

But the right- and leftmost terms are equal, since they are precisely the number of isoclasses of representations of Q of dimension vector d. So we have equality in all steps. □

The construction of a canonical basis for $\mathcal{U}^+_{\alpha,\beta}$ proceeds along the lines of the one-parameter case (see [L1]). We will give a short proof using the methods of section 2.

Proposition 6.4. *Writing* $\overline{E_{[X]}} = \sum_{[Y]} \omega^X_Y E_{[Y]}$, *we have*

a) $\omega^X_Y = 0$ *unless* $X \leq Y$, *and* $\omega^X_X = 1$,
b) *if* $X = M \oplus N$ *for representations* M, N *satisfying* $[N,M] = 0 = [M,N]^1$, *then*

$$\omega^X_Y = \sum_{\substack{M \leq M' \\ N \leq N'}} \omega^M_{M'} \omega^N_{N'} c^X_{M'N'},$$

c) *if* X *is an exponent of an indecomposable, then*

$$\omega^X_Y = \beta^{-[Y,Y]^1} - \sum_{X < Z \leq Y} \alpha^{[Z,Z]^1} \omega^Z_Y,$$

d) $\omega^X_Y \in \beta^{[X,X]-[Y,Y]}\mathbf{Z}[\alpha\beta]$.

Proof. We prove all the statements by induction on the dimension X. If this dimension equals 1, there is nothing to prove, since then, X is a simple representation E_i.

In general, we take a complete directed enumeration $X_1, \dots, X_\nu$ and write

$$X = \bigoplus_{k=1}^{\nu} X_k^{x_k}.$$

We distinguish two cases.

If there is more than one x_k which is non-zero, then we can write $X = M \oplus N$ for representations M, N satisfying $[N, M] = 0 = [M, N]^1$; for example, let k_0 be the minimal index such that $x_{k_0} \neq 0$ and define

$$M = \bigoplus_{k \leq k_0} X_k^{x_k}, \quad N = \bigoplus_{k > k_0} X_k^{x_k}.$$

Using Lemma 6.2 b) and the inductive assumption, we have

$$\begin{aligned} \overline{E_{[X]}} &= \overline{E_{[M]}} \cdot \overline{E_{[N]}} \\ &= \sum_{M \leq M'} \omega_{M'}^M E_{[M']} \cdot \sum_{N \leq N'} \omega_{N'}^N E_{[N']} \\ &= \sum_Y (\sum_{\substack{M \leq M' \\ N \leq N'}} \omega_{M'}^M \omega_{N'}^N c_{M',N'}^Y) E_{[Y]}. \end{aligned}$$

But on the other hand, $\overline{E_{[X]}}$ equals $\sum_Y \omega_Y^X E_{[Y]}$, thus we have proved part b).

Now if ω_Y^X is non-zero, there exists an exact sequence

$$0 \to N' \to Y \to M' \to 0$$

for degenerations M', N' of M, N, respectively. Thus, by Proposition 2.4, Y is a degeneration of $M * N = M \oplus N = X$, proving the first half of part a).

To prove the second half, we compute the right-hand side of the equation in part b) in exactly the same way as in the proof of Lemma 5.2. Since the statement holds for M and N by induction, it holds for X as well.

Turning to part d), we consider the 'normalized' coefficient

$$\widetilde{\omega}_Y^X := \beta^{[Y,Y]-[X,X]} \omega_Y^X$$

and describe it using part b). A short calculation using the definition of $c_{M',N'}^Y$ shows:

$$\begin{aligned} \widetilde{\omega}_Y^X &= \sum_{\substack{M \leq M' \\ N \leq N'}} \widetilde{\omega}_{M'}^M \widetilde{\omega}_{N'}^N \beta^{[X,X]-[M,N]^1-[M',M']-[N',N']} c_{M',N'}^Y \\ &= \widetilde{\omega}_{M'}^M \widetilde{\omega}_{N'}^N F_{M',N'}^Y(\alpha\beta). \end{aligned}$$

Part d) follows by induction.

As the second case, assume that there is precisely one x_k which is non-zero. Then $[X, X]^1 = 0$, and the orbit of X is dense in its variety $R(d)$ of representations. So the degenerations of X are the representations Z of the same dimension vector d which are not isomorphic to Z. Each such representation has to be a direct sum of representations of dimensions strictly smaller than $\dim X$. Thus, by our proof of the first case, we can assume that the above statements a) and d) already hold for all coefficients ω_Y^Z.

We apply Lemma 6.2 c) and calculate

$$\begin{aligned} \overline{E_{[X]}} &= \overline{E_1^{(d_1)} \ldots E_n^{(d_n)}} - \sum_{X < Z} \overline{\beta^{-[Z,Z]^1} E_{[Z]}} \\ &= E_1^{(d_1)} \ldots E_n^{(d_n)} - \sum_{X < Z} \alpha^{[Z,Z]^1} (\sum_{Z \leq Y} \omega_Y^Z E_{[Y]}) \\ &= E_{[X]} + \sum_{X < Y} (\beta^{-[Y,Y]^1} - \sum_{X < Z \leq Y} \alpha^{[Z,Z]^1} \omega_Y^Z) E_{[Y]}. \end{aligned}$$

Thus, $\omega_X^X = 1$, proving the second half of part a) (the first part is trivial). Moreover, for $X < Y$, we have

$$\omega_Y^X = \beta^{-[Y,Y]^1} - \sum_{X<Z\leq Y} \alpha^{[Z,Z]^1}\omega_Y^Z,$$

which proves part c).

To prove part d), we consider again the coefficient $\widetilde{\omega}_Y^X$. Using the identity $R(d,d) = [Y,Y] - [Y,Y]^1$, we get

$$\begin{aligned}\widetilde{\omega}_Y^X &= \beta^{[Y,Y]^1}\omega_Y^X \\ &= 1 - \sum_{X<Z\leq Y} (\alpha\beta)^{[Z,Z]^1}\widetilde{\omega}_Y^Z,\end{aligned}$$

yielding part d) by induction. □

Theorem 6.5. *For each isoclass* $[X]$*, there exists a unique element*

$$\mathcal{E}_{[X]} \in E_{[X]} + \beta^{-1}\mathbf{Z}[\alpha\beta^{-1},\alpha^{-1}\beta,\beta^{-1}](B\setminus\{E_{[X]}\})$$

such that $\overline{\mathcal{E}_{[X]}} = \mathcal{E}_{[X]}$.

Writing $\mathcal{E}_{[X]} = \sum_{[Y]}\zeta_Y^X E_{[Y]}$, *we have*

a) $\zeta_Y^X = 0$ *unless* $X \leq Y$, *and* $\zeta_X^X = 1$,
b) $\zeta_Y^X \in \beta^{[X,X]-[Y,Y]}\mathbf{Z}[\alpha\beta]$,
c) *denoting by* $\widehat{\zeta}_Y^X(v) \in \mathbf{Z}[v,v^{-1}]$ *the specialization of* ζ_Y^X *to* $\alpha = v = \beta$, *we have*

$$\zeta_Y^X = (\sqrt{\alpha\beta^{-1}})^{[Y,Y]-[X,X]}\widehat{\zeta}_Y^X(\sqrt{\alpha\beta}).$$

Proof. We define coefficients ζ_Y^X for $X \leq Y$ by the formula given in part c) of the theorem, and set $\mathcal{E}_{[X]} = \sum_{[Y]}\zeta_Y^X E_{[Y]}$ for all X. Then all the claimed properties follow immediately from the properties of the one-parameter canonical basis (see [L1]), using Proposition 6.4. □

Definition 6.6. The set $\mathcal{B} = \{\mathcal{E}_{[M]} : [M] \text{ an isoclass}\}$ is called the canonical basis of $\mathcal{U}_{\alpha,\beta}^+$.

The third part of the above theorem implies that the structure of the canoncial basis of $\mathcal{U}_{\alpha,\beta}^+$ (resp. $\mathcal{U}_{q,1}^+$) is completely determined by the canonical basis of $\mathcal{U}_{v,v}^+$.

7. The degenerate quantum group $\mathcal{U}_{0,1}^+$

Lemma 7.1. *Viewing the coefficients* ω_Y^X, ζ_Y^X *as Laurent polynomials in* α,β, *we have for all* X, Y*:*

$$\zeta_Y^X(0,1) = \omega_Y^X(0,1) = a_Y^X.$$

Proof. To prove the first equality, we fix the representation X and proceed by induction over $\leq$ for Y. If $Y \not\geq X$, then both coefficients in question are equal to 0 by Proposition 6.4 a) and Theorem 6.5 a), so there is nothing to prove in this case. Moreover, if $Y = X$, then both coefficients are equal to 1. So assume we know the equality for all Z satisfying $Z < Y$. By part b) of Theorem 6.5, we can write ζ_Z^X as

$$\zeta_Z^X = \sum_{k\geq 0} z_k\alpha^k\beta^{([Z,Z]-[X,X])+k}$$

for certain $z_k \in \mathbf{Q}$. Now the condition $\zeta_Z^X \in \mathbf{Z}[\alpha\beta^{-1}, \alpha^{-1}, \beta^{-1}]$ means

$$z_k = 0 \text{ unless } k \leq ([Y,Y] - [X,X])/2.$$

Thus, we have

$$\overline{\zeta_Z^X} = \sum_{0 \leq k \leq ([Z,Z]-[X,X])/2} z_k \alpha^{[Z,Z]-[X,X]-k} \beta^{-k},$$

which implies that $\overline{\zeta_Z^X}$ can be specialized to $\alpha = 0$, $\beta = 1$, and that

$$\overline{\zeta_Z^X}(0,1) = 0 \text{ unless } Z \simeq X.$$

We apply this observation to the equation

$$\zeta_Y^X - \overline{\zeta_Y^X} = \sum_{Z<Y} \overline{\zeta_Z^X} \omega_Y^Z = \omega_Y^X + \sum_{X<Z<Y} \overline{\zeta_Z^X} \omega_Y^Z,$$

which follows from developing both sides of the equation $\overline{\mathcal{E}_{[X]}} = \mathcal{E}_{[X]}$ into the basis B. This yields the equality $\zeta_Y^X = \omega_Y^X$.

To prove the second equality, we specialize the statements from Proposition 6.4 b) and c) to $\alpha = 0$, $\beta = 1$ and get

a) if $X = M \oplus N$ for representations M, N satisfying $[N,M] = 0 = [M,N]^1$, then

$$\omega_Y^X(0,1) = \sum_{\substack{M \leq M' \\ N \leq N'}} \omega_{M'}^M(0,1) \omega_{N'}^N(0,1) F_{M',N'}^Y(0),$$

b) if X is an exponent of an indecomposable, then $\omega_Y^X(0,1) = 1$ for all $Y \geq X$.

Thus, the coefficients $\omega_Y^X(0,1)$ satisfy the same recursive relations as the a_Y^X, by Corollary 5.3. Applying the usual induction over the dimension of a representation X, as in the proofs of Lemma 5.2 and Proposition 6.4, we see that $\omega_Y^X(0,1) = a_Y^X$ for all X, Y. □

This lemma immediately implies the following final result:

Theorem 7.2. *The multiplicative basis $\mathcal{A}$ of $\mathcal{U}_{0,1}^+$ arises as the specialization of the canonical basis $\mathcal{B}$ of $\mathcal{U}_{\alpha,\beta}^+$.*

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