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SEMISYMMETRIC POLYNOMIALS AND THE INVARIANT THEORY OF MATRIX VECTOR PAIRS

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ABSTRACT. We introduce and investigate a one-parameter family of multivariate polynomials R_λ. They form a basis of the space of semisymmetric polynomials, i.e., those polynomials which are symmetric in the variables with odd and even index separately. For two values of the parameter r, namely $r = \frac{1}{2}$ and $r = 1$, the polynomials have a representation theoretic meaning related to matrix-vector pairs. In general, they form the semisymmetric analogue of (shifted) Jack polynomials. Our main result is that the R_λ are joint eigenfunctions of certain difference operators. From this we deduce, among others, the Extra Vanishing Theorem, Triangularity, and Pieri Formulas.

1. INTRODUCTION

In this paper we investigate a new family of multivariable polynomials. These polynomials, denoted $R_\lambda(z_1, \ldots, z_n; r)$, depend on a parameter r and are indexed by a partition λ of length n. Up to a scalar, R_λ is characterized by the following elementary properties:

- R_λ is symmetric in the odd variables $z_1, z_3, z_5, \ldots$ as well as in the even variables $z_2, z_4, z_6, \ldots$. Polynomials having this kind of symmetry are called *semisymmetric*.
- For the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ define the odd degree as $|\lambda|_{\text{odd}} := \sum_{i \text{ odd}} \lambda_i$. Then the degree of $R_\lambda(z)$ is $|\lambda|_{\text{odd}}$.
- Consider the vector $\varrho := ((n-1)r, (n-2)r, \ldots, r, 0)$. Then $R_\lambda(z)$ vanishes at all points of the form $z = \varrho + \mu$ where μ is any partition with $\mu \neq \lambda$ and $|\mu|_{\text{odd}} \leq |\lambda|_{\text{odd}}$.

The simplest nontrivial example comes from the partition $(1) = (1, 0, \ldots, 0)$ in which case $R_\lambda(z; r) = \sum_{i=1}^n (-1)^{i-1} z_i - \lfloor n^2/4 \rfloor$. It is clearly semisymmetric, has degree $|\lambda|_{\text{odd}} = 1$ and vanishes at $z = \varrho + \mu$ where $\mu = (0)$ or $\mu = (1^2)$.

The polynomials $R_\lambda(z)$ are analogous to the polynomials $P_\lambda(z; r)$ which were previously introduced in [KS1]. In fact, the definition of P_λ is the same except that P_λ is symmetric in *all* variables $z_1, \ldots, z_n$ and the odd degree $|\lambda|_{\text{odd}}$ is replaced by the (full) degree $|\lambda| = \sum_i \lambda_i$. The P_λ are called *shifted Jack polynomials* since their highest degree components are the Jack polynomials. This is in contrast to their semisymmetric counterparts; even their highest degree components form a genuinely new class of multivariable homogeneous polynomials.

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All the polynomials mentioned above have a representation theoretic origin. Let G be a connected reductive group acting on a finite dimensional vector space V. We are interested in the case when this action is *multiplicity free,* i.e., every simple G-module occurs at most once in the algebra of polynomial functions of V. Then the algebra $\mathbb{PD}^G$ of G-invariant differential operators on V is commutative. Moreover, one can define a (Harish Chandra) isomorphism which identifies $\mathbb{PD}^G$ with the space $\mathbb{C}[\mathfrak{a}^*]^{W_V}$ of W_V-invariant polynomials on a finite dimensional vector space $\mathfrak{a}^*$ where W_V is a finite reflection group.

The point is now that there exist very particular invariant differential operators D_λ on V which form a basis of $\mathbb{PD}^G$. The idea of their construction goes back to Capelli. The operators D_λ correspond, via the Harish Chandra isomorphism, to polynomials $p_\lambda \in \mathbb{C}[\mathfrak{a}^*]^{W_V}$ and it is these polynomials which we would like to understand. Already the top homogeneous component $\overline{p}_\lambda$ of p_λ is very important since it has the following representation theoretic meaning. Consider the symbol E_λ of the differential operator D_λ. By construction, this is a G-invariant function on the cotangent bundle $T_V^* = V \oplus V^*$. It can be considered as a generalization of a zonal spherical function. On the other hand, one can define a (Chevalley) isomorphism of $\mathbb{C}[V \oplus V^*]^G$ with $\mathbb{C}[\mathfrak{a}^*]^{W_V}$ and under this isomorphism E_λ corresponds to $\overline{p}_\lambda$.

The investigation of the polynomials p_λ and $\overline{p}_\lambda$ is greatly facilitated by the fact that multiplicity free actions on vector spaces are classified. This is due to the efforts of Kac [Kac], Benson-Ratcliff [BR], and Leahy [Le]. The most important numerical invariant of a multiplicity free action is the dimension of $\mathfrak{a}^*$ which is called its *rank*. It follows from the classification that there are only seven series in which the rank is unbounded. These series are listed in the following table. More precisely, an (indecomposable) multiplicity free action which is not in the table has rank less or equal 7.

G		V	rank	r
Classical cases:				
$GL_p(\mathbb{C})$	$(p \geq 2)$	$S^2(\mathbb{C}^p)$	p	$\frac{1}{2}$
$GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$	$(p, q \geq 1)$	$\mathbb{C}^p \otimes \mathbb{C}^q$	$\min(p, q)$	1
$GL_p(\mathbb{C})$	$(p \geq 2)$	$\Lambda^2(\mathbb{C}^p)$	$\lfloor \frac{p}{2} \rfloor$	2
Semiclassical cases:				
$GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$	$(p, q \geq 1)$	$(\mathbb{C}^p \otimes \mathbb{C}^q) \oplus \mathbb{C}^q$	$\min(2p+1, 2q)$	$\frac{1}{2}$
$GL_p(\mathbb{C})$	$(p \geq 2)$	$\Lambda^2(\mathbb{C}^p) \oplus \mathbb{C}^p$	p	1
Quasiclassical cases:				
$GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$	$(p, q \geq 1)$	$(\mathbb{C}^p \otimes \mathbb{C}^q) \oplus (\mathbb{C}^q)^*$	$\min(2p+1, 2q)$	
$GL_p(\mathbb{C})$	$(p \geq 2)$	$\Lambda^2(\mathbb{C}^p) \oplus (\mathbb{C}^p)^*$	p	

As indicated in the table, the seven series fall into three classes: classical, semiclassical, and quasiclassical. The reason for this is that all the cases in each class can be treated uniformly: there are polynomials depending on a free parameter* r such that the polynomials p_λ of each particular case are obtained by specializing the parameter as indicated in the table.

In the classical class, the space V consists of matrices: symmetric, rectangular, or skewsymmetric. This case has been treated in [KS1] and the polynomials p_λ are basically the shifted Jack polynomials $P_\lambda(z; r)$.

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*In the odd rank quasiclassical case there are two parameters.

The purpose of the present paper is to study the semiclassical case. Here an element of V is a pair (A, v) where A is a rectangular or skewsymmetric matrix and v is a vector. The polynomials p_λ are our $R_\lambda(z; r)$ described in the beginning.

The quasiclassical class is left mostly to future research. Here an element of V is a pair (A, α) where A is a matrix as above and α is a covector (linear form). Preliminary investigations indicate that this case is much more involved than the other two cases. Nevertheless, the small cases, more precisely the cases with rank ≤ 4, are covered also in the present paper since for those the combinatorics of the quasiclassical and the semiclassical class turn out to be isomorphic. In particular, we can also say something about the action of $GL_1(\mathbb{C}) \times GL_q(\mathbb{C})$ and $\mathbb{C}^q \oplus (\mathbb{C}^q)^*$, a case already considered by Vilenkin–Šapiro [VS].

The zonal spherical functions E_λ have numerous different descriptions. This means that the results of this paper are also relevant for the action of $GL_{p-1}(\mathbb{C})$ on $GL_p(\mathbb{C})$ *by conjugation* or for the action of $Sp_{2p}(\mathbb{C})$ on $X := SL_{2p+1}(\mathbb{C})/Sp_{2p}(\mathbb{C})$. This is remarkable since the space X is only spherical and not symmetric.

Now we describe the results of the polynomials R_λ. The most important result is the construction of n commuting difference operators of which the R_λ are simultaneous eigenfunctions. These differential operators are defined by explicit formulas (4.4). An analogous result has already been the main statement of [KS1]. The result here is similar but much more involved.

Except for some elementary results, such as existence and uniqueness of the R_λ, most proofs hinge on the difference operators. The first immediate consequence is that the top homogeneous component $\overline{R}_\lambda$ of R_λ is a simultaneous eigenfunction of n commuting *differential operators* of order $1, 1, 2, 2, 3, 3, \ldots$. These are semisymmetric analogues of the Sekiguchi-Debiard operators which characterize Jack polynomials. Observe that Heckman and Opdam define analogues of the Sekiguchi-Debiard operators for any finite root system but our semisymmetric case is *not* covered by their construction.

Another rather immediate consequence of the difference operators is the *Extra Vanishing Theorem.* Remember, that R_λ is *defined* to vanish at all points of the form $z = \varrho + \mu$ where $\mu \neq \lambda$ and $|\mu|_{\text{odd}} \leq |\lambda|_{\text{odd}}$. It turns out, that R_λ actually vanishes at many more points. In section 4 we define an order relation $\lambda \sqsubseteq \mu$ on the set of partitions such that $R_\lambda(\varrho+\mu) = 0$ whenever $\lambda \not\sqsubseteq \mu$. This order relation should be regarded as a semisymmetric analogue of the familiar containment relation for partitions.

A property which can be considered as dual to extra vanishing is called *triangularity*. By *definition*, the polynomial R_λ can be expressed as a linear combination of monomials z^μ whose degree is less than or equal to $|\lambda|_{\text{odd}}$. As it turns out fewer monomials are needed. This phenomenon is called triangularity since it can be rephrased to say that the base change matrix from monomials to R_λ's is triangular. In section 6 we actually prove two versions of triangularity. For the first, we define a map $\lambda \mapsto [\lambda]$ from the set of partitions into the set $\mathbb{N}^n$ of compositions such that if z^μ appears in R_λ, then $\mu \leq [\lambda]$. Here "$\leq$" is the usual (inhomogeneous) dominance order on $\mathbb{N}^n$.

The monomials z^μ are not semisymmetric. Therefore, one can attempt to formulate triangularity strictly within the set of semisymmetric polynomials. To do this, define the *elementary semisymmetric polynomials* as $\mathbf{e}_1 := \overline{R}_{(1)}$, $\mathbf{e}_2 := \overline{R}_{(1^2)}$, etc. These can be computed explicitly: $\mathbf{e}_{2i-1} = e_i^{\text{odd}} - e_i^{\text{even}}$ and $\mathbf{e}_{2i} = e_i^{\text{even}}$ where

$e_i^{\mathrm{odd/even}}$ is the usual elementary symmetric function of degree i in the variables $\{z_{2j-1} \mid j\}$ or $\{z_{2j} \mid j\}$, respectively. Now consider all monomials in the $\mathbf{e}_i$:

$$\mathbf{e}_\mu := \mathbf{e}_1^{\mu_1-\mu_2}\mathbf{e}_2^{\mu_2-\mu_3}\ldots\mathbf{e}_{n-1}^{\mu_{n-1}-\mu_n}\mathbf{e}_n^{\mu_n} \tag{1.1}$$

where μ is a partition. In section 6 we define a new order relation $\mu \preceq \lambda$ on the set of partitions which is a semisymmetric analogue of the classical dominance order. Using the explicit form of the difference operators we are able to prove that each R_λ is a linear combinations of $\mathbf{e}_\mu$ with $\mu \preceq \lambda$. This is the second triangularity result, alluded to above. It should be noted that the second form easily implies the first one but not conversely. This has to be seen in contrast to the classical case of (shifted) Jack polynomials where both forms of triangularity are actually equivalent.

In section 7, we prove what could be considered as the second main result of this paper: the *duality formula* (7.6)

$$\frac{R_\lambda(-\underline{\alpha}-z)}{R_\lambda(-\underline{\alpha}-\varrho)} = \sum_\mu (-1)^{|\mu|_{\mathrm{odd}}}\frac{R_\mu(\varrho+\lambda)}{R_\mu(\varrho+\mu)}\frac{R_\mu(z)}{R_\mu(-\underline{\alpha}-\varrho)} \tag{1.2}$$

where α is an arbitrary parameter and $\underline{\alpha} = (\alpha, \ldots, \alpha)$. In other words, the formula expresses the transformation $z_i \mapsto -\alpha - z_i$ of the space of semisymmetric polynomials in terms of its basis R_λ. The classical analogue has been established by Okounkov [Ok] whose proof we follow closely. This holds also for some of its consequences described below. The key to the proof of the duality formula are again the difference operators.

A first consequence is an explicit interpolation formula (Theorem 7.6 iii)). It allows us to explicitly calculate the expansion of an arbitrary semisymmetric polynomial in terms of R_λ's. More precisely,

$$f(z) = \sum_\mu (-1)^{|\mu|_{\mathrm{odd}}}\frac{\hat{f}(\varrho+\mu)}{R_\mu(\varrho+\mu)}R_\mu(z) \tag{1.3}$$

where

$$\hat{f}(\varrho+\mu) = \sum_\nu (-1)^{|\nu|_{\mathrm{odd}}}\frac{R_\nu(\varrho+\mu)}{R_\nu(\varrho+\nu)}f(\varrho+\nu). \tag{1.4}$$

The key to this formula is the observation that the transformation $z \mapsto -\underline{\alpha} - z$ is involutory.

Another consequence of the duality theorem is the fact that the expression

$$\frac{R_\lambda(-\underline{\alpha}-\varrho-\nu)}{R_\lambda(-\underline{\alpha}-\varrho)} \tag{1.5}$$

is symmetric in λ and ν (just put $z = \varrho + \nu$ in (1.2)). This observation allows us sometimes to interchange in formulas the argument z with the index λ. For example, let D be one of the fundamental difference operators. Then the eigenvalue equation $D(R_\lambda) = c(\lambda)R_\lambda$ turns into a *Pieri type formula*, i.e., a formula which expresses the multiplication operator $p \mapsto fp$ on the space of semisymmetric polynomials (with fixed f) in terms of the basis R_λ. Thus, we obtain in section 9 the expansion of $f(z)R_\lambda(z)$ in terms of R_μ's where $f(z)$ is one of $e_i^{\mathrm{odd}}(z)$, $e_i^{\mathrm{even}}(z)$, or $R_{1^i}(z)$. As a byproduct of this investigation we prove in section 8 a formula for the value of $R_\lambda(z)$ in $z = -\underline{\alpha} - \varrho$.

Most of these results have consequences for the homogeneous polynomials $\overline{R}_\lambda$. The evaluation formula specializes to a formula for the value of $\overline{R}_\lambda(z)$ in $z = (1,\ldots,1)$. The duality formula implies the semisymmetric binomial theorem which expresses the effect of the transformation $z_i \mapsto \alpha + z_i$ in terms of the homogeneous basis $\overline{R}_\lambda$ (see section 8 for these two statements). Its classical analogue is due to Okounkov and Olshanski [OO]. Finally, we obtain expansions of $f(z)\overline{R}_\lambda$ in terms of $\overline{R}_\mu$'s where $f(z)$ is one of $e_i^{\mathrm{odd}}(z)$, $e_i^{\mathrm{even}}(z)$, or $\mathbf{e}_i(z)$

Finally, scattered throughout the paper, we derive several explicit formulas. More precisely, we determine $R_\lambda(z)$ when $\lambda = (a\,1^{m-1})$, $a, m \geq 1$ is a "hook" (Corollary 2.8 for $a = 1$, Corollary 4.10 for m odd, Corollary 9.5 for m even). Furthermore, we calculate $\overline{R}_\lambda(z)$ where $\lambda = (a\,b)$ is a two row diagram (Theorem 5.4) or any $\overline{R}_\lambda$ if $n = 3$. For $n = 3$ these are expressible in terms of Jacobi polynomials, for $n = 4$ we get one of Horn's hypergeometric functions.

Even though most of the theory is parallel to that of (shifted) Jack polynomials there are some differences. One of them is that the R_λ do not specialize for $r = 0$ to anything easy. It seems that matters become rather more involved. Also, neither R_λ nor its top homogeneous term $\overline{R}_\lambda$ seems to have any obvious positivity properties (see [KS1] and [KS2] for the classical case). Another remarkable difference occurs when the number of variables is even. Then the specialization of $\overline{R}_\lambda(z)$ at the point $z = (1,\ldots,1)$ may be zero. This has the consequence that the shifted polynomials $R_\lambda(z)$ cannot be defined via the binomial formula (8.17) since not all of them occur in this formula. A major open problem is orthogonality: Jack polynomials are most commonly defined by an orthogonalization process with respect to some explicit scalar product. Such a scalar product is still missing for the $\overline{R}_\lambda$'s.

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2. Shifted semisymmetric functions

Let k be a field of characteristic zero. Consider the polynomial ring $\mathcal{P} := k[z_1,\ldots,z_n]$. On it, the symmetric group S_n acts by permutation of the variables. The *semisymmetric group* is the subgroup W of S_n which doesn't mix even and odd entries: $\pi \in W$ if $\pi(i) \equiv i \bmod 2$ for all i. Throughout this paper, we are adopting the following notation: we put $\underline{n} := \lfloor n/2 \rfloor$ and $\overline{n} := n - \underline{n} = \lceil n/2 \rceil$. Then we have $W \cong S_{\overline{n}} \times S_{\underline{n}}$. For $z \in k^n$ we let $z_{\mathrm{odd}} := (z_1, z_3, \ldots, z_{2\overline{n}-1}) \in k^{\overline{n}}$ and $z_{\mathrm{even}} := (z_2, z_4, \ldots, z_{2\underline{n}}) \in k^{\underline{n}}$.

We are going to study the *ring of semisymmetric polynomials* $\mathcal{P}^W$. Clearly, as an algebra, $\mathcal{P}^W$ is a polynomial ring generated by $e_i(z_{\mathrm{odd}})$, $i = 1,\ldots,\overline{n}$ and $e_i(z_{\mathrm{even}})$, $i = 1,\ldots,\underline{n}$ where e_i is the i-th elementary symmetric polynomial.

Let Λ be the set of *partitions* of length n, i.e., n-tuples of integers $\lambda = (\lambda_i)$ with $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$. We are going to consider Λ as a subset of k^n. The degree of λ is $|\lambda| = \sum_i \lambda_i$. We also define the odd degree $|\lambda|_{\mathrm{odd}} := |\lambda_{\mathrm{odd}}|$ and the even degree $|\lambda|_{\mathrm{even}} := |\lambda_{\mathrm{even}}|$. The odd degree will be for semisymmetric polynomials what the degree is for symmetric polynomials.

Finally, we choose once and for all a parameter $r \in k$ with† $r \notin \mathbb{Q}_{<0}$ and put

$$\varrho := ((n-1)r, (n-2)r, \ldots, 2r, r, 0). \tag{2.1}$$

†In fact, only the slightly weaker condition $r \neq -\frac{p}{2q}$ where p and q are integers with $1 \leq p$ and $1 \leq q < \frac{n}{2}$ is needed.

2.1. Theorem. *For any $d \in \mathbb{N}$ let $\Lambda(d)$ be the set of $\lambda \in \Lambda$ with $|\lambda|_{\mathrm{odd}} \le d$. Let $\Lambda(d) \to k : \lambda \mapsto a_\lambda$ be any map. Then there is a unique $f \in \mathcal{P}^W$ with $\deg f \le d$ and $f(\varrho + \lambda) = a_\lambda$ for all $\lambda \in \Lambda(d)$.*

Proof. We are using induction on $d + n$. To make the dependence on the dimension n explicit we write it as an index. We have $\Lambda_{n-1} \hookrightarrow \Lambda_n$ by appending a zero. Moreover let $\mathcal{P}_{n-1}^{W_{n-1}} \to \mathcal{P}_n^{W_n} : g \mapsto g^+$ be the homomorphism which maps $e_i(z_{\mathrm{odd}\,/\,\mathrm{even}}) \in \mathcal{P}_{n-1}^{W_{n-1}}$ to $e_i(z_{\mathrm{odd}\,/\,\mathrm{even}}) \in \mathcal{P}_n^{W_n}$. Then $\deg g^+ = \deg g$ and $g^+(z_1, \ldots, z_{n-1}, 0) = g(z_1, \ldots, z_{n-1})$.

Let $e(z) := \prod_{i:\, n-i \text{ even}} z_i$. Observe $\deg e = \overline{n}$. Since $e(z)$ is the one generator of $\mathcal{P}_n^{W_n}$ which is not in the image of $\mathcal{P}_{n-1}^{W_{n-1}}$, every $f \in \mathcal{P}_n^{W_n}$ can be uniquely expressed as $f(z) = g^+(z) + e(z)h(z)$ with $g \in \mathcal{P}_{n-1}^{W_{n-1}}$, $\deg g \le \deg f$, $h \in \mathcal{P}_n^{W_n}$, and $\deg h \le \deg f - \overline{n}$.

Now we split $\Lambda_n(d)$ into two parts $\Lambda_n(d)^0$ and $\Lambda_n(d)^1$ according to whether the last component λ_n is zero or not.

For any $g \in \mathcal{P}_{n-1}$ let $g_0(z) := g(z_1 + r, \ldots, z_{n-1} + r)$. Clearly, we can identify $\Lambda_n(d)^0$ with $\Lambda_{n-1}(d)$. Then for any $\lambda \in \Lambda_n(d)^0$ we have

$$g^+(\varrho_n + \lambda) = g(\lambda_1 + (n-1)r, \ldots, \lambda_{n-1} + r) = g_0(\varrho_{n-1} + \lambda). \tag{2.2}$$

Since $e(\varrho+\lambda) = 0$, for every λ with $\lambda_n = 0$ the system of linear equations $f(\varrho_n+\lambda) = a_\lambda, \lambda \in \Lambda_n(d)^0$ is equivalent to the system $g_0(\varrho_{n-1} + \lambda) = a_\lambda, \lambda \in \Lambda_{n-1}(d)$. By induction on the number of variables we conclude that it has a unique solution.

For any $\lambda \in \Lambda_n(d)^1$ holds $e(\varrho + \lambda) \ne 0$ since, by assumption, $r \notin \mathbb{Q}_{<0}$. Thus, we can define $a'_\lambda := (a_\lambda - g^+(\varrho + \lambda))/e(\varrho + \lambda)$. The map $\lambda \mapsto \tilde\lambda := (\lambda_1 - 1, \ldots, \lambda_n - 1)$ identifies $\Lambda_n(d)^1$ with $\Lambda_n(d - \overline{n})$. Thus the system of linear equations $f(\varrho + \lambda) = a_\lambda, \lambda \in \Lambda_n(d)^1$ is equivalent to the system $\tilde h(\varrho + \tilde\lambda) = a'_\lambda, \tilde\lambda \in \Lambda_n(d - \overline{n})$ where $\tilde h(z) = h(z_1 - 1, \ldots, z_n - 1)$. By induction on the degree we conclude that it has a unique solution, as well. □

Now, we can define interpolation polynomials as follows:

Definition. For every $\lambda \in \Lambda$ let $r_\lambda(z; r)$ be the unique polynomial such that

- it is W-invariant,
- its degree is $d := |\lambda|_{\mathrm{odd}}$,
- for all $\mu \in \Lambda$ with $|\mu|_{\mathrm{odd}} \le d$ holds $r_\lambda(\varrho + \mu; r) = \delta_{\lambda\mu}$ (Kronecker delta).

The normalization $r_\lambda(\varrho + \lambda; r) = 1$ is very natural but there is one which is often more convenient: the "leading" coefficient should be equal to one. To define what that means, observe that every W-orbit of a monomial contains exactly one monomial, say z^ν, such that both ν_{odd} and ν_{even} are partitions. These ν are in bijection with Λ. In fact, for every partition $\lambda \in \Lambda$ we define the composition $[\lambda] \in \mathbb{N}^n$ by

$$[\lambda]_m := \lambda_m - \lambda_{m+1} + \ldots + (-1)^{n-m}\lambda_n. \tag{2.3}$$

Since $[\lambda]_m = (\lambda_m - \lambda_{m+1}) + [\lambda]_{m+2}$, both $[\lambda]_{\mathrm{odd}}$ and $[\lambda]_{\mathrm{even}}$ are in fact partitions. Conversely, let ν be a composition such that both ν_{odd} and ν_{even} are partitions. Then

$$\lambda = (\nu_1 + \nu_2, \nu_2 + \nu_3, \nu_3 + \nu_4, \nu_4 + \nu_5, \ldots) \tag{2.4}$$

is in Λ. One easily checks that these two maps are inverse to each other. Of special interest is the first component of $[\lambda]$ since

$$[\lambda]_1 = |\lambda|_{\mathrm{odd}} - |\lambda|_{\mathrm{even}} = \lambda_1 - \lambda_2 + \lambda_3 - + \ldots \tag{2.5}$$

In particular,

$$[\lambda]_1 = 0 \text{ if and only if } \lambda_1 = \lambda_2, \lambda_3 = \lambda_4, \ldots, \text{ and } \lambda_n = 0 \text{ in case } n \text{ is odd.} \tag{2.6}$$

Moreover, we have $|[\lambda]| = |\lambda|_{\mathrm{odd}}$ and therefore $\deg r_\lambda(z;r) = \deg z^{[\lambda]}$.

2.2. Proposition. *The coefficient $C_\lambda(r)$ of $z^{[\lambda]}$ in $r_\lambda(z;r)$ is non-zero.*

With this result we can define the renormalized polynomial

$$R_\lambda(z;r) := \frac{1}{C_\lambda(r)} r_\lambda(z;r) = z^{[\lambda]} + \ldots \tag{2.7}$$

We prove the proposition by computing $C_\lambda(r)$ explicitly. For this we need further notation. A partition λ can be represented by its diagram, i.e., the set of all $(i,j) \in \mathbb{N}^2$ (called boxes) such that $1 \le i \le n$ and $1 \le j \le \lambda_i$. The dual partition λ' is defined by the transposed diagram $\{(j,i) \mid (i,j) \in \lambda\}$. For every box $s = (i,j) \in \lambda$ we define the arm-length $a_\lambda(s) := \lambda_i - j$ and the leg-length $l_\lambda(s) := \lambda'_j - i$. Then we define

$$[c'_\lambda(r)]_{\mathrm{even}} := \prod_{\substack{s \in \lambda \\ l_\lambda(s) \text{ even}}} (a_\lambda(s) + 1 + l_\lambda(s) r). \tag{2.8}$$

For example, we have $[c'_{(a)}(r)]_{\mathrm{even}} = a!$, $[c'_{(a\,b)}(r)]_{\mathrm{even}} = (a-b)!b!$, and $[c'_{(1^m)}(r)]_{\mathrm{even}} = \prod_{\substack{1 \le i < m \\ i \text{ even}}} (1 + ir)$.

2.3. Lemma. *For every $\lambda \in \Lambda$ holds $C_\lambda(r) = [c'_\lambda(r)]_{\mathrm{even}}^{-1}$. In particular, we have*

$$R_\lambda(\varrho + \lambda; r) = [c'_\lambda(r)]_{\mathrm{even}}. \tag{2.9}$$

Proof. We retain the notation of the proof of Theorem 2.1 and prove the lemma by a similar induction. In particular, we have an expression $r_\lambda(z) = g^+(z) + e(z)h(z)$.

If $\lambda_n = 0$, then $e(\varrho + \lambda) = 0$ and therefore $g(z) = r_{\lambda'}(z_1 - r, \ldots, z_{n-1} - r)$ where the prime means "drop the last component". Moreover, the coefficient of $z^{[\lambda]}$ in r_λ equals the one in g (observe $[\lambda]_n = 0$). Thus, we get by induction $C_\lambda(r) = C_{\lambda'}(r) = [c'_{\lambda'}(r)]_{\mathrm{even}}^{-1}$. But we also have $[c'_\lambda(r)]_{\mathrm{even}} = [c'_{\lambda'}(r)]_{\mathrm{even}}$ which finishes this case.

If $\lambda_n \ge 1$, then $g(z) = 0$ and $h(z) = e(\varrho + \lambda)^{-1} r_{\tilde\lambda}(z_1 - 1, \ldots, z_n - 1)$. One checks $z^{[\lambda]} = e(z) z^{[\tilde\lambda]}$. Thus, by induction, the coefficient of $z^{[\lambda]}$ is $e(\varrho + \lambda)^{-1} [c'_{\tilde\lambda}(r)]_{\mathrm{even}}^{-1}$. But $e(\varrho + \lambda)$ is the contribution of the first column of λ to $[c'_\lambda(r)]_{\mathrm{even}}$. Thus we get $C_\lambda(r) = [c'_\lambda(r)]_{\mathrm{even}}^{-1}$, as claimed. □

The second case of the preceding proof gives the following recursion formula which allows us to reduce the computation of R_λ to the case $\lambda_n = 0$.

2.4. Corollary. *Let $\delta := (1, \ldots, 1)$. Then for every $\lambda \in \Lambda$ with $\lambda_n \ge 1$ holds*

$$R_\lambda(z;r) = \Big(\prod_{n-i \text{ even}} z_i \Big) \cdot R_{\lambda - \delta}(z - \delta; r). \tag{2.10}$$

We also have the following stability result:

2.5. Proposition. *For $z = (z_1, \dots, z_n) \in k^n$ let $z' := (z_1, \dots, z_{n-1}) \in k^{n-1}$. Then we have for any $\lambda \in \Lambda$:*

$$R_\lambda(z_1, \dots, z_{n-1}, 0) = \begin{cases} R_{\lambda'}(z_1 - r, \dots, z_{n-1} - r) & \text{if } \lambda_n = 0; \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

Proof. If $\lambda_n \geq 1$, then R_λ is divisible by z_n (Corollary 2.4), hence $R_\lambda|_{z_n=0} = 0$. Otherwise, $R_\lambda|_{z_n=0}$ satisfies the definition of $R_{\lambda'}(z_1 - r, \dots, z_{n-1} - r)$. □

Remark. In many circumstances it is more convenient to consider the polynomials $\tilde{R}_\lambda(u; r) := R_\lambda(\varrho + u; r)$. Their main advantage is that the stability result above can now be expressed as

$$\tilde{R}_\lambda(u_1, \dots, u_{n-1}, 0) = \tilde{R}_{\lambda'}(u_1, \dots, u_{n-1}) \tag{2.12}$$

whenever $\lambda_n = 0$. This means that one can form a theory of shifted semisymmetric polynomials which is independent of the dimension n. For this, one defines them in infinitely many variables as follows. Let $\mathcal{P}_\infty$ be the projective limit of the polynomial rings $k[u_1, \dots, u_n]$ in the category of filtered algebras. An element of $\mathcal{P}_\infty$ is a possibly infinite linear combination of monomials in $u_1, u_2, \dots$ whose degrees are uniformly bounded. Let Λ_∞ be the set of all descending sequences of integers $\lambda_1 \geq \lambda_2 \geq \dots$ with $\lambda_n = 0$ for $n \gg 0$. The stability result above says that for any λ the sequence $(\tilde{R}_{(\lambda_1, \dots, \lambda_n)}(u_1, \dots, u_n))_{n \gg 0}$ is an element of $\mathcal{P}_\infty$. It is denoted by $\tilde{R}_\lambda$.

The drawback of this method is that the action of the semisymmetric group gets distorted. More precisely, W acts now by $\pi \bullet u := \pi(u + \varrho) - \varrho$. For example, the simple reflection $s_{i\,i+2}$ acts as $u_i \mapsto u_{i+2} - 2r$ and $u_{i+2} \mapsto u_i + 2r$. This action extends to an action of W_∞ on $\mathcal{P}_\infty$ where W_∞ is the group of parity preserving permutations of $\mathbb{N}$ with finite support. It is easy to show that the $\tilde{R}_\lambda$, $\lambda \in \Lambda_\infty$ form a linear basis of $\mathcal{P}_\infty^{W_\infty}$.

Next, we present some compatibility results with shifted Jack polynomials. First, we recall their definition from [KS1]. More or less, one has to replace the semisymmetric group by the full symmetric group and the odd degree by the full degree. More precisely, for each $\lambda \in \Lambda$ we define $P_\lambda(z_1, \dots, z_n; r)$ as the unique polynomial having the following properties:

- P_λ is invariant under the full symmetric group S_n;
- $\deg P_\lambda = |\lambda|$;
- the coefficient of z^λ is 1;
- $P_\lambda(\varrho + \mu) = 0$ for all $\mu \in \Lambda$ with $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$.

Analogously, we define $\tilde{P}_\lambda(u; r) := P_\lambda(\varrho + u; r)$.

Now we show that the symmetric polynomials $P_\lambda(z; r)$ are in two ways special cases of the semisymmetric polynomials $R_\lambda(z; r)$.

2.6. Theorem. *Let $\lambda \in \Lambda$ with $[\lambda]_1 = 0$ (see (2.6)). Then*

$$\tilde{R}_\lambda(u; r) = \tilde{P}_{\lambda_{\text{even}}}(u_{\text{even}}; 2r). \tag{2.13}$$

Proof. We show that the polynomial P on the right-hand side matches the definition of $\tilde{R}_\lambda$. First, observe that the shifted action of W induces on the even coordinates the shifted action of W_{even} with parameter $2r$. Thus, P is shifted semisymmetric. Moreover, we have

$$|\lambda|_{\text{odd}} = |\lambda_{\text{odd}}| = |\lambda_{\text{even}}| \tag{2.14}$$

which shows that the degree of P is correct. Since $[\lambda]=(0,\lambda_2,0,\lambda_4,\dots)$ we have $z^{[\lambda]}=z_{\rm even}^{\lambda_{\rm even}}$. This shows that the normalization of P is correct, as well.

It remains to check the vanishing conditions. For this let $\mu\in\Lambda$ with $|\mu|_{\rm odd}\le|\lambda|_{\rm odd}=|\lambda_{\rm even}|$. Then

$$|\mu_{\rm even}|\le|\mu_{\rm odd}|\le|\lambda_{\rm even}|. \tag{2.15}$$

This implies $P(\mu)=0$ unless $\mu_{\rm even}=\lambda_{\rm even}$. But then $0\le[\mu]_1=|\mu|_{\rm odd}-|\mu_{\rm even}|\le|\lambda|_{\rm odd}-|\lambda_{\rm even}|=0$ which implies $\mu_1=\mu_2$, $\mu_3=\mu_4$, etc., i.e., $\mu=\lambda$. □

The other connection between R_λ and P_λ is:

2.7. Theorem. *For every $\mu=(\mu_1,\dots,\mu_{\overline{n}})\in\Lambda_{\overline{n}}$ holds*

$$\sum_{\lambda:\,\lambda_{\rm odd}=\mu}\frac{\tilde R_\lambda(u;r)}{\tilde R_\lambda(\lambda;r)}=\frac{\tilde P_\mu(u_{\rm odd};2r)}{\tilde P_\mu(\mu;2r)}. \tag{2.16}$$

Proof. Let P be the polynomial on the right-hand side of (2.16). Again, the shifted action of W induces on the odd coordinates the shifted action of $W_{\rm odd}$ with parameter $2r$. Thus, P is shifted semisymmetric. In particular, we have an expression

$$P=\sum_\lambda c_\lambda\tilde R_\lambda,\quad\text{with }|\lambda|_{\rm odd}\le|\mu|. \tag{2.17}$$

Suppose there is λ with $c_\lambda\ne0$ and $\lambda_{\rm odd}\ne\mu$. If we choose one of minimal degree, the left-hand side of (2.17) evaluates at $u=\lambda$ to c_λ while $P(\lambda_{\rm odd})=0$. Contradiction. Thus $c_\lambda=0$ unless $\lambda_{\rm odd}=\mu$. In that case, the value of c_λ is immediately obtained by evaluating both sides of (2.17) at $u=\lambda$. □

As a corollary we get a formula for the elementary semisymmetric polynomials:

2.8. Corollary.

$$\tilde R_{(1^{2m-1})}(u;r)=\tilde P_{(1^m)}(u_{\rm odd};2r)-\tilde P_{(1^m)}(u_{\rm even};2r), \tag{2.18}$$

$$\tilde R_{(1^{2m})}(u;r)=\tilde P_{(1^m)}(u_{\rm even};2r). \tag{2.19}$$

Proof. Formula (2.19) is a special case of (2.13). If we put $\lambda=(1^{2m-1})$ in (2.16) and use (2.19), we get

$$\tilde R_{(1^{2m-1})}(u;r)=\alpha\tilde P_{(1^m)}(u_{\rm odd};2r)-\beta\tilde P_{(1^m)}(u_{\rm even};2r) \tag{2.20}$$

with two constants α and β. Comparison of the coefficient of $u^{[(1^{2m-1})]}=u_1u_3\dots u_{2m-1}$ implies $\alpha=1$. Next we evaluate (2.20) at $u=(1^{2m})$. The left-hand side is zero by definition. Then $(1^{2m})_{\rm odd}=(1^m)=(1^{2m})_{\rm even}$ implies $\beta=1$. □

Explicit formulas for $\tilde P_{(1^m)}(u;r)$ can be found in [KS1], 3.1. One them is:

$$\tilde P_{(1^m)}(u;r)=\sum_{n\ge i_1>i_2>\ldots>i_m\ge1}\prod_{j=1}^m(u_{i_j}+(j-1)r). \tag{2.21}$$

Thus, the first few elementary semisymmetric polynomials are

$$\tilde{R}_{(1)}(u;r) = e_1(u_{\text{odd}}) - e_1(u_{\text{even}}) = (u_1+u_3+\dots) - (u_2+u_4+\dots), \tag{2.22}$$

$$\tilde{R}_{(11)}(u;r) = e_1(u_{\text{even}}) = u_2+u_4+\dots, \tag{2.23}$$

$$\tilde{R}_{(111)}(u;r) = e_2(u_{\text{odd}}) - e_2(u_{\text{even}}) + r\sum_{i\text{ odd}}(i-1)u_i - r\sum_{i\text{ even}}(i-2)u_i, \tag{2.24}$$

$$\tilde{R}_{(1111)}(u;r) = e_2(u_{\text{even}}) + r\sum_{i\text{ even}}(i-2)u_i. \tag{2.25}$$

Let $\overline{R}_\lambda(z;r)$ be the top homogeneous component of $R_\lambda(z;r)$. Since the highest degree component of $\tilde{P}_{(1^m)}$ is the elementary symmetric function e_m we obtain:

2.9. Corollary. *Let $\overline{R}_\lambda(z;r)$ be the highest degree component of $R_\lambda(z;r)$. Then*

$$\overline{R}_{(1^{2m-1})}(z;r) = e_m(z_{\text{odd}}) - e_m(z_{\text{even}}), \tag{2.6}$$

$$\overline{R}_{(1^{2m})}(z;r) = e_m(z_{\text{even}}). \tag{2.7}$$

We conclude this section with a list of all polynomials R_λ which are non-elementary of degree at most 3, i.e., with $|\lambda|_{\text{odd}} \le 3$ and $\lambda_1 > 1$. Each R_λ is expressed as a polynomial in the $R_{(1^i)}$. This means, that the formulas are valid *for all* n with the convention that $R_\lambda = 0$ if the length of λ is greater than n.

$$\begin{aligned}
R_{(2)} &= R_{(1)}^2 - R_{(1)},\\
R_{(21)} &= R_{(1)}R_{(11)} - \frac{1}{1+2r}R_{(111)},\\
R_{(22)} &= R_{(11)}^2 - \frac{2}{1+2r}R_{(1111)} - R_{(11)},\\
R_{(211)} &= R_{(1)}R_{(111)} - R_{(111)},\\
R_{(2111)} &= R_{(1)}R_{(1111)} - \frac{1}{1+4r}R_{(11111)},\\
R_{(221)} &= R_{(11)}R_{(111)} - \frac{1}{1+2r}R_{(1)}R_{(1111)} - \frac{1}{1+2r}R_{(11111)} - R_{(111)},\\
R_{(2211)} &= R_{(11)}R_{(1111)} - \frac{3}{4r+1}R_{(111111)} - 2R_{(1111)},\\
R_{(3)} &= R_{(1)}^3 - 3R_{(1)}^2 + 2R_{(1)},\\
R_{(31)} &= R_{(1)}^2R_{(11)} - \frac{1}{1+r}R_{(1)}R_{(111)} - R_{(1)}R_{(11)} + \frac{1}{1+r}R_{(111)},\\
R_{(32)} &= R_{(1)}R_{(11)}^2 - \frac{1}{1+r}R_{(11)}R_{(111)} - \frac{1}{1+r}R_{(1)}R_{(1111)}\\
&\quad + \frac{1}{(1+r)(1+2r)}R_{(11111)} - R_{(1)}R_{(11)} + \frac{1}{1+r}R_{(111)},\\
R_{(33)} &= R_{(11)}^3 - \frac{3}{1+r}R_{(11)}R_{(1111)} + \frac{3}{(1+r)(1+2r)}R_{(111111)}\\
&\quad - 3R_{(11)}^2 + \frac{6}{1+r}R_{(1111)} + 2R_{(11)}.
\end{aligned} \tag{2.28}$$

These formulas were obtained with the help of a computer.

3. Representation theoretic interpretation

Before we study the polynomials $R_\lambda(z;r)$ further, we describe the representation theoretic interpretation of the three special cases which are mentioned in the introduction. For this, we recall some basic facts about multiplicity free representations. Details appear, for example, in [Kn1].

Let G be a connected complex reductive group. A finite dimensional representation V of G is called *multiplicity free* if every simple G-module appears in $\mathbb{P} := \mathbb{C}[V]$ at most once. Equivalent to this condition is that a Borel subgroup of G has a dense orbit in V. Thus, as a G-module, we have a decomposition $\mathbb{P} = \bigoplus_{\lambda\in\Lambda_V} \mathbb{P}_\lambda$ where $\mathbb{P}_\lambda$ is the simple module with *lowest* weight $-\lambda$. Then Λ_V is a set of dominant weights which can be shown to be a free abelian monoid (i.e., isomorphic to $\mathbb{N}^r$). Clearly, all non-zero polynomials in $\mathbb{P}_\lambda$ have the same degree, denoted by $|\lambda|$.

The symmetric algebra $\mathbb{D} := S^*(V)$ then decomposes accordingly as $\mathbb{D} = \bigoplus_{\lambda\in\Lambda_V} \mathbb{D}_\lambda$ where $\mathbb{D}_\lambda$ is isomorphic to $\mathbb{P}_\lambda^*$. In particular, λ is its highest weight. The space $\mathbb{D}$ can be interpreted either as polynomial functions on V^* or as constant coefficient differential operators on V. Accordingly, $\mathbb{P}\otimes\mathbb{D}$ can be identified with either the algebra of polynomial functions on $V\oplus V^*$ or the algebra $\mathbb{PD}$ of linear differential operators on V with polynomial coefficients.

The point is now that the space of G-invariants $(\mathbb{P}\otimes\mathbb{D})^G$ comes with a (up to scalars) distinguished basis: we have

$$(\mathbb{P}\otimes\mathbb{D})^G = \bigoplus_{\lambda,\mu\in\Lambda_V} (\mathbb{P}_\lambda\otimes\mathbb{D}_\mu)^G. \tag{3.1}$$

Each summand is zero unless $\lambda=\mu$ in which case it is one-dimensional (Schur's Lemma). We denote a generator as E_λ if regarded as a function on $V\oplus V^*$ (called a *zonal spherical function*) and D_λ if regarded as a differential operator (called a *Capelli operator*).

The Capelli operators are easily accessible, whence we start with them. Each differential operator $D\in(\mathbb{PD})^G$ acts on $\mathbb{P}_\lambda$ by a scalar denoted $c_D(\lambda)$. Recall that Λ_V is a set of weights and therefore sits in $\mathfrak{t}^*$, the dual of the Cartan subalgebra. Let $\mathfrak{a}^*$ be its $\mathbb{C}$-span. Let $W\subseteq GL(\mathfrak{t}^*)$ be the Weyl group and let $\overline{\varrho}\in\mathfrak{t}^*$ be the half-sum of the positive roots. Then the shifted action of W on $\mathfrak{t}^*$ is defined by $w\bullet\chi = w(\chi+\overline{\varrho})-\overline{\varrho}$.

3.1. Theorem ([Kn1], 4.4, 4.8, 4.9, 4.7)**.** *Let V be a multiplicity free representation.*

a) *Each c_D is the restriction of a unique polynomial (also denoted c_D) on $\mathfrak{a}^*$.*
b) *There is a subgroup $W_V\subseteq W$ such that $\mathfrak{a}^*\subseteq\mathfrak{t}^*$ is W_V-stable with respect to the shifted action and such that $D\mapsto c_D$ is an algebra isomorphism of $(\mathbb{PD})^G$ with $\mathbb{C}[\mathfrak{a}^*]^{W_V^\bullet}$, the space of shifted W_V-invariant polynomials on $\mathfrak{a}^*$.*
c) *The "little Weyl group" W_V acts as a reflection group on $\mathfrak{a}^*$. In particular, $(\mathbb{PD})^G$ and $\mathbb{C}[\mathfrak{a}^*]^{W_V^\bullet}$ are polynomial rings.*

Since $(\mathbb{PD})^G$ has a distinguished basis, we obtain a basis $c_\lambda = c_{D_\lambda}$ of $\mathbb{C}[\mathfrak{a}^*]^{W_V^\bullet}$. There is a purely combinatorial characterization of the c_λ:

3.2. Theorem ([Kn1], 4.10)**.** *The polynomial $c_\lambda\in\mathbb{C}[\mathfrak{a}^*]^{W_V^\bullet}$ is, up to a scalar factor, characterized by the vanishing condition $c_\lambda(\mu)=0$ for all $\mu\in\Lambda_V$ with $|\mu|\le|\lambda|$ and $\mu\neq\lambda$.*

One can eliminate the shifted action of W_V as follows: choose a W_V-stable complement $\mathfrak{a}_0$ of $\mathfrak{a}^*$ in $\mathfrak{t}^*$ and let $\overline{\varrho} = \varrho + \varrho_0$ with $\varrho \in \mathfrak{a}^*$ and $\varrho_0 \in \mathfrak{a}_0$. The condition that $\mathfrak{a}^*$ is shifted W_V-stable means $w\overline{\varrho} - \overline{\varrho} \in \mathfrak{a}^*$ for all $w \in W_V$. Thus, ϱ_0 is W_V-fixed. Therefore, we can define the shifted W_V-action as well with $\overline{\varrho}$ replaced by ϱ. Actually, one can add to ϱ any fixed vector in $\mathfrak{a}^*$ without changing the shifted action. The point is now that $p_\lambda(\chi) := c_\lambda(\chi - \varrho)$ is a truly W_V-invariant polynomial on $\mathfrak{a}^*$.

3.3. Corollary. *The polynomial $p_\lambda \in \mathbb{C}[\mathfrak{a}^*]^{W_V}$ is, up to a scalar factor, characterized by the vanishing condition $p_\lambda(\varrho + \mu) = 0$ for all $\mu \in \Lambda_V$ with $|\mu| \le |\lambda|$ and $\mu \ne \lambda$.*

Now we say a few words about the zonal spherical functions E_λ. One of their main features is that they have many different interpretations. First, we can consider $V \oplus V^*$ as the cotangent bundle of V. Then *the symbol of D_λ is E_λ*. This is our principal method for their study.

It is possible to define E_λ without reference to Capelli operators. Every differential operator $D \in \mathbb{PD}(V)$ is also a differential operator on $V \oplus V^*$ by acting on the first argument. As such it is denoted $D^{(1)}$. Observe, that the eigenspaces of $\mathbb{PD}^G$ are then just the spaces $\mathbb{P}_\lambda \otimes \mathbb{D}$. Therefore one can characterize E_λ as *the (up to scalar) unique G-invariant function f on $V \oplus V^*$ with $D^{(1)}(f) = c_D(\lambda) f$ for all $D \in \mathbb{PD}^G$*. Clearly, it suffices to let D run through a set of generators of $\mathbb{PD}^G$.

There is also a "Chevalley isomorphism" for G-invariant functions on $V \oplus V^*$:

3.4. Theorem ([Kn1], 4.2, 4.8, 4.5)**.** *There is $v^* \in V^*$ and a linear embedding $\mathfrak{a}^* \hookrightarrow V$ such that the restriction map $f \mapsto f|_{\mathfrak{a}^* \times v^*}$ induces an isomorphism $(\mathbb{P} \otimes \mathbb{D})^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{a}^*]^{W_V}$. Moreover, the image of the symbol of $D \in (\mathbb{PD})^G$ is the highest degree component $\overline{c}_D$ of c_D. In particular, E_λ is mapped to $\overline{c}_\lambda$.*

The subspace $\mathfrak{a}^*$ is constructed as follows: choose $v^* \in V^*$ in the open G-orbit. Then choose a Borel subalgebra $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u} \subseteq \operatorname{Lie} G$ such that $\mathfrak{b}v^* = V^*$. This is possible, since V^* also has a dense orbit for any Borel subgroup. The surjective map $\mathfrak{b} \twoheadrightarrow V^* : \xi \mapsto \xi v^*$ induces the dual injective map $\iota : V \hookrightarrow \mathfrak{b}^*$. Via the projection $\mathfrak{b} \twoheadrightarrow \mathfrak{b}/\mathfrak{u} = \mathfrak{t}$ we have $\mathfrak{t}^* \subseteq \mathfrak{b}^*$. Now, one can show that $\iota(V) \cap \mathfrak{t}^* = \mathfrak{a}^*$ which furnishes us with the desired embedding $\mathfrak{a}^* \hookrightarrow V$.

Theorem 3.4 indicates another way to interpret E_λ. Let $H^* \subseteq G$ be the isotropy subgroup of v^*. Then the orbit Gv^* is open in V^* and isomorphic to G/H^*. Therefore, a function f on $V \oplus V^*$ is G-invariant if and only if its restriction to $V \times v^*$ is H^*-invariant. Thus the restriction $E_{v^*,\lambda}(v) := E_\lambda(v, v^*)$ is *the (up to a scalar) unique H^*-invariant function f on V with $D(f) = c_D(\lambda) f$ for all $D \in \mathbb{PD}^G$*. The restriction map from V to $\mathfrak{a}^*$ defines now an isomorphism of the algebra of H^*-invariants with $\mathbb{C}[\mathfrak{a}^*]^{W_V}$. Thereby, the function $E_{v^*,\lambda}$ is mapped to the highest degree component $\overline{c}_\lambda$ of c_λ.

Another interpretation is as follows: let $K \subseteq G$ be a maximal compact subgroup. Let $V_\mathbb{R}$ be V regarded as a real vector space. It is equipped with a complex conjugation $v \mapsto \overline{v}$. Then we can regard $\mathbb{D}$ as the algebra of polynomials in the antiholomorphic variables $\overline{z}_i$ and $\mathbb{P} \otimes \mathbb{D}$ is the algebra of all $\mathbb{C}$-valued polynomials on $V_\mathbb{R}$. Thus, the polynomial $v \mapsto E_\lambda(v, \overline{v})$ is *the (up to a scalar) unique K-invariant function f on $V_\mathbb{R}$ with $D(f) = c_D(\lambda) f$ for all $D \in \mathbb{PD}^G$*.

Observe that V also has a dense G-orbit Gv with isotropy group denoted by H. By restriction we can interpret E_λ also as a G-invariant function on $G/H \times G/H^*$, as an H^*-invariant function on G/H, or as a function on G which is constant on double cosets for H and H^*. In this last form, E_λ can be interpreted purely representation theoretically: Let M be a simple G-module which is isomorphic to $\mathbb{P}_\lambda$ for some λ. Then M^H and $(M^*)^{H^*}$ are both one-dimensional, generated by vectors m_λ and α_λ, respectively. Then $g \mapsto E_\lambda(gv, v^*)$ equals (up to a scalar) *the matrix coefficient* $g \mapsto \alpha_\lambda(gm_\lambda)$. In fact, if we identify M with $\mathcal{P}_\lambda$, then m is just the evaluation $f \mapsto f(v)$. Similarly, $M^* \cong \mathbb{D}_\lambda$ and α is an evaluation in v^*. Finally, E_λ corresponds to the canonical pairing $M \times M^* \to \mathbb{C}$.

Now we are in a position to explain the representation theoretic relevance of the polynomials $R_\lambda(z;r)$.

The case of $G = GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$ *acting on* $V := (\mathbb{C}^p \otimes \mathbb{C}^q) \oplus \mathbb{C}^q$. The following data are taken from [Kn1], p. 315. Put $n := \min(2p+1, 2q)$. Then $\overline{n} = \min(p+1, q)$ and $\underline{n} = \min(p,q)$. Let ε_i and ε'_i be the weights of the defining representation of $GL_p(\mathbb{C})$ and $GL_q(\mathbb{C})$, respectively. Moreover, let $\omega_i := \sum_{j=1}^i \varepsilon_i$, $\omega'_i := \sum_{j=1}^i \varepsilon'_i$. Then Λ_V is the free abelian monoid generated by $\omega_{i-1} + \omega'_i$ for $i = 1, \ldots, \overline{n}$ and $\omega_i + \omega'_i$ for $i = 1, \ldots, \underline{n}$. Thus, if we put $e_{2i-1} := \varepsilon'_i$ for $i = 1, \ldots, \overline{n}$ and $e_{2i} := \varepsilon_i$ for $i = 1, \ldots, \underline{n}$, then Λ_V consists of all $\chi = \sum_{i=1}^n \lambda_i e_i$ where $(\lambda_1, \ldots, \lambda_n)$ is a partition. The degree function is such that $|\omega_i| = 0$ and $|\omega'_i| = i$ which translates into $|\chi| = |\lambda|_{\mathrm{odd}}$. The little Weyl group consists of all permutations of the ε_i and ε'_i separately, i.e., W_V is the semisymmetric group. Finally, we have $\overline{\varrho} = (\frac{p-1}{2}, \frac{p-3}{2}, \ldots; \frac{q-1}{2}, \frac{q-3}{2}, \ldots)$. Thus, if we project it to the first $\overline{n} + \underline{n}$ coordinates and shift it by a suitable W_V-fixed vector, we arrive at $\varrho = (\frac{n-1}{2}, \frac{n-3}{2}, \ldots; \frac{n-2}{2}, \frac{n-4}{2}, \ldots) = \frac{1}{2}\sum_i (n-i)e_i$. This shows $r = \frac{1}{2}$. In particular, $c_\chi(x)$ is a multiple of $\tilde{R}_\lambda(x; \frac{1}{2})$.

Now we describe the combinatorics in more classical terms. For this, it is convenient to write

$$V = (\mathbb{C}^p \oplus \mathbb{C}) \otimes \mathbb{C}^q = \mathbb{C}^{p+1} \otimes \mathbb{C}^q, \tag{3.2}$$

i.e., V is the space of $(p+1) \times q$-matrices X acted upon by

$$G = GL_p(\mathbb{C}) \times GL_q(\mathbb{C}) \subseteq \overline{G} := GL_{p+1}(\mathbb{C}) \times GL_q(\mathbb{C}) \tag{3.3}$$

by $X \mapsto AXB^t$ with $A \in GL_p(\mathbb{C}) \subseteq GL_{p+1}(\mathbb{C})$ and $B \in GL_q(\mathbb{C})$.

Let Λ_∞ be the set of infinite partitions, i.e., descending sequences of integers $\tau_1 \geq \tau_2 \geq \ldots$ with $\tau_i = 0$ for $i \gg 0$. The length $\ell(\tau)$ is the maximal i with $\tau_i \neq 0$. Every $\tau \in \Lambda_\infty$ with $\ell(\tau) \leq p$ parametrizes an irreducible (polynomial) representation $M_\tau^{(p)}$ of $GL_p(\mathbb{C})$.

Let $\overline{n} := \min(p+1, q)$. Then it is well known (see e.g. [GW] Thm. 5.2.7) that there is a decomposition of $\overline{G}$-modules:

$$\mathbb{P} = \sum_{\substack{\tau \in \Lambda_\infty \\ \ell(\tau) \leq \overline{n}}} M_\tau^{(p+1)} \otimes M_\tau^{(q)}. \tag{3.4}$$

Recall also the branching law of $GL_{p+1}(\mathbb{C})$ to $GL_p(\mathbb{C})$ (see e.g. [GW] Thm. 8.1.1): as a $GL_p(\mathbb{C})$-module we have

$$M_\tau^{(p+1)} = \sum_\sigma M_\sigma^{(p)} \tag{3.5}$$

where σ runs through all partitions with $\ell(\sigma) \leq p$ and which are "interlaced" with τ, i.e., with $\tau_1 \geq \sigma_1 \geq \tau_2 \geq \sigma_2 \geq \ldots$. Thus we have $\ell(\sigma) \leq \underline{n} := \min(p,q)$. Now we can make the decomposition of $\mathbb{P}$ into simple G-modules more explicit. Combine τ and σ to a single partition λ by putting $\lambda_{2i-1} := \tau_i$ and $\lambda_{2i} := \sigma_i$. Then, as a G-module, we have

$$\mathbb{P} = \sum_{\lambda\in\Lambda} \mathbb{P}_\lambda \qquad \text{with} \qquad \mathbb{P}_\lambda = M^{(p)}_{\lambda_{\text{even}}} \otimes M^{(q)}_{\lambda_{\text{odd}}}. \tag{3.6}$$

Here, we use the fact that $\ell(\lambda) \le n := \overline{n} + \underline{n} = \min(2p+1, 2q)$. Therefore, one can regard λ as an element of $\Lambda = \Lambda_n$.

This gives also a nice interpretation of the comparison Theorems 2.6 and 2.7. Let $V' = \mathbb{C}^p \otimes \mathbb{C}^q$ be the space of $p \times q$ matrices. Since V projects onto V', we have $\mathbb{C}[V'] \subseteq \mathbb{P}$. More precisely,

$$\mathbb{C}[V'] = \sum_{\substack{\lambda\in\Lambda\\ \lambda_{\text{even}}=\lambda_{\text{odd}}}} \mathbb{P}_\lambda. \tag{3.7}$$

Thus, every Capelli operator on V' can be regarded as a Capelli operator on V. This is reflected in formula (2.13).

On the other hand, each $\overline{G}$-invariant Capelli operator on V decomposes as a sum of G-invariant Capelli operators on V. This is the origin of formula (2.16).

We can make this fully explicit for the generators $D_{(1^a)}$. Let $A \in V$ be a $(p+1)\times q$-matrix. For subsets $I \subseteq [p+1] := \{1,\ldots,p+1\}$ and $J \subseteq [q] := \{1,\ldots,q\}$ of the same size i let

$$\det{}_I^J(A) = \det\left(a_{ij}\right)_{\substack{i\in I\\ j\in J}} \tag{3.8}$$

be the corresponding minor. These form a basis of $M^{(p+1)}_{(1^i)} \otimes M^{(q)}_{(1^i)} = \bigwedge^i(\mathbb{C}^{p+1})^* \otimes \bigwedge^i(\mathbb{C}^q)^*$. If V is parametrized by the coordinate functions a_{ij}, let ∂_A be the matrix with entries $\frac{\partial}{\partial a_{ij}}$. Then the classical $\overline{G}$-invariant Capelli operators on V are

$$C_i := \sum_{\substack{I\subseteq[p+1],J\subseteq[q]\\ |I|=|J|=i}} \det{}_I^J(A)\det{}_I^J(\partial_A), \qquad i = 1,\ldots,\overline{n}. \tag{3.9}$$

Now each $\bigwedge^i(\mathbb{C}^{p+1})^*$ decomposes as a G-module into two pieces:

$$\bigwedge^i(\mathbb{C}^{p+1})^* = \bigwedge^i(\mathbb{C}^p \oplus \mathbb{C})^* = \bigwedge^i(\mathbb{C}^p)^* \oplus \bigwedge^i -1(\mathbb{C}^p)^*. \tag{3.10}$$

Thus C_i also decomposes as $C_i = D_{(1^{2i})} + D_{(1^{2i-1})}$ with

$$\begin{aligned} D_{(1^{2i})} &= \sum_{\substack{I\subseteq[p],J\subseteq[q]\\ |I|=|J|=i}} \det{}_I^J(A)\det{}_I^J(\partial_A),\\ D_{(1^{2i-1})} &= \sum_{\substack{I\subseteq[p+1],J\subseteq[q]\\ p+1\in I,|I|=|J|=i}} \det{}_I^J(A)\det{}_I^J(\partial_A). \end{aligned} \tag{3.11}$$

For example, for $n = 3$, i.e. $p+1 = q = 2$, we have

$$D_{(1)} = a_{21}\frac{\partial}{\partial a_{21}} + a_{22}\frac{\partial}{\partial a_{22}}, \tag{3.12}$$

$$D_{(11)} = a_{11}\frac{\partial}{\partial a_{11}} + a_{12}\frac{\partial}{\partial a_{12}}, \tag{3.13}$$

$$D_{(111)} = (a_{11}a_{22} - a_{12}a_{21})\left(\frac{\partial}{\partial a_{11}}\frac{\partial}{\partial a_{22}} - \frac{\partial}{\partial a_{12}}\frac{\partial}{\partial a_{21}}\right). \tag{3.14}$$

Now, we explain the zonal spherical functions. We identify V^* with the space of $q \times (p+1)$-matrices and the pairing $V \times V^* \to \mathbb{C}$ is given by $(A, A^*) \mapsto \operatorname{tr}(AA^*)$. By definition, E_λ is a G-invariant function on $V \oplus V^*$ which is a joint eigenvector of the differential operators $D_{(1^i)}$ (see (3.11)) acting on the first factor. To make the Chevalley isomorphism from Theorem 3.4 explicit, we define for $p, q \geq 1$ the following two matrices $\Xi_{p,q} = \Xi_{p,q}(z_1, \ldots, z_n) \in V$ and $\Xi^*_{p,q} \in V^*$:

$$(3.15)\qquad \begin{aligned} (\Xi_{p,q})_{ij} &= \begin{cases} z_{2i} & \text{if } i = j,\ i \leq p, \\ z_{2i} - z_{2i+1} & \text{if } i < j,\ i \leq p, \\ u_{2j} & \text{if } i = p+1, \\ 0 & \text{otherwise;} \end{cases} \\ (\Xi^*_{p,q})_{ij} &= \begin{cases} 1 & \text{if } i = j, \\ 1 & \text{if } i = q < j = p+1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here, we put $u_i := z_1 - z_2 + z_3 - + \ldots \pm z_i$ and $z_i = 0$ for $i > 2p+1$.

For example, for $p = 3 < q$, $n = 7$ we have

$$(3.16)\qquad \begin{aligned} \Xi_{3,q}(z) &= \begin{pmatrix} z_2 & z_2 - z_3 & z_2 - z_3 & z_2 - z_3 & \cdots \\ 0 & z_4 & z_4 - z_5 & z_4 - z_5 & \cdots \\ 0 & 0 & z_6 & z_6 - z_7 & \cdots \\ u_2 & u_4 & u_6 & u_7 & \cdots \end{pmatrix}, \\ \Xi^*_{3,q} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \end{aligned}$$

while for $p \geq q = 3$, $n = 6$ we have

$$(3.17)\qquad \begin{aligned} \Xi_{p,3}(z) &= \begin{pmatrix} z_2 & z_2 - z_3 & z_2 - z_3 \\ 0 & z_4 & z_4 - z_5 \\ 0 & 0 & z_6 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ u_2 & u_4 & u_6 \end{pmatrix}, \\ \Xi^*_{p,3} &= \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 1 \end{pmatrix}. \end{aligned}$$

Let $B_p \subseteq GL_p(\mathbb{C})$ be the subgroup of upper triangular matrices. Let $B_q \subseteq GL_q(\mathbb{C})$ be the stabilizer of the flag $\langle v_1 \rangle, \langle v_1, v_2 \rangle, \ldots$ where $v_i := e_i + \ldots + e_q$ and where e_i is the i-th canonical basis vector of $\mathbb{C}^q$. Then $B := B_p \times B_q$ is a Borel subgroup of G. One can verify by a straightforward but tedious calculation that $z \mapsto \Xi_{p,q}(z)$ is the embedding $\mathfrak{a}^* \hookrightarrow V$ when one follows the recipe described after Theorem 3.4 using $v^* = \Xi^*_{p,q}$ and the Borel subgroup B.

It follows that for every zonal spherical function $E_\lambda(A, A^*)$ the restriction $E_\lambda(\Xi(z), \Xi^*)$ is proportional to $\overline{R}_\lambda(z; \frac{1}{2})$. Since E_λ is the symbol of D_λ, we obtain $\overline{R}_\lambda(z; \frac{1}{2})$ also from D_λ by replacing all coordinate functions a_{ij} by $\Xi_{ij}(z)$ and all derivations $\frac{\partial}{\partial a_{ij}}$ by 1 if $i = j$ or $i = p+1 > j = q$ and 0 otherwise. For example, in the case $p + 1 = q = 2$ according to (3.12)–(3.14) we get

$$D_{(1)} \mapsto u_3 = z_1 - z_2 + z_3 = \overline{R}_{(1)}(z), \qquad D_{(11)} \mapsto z_2 = \overline{R}_{(11)}(z), \tag{3.18}$$

$$D_{(111)} \mapsto z_2u_3 - (z_2 - z_3)u_2 = z_1z_3 = \overline{R}_{(111)}(z). \tag{3.19}$$

For the other interpretations of zonal spherical functions we just mention the case when $p + 1 = q$ since that is the only case when the isotropy groups H and H^* are reductive. In fact, in that case we have $H = H^* = GL_{q-1}(\mathbb{C})$ embedded diagonally into G. Thus, the action of H^* on V is just the action of $GL_{q-1}(\mathbb{C}) \subseteq GL_q(\mathbb{C})$ by conjugation on $q \times q$-matrices. The matrix $\Xi^*_{p,q}$ is the identity matrix I_q. Thus, the function $E_\lambda(A, I_q)$ is a joint eigenfunction of the differential operators $D_{(1^i)}$ which is invariant under conjugation by $GL_{q-1}(\mathbb{C})$. Any conjugation invariant function is uniquely determined by its value in $\Xi_{p,q}(z)$ and we have $E_\lambda(\Xi_{p,q}(z), I_q) = \overline{R}_\lambda(z; \frac{1}{2})$.

The case of $G = GL_n(\mathbb{C})$ acting on $V = \bigwedge^2 \mathbb{C}^n \oplus \mathbb{C}^n$. We keep the notation of the previous example. According to the data in [Kn1], p. 314, the weight monoid Λ_V is freely generated by ω_i for $i = 1, \ldots, n$. Thus, if we set $e_i := \varepsilon_i$ for all i, then Λ_V consists of all $\chi = \sum_{i=1}^n \lambda_i e_i$ where $(\lambda_1, \ldots, \lambda_n)$ is a partition. The degree function is given by $|\omega_i| = \lceil \frac{i}{2} \rceil$. Thus $|\chi| = |\lambda|_{\mathrm{odd}}$. The little Weyl group permutes the ε_i with even and odd indices separately and therefore equals the semisymmetric group. Finally, $\overline{\varrho} = (\frac{n-1}{2}, \frac{n-3}{2}, \ldots)$. Thus we can choose $\varrho = \sum_i (n-i)e_i$ which shows $r = 1$. In particular, $c_\chi(x)$ is a multiple of $\tilde{R}_\lambda(x; 1)$.

Again, this can be made more explicit. Observe, that G is a subgroup of $\overline{G} := GL_{n+1}(\mathbb{C})$ and V is the restriction of the natural $\overline{G}$-action on $\bigwedge^2 \mathbb{C}^{n+1}$ to G. It is known (see e.g. [GW] Thm. 5.2.11) that as a $\overline{G}$-module

$$\mathbb{P} = \sum_\tau M_\tau^{(n+1)} \tag{3.20}$$

where τ runs through all partitions with $\tau_1 = \tau_2$, $\tau_3 = \tau_4, \ldots$ and $\ell(\tau) \le n+1$. Now, we use again the $GL_{n+1} - GL_n$ branching rule. For λ to be interlaced with τ means $\lambda_1 = \tau_1$, $\lambda_3 = \tau_3, \ldots$. Thus, as a G-module, we obtain

$$\mathbb{P} = \sum_\lambda M_\lambda^{(n)} \tag{3.21}$$

where λ runs through all partitions with $\ell(\lambda) \le n$. Here $M_\lambda^{(n)}$ is sitting in $M_{\lambda^*}^{(n+1)}$ where $\lambda^* = (\lambda_1, \lambda_1, \lambda_3, \lambda_3, \ldots)$.

An element of V is represented by a skewsymmetric matrix $A = \big(a_{ij}\big)$ of size $n+1$. For $I \subseteq [n+1]$ of *even* size $2m$ let

$$\mathrm{Pf}_I(A) := \mathrm{Pfaffian}\,\big(a_{ij}\big)_{\substack{i\in I\\ j\in I}}. \tag{3.22}$$

Then, the Capelli operators for $\overline{G}$ corresponding to simple weights are

$$C_m := \sum_{\substack{I\subseteq[n+1]\\ |I|=2m}} \mathrm{Pf}_I(A)\,\mathrm{Pf}_I(\partial_A). \tag{3.23}$$

Each $\overline{G}$-module $M^{(n+1)}_{(1^{2m})}$ decomposes into (at most) two components, namely, $M^{(n)}_{(1^{2m-1})}$ and $M^{(n)}_{(1^{2m})}$. Therefore, C_m also decomposes as $C_m = D_{(1^{2m-1})} + D_{(1^{2m})}$ where

$$\begin{aligned} D_{(1^{2m})} &= \sum_{\substack{I\subseteq[n]\\ |I|=2m}} \operatorname{Pf}_I(A)\operatorname{Pf}_I(\partial_A),\\ D_{(1^{2m-1})} &= \sum_{\substack{I\subseteq[n+1]\\ n+1\in I,|I|=2m}} \operatorname{Pf}_I(A)\operatorname{Pf}_I(\partial_A). \end{aligned} \tag{3.24}$$

For example, for $n=3$ we get

$$D_{(1)} = a_{14}\frac{\partial}{\partial a_{14}} + a_{24}\frac{\partial}{\partial a_{24}} + a_{34}\frac{\partial}{\partial a_{34}} \tag{3.25}$$

$$D_{(11)} = a_{12}\frac{\partial}{\partial a_{12}} + a_{31}\frac{\partial}{\partial a_{31}} + a_{23}\frac{\partial}{\partial a_{23}} \tag{3.26}$$

$$\begin{aligned} D_{(111)} = (a_{12}a_{34} - a_{13}a_{24} + a_{23}a_{14})\Big(&\frac{\partial}{\partial a_{12}}\frac{\partial}{\partial a_{34}}\\ &- \frac{\partial}{\partial a_{13}}\frac{\partial}{\partial a_{24}} + \frac{\partial}{\partial a_{23}}\frac{\partial}{\partial a_{14}}\Big) \end{aligned} \tag{3.27}$$

To describe the zonal spherical functions we identify V^* also with skewsymmetric matrices of size $n+1$ and pairing $V\times V^*\to\mathbb{C}:(A,A^*)\mapsto\frac{1}{2}\operatorname{tr}(AA^*)$. The function E_λ is a G-invariant function on $V\oplus V^*$ which is a joint eigenvector of the differential operators $D_{(1^i)}$ defined in (3.24) acting on the first argument.

To make the Chevalley isomorphism explicit we define skewsymmetric matrices $T(z)$ and T^*:

$$T(z) := \begin{pmatrix} 0 & -\Xi^t_{\underline{n},\overline{n}}\\ \Xi_{\underline{n},\overline{n}} & 0\end{pmatrix},\quad T^* := \begin{pmatrix} 0 & -\Xi_{\underline{n},\overline{n}}\\ \Xi^t_{\underline{n},\overline{n}} & 0\end{pmatrix} \tag{3.28}$$

where Ξ and Ξ^* are defined in (3.15). For example, for $n=4$ we get (again setting $u_i := z_1 - z_2 + z_3 - + \ldots \pm z_i$)

$$T(z) = \begin{pmatrix} 0 & 0 & -z_2 & 0 & -u_2\\ 0 & 0 & -z_2+z_3 & -z_4 & -u_4\\ z_2 & z_2-z_3 & 0 & 0 & 0\\ 0 & z_4 & 0 & 0 & 0\\ u_2 & u_4 & 0 & 0 & 0\end{pmatrix},\quad T^* = \begin{pmatrix} 0&0&-1&0&0\\ 0&0&0&-1&-1\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&1&0&0&0\end{pmatrix} \tag{3.29}$$

while for $n=5$ one has

$$
\begin{aligned}
T(z) &= \begin{pmatrix}
0 & 0 & 0 & -z_2 & 0 & -u_2 \\
0 & 0 & 0 & -z_2+z_3 & -z_4 & -u_4 \\
0 & 0 & 0 & -z_2+z_3 & -z_4+z_5 & -u_5 \\
z_2 & z_2-z_3 & z_2-z_3 & 0 & 0 & 0 \\
0 & z_4 & z_4-z_5 & 0 & 0 & 0 \\
u_2 & u_4 & u_5 & 0 & 0 & 0
\end{pmatrix}, \\
T^* &= \begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.
\end{aligned}
\tag{3.30}
$$

Let $B \subseteq G$ be the Borel subgroup which is the stabilizer of the flag $\langle v_1\rangle$, $\langle v_1, v_2\rangle, \ldots$ where $v_i = \sum_{\frac{i}{2}\le j\le \overline{n}+\frac{i}{2}} e_j$. Then one can verify that $z \mapsto T(z)$ is the embedding of $\mathfrak{a}^*$ into V which is induced from $v^* = T^*$ and the Borel subgroup B. Thus, we get that the restricted zonal spherical function $E_\lambda(T(z), T^*)$ is proportional to $\overline{R}_\lambda(z; 1)$.

The other interpretations of spherical functions are most interesting when n is odd. Then $H = H^* = Sp_{n-1}(\mathbb{C})$ and $A \mapsto E_\lambda(A, T^*)$ is an $Sp_{n-1}(\mathbb{C})$-invariant function on the space V of skewsymmetric matrices of size $n+1$ which is a joint eigenfunction for the differential operators $D_{(1^i)}$. Any invariant function is uniquely determined by its value at $T(z)$ and we have $E_\lambda(T(z), T^*) = \overline{R}_\lambda(z; 1)$.

The open G-orbit in V is isomorphic to $G/H = GL_n(\mathbb{C})/Sp_{n-1}(\mathbb{C})$. Thus, the pullback of an $Sp_{n-1}(\mathbb{C})$-invariant function on V leads to an $Sp_{n-1}(\mathbb{C})$-bi-invariant function on $GL_n(\mathbb{C})$. Clearly, not all of them are of this form. For this, we have to make the function $A \mapsto \mathrm{Pf}(A)$ invertible since its zero-set is the complement of the open orbit. Since $E_{(1^n)}(A, T^*) = \mathrm{Pf}(A)$, this corresponds with $\mathfrak{a}^*$ to make the function $\pi(z) := \overline{R}_{(1^n)}(z) = \prod_{i \text{ odd}} z_i$ invertible. We can extend the definition of $\overline{R}_\lambda(z)$ to every element $\lambda \in \Lambda' := \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \ge \ldots \ge \lambda_n\}$ by $\overline{R}_\lambda := \pi^{-m}\overline{R}_{\lambda+m(1^n)}$ for $m \gg 0$ (by Corollary 2.4, this is independent of the choice of m). Then $\overline{R}_\lambda(z; 1)$, $\lambda \in \Lambda'$ is the radial part of an $Sp_{n-1}(\mathbb{C})$-bi-invariant function on $GL_n(\mathbb{C})$ which is a joint eigenfunction for all $GL_n(\mathbb{C})$-bi-invariant differential operators. A similar result holds if $GL_n(\mathbb{C})$ is replaced by $SL_n(\mathbb{C})$. Then $\mathfrak{a}^*$ should be replaced by $\{z \in \mathfrak{a}^* \mid \delta(z) = 1\}$ and λ should be an element of $\Lambda'/\mathbb{Z}(1^n)$.

The case of $G = GL_p(\mathbb{C}) \times GL_1(\mathbb{C})$ *acting on* $(\mathbb{C}^p \otimes \mathbb{C}) \oplus (\mathbb{C}^p)^*$. The action of $(A, s) \in G$ on a pair of vectors (u, v) is $(sAu, (A^t)^{-1}v)$. Here Λ_V is generated by $\varepsilon_1 + \varepsilon'$, $-\varepsilon_p$, and ε' with degrees 1, 1, and 2, respectively. Thus, if we put

$$
e_1 = -\varepsilon_p,\ e_2 = \varepsilon_1 + \varepsilon_p + \varepsilon',\ e_3 = -\varepsilon_1,
\tag{3.31}
$$

then the generating weights become e_1, $e_1 + e_2$, and $e_1 + e_2 + e_3$. In particular, the degree of e_i is 1, 0, 1, respectively. The little Weyl group is generated by the permutation which swaps ε_1 and ε_p, and therefore e_1 and e_3. Thus, the Capelli operators are described by semisymmetric polynomials in $n = 3$ variables. The vector $\overline{\varrho} = (\frac{p-1}{2}, \ldots, -\frac{p-1}{2}; 0)$ equals, up to a W_V-fixed vector $(\frac{p-1}{2}, 0, \ldots, 0, -\frac{p-1}{2}; \frac{p-1}{2}) = \frac{p-1}{2}(\varepsilon_1 - \varepsilon_p + \varepsilon) = \frac{p-1}{2}(2e_1 + e_2)$. This shows $r = \frac{p-1}{2}$.

The concrete decomposition of $\mathbb{P}$ as a G-module has been worked out in [VS]; see also [VK], §11.1–11.2. Here, we give only the fundamental Capelli operators.

Denote the coordinates of V by $u_1, \ldots, u_p; v_1, \ldots, v_p$. Then

$$D_{(1)} = \sum_{i=1}^{p} v_i \frac{\partial}{\partial v_i}, \ D_{(11)} = \sum_{i=1}^{p} u_i \frac{\partial}{\partial u_i}, \ D_{(111)} = (\sum_i u_i v_i)(\sum_i \frac{\partial}{\partial u_i}\frac{\partial}{\partial v_i}). \tag{3.32}$$

The zonal spherical functions have been investigated by Vilenkin–Šapiro [VS] (see also [VK] 11.3.2). They are eigenfunctions for the three differential operators $D_{(1)}$, $D_{(11)}$, and $D_{(111)}$ above. For the Chevalley isomorphism we define

$$\begin{aligned} u(z) &= (z_2, 0, \ldots, 0, z_2 - z_3); \\ v(z) &= (z_1 - z_2 + z_3, 0, \ldots, 0, -z_1 + z_2); \\ u_0^* &= (1, 0, \ldots, 0); \\ v_0^* &= (1, 0, \ldots, 0). \end{aligned} \tag{3.33}$$

Then $z \mapsto (u(z), v(z))$ is the embedding $\mathfrak{a}^* \hookrightarrow V$ which corresponds to $(u_0^*, v_0^*) \in V^*$ and the Borel subgroup which stabilizes the flag $\langle e_1 + e_n \rangle, \langle e_1 + e_n, e_2 \rangle, \ldots, \langle e_1 + e_n, e_2, \ldots, e_n \rangle$.

To be eigenfunction for $D_{(1)}$ and $D_{(11)}$ simply means to be bihomogeneous in the u- and v-coordinates. Since $H^* = GL_{n-1}(\mathbb{C})$, we get the following interpretation of zonal spherical functions: they are bihomogeneous $GL_{n-1}(\mathbb{C})$-invariant functions on $\mathbb{C}^n \oplus (\mathbb{C}^n)^*$ which are eigenfunctions for the "Laplace operator" $D_{(111)}$.

This interpretation has also a real form: The complexification of $U(n-1)$ is $H^* = GL_{n-1}(\mathbb{C})$ while V is the complexification of $\mathbb{C}^n$, considered as an $\mathbb{R}$-vector space. The coordinate function v_i is then simply the complex conjugate $\overline{u}_i$ of u_i. Thus, $E_\lambda(u, \overline{u}, u_0^*, v_0^*)$ is a $U(n-1)$-invariant function on $\mathbb{C}^n$ which is bihomogeneous in the holomorphic and the antiholomorphic variables and which is an eigenfunction of the (now genuine) Laplace operator $D_{(111)} = \sum_i \frac{\partial^2}{\partial u_i \partial \overline{u}_i}$. In this form, the E_λ have been studied by Vilenkin–Šapiro [VS].

4. Difference operators

For $\lambda \in k^n$ let T_λ be the shift operator $T_\lambda f(z) := f(z - \lambda)$. Let ε_i be the i-th canonical basis vector of k^n and $T_i := T_{\varepsilon_i}$. For reasons of clarity we adopt the following notation: $x_i := z_{2i-1}$, $y_i := z_{2i}$, $T_{x,i} := T_{2i-1}$, and $T_{y,i} := T_{2i}$. Then we define the following block matrices whose entries are difference operators (where t is an indeterminate):

$$\mathfrak{X}(t) := \begin{pmatrix} \Big[(x_i{+}t)(x_i{+}r)^{\overline{n}-j} - x_i^{\overline{n}+1-j} T_{x,i}\Big]_{\substack{i=1\ldots\overline{n}\\ j=1\ldots\overline{n}}} & \Big[-x_i^{\overline{n}-j} T_{x,i}\Big]_{\substack{i=1\ldots\overline{n}\\ j=1\ldots\underline{n}}} \\ \Big[(y_i{+}r)^{\underline{n}+1-j} - y_i^{\underline{n}+1-j} T_{y,i}\Big]_{\substack{i=1\ldots\underline{n}\\ j=1\ldots\overline{n}}} & \Big[(y_i{+}r)^{\underline{n}-j}\Big]_{\substack{i=1\ldots\underline{n}\\ j=1\ldots\underline{n}}} \end{pmatrix}, \tag{4.1}$$

$$\mathfrak{Y}(t) := \begin{pmatrix} \Big[(x_i{+}r)^{\overline{n}-j}\Big]_{\substack{i=1\ldots\overline{n}\\ j=1\ldots\overline{n}}} & \Big[(x_i{+}r)^{\overline{n}-j} - x_i^{\overline{n}-j} T_{x,i}\Big]_{\substack{i=1\ldots\overline{n}\\ j=1\ldots\underline{n}}} \\ \Big[-y_i^{\underline{n}+1-j} T_{y,i}\Big]_{\substack{i=1\ldots\underline{n}\\ j=1\ldots\overline{n}}} & \Big[(y_i{+}t)(y_i{+}r)^{\underline{n}-j} - y_i^{\underline{n}+1-j} T_{y,i}\Big]_{\substack{i=1\ldots\underline{n}\\ j=1\ldots\underline{n}}} \end{pmatrix}. \tag{4.2}$$

The semisymmetric Vandermonde determinant is

$$\varphi(z) := \prod_{\substack{1 \le i < j \le n \\ j-i \text{ even}}} (z_i - z_j) = \prod_{1 \le i < j \le \overline{n}} (x_i - x_j) \prod_{1 \le i < j \le \underline{n}} (y_i - y_j). \tag{4.3}$$

Now we define the operators

$$X(t) := \varphi(z)^{-1} \det \mathfrak{X}(t) \text{ and } Y(t) := \varphi(z)^{-1} \det \mathfrak{Y}(t). \tag{4.4}$$

First observe, that the entries of $\mathfrak{X}(t)$ and $\mathfrak{Y}(t)$ commute if they are in different rows. Thus, the determinants are well defined.

4.1. Lemma. *Both $X(t)$ and $Y(t)$ act on $\mathcal{P}^W$.*

Proof. Let $f \in \mathcal{P}^W$. Then both $\mathfrak{X}(t)f$ and $\mathfrak{Y}(t)f$ are polynomials which are skewsymmetric with respect to both factors $S_{\overline{n}}$, $S_{\underline{n}}$ of W. Therefore, they are divisible by $\varphi(z)$ and the quotient is W-symmetric. □

4.2. Lemma. *For $f \in \mathcal{P}^W$ holds $\deg X(t)f \le \deg f$ and $\deg Y(t)f \le \deg f$.*

Proof. We use the following elementary fact: let $A = (a_{ij})$ be a matrix with entries in a filtered ring such that $\deg a_{ij} \le d'_i + d''_j$ for integers d'_i and d''_j. Then $\deg \det A \le \sum_i (d'_i + d''_i)$.

We apply this to $\mathfrak{X}(t)$. Using that the operator $1 - T_i$ has degree -1 the entries of $\mathfrak{X}(t)$ have degree $d'_i + d''_j$ with $\varepsilon := \overline{n} - \underline{n}$ and

$$\begin{aligned} (d'_1, \dots, d'_n) &= (0, \dots, 0, 0, -\varepsilon, -\varepsilon, \dots, -\varepsilon), \\ (d''_1, \dots, d''_n) &= (\bar{n} - 1, \dots, 1, 0, \bar{n} - 1, \bar{n} - 2, \dots, \varepsilon). \end{aligned} \tag{4.5}$$

Thus $\deg \det \mathfrak{X}(t) \le \sum_i (d'_i + d''_i) = \deg \varphi(z)$.

For $\mathfrak{Y}(t)$ one argues in the same way with

$$\begin{aligned} (d'_1, \dots, d'_n) &= (0, \dots, 0, 0, 1 - \varepsilon, 1 - \varepsilon, \dots, 1 - \varepsilon), \\ (d''_1, \dots, d''_n) &= (\bar{n} - 1, \dots, 1, 0, \bar{n} - 2, \bar{n} - 3, \dots, \varepsilon - 1). \end{aligned} \tag{4.6}$$

□

Next we derive an explicit formula for $X(t)$ and $Y(t)$. For $I \subseteq \{1, \dots, n\}$ put $\varepsilon_I := \sum_{i \in I} \varepsilon_i$ and $T_I f(z) := f(z - \varepsilon_I)$. Thus $T_I = \prod_{i \in I} T_i$. With this notation we define

$$D_I := \prod_{\substack{i \in I \\ n - i \text{ even}}} z_i \prod_{\substack{i \in I, j \notin I \\ j - i \text{ odd}}} (z_i - z_j - r) \prod_{\substack{i \in I, j \notin I \\ j - i \text{ even}}} (z_i - z_j)^{-1} \, T_I. \tag{4.7}$$

Let P_{odd} be the set of subsets $I \subseteq \{1, \dots, n\}$ such that there is a $w \in W$ with $I = w\{1, \dots, m\}$ (where $m = |I|$). Thus $I \in P_{\text{odd}}$ if and only if the number of its odd members is equal or one more than the number of its even members. Let $P_{\text{even}} \subseteq P_{\text{odd}}$ consist of those sets where these numbers are equal. This is equivalent to $|I|$ being even. Finally, we set

$$|I|_{\text{o}} := |\{ i \in I \mid i \text{ odd} \}| = \left\lceil \frac{|I|}{2} \right\rceil = |\varepsilon_I|_{\text{odd}}. \tag{4.8}$$

4.3. Proposition. *We have*

$$X(t) = \sum_{I \in P_{\text{odd}}} (-1)^{|I|_{\text{o}}} \prod_{\substack{i \notin I \\ i \text{ odd}}} (t + z_i) \; D_I, \tag{4.9}$$

$$Y(t) = \sum_{I \in P_{\text{even}}} (-1)^{|I|_{\text{o}}} \prod_{\substack{i \notin I \\ i \text{ even}}} (t + z_i) \; D_I. \tag{4.10}$$

Proof. Clearly both $X(t)$ and $Y(t)$ have expansions of the form $\sum_I c_I T_I$ where I runs through all subsets of $\{1,\dots,n\}$. First we show that in $Y(t)$ only the $I \in P_{\text{even}}$ contribute. For this, we subtract in (4.2) the columns $1,\dots,\underline{n}$ from the columns $\overline{n}+1,\dots,\overline{n}+\underline{n}=n$, respectively and obtain

$$\det \mathfrak{Y}(t) = \det \begin{pmatrix} \Big[(x_i+r)^{\overline{n}-j}\Big]_{\substack{i=1\dots\overline{n}\\ j=1\dots\overline{n}}} & \Big[-x_i^{\overline{n}-j}T_{x,i}\Big]_{\substack{i=1\dots\overline{n}\\ j=1\dots\underline{n}}} \\ \Big[-y_i^{\underline{n}+1-j}T_{y,i}\Big]_{\substack{i=1\dots\underline{n}\\ j=1\dots\overline{n}}} & \Big[(y_i+t)(y_i+r)^{\underline{n}-j}\Big]_{\substack{i=1\dots\underline{n}\\ j=1\dots\underline{n}}} \end{pmatrix}. \tag{4.11}$$

Next, we use the well known fact that the determinant of a block matrix is

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C)\det D \tag{4.12}$$

if D is invertible. In our case, the entries of $A-BD^{-1}C$ are linear combinations (over the field of rational functions in z) of 1 and the $T_{x,i}T_{y,j}$ which proves the claim.

Since the given form of the operator $Y(t)$ is W-symmetric, it suffices to check the coefficient of $T_I = T_{x,1}\dots T_{x,l}T_{y,1}\dots T_{y,l}$ where $I=\{1,\dots,2l\}$, $l=0,\dots,\underline{n}$. Every entry of the matrix (4.11) is of the form $a+bT_{x,i}$ or $a+bT_{y,i}$. Thus the required coefficient is the determinant of the matrix obtained by replacing that entry by b if $i\le l$ or a if $i>l$. The ensuing matrix has the following form with dimensions as indicated:

$$\begin{pmatrix} 0_{l\times\overline{n}} & *_{l\times\underline{n}} \\ *_{\overline{n}-l\times\overline{n}} & 0_{\overline{n}-l\times\underline{n}} \\ *_{l\times\overline{n}} & 0_{l\times\underline{n}} \\ 0_{\underline{n}-l\times\overline{n}} & *_{\underline{n}-l\times\underline{n}} \end{pmatrix}. \tag{4.13}$$

We interchange the first with the third block of rows. Then the determinant gets multiplied by $(-1)^l=(-1)^{|I|_\circ}$ and the matrix acquires block diagonal form. The blocks are, up to a common factor in each row, in Vandermonde form. Thus the formula given in the theorem is easily established.

The case of $X(t)$ is similar but a bit more complicated. Here we can subtract in (4.1) column number $\overline{n}+1$ through $2\overline{n}-1$ from columns 2 through $\overline{n}$, respectively. Then we obtain

(4.14)

$$\det \mathfrak{X}(t) = \det \begin{pmatrix} [u_i]_{i=1\dots\overline{n}} & \Big[(x_i+t)(x_i+r)^{\overline{n}-j}\Big]_{\substack{i=1\dots\overline{n}\\ j=2\dots\overline{n}}} & \Big[-x_i^{\overline{n}-j}T_{x,i}\Big]_{\substack{i=1\dots\overline{n}\\ j=1\dots\underline{n}}} \\ [v_i]_{i=1\dots\underline{n}} & \Big[-y_i^{\underline{n}+1-j}T_{y,i}\Big]_{\substack{i=1\dots\underline{n}\\ j=2\dots\overline{n}}} & \Big[(y_i+r)^{\underline{n}-j}\Big]_{\substack{i=1\dots\underline{n}\\ j=1\dots\underline{n}}} \end{pmatrix}$$

with $u_i := (x_i+t)(x_i+r)^{\overline{n}-1} - x_i^{\overline{n}}T_{x,i}$ and $v_i := (y_i+r)^{\underline{n}} - y_i^{\underline{n}}T_{y,i}$. Arguing as above, one notes that all entries of $A-BD^{-1}C$ are linear combinations of 1 and $T_{x,i}T_{y,j}$ except those in the first column where $T_{x,i}$ also appears. Thus, if $c_I\ne 0$, then the number of odd elements is equal or one more than the number of even elements, i.e., $I\in P_{\text{odd}}$.

To determine the correct coefficient we proceed as above. The case $I=\{1,\dots,2l\}$ is the same. In the case $I=\{1,\dots,2l+1\}$ one has to move in (4.14) the first column to the $\overline{n}+1$-st place (i.e., between the two other blocks). $\square$

The main feature of the difference operators is the following cut-off property:

4.4. Lemma. *Expand $X(t)$ or $Y(t)$ as $\sum_I c_I(z)T_I$. Assume $r \neq 0$. Then for any $\mu \in \Lambda$ holds: if $\mu - \varepsilon_I \notin \Lambda$, then $c_I(\varrho + \mu) = 0$.*

Proof. Since $r \neq 0$ (and, as always, ϱ dominant), the denominator of c_I does not vanish at $\varrho+\mu$. If $\mu-\varepsilon_I \notin \Lambda$, then either $\mu_n = 0$ and $n \in I$ or there is an $i < n$ with $\mu_i = \mu_{i+1}$ and $i \in I$, $i+1 \notin I$. Now we use the precise form of $c_I(z)$ established in Proposition 4.3. From the definition of D_I, (4.7), we obtain that c_I is a multiple of $z_n(z_i - z_{i+1} - r)$, hence $c_I(\varrho + \mu) = 0$. ☐

Combining all results, we obtain the main result of this paper.

4.5. Theorem. *Every R_λ, $\lambda \in \Lambda$, is an eigenvector of both $X(t)$ and $Y(t)$. More precisely*

$$X(t)R_\lambda = \prod_{i \text{ odd}} (t + \varrho_i + \lambda_i) \cdot R_\lambda, \tag{4.15}$$

$$Y(t)R_\lambda = \prod_{i \text{ even}} (t + \varrho_i + \lambda_i) \cdot R_\lambda. \tag{4.16}$$

Proof. We may assume $r \neq 0$. The case $r = 0$ then follows by continuity. Let $R := X(t)R_\lambda$. Lemma 4.2 implies $\deg R \leq \deg R_\lambda = |\lambda|_{\text{odd}}$. Let $\mu \in \Lambda$ with $|\mu|_{\text{odd}} \leq |\lambda|_{\text{odd}}$ and $\mu \neq \lambda$. If $X(t) = \sum_I c_I(z)T_I$, then $R(\varrho + \mu) = \sum_I c_I(\varrho + \mu)R_\lambda(\varrho + (\mu - \varepsilon_I))$. If $\mu - \varepsilon_I \in \Lambda$, then $R_\lambda(\varrho + (\mu - \varepsilon_I)) = 0$ by definition of R_λ. Otherwise, $c_I(\varrho + \mu) = 0$ by Lemma 4.4. Hence, $R(\varrho + \mu) = 0$ which shows that R is a multiple of R_λ. The coefficient of $T_\emptyset$ in $X(t)$ is $c(z) = \prod_{i=1}^{\overline{n}}(t + z_{2i-1})$. Thus, evaluation in $z = \varrho + \lambda$ gives $R = c(\varrho + \lambda)R_\lambda$. The same argument works for $Y(t)$. ☐

4.8. Corollary. *Let*

$$\begin{aligned} X(t) &= t^{\overline{n}} + X_1 t^{\overline{n}-1} + \ldots + X_{\overline{n}}, \\ Y(t) &= t^{\underline{n}} + Y_1 t^{\underline{n}-1} + \ldots + Y_{\underline{n}}. \end{aligned} \tag{4.17}$$

Then $X_1, \ldots, X_{\overline{n}}, Y_1, \ldots, Y_{\underline{n}}$ are pairwise commuting difference operators.

Example. We compute X_1 and Y_1. Any contribution to the coefficient of $t^{\overline{n}-1}$ in (4.9) comes from $I = \emptyset$, $I = \{i\}$ with i odd, and $I = \{i, j\}$ with i odd, j even. Since $D_\emptyset = 1$ we get

$$X_1 = \sum_{i \text{ odd}} z_i - \sum_{i \text{ odd}} D_{\{i\}} - \sum_{\substack{i \text{ odd} \\ j \text{ even}}} D_{\{i,j\}}. \tag{4.18}$$

Similarly, in (4.10), the contribution for $t^{\underline{n}-1}$ comes from $I = \emptyset$ and $I = \{i, j\}$ with i odd, j even. Thus,

$$Y_1 = \sum_{i \text{ even}} z_i - \sum_{\substack{i \text{ odd} \\ j \text{ even}}} D_{\{i,j\}}. \tag{4.19}$$

The operators $X_1, \ldots, X_{\overline{n}}, Y_1, \ldots, Y_{\underline{n}}$ defined in (4.17) generate a polynomial algebra $\mathcal{R} \subseteq \operatorname{End} \mathcal{P}^W$. We show that it is canonically isomorphic to $\mathcal{P}^W$. More precisely:

4.7. Proposition. a) *Every element $D \in \mathcal{R}$ has an expansion*

$$D = \sum_{\mu \in \Psi_0} c_\mu^D(z) T_\mu, \tag{4.20}$$

where Ψ_0 is the smallest W-stable submonoid of $\mathbb{Z}^n$ containing Λ and where the coefficients c_μ^D are rational functions in $z_1, \ldots, z_n$ with poles along the hyperplanes $z_i - z_j = a$ where $i - j \neq 0$ is even and $a \in \mathbb{Z}$.

b) *The coefficient $c_0^D(z)$ is in $\mathcal{P}^W$ and the map $\mathcal{R} \to \mathcal{P}^W : D \mapsto c_0^D(z)$ is an algebra isomorphism.*

c) *For every $D \in \mathcal{R}$ and $\lambda \in \Lambda$ holds $D(R_\lambda) = c_0^D(\varrho + \lambda)R_\lambda$.*

d) *If $r \notin \mathbb{Q}$, the coefficients $c_\mu^D(z)$ have the cut-off property: Let $\lambda \in \Lambda$ with $\lambda - \mu \notin \Lambda$. Then $c_\mu^D(\varrho + \lambda) = 0$.*

Proof. a) Since Ψ_0 contains all W-translates of elements of Λ it contains all ε_I with $I \in P_{\mathrm{odd}}$. Hence, by Proposition 4.3, the generators $X_1, \ldots, X_{\overline{n}}, Y_1, \ldots, Y_{\underline{n}}$ of $\mathcal{R}$ have an expansion as claimed. This implies the result easily for all $D \in \mathcal{R}$.

b) Since Ψ_0 is contained in $\mathbb{N}^n$ it is a pointed cone, i.e., $\lambda, \mu \in \Psi_0$ with $\lambda + \mu = 0$ implies $\lambda = \mu = 0$. Looking at how two operators with an expansion as in (4.20) multiply this implies that $D \mapsto c_0^D$ is an algebra homomorphism. It is an isomorphism, since the generators X_i and Y_i of $\mathcal{R}$ are mapped to free generators of $\mathcal{P}^W$.

c) The assertion needs to be checked just for the generators of $\mathcal{R}$ and there it is the content of Theorem 4.5.

d) The algebra $\mathcal{R}$ acts on the dual space $(\mathcal{P}^W)^*$ on the right by $(\delta D)(f) := \delta(D(f))$. Let $v = (v_i) \in k^n$ such that $v_i - v_j \notin \mathbb{Z}$ whenever $i - j \neq 0$ is even (e.g. $v \in \varrho + \Lambda$, since $r \notin \mathbb{Q}$) and let $\delta_v : f \mapsto f(v)$ be the corresponding evaluation function. Then $\delta_v D = \sum_\mu c_\mu^D(v)\delta_{v-\mu}$. Thus the cut-off property is equivalent to the statement that $\bigoplus_{\lambda\in\Lambda} k\delta_{\varrho+\lambda} \subseteq (\mathcal{P}^W)^*$ is $\mathcal{R}$-stable. It suffices to check this for generators of $\mathcal{R}$ which is the content of Lemma 4.4. $\square$

Next we study the monoid Ψ_0 more closely.

4.8. Lemma. *The monoid Ψ_0 also has the following descriptions:*

a) *It is generated by $\{\varepsilon_i \mid i \text{ odd}\} \cup \{\varepsilon_i + \varepsilon_j \mid i \text{ odd}, j \text{ even}\}$.*

b) *It consists of all $\lambda \in \mathbb{N}^n$ with $[\lambda]_1 \geq 0$.*

Proof. a) Since Λ is generated by all $\varepsilon_{\{1,\ldots,m\}}$, $m = 1, \ldots, n$, the monoid Ψ_0 is generated by all ε_I, $I \in P_{\mathrm{odd}}$. But those can be obtained from the given subset.

b) Let Ψ_0' be the set of all $\lambda \in \mathbb{N}^n$ with $[\lambda]_1 \geq 0$, i.e., $|\lambda_{\mathrm{odd}}| \geq |\lambda_{\mathrm{even}}|$. We have to show $\Psi_0 = \Psi_0'$. The inclusion $\Psi_0 \subseteq \Psi_0'$ follows, e.g., from a). For the converse, let $\lambda \in \Psi_0$. We show $\lambda \in \Psi_0'$ by induction on $|\lambda|_{\mathrm{odd}}$. If $|\lambda|_{\mathrm{odd}} = 0$, then also $|\lambda_{\mathrm{even}}| = 0$. Thus $\lambda = 0 \in \Psi_0$. For $|\lambda|_{\mathrm{odd}} > 0$ there are two cases. If $|\lambda_{\mathrm{odd}}| > |\lambda_{\mathrm{even}}|$, then choose any odd i such that $\lambda_i > 0$. Then $\lambda' := \lambda - \varepsilon_i$ is in Ψ_0' hence, by induction, in Ψ_0. Thus also $\lambda = \lambda' + \varepsilon_i \in \Psi_0$. If $|\lambda_{\mathrm{odd}}| = |\lambda_{\mathrm{even}}| > 0$, then there is i odd and j even such that $\lambda_i > 0$ and $\lambda_j > 0$. Then $\lambda' := \lambda - \varepsilon_i - \varepsilon_j$ is in Ψ_0', hence in Ψ_0 by induction. We conclude $\lambda \in \Psi_0$, as well. $\square$

In the theory of symmetric polynomials, the containment relation $\lambda \subseteq \mu$ for $\lambda, \mu \in \Lambda$ is defined as $\mu - \lambda \in \mathbb{N}^n$. The semisymmetric analogue is $\mu - \lambda \in \Psi_0$ or, equivalently,

$$\lambda \sqsubseteq \mu \quad \overset{\text{def}}{\Longleftrightarrow} \quad \lambda \subseteq \mu \text{ and } [\lambda]_1 \leq [\mu]_1. \tag{4.21}$$

Example. We have $(1,0,0) \subseteq (1,1,0)$ but $(1,0,0) \not\sqsubseteq (1,1,0)$. Moreover, we have

$$(1,0,0), (1,1,0) \sqsubseteq (2,1,0), (1,1,1). \tag{4.22}$$

This implies in particular that $(1,0,0)$ and $(1,1,0)$ have no supremum. Therefore, as opposed to the classical containment relation, its semisymmetric analogue does not form a lattice.

Now we prove that the polynomial R_λ vanishes at many more points than it is supposed to by definition (Extra Vanishing Theorem).

4.9. Theorem. *For $\lambda, \mu \in \Lambda$ holds $R_\lambda(\varrho+\mu) = 0$ unless $\lambda \sqsubseteq \mu$.*

Proof. We may assume $r \notin \mathbb{Q}$ since the general case follows by continuity. For fixed λ let μ be a counterexample (i.e., $R_\lambda(\varrho+\mu) \neq 0$ and $\lambda \not\sqsubseteq \mu$) which is minimal with respect to "$\sqsubseteq$". Since $\varrho+\mu$ is not in the W-orbit of $\varrho+\lambda$, there is $D \in \mathcal{R}$ such that $c_0^D(\varrho+\lambda) \neq c_0^D(\varrho+\mu)$. From $D(R_\lambda) = c_0^D(\varrho+\lambda)R_\lambda$ we obtain, after substituting $z = \varrho+\mu$,

$$(c_0^D(\varrho+\lambda) - c_0^D(\varrho+\mu))R_\lambda(\varrho+\mu) = \sum_{\eta\in\Psi_0} c_\eta^D(\varrho+\mu)R_\lambda(\varrho+\mu-\eta). \tag{4.23}$$

If $\mu-\eta \in \Lambda$, then $R_\lambda(\varrho+\mu-\eta) = 0$ by minimality of μ. Otherwise $c_\eta^D(\varrho+\mu) = 0$ by Proposition 4.7 d). Contradiction. □

As an application we derive an explicit formula for R_λ when λ is a particular kind of "hook".

4.10. Corollary. *Let $a, m \geq 1$ be integers with m odd. Then*

$$R_{(a\,1^{m-1})} = (R_{(1)}-1)(R_{(1)}-2)\ldots(R_{(1)}-a+1)R_{(1^m)}. \tag{4.24}$$

Proof. Denote the right-hand side by f. Let $\lambda = (a\,1^{m-1})$ and $\mu \in \Lambda$ with $|\mu|_{\text{odd}} \leq |\lambda|_{\text{odd}} = a + \frac{m-1}{2}$ and $f(\varrho+\mu) \neq 0$. Then $R_{(1^m)}(\varrho+\mu) \neq 0$ and $R_{(1)}(\varrho+\mu) \neq 1, 2, \ldots, a-1$. The Extra Vanishing Theorem 4.9 implies $(1^m) \sqsubseteq \mu$, hence $\mu_m \geq 1$ and $[\mu]_1 \geq [(1^m)]_1 = 1$. From $R_{(1)}(\varrho+\mu) = [\mu]_1$ (Corollary 2.8) we obtain $[\mu]_1 \geq a$. Thus $\mu_1 = [\mu]_1 + (\mu_2-\mu_3) + \ldots \geq a$. We know already $\mu_3, \mu_5, \ldots, \mu_m \geq 1$. Since $|\mu|_{\text{odd}} \leq a + \frac{m-1}{2}$, equality holds throughout. This implies easily $\mu = \lambda$. Therefore, f must be a multiple of R_λ. Equality follows from the fact that the coefficient of $z^{[\lambda]} = z_1^a z_3 z_5 \ldots z_m$ is 1 in both cases. □

Remarks. 1. For even m, the polynomials $R_{(a\,1^{m-1})}$ will be calculated in Corollary 9.5.

2. For $m = 1$ one obtains in particular $R_{(a)} = R_{(1)}(R_{(1)}-1)\ldots(R_{(1)}-a+1)$ which clearly do not generate $\mathcal{P}^W$. Therefore, the polynomials $R_{(a)}$ are not a semisymmetric analogue of the complete symmetric functions.

5. The top homogeneous components

The highest degree components $\overline{R}_\lambda(z;r)$ of $R_\lambda(z;r)$ are also of high representation theoretic interest (see Theorem 3.4). We show that they are eigenfunctions of *differential* equations. More precisely, put

$$\overline{\mathfrak{X}}(t) := \begin{pmatrix} \Big[x_i^{\overline{n}-j}\big(x_i\partial_{x_i} + (\overline{n}-j)r + t\big)\Big]_{\substack{i=1\ldots\overline{n}\\ j=1\ldots\overline{n}}} & \Big[-x_i^{\overline{n}-j}\Big]_{\substack{i=1\ldots\overline{n}\\ j=1\ldots\underline{n}}} \\ \Big[y_i^{\underline{n}-j}\big(y_i\partial_{y_i} + (\underline{n}+1-j)r\big)\Big]_{\substack{i=1\ldots\underline{n}\\ j=1\ldots\overline{n}}} & \Big[y_i^{\underline{n}-j}\Big]_{\substack{i=1\ldots\underline{n}\\ j=1\ldots\underline{n}}} \end{pmatrix}, \tag{5.1}$$

$$\overline{\mathfrak{Y}}(t) := \begin{pmatrix} \left[x_i^{\overline{n}-j}\right]_{\substack{i=1\ldots\overline{n}\\ j=1\ldots\overline{n}}} & \left[x_i^{\overline{n}-j-1}\big(x_i\partial_{x_i}+(\overline{n}-j)r\big)\right]_{\substack{i=1\ldots\overline{n}\\ j=1\ldots\underline{n}}} \\ \left[-y_i^{\underline{n}+1-j}\right]_{\substack{i=1\ldots\underline{n}\\ j=1\ldots\overline{n}}} & \left[y_i^{\underline{n}-j}\big(y_i\partial_{y_i}+(\underline{n}-j)r+t\big)\right]_{\substack{i=1\ldots\underline{n}\\ j=1\ldots\underline{n}}} \end{pmatrix} \tag{5.2}$$

and $\overline{X}(t) := \varphi(z)^{-1}\det\overline{\mathfrak{X}}(t)$, $\overline{Y}(t) := \varphi(z)^{-1}\det\overline{\mathfrak{Y}}(t)$ where $\partial_{x_i} = \partial/\partial x_i$ and $\partial_{y_i} = \partial/\partial y_i$. These are linear differential operators with rational coefficients.

5.1. Theorem. *Every $\overline{R}_\lambda$, $\lambda \in \Lambda$, is an eigenvector of both $\overline{X}(t)$ and $\overline{Y}(t)$. More precisely*

$$\overline{X}(t)\overline{R}_\lambda = \prod_{i\ \mathrm{odd}} (t+\varrho_i+\lambda_i)\,\overline{R}_\lambda, \tag{5.3}$$

$$\overline{Y}(t)\overline{R}_\lambda = \prod_{i\ \mathrm{even}} (t+\varrho_i+\lambda_i)\,\overline{R}_\lambda. \tag{5.4}$$

Proof. Let $f \in \mathcal{P}^W$ be homogeneous of degree d. For each entry a_{ij} of $\mathfrak{X}(t)$ or $\mathfrak{Y}(t)$ we know that $a_{ij}(f)$ is a polynomial of degree $\le d+d_i'+d_j''$ (with d_i', d_j'' as in (4.5) or (4.6)). Using the fact that

$$\begin{aligned}(1-T_{x,i})f(z) =& \partial_{x_i}f + \text{ lower order terms,}\\ (1-T_{y,i})f(z) =& \partial_{y_i}f + \text{ lower order terms}\end{aligned} \tag{5.5}$$

one easily calculates that the $d+d_i'+d_j''$-degree component of $a_{ij}(f)$ is $\overline{a}_{ij}(f)$ where $\overline{a}_{ij}$ is the ij-entry of $\overline{\mathfrak{X}}(t)$ or $\overline{\mathfrak{Y}}(t)$, respectively. Since $\sum_i(d_i'+d_i'') = \deg\varphi(z)$, we get that $\overline{X}(t)f$ or $\overline{Y}(t)f$ is the d-degree homogeneous component of $X(t)f$ or $Y(t)f$, respectively. Now the assertion follows from Theorem 4.5. $\square$

Remark. The operators $\overline{X}(t)$ and $\overline{Y}(t)$ are the semisymmetric analogues of the Sekiguchi-Debiard operators, [Se], [De], which characterize Jack polynomials.

If we expand $\overline{X}(t)$ and $\overline{Y}(t)$ as a polynomial in t,

$$\begin{aligned}\overline{X}(t) &= t^{\overline{n}} + \overline{X}_1 t^{\overline{n}-1} + \ldots + \overline{X}_{\overline{n}},\\ \overline{Y}(t) &= t^{\underline{n}} + \overline{Y}_1 t^{\underline{n}-1} + \ldots + \overline{Y}_{\underline{n}},\end{aligned} \tag{5.6}$$

we obtain as in Corollary 4.6 pairwise commuting differential operators $\overline{X}_1, \ldots, \overline{X}_{\overline{n}}$, $\overline{Y}_1, \ldots, \overline{Y}_{\underline{n}}$ with $\overline{R}_\lambda$ as common eigenvectors. In general, these operators seem to be more difficult to compute explicitly than their difference counterparts. We give a formula for the most important ones, namely those of order one. For odd i we define the following rational function:

$$u_i := v_i\,\frac{\prod_{j\ \mathrm{even}}(z_i-z_j)}{\prod_{j\neq i\ \mathrm{odd}}(z_i-z_j)} \text{ where } v_i := \begin{cases} z_i & \text{for } n \text{ odd,}\\ 1 & \text{for } n \text{ even.}\end{cases} \tag{5.7}$$

5.2. Theorem. *The following equations hold:*

$$\eta := \overline{X}_1 - \overline{n}\underline{n}r = \sum_i z_i\frac{\partial}{\partial z_i} \qquad \textit{(Euler vector field)}, \tag{5.8}$$

$$\eta' := \overline{X}_1 - \overline{Y}_1 - \underline{n}r = \sum_{i\ \mathrm{odd}} u_i\frac{\partial}{\partial z_i}. \tag{5.9}$$

Moreover, for all $\lambda \in \Lambda$ holds $\eta(\overline{R}_\lambda) = |\lambda|_{\mathrm{odd}}\overline{R}_\lambda$, $\eta'(\overline{R}_\lambda) = [\lambda]_1\overline{R}_\lambda$.

Proof. Let E be the Euler vector field. By (5.3), we have $\overline{X}_1(\overline{R}_\lambda) = |\varrho + \lambda|_{\mathrm{odd}}\overline{R}_\lambda$. From $|\lambda|_{\mathrm{odd}} = \deg \overline{R}_\lambda$ and $|\varrho|_{\mathrm{odd}} = \overline{n}\underline{n}r$ it follows that $\eta - E$ kills every $\overline{R}_\lambda$ and therefore every W-invariant. The (non-symmetric) polynomials are all algebraic functions of the semisymmetric ones. Since $\eta - E$ is a derivation, it kills all polynomials, i.e., $\eta - E = 0$.

By (4.18) and (4.19), we have

$$E' := X_1 - Y_1 = [z]_1 - \sum_{i \text{ odd}} D_{\{i\}} = [z]_1 - \sum_{i \text{ odd}} u_i' T_i \tag{5.10}$$

where

$$u_i' := v_i \frac{\prod\limits_{j \text{ even}} (z_i - z_j - r)}{\prod\limits_{j \neq i \text{ odd}} (z_i - z_j)}. \tag{5.11}$$

From (4.15) and (4.16) we obtain $E'(1) = [\varrho]_1 = \underline{n}r$. Thus, $[z]_1 - \sum_{i \text{ odd}} u_i' = \underline{n}r$ and we get

$$X_1 - Y_1 - \underline{n}r = \sum_{i \text{ odd}} u_i'(1 - T_i). \tag{5.12}$$

This implies (5.9) since $(1 - T_i)(f) = \frac{\partial f}{\partial z_i} + \text{lower order terms}$. □

The derivations η, η' induce a bigrading on $\mathcal{P}^W$. More precisely, for integers a, b let

$$\mathcal{P}^W_{a,b} := \{f \in \mathcal{P}^W \mid \eta(f) = af, \eta'(f) = bf\}. \tag{5.13}$$

Then $\mathcal{P}^W = \bigoplus_{a \geq b \geq 0} \mathcal{P}^W_{a,b}$. To describe $\mathcal{P}^W_{a,b}$ explicitly, we have to find bihomogeneous generators of $\mathcal{P}^W$. For this, we introduce the semi-symmetric analogue of the elementary symmetric polynomials, namely $\mathbf{e}_m(z) := \overline{R}_{(1^m)}(z; r)$. More explicitly we have by Corollary 2.9,

$$\begin{aligned} \mathbf{e}_{2m-1}(z) &= e_m(z_{\mathrm{odd}}) - e_m(z_{\mathrm{even}}) \quad m = 1, \ldots, \overline{n}; \\ \mathbf{e}_{2m}(z) &= e_m(z_{\mathrm{even}}) \quad m = 1, \ldots, \underline{n}. \end{aligned} \tag{5.14}$$

Now, we consider the basis of $\mathcal{P}^W$ which consists of all monomials in the $\mathbf{e}_m$. More precisely, we define[‡] for any $\lambda \in \Lambda$

$$\mathbf{e}_\lambda := \mathbf{e}_1^{\lambda_1 - \lambda_2} \mathbf{e}_2^{\lambda_2 - \lambda_3} \ldots \mathbf{e}_{n-1}^{\lambda_{n-1} - \lambda_n} \mathbf{e}_n^{\lambda_n}. \tag{5.15}$$

This parametrization is chosen such that the leading term of $\mathbf{e}_\lambda$ is $z^{[\lambda]}$. Then $\mathcal{P}^W_{a,b}$ is spanned by all $\mathbf{e}_\lambda$ with $|\lambda|_{\mathrm{odd}} = a$ and $[\lambda]_1 = b$.

5.3. Corollary. *For $\lambda \in \Lambda$ consider the expansion $\overline{R}_\lambda = \sum_\mu a_{\lambda\mu} \mathbf{e}_\mu$. Then only those $\mathbf{e}_\mu$ occur for which $|\mu|_{\mathrm{odd}} = |\lambda|_{\mathrm{odd}}$ and $[\mu]_1 = [\lambda]_1$.*

Remark. This result will be generalized in Theorem 6.6.

We use Corollary 5.3 to compute $\overline{R}_\lambda$ for all two-row diagrams. We use the multinomial coefficient $\binom{a}{k_1, \ldots, k_n} := \frac{a!}{k1! \ldots k_n!}$ where $a = k_1 + \ldots + k_n$.

[‡]In this notation, one has to be careful to distinguish between $\mathbf{e}_a$ (the index is a number) and $\mathbf{e}_{(a)}$ (the index is a partition). The latter equals $\mathbf{e}_1^a$.

5.4. Theorem. *For integers $a \geq b \geq 0$ let $c_{ab} = \binom{a}{b}\binom{-2r}{a}$. Then*

$$\overline{R}_{(a\,b)} = \frac{1}{c_{ab}} \sum_\mu \binom{-2r}{\mu_1} \binom{\mu_1}{\mu_1-\mu_2, \ldots, \mu_{n-1}-\mu_n, \mu_n} \mathbf{e}_\mu \tag{5.16}$$

where the sum runs through all $\mu \in \Lambda$ with $|\mu_{\mathrm{odd}}| = a$ and $|\mu_{\mathrm{even}}| = b$.

Proof. We use a result for the usual Jack polynomials $\overline{P}_\lambda(z;r)$. Stanley ([St], see also [KS1], Prop. 3.4 for a proof in the spirit of this paper), has shown that there is a generating series

$$\sum_{a=0}^\infty v_a \overline{P}_{(a)}(z;r) = \prod_i (1+z_i)^{-r} \tag{5.17}$$

where the $v_a = \binom{-r}{a} \neq 0$. Then, by the Comparison Theorem 2.7 (with $\mu = (a, 0, \ldots)$) there are constants $w_{a,b} \neq 0$ such that

$$\sum_{a \geq b \geq 0} w_{a,b} \overline{R}_{(a\,b)}(z) = \prod_{i \text{ odd}} (1+z_i)^{-2r}. \tag{5.18}$$

Now, we expand the right-hand side in bihomogeneous components. For this observe

$$\prod_{i \text{ odd}} (1+z_i) = 1 + \sum_{i \geq 1} e_i(z_{\mathrm{odd}}) = 1 + \sum_{i \geq 1} \mathbf{e}_i(z). \tag{5.19}$$

Thus, we get

$$\begin{aligned} \sum_{a \geq b \geq 0} w_{a,b} \overline{R}_{(a\,b)}(z) &= \sum_d \binom{-2r}{d} \Big(\sum_{i \geq 1} \mathbf{e}_i \Big)^d \\ &= \sum_d \sum_{k_1 + \ldots + k_n = d} \binom{-2r}{d} \binom{d}{k_1, \ldots, k_n} \mathbf{e}_1^{k_1} \mathbf{e}_2^{k_2} \ldots \mathbf{e}_n^{k_n} \\ &= \sum_{\mu \in \Lambda} \binom{-2r}{\mu_1} \binom{\mu_1}{\mu_1-\mu_2, \ldots, \mu_{n-1}-\mu_n, \mu_n} \mathbf{e}_\mu. \end{aligned} \tag{5.20}$$

Now, we compare the bihomogeneous components of bidegree $(a, a-b)$ of both sides and get formula (5.16) up to the scalar c_{ab}. But that scalar is easily obtained by the requirement that the coefficient of $\mathbf{e}_{(a\,b)}$ should be 1. □

Examples. 1. *The case $n = 3$.* The summation in (5.16) runs through all $\mu \in \Lambda_3$ with $\mu_1 + \mu_3 = a$ and $\mu_2 = b$. If we put $\mu_3 = k$, we get $\mu = (a-k, b, k)$ with $0 \leq k \leq \min(a-b, b)$. Thus,

$$\overline{R}_{a,b,0} = \sum_k \frac{\binom{a-k}{a-b-k, b-k, k}\binom{-2r}{a-k}}{\binom{a}{b}\binom{-2r}{a}} \mathbf{e}_{a-k,b,k} = \sum_k (-1)^k \frac{\binom{a-b}{k}\binom{b}{k}}{\binom{a+2r-1}{k}} \mathbf{e}_{a-k,b,k}. \tag{5.21}$$

Now, the recursion formula (2.10) implies $\overline{R}_{a+1,b+1,c+1} = \mathbf{e}_3 \overline{R}_{a,b,c}$. Thus, we obtain a formula for $\overline{R}_\mu$ for arbitrary $\mu \in \Lambda_3$:

$$\overline{R}_{\mu_1,\mu_2,\mu_3} = \sum_k (-1)^k \frac{\binom{\mu_1-\mu_2}{k}\binom{\mu_2-\mu_3}{k}}{\binom{\mu_1-\mu_3+2r-1}{k}} \mathbf{e}_1^{\mu_1-\mu_2-k} \mathbf{e}_2^{\mu_2-\mu_3-k} \mathbf{e}_3^{\mu_3+k} \tag{5.22}$$

with

$$\mathbf{e}_1 = z_1 - z_2 + z_3, \quad \mathbf{e}_2 = z_2, \quad \mathbf{e}_3 = z_1 z_3. \tag{5.23}$$

This formula can be rewritten in two ways. First, as a hypergeometric function

$$\overline{R}_{\mu_1,\mu_2,\mu_3} = \mathbf{e}_1^{\mu_1-\mu_2}\mathbf{e}_2^{\mu_2-\mu_3}\mathbf{e}_3^{\mu_3}\cdot {}_2F_1\Big(\begin{matrix}\mu_2-\mu_1,\,\mu_3-\mu_2\\ \mu_3-\mu_1-2r+1\end{matrix}\Big|\frac{\mathbf{e}_3}{\mathbf{e}_1\,\mathbf{e}_2}\Big). \tag{5.24}$$

Secondly, we can express the sum (5.22) as a Jacobi polynomial which is defined as

$$P_n^{\alpha,\beta}(x) := \binom{\alpha+n}{n}\cdot {}_2F_1\Big(\begin{matrix}-n,\,n+\alpha+\beta+1\\ \alpha+1\end{matrix}\Big|\frac{1-x}{2}\Big). \tag{5.25}$$

For this, we invert the order of the summands. Since k runs from 0 to the smaller of $\mu_1-\mu_2$ and $\mu_2-\mu_3$, we have two cases. Set $\mu=(k_1+k_2+k_3,k_2+k_3,k_3)$. Then the first case is $k_1\le k_2$. The substitution $k=k_1-l$ gives

$$(-1)^k\frac{\binom{k_1}{k}\binom{k_2}{k}}{\binom{k_1+k_2+2r-1}{k}} = (-1)^{k_1}\frac{\binom{k_2}{k_1}}{\binom{k_1+k_2+2r-1}{k_1}}\frac{(-k_1)_l\,(k_2+2r)_l}{(k_2-k_1+1)_l\,l!} \tag{5.26}$$

where $(a)_l=a(a+1)\dots(a+l-1)$ is the Pochhammer symbol. Thus,

$$\overline{R}_\mu = \binom{k_2}{k_1}\binom{-k_2-2r}{k_1}^{-1}\mathbf{e}_2^{k_2-k_1}\mathbf{e}_3^{k_1+k_3}\cdot {}_2F_1\Big(\begin{matrix}-k_1,\,k_2+2r\\ k_2-k_1+1\end{matrix}\Big|\frac{\mathbf{e}_1\mathbf{e}_2}{\mathbf{e}_3}\Big). \tag{5.27}$$

This, and a similar computation for $k_1\ge k_2$ gives

(5.28)

$$\overline{R}_\mu(z) = \begin{cases}\binom{-k_2-2r}{k_1}^{-1}\cdot\mathbf{e}_2^{k_2-k_1}\mathbf{e}_3^{k_1+k_3}\cdot P_{k_1}^{k_2-k_1,2r-1}(1-2\frac{\mathbf{e}_1\mathbf{e}_2}{\mathbf{e}_3}) & \text{for } k_1\le k_2,\\ \binom{-k_1-2r}{k_2}^{-1}\cdot\mathbf{e}_1^{k_1-k_2}\mathbf{e}_3^{k_2+k_3}\cdot P_{k_2}^{k_1-k_2,2r-1}(1-2\frac{\mathbf{e}_1\mathbf{e}_2}{\mathbf{e}_3}) & \text{for } k_1\ge k_2.\end{cases}$$

These formulas are essentially due to Vilenkin–Šapiro [VS]; see also [VK] 11.3.2.

2. *The case* $n=4$. In this case, we put $\mu_2-\mu_3=k$ and $\mu_4=l$. Then $\mu=(a-k-l,b-l,k+l,l)$ and we get

$$\begin{aligned}\overline{R}_{a,b,0,0} &= \frac{1}{c_{ab}}\sum_{k,l}\binom{a-k-l}{a-b-k,\,b-k-2l,\,k,\,l}\binom{-2r}{a-k-l}\mathbf{e}_\mu\\ &= \sum_{k,l}\frac{(-a+b)_k\,(-b)_{k+2l}}{(-a-2r+1)_{k+l}\,k!\,l!}\mathbf{e}_1^{a-b-k}\mathbf{e}_2^{b-l-2l}\mathbf{e}_3^k\mathbf{e}_4^l\end{aligned} \tag{5.29}$$

where

$$\mathbf{e}_1=z_1-z_2+z_3-z_4,\quad \mathbf{e}_2=z_2+z_4,\quad \mathbf{e}_3=z_1z_3-z_2z_4,\quad \mathbf{e}_4=z_2z_4.$$

This can be expressed in terms of one of Horn's hypergeometric functions (see e.g. [Ba] §5.7.1):

$$\overline{R}_{a,b,0,0} = \mathbf{e}_1^{a-b}\mathbf{e}_2^b\cdot H_3(-b,-a+b,-a-2r+1;\frac{\mathbf{e}_4}{\mathbf{e}_2^2},\frac{\mathbf{e}_3}{\mathbf{e}_1\mathbf{e}_2}). \tag{5.30}$$

6. Triangularity

In this section, we investigate vanishing properties of the coefficients of $R_\lambda(z;r)$. For this, we consider the *inhomogeneous dominance order*: for $\mu,\lambda\in\mathbb{N}^n$ define

$$\mu\le\lambda \quad\overset{\text{def}}{\Longleftrightarrow}\quad \mu_1+\ldots+\mu_m\le\lambda_1+\ldots+\lambda_m \quad\text{for all } m=1,\ldots,n. \tag{6.1}$$

The homogeneous dominance order, commonly considered in the theory of symmetric functions, is

$$\mu \leqq \lambda \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \mu \leq \lambda \text{ and } |\mu| = |\lambda|. \tag{6.2}$$

Recall, that we defined the leading term of R_λ as $z^{[\lambda]}$ where $[\lambda]$ is defined in (2.3). The next theorem justifies this terminology.

6.1. Theorem. *For every $\lambda \in \Lambda$ there are expansions*

$$R_\lambda(z) = \sum_{\mu \in \mathbb{N}^n : \mu \leq [\lambda]} a_{\lambda\mu} z^\mu \quad \textit{and} \quad \overline{R}_\lambda(z) = \sum_{\mu \in \mathbb{N}^n : \mu \leqq [\lambda]} a_{\lambda\mu} z^\mu. \tag{6.3}$$

Proof. For $1 \leq m \leq n$ and $f \in \mathcal{P}$ denote the total degree of f in $z_1, \ldots, z_m$ by $\deg_m f$. Let $\underline{m} := \lfloor m/2 \rfloor$ and $\overline{m} := m - \underline{m} = \lceil m/2 \rceil$. We show first $\deg_m X(t) f \leq \deg_m f$ and $\deg_m Y(t) f \leq \deg_m f$ for all $f \in \mathcal{P}^W$.

Lemma 4.2 is nothing more than the case $m = n$. The general case is the same except that the entries in the rows involving $x_{\overline{m}+1}, \ldots, x_{\overline{n}}$ and $y_{\underline{m}+1}, \ldots, y_{\underline{n}}$ have degree 0. Thus the degree of $\mathfrak{X}(t)$ can be computed by taking in (4.5) or (4.6) the m largest entries of the d''_j and the entries of d'_i which correspond to $x_1, \ldots, x_{\overline{m}}, y_1, \ldots, y_{\underline{m}}$. Thus

$$\begin{aligned} \deg_m \det \mathfrak{X}(t) &= \sum_{i=1}^{\overline{m}} (\overline{n} - i) + \sum_{i=1}^{\underline{m}} (\overline{m} - i) + \underline{m}(\underline{n} - \overline{n}) \\ &= \sum_{i=1}^{\overline{m}} (\overline{n} - i) + \sum_{i=1}^{\underline{m}} (\underline{m} - i), \end{aligned} \tag{6.4}$$

$$\begin{aligned} \deg_m \det \mathfrak{Y}(t) &= \sum_{i=1}^{\overline{m}} (\overline{n} - i) + \sum_{i=1}^{\underline{m}} (\overline{m} - i - 1) + \underline{m}(\underline{n} - \overline{n} + 1) \\ &= \sum_{i=1}^{\overline{m}} (\overline{n} - i) + \sum_{i=1}^{\underline{m}} (\underline{m} - i). \end{aligned} \tag{6.5}$$

On the other hand, $\deg_m \varphi(z) = \sum_{i=1}^{\overline{m}} (\overline{n} - i) + \sum_{i=1}^{\underline{m}} (\underline{m} - i)$ which proves the claim.

For $\lambda \in \Lambda$ let $\mathcal{P}_\lambda$ and $\mathcal{P}^\circ_\lambda$ be the intersection of $\mathcal{P}^W$ with the span of all z^μ with $\mu \leq [\lambda]$ and $\mu < [\lambda]$, respectively. Then, by what we have proved above, both $\mathcal{P}_\lambda$ and $\mathcal{P}^\circ_\lambda$ are stable under $X(t)$ and $Y(t)$. The monomial symmetric polynomial $m_{[\lambda]}(z)$ is in $\mathcal{P}_\lambda$ but not in $\mathcal{P}^\circ_\lambda$. Thus, $\mathcal{P}^\circ_\lambda$ is of codimension one in $\mathcal{P}_\lambda$. Because the action of $X(t), Y(t)$ is diagonalizable there is exactly one $\nu_\lambda \in \Lambda$ such that R_{ν_λ} is in $\mathcal{P}_\lambda$ but not in $\mathcal{P}^\circ_\lambda$.

It remains to show $\nu_\lambda = \lambda$ for all λ. If there exists a counterexample, then choose one which is minimal with respect to the order relation $[\nu] \leq [\lambda]$. Since R_{ν_λ} contains $z^{[\nu_\lambda]}$ (Lemma 2.3) we have $[\nu_\lambda] < [\lambda]$. Thus, by minimality, $R_{\nu_\lambda} \in \mathcal{P}_{\nu_\lambda} \subseteq \mathcal{P}^\circ_\lambda$ in contradiction to the definition of ν_λ. ☐

Examples. 1. If λ is of the form $(a, a, b, b, c, c, \ldots)$, then we know from Theorem 2.6 that R_λ is a polynomial in the even variables $z_2, z_4, \ldots$ only. This can also be seen from triangularity: since $[\lambda]_1 = 0$, we have $\mu_1 = 0$ for every z^μ which occurs in R_λ. Hence z_1 does not occur. By symmetry, no odd variable occurs.

2. If λ is of the form $(a, b, b, c, c, \ldots)$, then $[\lambda] = (a, 0, b, 0, \ldots)$. Hence triangularity prohibits, e.g., the occurrence of monomials $z_1^{\mu_1} z_2^{\mu_2} \ldots$ with $\mu_1 + \mu_2 > a$.

This form of triangularity seems to be optimal when the expansion of R_λ in monomials is considered but, since monomials are not bihomogeneous, it does not cover the bigrading result of Corollay 5.3. Therefore, we expand R_λ in elementary semisymmetric symmetric functions $\mathbf{e}_\mu$ defined in (5.15). Then, an equivalent form of Theorem 6.1 is that for every $\lambda \in \Lambda$ there are expansions

$$R_\lambda(z) = \sum_{\mu\in\mathbb{N}^n:[\mu]\le[\lambda]} a_{\lambda\mu}\mathbf{e}_\mu \quad\text{and}\quad \overline{R}_\lambda(z) = \sum_{\mu\in\mathbb{N}^n:[\mu]\leqq[\lambda]} a_{\lambda\mu}\mathbf{e}_\mu. \tag{6.6}$$

The point is now to define an order relation on Λ which is stronger than $[\mu] \le [\lambda]$.

For this, let $\varphi^+ \subseteq \mathbb{Z}^n$ be the submonoid generated by all simple roots $\varepsilon_i - \varepsilon_{i+2}$, $1 \le i \le n-2$. Recall from Proposition 4.7, that $\Psi_0 \subseteq \mathbb{Z}^n$ was defined to be the smallest W-stable monoid containing Λ. We define the semisymmetric analogue of the inhomogeneous dominance order on Λ as

$$\mu \preceq \lambda \quad \overset{\text{def}}{\Longleftrightarrow} \quad \lambda - \mu \in \Psi_1 := \Psi_0 + \varphi^+. \tag{6.7}$$

6.2. Lemma. *The monoid Ψ_1 also has the following descriptions:*

a) *It is generated by $\{\varepsilon_i - \varepsilon_{i+2} \mid 1 \le i \le n-2\} \cup \{\varepsilon_{n-1} + \varepsilon_n, \varepsilon_{2\overline{n}-1}\}$.*
b) *It consists of all $\lambda \in \mathbb{Z}^n$ with $0 \le \lambda_{\text{odd}}$, $0 \le \lambda_{\text{even}}$, and $|\lambda_{\text{even}}| \le |\lambda_{\text{odd}}|$.*

Proof. a) By Lemma 4.8, the monoid Ψ_0 is generated by all elements of the form $\varepsilon_i + \varepsilon_j$, ($i$ odd, j even) and ε_i, (i odd). Using the generators of φ^+, one can obtain all these generators from either $\varepsilon_{n-1} + \varepsilon_n$ or $\varepsilon_{2\overline{n}-1}$ alone. This shows the claim.

b) First observe that the set of generators Σ in a) forms in fact a linear basis of $\mathbb{Z}^n$. Now consider the set Σ' consisting of the linear forms $\lambda_1, \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_4, \lambda_1 + \lambda_3 + \lambda_5, \ldots$ and $\lambda_1 - \lambda_2 + \lambda_3 - + \ldots$. Then the conditions in b) can be rephrased as $\ell(\lambda) \ge 0$ for all $\ell \in \Sigma'$. Observe that Σ' contains $|\lambda_{\text{odd}}|$ which is a sum of two other elements, thus redundant. When we remove it from Σ' we obtain a set Σ^* which turns out to be the dual basis of Σ. Thus Ψ_1 equals the set $\lambda \in \mathbb{Z}^n$ with $\ell(\lambda) \ge 0$ for all $\ell \in \Sigma^*$. $\square$

Since $[\lambda]_1 = |\lambda_{\text{odd}}| - |\lambda_{\text{even}}|$, we have in particular

$$\mu \preceq \lambda \quad \Longleftrightarrow \quad \mu_{\text{odd}} \le \lambda_{\text{odd}},\ \mu_{\text{even}} \le \lambda_{\text{even}}, \text{ and } [\mu]_1 \le [\lambda]_1. \tag{6.8}$$

Next, we compare $\mu \preceq \lambda$ with $[\mu] \le [\lambda]$.

6.3. Lemma. *The monoid $\tilde{\Psi}_1 := \{\lambda \in \mathbb{Z}^n \mid 0 \le [\lambda]\}$ is generated by*

$$\{\varepsilon_i - \varepsilon_{i+2} \mid 1 \le i \le n-2\} \cup \{\varepsilon_{n-1} + \varepsilon_n, -\varepsilon_2\}. \tag{6.9}$$

Proof. One easily checks that the proposed set of generators is the dual basis to

$$\{[\lambda]_1 + \ldots + [\lambda]_m \mid 1 \le m \le n\}. \tag{6.10}$$

$\square$

The new order relation is indeed stronger than the one considered before:

6.4. Corollary. *$\mu \preceq \lambda$ implies $[\mu] \le [\lambda]$.*

Proof. We have $-\varepsilon_{2\underline{n}} = -\varepsilon_2 + (\varepsilon_2 - \varepsilon_4) + \ldots + (\varepsilon_{2\underline{n}-2} - \varepsilon_{2\underline{n}})$. Since $2\underline{n}$ (resp. $2\overline{n}-1$) is the largest even (resp. odd) integer in $1, \ldots, n$ we have $\varepsilon_{2\overline{n}-1} = -\varepsilon_{2\underline{n}} + (\varepsilon_{n-1} + \varepsilon_n)$. This implies $\Psi_1 \subseteq \tilde{\Psi}_1$ which is equivalent to the assertion. $\square$

The homogeneous version of "$\preceq$" is defined as

$$\mu \leqq\!\!\!\!\preceq \lambda \quad \overset{\text{def}}{\iff} \quad \mu \preceq \lambda \text{ and } |\mu|_{\text{odd}} = |\lambda|_{\text{odd}}. \tag{6.11}$$

Since $|\lambda|_{\text{odd}} = 0$ for all $\lambda \in \varphi^+$ and $|\lambda|_{\text{odd}} > 0$ for all $\lambda \in \Psi_0 \setminus \{0\}$ the definition of $\mu \leqq\!\!\!\!\preceq \lambda$ simplifies to $\lambda - \mu \in \varphi^+$. Thus, we get

$$\mu \leqq\!\!\!\!\preceq \lambda \quad \iff \quad \mu_{\text{odd}} \leqq \lambda_{\text{odd}} \text{ and } \mu_{\text{even}} \leqq \lambda_{\text{even}}. \tag{6.12}$$

Now we are looking at expansions of elements in $\mathcal{P}^W$ in the form $\sum_\mu a_{\lambda\mu}\mathbf{e}_\mu$. For technical reasons we need a version which works for all elements in $\mathcal{P}$.

6.5. Lemma. *For $f \in \mathcal{P}^W$ and $\lambda \in \Lambda$ the following statements are equivalent:*

a) *In the expansion*

$$f(u_1 + u_2, u_2, u_3 + u_4, u_4, \ldots) = \sum_{\mu \in \mathbb{N}^n} a_\mu u^\mu \tag{6.13}$$

(where $u_{n+1} := 0$ if n is odd) only monomials u^μ with $\mu \leq [\lambda]$ and $\mu_{\text{even}} \leq \lambda_{\text{even}}$ occur.

b) *There is an expansion*

$$f(z) = \sum_{\mu \in \Lambda : \mu \preceq \lambda} b_\mu \mathbf{e}_\mu. \tag{6.14}$$

Proof. "*b) $\Rightarrow$ a)*": Let $\deg_m f$ be the total degree of f in $u_1, \ldots, u_m$ which is the same as the degree in $z_1, \ldots, z_m$. Then one calculates

$$\deg_m \mathbf{e}_\lambda = \sum_{i=1}^m [\lambda]_i = \begin{cases} \sum_{i=1}^{m/2} \lambda_{2i-1} & \text{if } m \text{ is even,} \\ \sum_{i=1}^{(m-1)/2} \lambda_{2i} + [\lambda]_1 & \text{if } m \text{ is odd.} \end{cases} \tag{6.15}$$

We conclude that $\mu \preceq \lambda$ implies $\deg_m \mathbf{e}_\mu \leq \deg_m \mathbf{e}_\lambda$. Thus, if u^μ occurs in f, then

$$\mu_1 + \ldots + \mu_m = \deg_m u^\mu \leq \deg_m f \leq \deg_m \mathbf{e}_\lambda = [\lambda]_1 + \ldots + [\lambda]_m, \tag{6.16}$$

i.e., $\mu \leq [\lambda]$.

Now let $\deg_m^u$ be the total degree of f in $u_2, u_4, \ldots, u_{2m}$. Then, due to cancellations, one has $\deg_m^u \mathbf{e}_\lambda = \sum_{i=1}^m \lambda_{2i}$. The same reasoning as above implies $\mu_{\text{even}} \leq \lambda_{\text{even}}$ whenever u^μ occurs in $f(u)$.

"a) $\Rightarrow$ b)": Assume $\mathbf{e}_\mu$ occurs in the expansion of f. Then, by the calculations above, we have to show $\deg_m \mathbf{e}_\mu \leq \deg_m f$ and $\deg_m^u \mathbf{e}_\mu \leq \deg_m^u f$ for all m (actually it suffices to consider in the first case only $m = 1$ and all even m).

We treat $\deg_m$ first. For this define a total order on the monomials u^λ: first we order them by $\deg_m$ and then by the lexicographic order on $(\lambda_1, \lambda_3, \ldots, \lambda_2, \lambda_4, \ldots)$. One checks that $\mathbf{e}_p$ has the leading monomial $u_1 u_3 \ldots u_p$ if p is odd and $u_2 u_4 \ldots u_p$ if p is even. Thus, the leading monomial of $\mathbf{e}_\lambda$ is $u^{[\lambda]}$. This shows in particular that if the leading monomials of $\mathbf{e}_\lambda$ and $\mathbf{e}_\mu$ coincide, then $\lambda = \mu$. Therefore, if there were an $\mathbf{e}_\mu$ occurring in f with $\deg_m \mathbf{e}_\mu > \deg_m f$, then we take a maximal one. Its leading monomial u^ν would not cancel out and would satisfy $\deg_m u^\nu > \deg_m f$ in contradiction to a).

For $\deg_m^u$ we argue similarly. This time the total order on the monomials u^λ is by lexicographic order on $(\deg_m^u u^\lambda, \lambda_2, \lambda_4, \ldots, \lambda_1, \lambda_3, \ldots)$. Then the leading term of $\mathbf{e}_p$ is $u_2 u_4 \ldots u_p$ if p is even and $u_2 u_4 \ldots u_{p-1} u_p$ if p is odd. Hence the leading term of $\mathbf{e}_\lambda$ is $u_1^{\lambda_1 - \lambda_2} u_2^{\lambda_2} u_3^{\lambda_3 - \lambda_4} \ldots u_n^{\lambda_n}$. These terms are again distinct for different $\mathbf{e}_\lambda$'s. The rest of the argument is as above. $\square$

Now we can state the better triangularity result announced earlier.

6.6. Theorem. *For every $\lambda \in \Lambda$ there are expansions*

a) $R_\lambda(u_1+u_2, u_2, u_3+u_4, u_4, \dots) = \sum_{\mu\in\mathbb{N}^n} a_{\lambda\mu} u^\mu$ *where* $\mu \le [\lambda]$, $\mu_{\rm even} \le \lambda_{\rm even}$*;*

b) $R_\lambda(z) = \sum_{\mu\in\Lambda:\mu\preceq\lambda} b_{\lambda\mu}\mathbf{e}_\mu(z)$ *and* $\overline{R}_\lambda(z) = \sum_{\mu\in\Lambda:\mu\,\overline{\preceq}\,\lambda} b_{\lambda\mu}\mathbf{e}_\mu(z)$.

Proof. By Lemma 6.5, it suffices to prove a). The degree function $\deg_m$ (see last proof) is invariant under upper triangular linear coordinate transformations. Thus $\mu \le [\lambda]$ follows from Theorem 6.1.

To prove $\mu_{\rm even} \le \lambda_{\rm even}$ recall that the total degree of $f(u_1+u_2, u_2, u_3+u_4, u_4, \dots)$ in the coordinates $u_2, u_4, \dots u_{2m}$ is denoted by $\deg^u_m f$. We show first that the operators $X(t)$ and $Y(t)$ preserve $\deg^u_m$.

The substitution $z_{2i-1} \to u_{2i-1} + u_{2i}$ corresponds to $x_i \to x_i + y_i$ and $T_{y,i} \to T_{y,i}T_{x,i}^{-1}$ in $\mathfrak{X}(t)$ and $\mathfrak{Y}(t)$. The same reasoning as in the proof of Theorem 6.1 shows that $\deg^u_m \det\mathfrak{X}(t)$ and $\deg\det\mathfrak{Y}(t)$ are bounded by $\deg\varphi + m$. Thus we have to find a way to decrease this estimate for the degree of $\det\mathfrak{X}(t)$ and $\det\mathfrak{Y}(t)$ by m.

The idea is to add a multiple of the y_i-row to the x_i-row. Since the entries are in a non-commutative ring some care is advised. For this, we develop $\det\mathfrak{X}(t)$ and $\det\mathfrak{Y}(t)$ as

$$\sum \pm \det A^{1,\overline{n}+1}_{i_1,j_1} \dots \det A^{m,\overline{n}+m}_{i_m,j_m} \det A^{m+1,\dots,\overline{n},\overline{n}+1+m,\dots,n}_S \tag{6.17}$$

where $\det A^U_V$ is the minor with row index in U and column index in V and where the sum runs through all partitions $\{1,\dots,n\} = \{i_1,j_1\}\dot\cup\dots\dot\cup\{i_m,j_m\}\dot\cup S$. The degree of the last factor (involving S) is zero. Now we show that the degree of each 2×2-minor is one less than expected which would prove the claim.

For this we write

(6.18)

$$\det A^{l,\overline{n}+l}_{i_l,j_l} = \det\begin{pmatrix} x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix} = (x_{11}+\alpha x_{21})x_{22} - x_{21}(x_{12}+\alpha x_{22}) - [\alpha, x_{21}]x_{22}$$

where

$$\alpha := \begin{cases} T_{x,l} & \text{if } A = \mathfrak{X}(t) \text{ and } n \text{ is even,}\\ y_l T_{x,l} & \text{if } A = \mathfrak{X}(t) \text{ and } n \text{ is odd,}\\ y_l^{-1}T_{x,l} & \text{if } A = \mathfrak{Y}(t) \text{ and } n \text{ is even,}\\ T_{x,l} & \text{if } A = \mathfrak{Y}(t) \text{ and } n \text{ is odd.}\end{cases} \tag{6.19}$$

As mentioned above this amounts to add α times row $\#\overline{n}+l$ to row $\#l$ of (4.11) or (4.14), respectively. Then it is easy to check that $\deg^u_m(x_{1j}+\alpha x_{2j}) \le \deg^u_m x_{1j} - 1$ and $\deg^u_m[\alpha, x_{21}] \le \deg^u_m x_{11} - 1$ which proves the claim.

The rest of the proof is the same as for Theorem 6.1. For $\lambda\in\Lambda$ let $\mathcal{P}_\lambda$ be the space of all semisymmetric functions f in which only monomials u^μ with $\mu \le [\lambda]$ and $\mu_{\rm even} \le \lambda_{\rm even}$ occur. Let $\mathcal{P}^\circ_\lambda$ be the same with additionally $\mu \ne \lambda$. Then both spaces are stable under $X(t)$ and $Y(t)$. Moreover $\mathbf{e}_\lambda \in \mathcal{P}_\lambda \setminus \mathcal{P}^\circ_\lambda$. We conclude as in Theorem 6.1. ∎

Examples. The improvement of strong triangularity over the weak one is the more significant the smaller $\lambda_{\rm even}$ is. The most extreme case is $\lambda = (a)$ where Theorem 6.1 doesn't give any restriction. But Theorem 6.6 states $R_{(a)} = \sum_{i=0}^a c_i\mathbf{e}_{(i)}$ which is of course also a consequence of the direct calculation in Corollary 4.10. A more specific example is $\lambda = (5, 2, 0, \dots)$ for $n \geq 10$. In that case, R_λ has, according to Theorem 6.1, 70 independent coefficients while Theorem 6.6 boils that down to 27.

Remark. Part b) of the theorem is entirely analogous to a similar theorem for (shifted) Jack polynomials but part a) is a bit strange since the pretty asymmetric coordinates u_i appear. A conceptual explanation for their appearance would be very desirable.

Now we can prove a triangularity property which is completely intrinsic for the polynomials R_λ:

6.7. Theorem. *For every $\lambda, \mu \in \Lambda$ consider the expansion $R_\lambda R_\mu = \sum_\tau a_{\lambda\mu}^\tau R_\tau$. Then $a_{\lambda\mu}^\tau = 0$ unless $\lambda, \mu \sqsubseteq \tau \preceq \lambda + \mu$.*

Proof. First we show $\tau \preceq \lambda + \mu$ whenever $a_{\lambda\mu}^\tau \neq 0$. We have

$$R_\lambda R_\mu = \sum_{\tau_1, \tau_2} b_{\lambda\tau_1} b_{\mu\tau_2} \mathbf{e}_{\tau_1} \mathbf{e}_{\tau_2} = \sum_\nu c_\nu \mathbf{e}_\nu \quad \text{with} \quad c_\nu = \sum_{\tau_1 + \tau_2 = \nu} b_{\lambda\tau_1} b_{\mu\tau_2}. \tag{6.20}$$

Moreover, $\tau_1 \preceq \lambda$ and $\tau_2 \preceq \mu$ imply $\nu = \tau_1 + \tau_2 \preceq \lambda + \mu$. Now, observe that the transformation matrix $b_{\lambda\mu}$ is upper unitriangular. Thus, its inverse matrix has the same property, i.e., we have expansions $\mathbf{e}_\lambda = \sum_{\mu \preceq \lambda} b'_{\lambda\mu} R_\mu$. Hence

$$R_\lambda R_\mu = \sum_{\nu, \tau} c_\nu b'_{\nu\tau} R_\tau = \sum_\tau a_{\lambda\mu}^\tau R_\tau \tag{6.21}$$

with $\tau \preceq \nu \preceq \lambda + \mu$.

Now we show $\lambda \sqsubseteq \tau$ whenever $a_{\lambda\mu}^\tau \neq 0$. The relation $\mu \sqsubseteq \tau$ follows then by symmetry. Let τ_0 be a $\sqsubseteq$-minimal counterexample. Since $\lambda \not\sqsubseteq \tau_0$, the Extra Vanishing Theorem 4.9 implies $R_\lambda(\varrho + \tau_0) = 0$. Hence $\sum_\tau a_{\lambda\tau}^\tau R_\tau(\varrho + \tau_0) = 0$. Again by the Extra Vanishing Theorem, only those τ with $\tau \sqsubseteq \tau_0$ contribute to this sum. For those we have $\lambda \not\sqsubseteq \tau$. The minimality of τ_0 implies $\tau = \tau_0$ unless $a_{\lambda\mu}^\tau = 0$. From this we derive the contradiction $a_{\lambda\mu}^{\tau_0} R_{\tau_0}(\varrho + \tau_0) = 0$. $\square$

7. The binomial theorem

In this section we derive a binomial type theorem for semisymmetric functions. The proof is similar to that for symmetric functions in [Ok].

So far, we considered values of R_λ in the points $z = \varrho + \lambda$. Now we use that the difference operators also have a dual vanishing property.

Recall that $\mathcal{R}$ is the algebra generated by the X_i and Y_i where $X(t) = \sum_i X_i t^i$ and $Y(t) = \sum_i Y_i t^i$. We introduce a degree function on $\mathcal{R}$ by letting $\deg z_i = 0$ and $\deg T_\lambda := |\lambda|_{\text{odd}}$. Thus, $\deg X_i = \deg Y_i = i$. Observe that $(\mathcal{P}^W)^*$ is a right $\operatorname{End} \mathcal{P}^W$-module, hence a right $\mathcal{R}$-module. For $\tau \in k^n$ let $\delta_\tau \in (\mathcal{P}^W)^*$ be the evaluation map $f \mapsto f(\tau)$. Then, as explained in its proof, Proposition 4.7 d) amounts to $\bigoplus_{\lambda \in \Lambda} k\delta_{\varrho+\lambda}$ being an $\mathcal{R}$-submodule of $(\mathcal{P}^W)^*$. Now, for any $\alpha \in k$, let $\underline{\alpha} := (\alpha, \dots, \alpha) \in k^n$ and $\varrho_\alpha := \varrho + \underline{\alpha} = ((n-i)r + \alpha)_i$. Then we have

7.1. Proposition. *Assume $r \neq 0$. Consider the space $M := \bigoplus_{\lambda \in \Lambda} k\, \delta_{-\varrho_\alpha - \lambda}$.*

a) *M is an $\mathcal{R}$-submodule of $(\mathcal{P}^W)^*$.*
b) *Define a filtration on M by putting $\deg \delta_{-\varrho_\alpha - \lambda} := |\lambda|_{\text{odd}}$. Assume $\alpha \notin -\mathbb{N} - \mathbb{N} \cdot 2r$. Then the map $\mathcal{R} \to M : D \mapsto \delta_{-\varrho_\alpha} D$ is an isomorphism of filtered k-vector spaces.*

Proof. Let $\lambda \in \Lambda$ and D be either X_i or Y_i. Then, the assumption $r \neq 0$ makes sure that $\delta_{-\varrho_\alpha-\lambda}D$ can be computed in the obvious way since then the denominator of D does not vanish at $-\varrho_\alpha - \lambda$.

a) It suffices to show that $\delta_{-\varrho_\alpha-\lambda}D_I \in M$ for any $I \subseteq \{1, \ldots, n\}$. This is no problem if $\lambda + \varepsilon_I \in \Lambda$. Otherwise, there is an index j with $j \notin I$, $i := j+1 \in I$, and $\lambda_i = \lambda_j$. But then the factor $z_i - z_j - r$ in D_I vanishes at $z = -\varrho_\alpha - \lambda$.

b) The map clearly preserves filtrations. Since corresponding filtration spaces on both sides have the same dimension, it suffices to show surjectivity. We do that by induction on the degree. By the explicit formulas (4.9), (4.10) we have

$$X_m = (-1)^m \sum_{\substack{I \in P_{\text{odd}} \\ |I|_o = m}} D_I + \sum_{I:|I|_o<m} c_I^{(1)}(z) D_I, \tag{7.1}$$

$$Y_m = (-1)^m \sum_{\substack{I \in P_{\text{even}} \\ |I|_o = m}} D_I + \sum_{I:|I|_o<m} c_I^{(2)}(z) D_I. \tag{7.2}$$

Thus, if we put $Z_{2m-1} := (-1)^m (X_m - Y_m)$ for $m = 1, \ldots, \overline{n}$ and $Z_{2m} := (-1)^m Y_m$ for $m = 0, \ldots, \underline{n} - 1$ we obtain operators with an expansion

$$Z_p = \sum_{\mu \in \mathbb{N}^n : \mu \preceq (1^p)} c_\mu(z) T_\mu \tag{7.3}$$

where

$$c_{(1^p)}(z) = \prod_{\substack{i \leq p \\ n-i \text{ even}}} z_i \prod_{\substack{i \leq p < j \\ j-i \text{ odd}}} (z_i - z_j - r) \prod_{\substack{i \leq p < j \\ j-i \text{ even}}} (z_i - z_j)^{-1}. \tag{7.4}$$

Let $\lambda \in \Lambda$ be non-zero. Let p be maximal with $\lambda_p \neq 0$. Put $\mu := \lambda - (1^p) \in \Lambda$. Then we have

$$\delta_{-\varrho_\alpha-\mu} Z_r \in c_{(1^p)}(-\varrho_\alpha - \mu)\delta_{-\varrho_\alpha-\lambda} + \sum_{\nu \prec \lambda} k \delta_{-\varrho_\alpha-\nu}. \tag{7.5}$$

The assumptions on r and α ensure that $c_{(1^p)}(-\varrho_\alpha - \mu) \neq 0$. The induction hypothesis implies that $\delta_{-\varrho_\alpha-\lambda}$ is in the image. $\square$

7.2. Lemma. *Assume $r \neq 0$ and $\alpha \notin -\mathbb{N} - \mathbb{N} \cdot 2r$. Then $R_\lambda(-\varrho_\alpha) \neq 0$ for all $\lambda \in \Lambda$.*

Proof. Suppose $R_\lambda(-\varrho_\alpha) = 0$. For $\mu \in \Lambda$ let $D \in \mathcal{R}$ with $\delta_{-\varrho_\alpha-\mu} = \delta_{-\varrho_\alpha}D$. Since R_λ is an eigenvector of D we also have $R_\lambda(-\varrho_\alpha - \mu) = 0$. This contradicts the fact that $-\varrho_\alpha - \Lambda$ is Zariski dense in k^n. $\square$

Remark. This lemma is only preliminary. Later, we prove the explicit formula (8.9) for $R_\lambda(-\varrho_\alpha)$.

The binomial type theorem, announced in the beginning, is:

7.3. Theorem. *Assume $r \neq 0$ and $\alpha \notin -\mathbb{N} - \mathbb{N} \cdot 2r$. Then for every $\lambda \in \Lambda$ the following formula holds:*

$$\frac{R_\lambda(-\underline{\alpha} - z)}{R_\lambda(-\varrho_\alpha)} = \sum_{\mu \in \Lambda} (-1)^{|\mu|_{\text{odd}}} \frac{R_\mu(\varrho + \lambda)}{R_\mu(\varrho + \mu)} \frac{R_\mu(z)}{R_\mu(-\varrho_\alpha)}. \tag{7.6}$$

Proof. The polynomials $R_\lambda(-\underline{\alpha}-z)$ form also a basis of $\mathcal{P}^W$. Hence, every $f\in\mathcal{P}^W$ has an expansion

$$f(z)=\sum_\mu a_\mu(f)R_\mu(-\underline{\alpha}-z) \tag{7.7}$$

with $a_\mu\in(\mathcal{P}^W)^*$. We claim $a_\mu\in M$ with $\deg a_\mu\le|\mu|_{\rm odd}$. To show this, we evaluate (7.7) at $z=-\varrho_\alpha-\mu$ and get

$$\delta_{-\varrho_\alpha-\mu}(f)=\sum_\tau a_\tau(f)R_\tau(\varrho+\mu)=a_\mu(f)+\sum_{|\tau|_{\rm odd}<|\mu|_{\rm odd}}a_\tau(f)R_\tau(\varrho+\mu). \tag{7.8}$$

Then the claim follows by induction on $|\mu|_{\rm odd}$.

It follows from Proposition 7.1 that there is $D_\mu\in\mathcal{R}$ with $\deg D_\mu\le|\mu|_{\rm odd}$ such that $a_\mu(f)=(D_\mu f)(-\varrho_\alpha)$. We apply this to $f=R_\lambda$. Then

$$a_\mu(R_\lambda)=(D_\mu R_\lambda)(-\varrho_\alpha)=p_\mu(\varrho+\lambda)R_\lambda(-\varrho_\alpha) \tag{7.9}$$

where $p_\mu:=c_0^{D_\mu}\in\mathcal{P}^W$ by Proposition 4.7 c). We have $\deg p_\mu=\deg D_\mu\le|\mu|_{\rm odd}$. On the other side, we see directly from (7.7) that

$$a_\mu(R_\lambda)=\begin{cases}0 & \text{if } |\lambda|_{\rm odd}\le|\mu|_{\rm odd} \text{ and } \lambda\ne\mu;\\ (-1)^{|\mu|_{\rm odd}} & \text{if } \lambda=\mu.\end{cases} \tag{7.10}$$

Thus (7.9), (7.10) together and the very definition of $R_\mu(z)$ imply

$$R_\mu(-\varrho_\alpha)p_\mu(z)=\frac{(-1)^{|\mu|_{\rm odd}}}{R_\mu(\varrho+\mu)}R_\mu(z) \tag{7.11}$$

and therefore

$$a_\mu(R_\lambda)=(-1)^{|\mu|_{\rm odd}}\frac{R_\mu(\varrho+\lambda)}{R_\mu(\varrho+\mu)}\frac{R_\lambda(-\varrho_\alpha)}{R_\mu(-\varrho_\alpha)}. \tag{7.12}$$

Now, we insert this into (7.7), replace z by $-\underline{\alpha}-z$ and obtain (7.6). □

Remarks. 1. By Theorem 4.9, only those μ with $\mu\sqsubseteq\lambda$ contribute to the sum in formula (7.6). In particular, the sum is finite.

2. The normalizing factor $R_\lambda(-\varrho_\alpha)$ in the denominator renders the formula more symmetric but causes the restriction on α. Of course, for every α there is an expansion of $R_\lambda(-\underline{\alpha}-z)$ in terms of $R_\mu(z)$. It can be easily obtained by using the explicit formula (8.9) to calculate the ratio $R_\lambda(-\varrho_\alpha)/R_\mu(-\varrho_\alpha)$.

There are two immediate applications of the binomial formula (7.6).

7.4. Corollary. *Assume $r\ne0$ and $\alpha\notin-\mathbb{N}-\mathbb{N}\cdot2r$. Then the matrix*

$$\left(\frac{R_\lambda(-\varrho_\alpha-\nu)}{R_\lambda(-\varrho_\alpha)}\right)_{\lambda,\nu\in\Lambda} \tag{7.13}$$

is symmetric.

Proof. Substitute $z=\varrho+\nu$ in (7.6). Then the right-hand side becomes clearly symmetric in λ and ν. □

7.5. Corollary. *The matrix*

$$\left((-1)^{|\mu|_{\rm odd}}\frac{R_\mu(\varrho+\lambda)}{R_\mu(\varrho+\mu)}\right)_{\mu,\lambda\in\Lambda} \tag{7.14}$$

is an involution.

Proof. By the binomial formula (7.6), the matrix (7.14) expresses the effect of the involution $z \mapsto -\underline{\alpha} - z$ on $\mathcal{P}^W$ with respect to the basis $\frac{R_\mu(z)}{R_\mu(-\varrho_\alpha)}$, at least if $r \neq 0$. For $r = 0$ we argue by continuity. □

The involutory matrix (7.14) can be used to derive an explicit interpolation formula (Theorem 7.6 iii) below). For this, let $\mathcal{C}(\varrho + \Lambda)$ be the set of k-valued functions on $\varrho + \Lambda$. For $f \in \mathcal{C}(\varrho + \Lambda)$ we define $\hat{f} \in \mathcal{C}(\varrho + \Lambda)$ by

$$\hat{f}(\varrho + \lambda) := \sum_{\mu \in \Lambda} (-1)^{|\mu|_{\mathrm{odd}}} \frac{R_\mu(\varrho + \lambda)}{R_\mu(\varrho + \mu)} f(\varrho + \mu). \tag{7.15}$$

For any fixed λ the sum is finite by the Extra Vanishing Theorem 4.9. Let $\mathcal{C}_0(\varrho + \Lambda) \subseteq \mathcal{C}(\varrho + \Lambda)$ be the functions with finite support. We consider, via restriction, $\mathcal{P}^W$ as a subspace of $\mathcal{C}(\varrho + \Lambda)$. By Lemma 4.4, $\mathcal{C}(\varrho + \Lambda)$ is a left $\mathcal{R}$-module, provided $r \neq 0$.

7.6. Theorem. *The transformation $f \mapsto \hat{f}$ has the following properties:*

i) $\hat{\hat{f}} = f$.
ii) $f \in \mathcal{P}^W \Leftrightarrow \hat{f} \in \mathcal{C}_0(\varrho + \Lambda)$.
iii) *For $f \in \mathcal{P}^W$ holds $f(z) = \sum_{\mu \in \Lambda} (-1)^{|\mu|_{\mathrm{odd}}} \hat{f}(\varrho + \mu) \frac{R_\mu(z)}{R_\mu(\varrho + \mu)}$.*
iv) *Assume $r \neq 0$. For every $D \in \mathcal{R}$ holds $\widehat{D(f)} = c_0^D \hat{f}$ and $\widehat{c_0^D f} = D(\hat{f})$.*

Proof. i) follows from Corollary 7.5 and the fact that the transpose of an involutive matrix is involutive. Let $\chi_{\varrho+\nu} \in \mathcal{C}_0$ be the characteristic function of the one-point set $\{\varrho + \nu\}$. Then $\hat{\chi}_{\varrho+\nu} = (-1)^{|\nu|_{\mathrm{odd}}} R_\nu(\varrho + \nu)^{-1} R_\nu$. Hence, $f \mapsto \hat{f}$ maps a basis of $\mathcal{C}_0$ to a basis of $\mathcal{P}^W$ which proves ii). Part iii) is a direct consequence of i) and ii).

Finally, let $D \in \mathcal{R}$. The second formula in iv) follows from the first by i). Thus we have to prove

$$\widehat{D(f)}(\varrho + \lambda) = c_0^D(\varrho + \lambda) \hat{f}(\varrho + \lambda) \tag{7.16}$$

for every $D \in \mathcal{R}$, $\lambda \in \Lambda$, and $f \in \mathcal{C}$. If we fix D and λ, then there is a finite subset $S \subset \varrho + \Lambda$ such that both sides of (7.16) depend only on values of f in S. Since on S every $f \in \mathcal{C}$ can be interpolated by an element of $\mathcal{P}^W$ it suffices to prove (7.16) for $f = R_\nu$. But then we have

$$\widehat{D(R_\nu)} = [c_0^D(\varrho + \nu) R_\nu]^\wedge = c_0^D(\varrho + \nu) \hat{R}_\nu = c_0^D \hat{R}_\nu. \tag{7.17}$$

The last equality holds since $\hat{R}_\nu$ is a multiple of the characteristic function $\chi_{\varrho+\nu}$. □

7.7. Corollary. *Assume $r \neq 0$. Let $\mathcal{A} \subseteq \mathrm{End}_k \mathcal{P}^W$ be the algebra generated by $\mathcal{P}^W$ and $\mathcal{R}$. Then there is an involutory automorphism of $\mathcal{A}$ which interchanges $\mathcal{P}^W$ and $\mathcal{R}$.*

Proof. The automorphism is $D \mapsto \hat{D}$, where $\hat{D}(f) := \widehat{D(\hat{f})}$. Then Theorem 7.6 i) implies that this is an involution and part iv) implies $\hat{D} = c_0^D$ for every $D \in \mathcal{R}$. □

8. The evaluation formula

The symmetry of the matrix (7.13) allows us to switch the index with the argument. Using this, we obtain Pieri type formulas:

8.1. Theorem. *Assume $r \neq 0$ and $\alpha \notin -\mathbb{N} - \mathbb{N}\cdot 2r$. Let $D = \sum_\eta c_\eta^D(z) T_\eta \in \mathcal{R}$. Then for all $\mu \in \Lambda$ holds*

$$c_0^D(-\underline{\alpha} - z)\frac{R_\mu(z)}{R_\mu(-\varrho_\alpha)} = \sum_{\lambda\in\Lambda} c_{\lambda-\mu}^D(-\varrho_\alpha - \mu)\frac{R_\lambda(z)}{R_\lambda(-\varrho_\alpha)}. \tag{8.1}$$

Proof. We substitute $z = -\varrho_\alpha - \mu$ in the equation $c_0^D(\varrho+\nu)R_\nu(z) = D(R_\nu)(z)$ and apply symmetry (i.e., Corollary 7.4) on both sides. Thus we obtain

$$c_0^D(\varrho+\nu)R_\nu(-\varrho_\alpha)\frac{R_\mu(-\varrho_\alpha-\nu)}{R_\mu(-\varrho_\alpha)} = \sum_\eta c_\eta^D(-\varrho_\alpha - \mu)R_\nu(-\varrho_\alpha)\frac{R_{\mu+\eta}(-\varrho_\alpha-\nu)}{R_{\mu+\eta}(-\varrho_\alpha)}. \tag{8.2}$$

After canceling $R_\nu(-\varrho_\alpha)$, both sides of (8.2) become polynomials in ν. Hence we may replace ν by $-\varrho_\alpha - z$. Then putting $\eta = \lambda - \mu$ yields the desired formula. □

We are applying this to $D = X(t)$ and $D = Y(t)$. By Proposition 4.3, the non-zero coefficients are

$$\begin{aligned} c_{\varepsilon_I}^{X(t)}(z) &= (-1)^{|I|_o}\, u_I^{\mathrm{odd}}(z,t)\, v_I(z)\, w_I(z;r), \quad I \in P_{\mathrm{odd}},\\ c_{\varepsilon_I}^{Y(t)}(z) &= (-1)^{|I|_o}\, u_I^{\mathrm{even}}(z,t)\, v_I(z)\, w_I(z;r), \quad I \in P_{\mathrm{even}}. \end{aligned} \tag{8.3}$$

where

$$u_I^{\mathrm{odd/even}}(z,t) = \prod_{\substack{i\notin I\\ i \text{ odd/even}}} (t+z_i), \qquad v_I(z) = \prod_{\substack{i\in I\\ n-i \text{ even}}} z_i \tag{8.4}$$

$$w_I(z;r) = \prod_{\substack{i\in I, j\notin I\\ j-i \text{ odd}}} (z_i - z_j - r)\cdot \prod_{\substack{i\in I, j\notin I\\ j-i \text{ even}}} (z_i - z_j)^{-1}. \tag{8.5}$$

After replacing t by $\alpha - t$ we obtain

$$\begin{aligned} \prod_{i \text{ odd/even}}(t+z_i)\ R_\mu(z) = \sum_{I\in P_{\mathrm{odd/even}}} & u_I^{\mathrm{odd/even}}(\varrho+\mu, t)\\ &\times v_I(\varrho_\alpha+\mu) w_I(\varrho_\alpha+\mu; -r)\frac{(-1)^{|I|_o}R_\mu(-\varrho_\alpha)}{R_{\mu+\varepsilon_I}(-\varrho_\alpha)}R_{\mu+\varepsilon_I}(z). \end{aligned} \tag{8.6}$$

We postpone the simplification of this formula until section 9. Instead, we use (8.6) to derive an explicit formula for $R_\lambda(-\varrho_\alpha)$. To state the result let

$$[x\uparrow m] := x(x+1)\dots(x+m-1) \tag{8.7}$$

be the rising factorial polynomial. For a box $s = (i,j) \in \mathbb{N}^2$ of a partition λ recall the following notation:

$$\begin{aligned} a_\lambda(s) &:= \lambda_i - j \quad \text{(arm-length)} \quad & a'(s) &:= j-1 \quad \text{(arm-colength)},\\ l_\lambda(s) &:= \lambda'_j - i \quad \text{(leg-length)} \quad & l'(s) &:= i-1 \quad \text{(leg-colength)}. \end{aligned} \tag{8.8}$$

8.2. Theorem. *Assume $r \neq 0$. Then for every $\lambda \in \Lambda$ the evaluation formula*

$$R_\lambda(-\varrho_\alpha) = (-1)^{|\lambda|_{\mathrm{odd}}} A_\lambda(\alpha) B_\lambda, \tag{8.9}$$

holds where

$$A_\lambda(\alpha) := \prod_{\substack{1\le i\le n\\ n-i \text{ even}}} [\alpha + (n-i)r \uparrow \lambda_i] = \prod_{\substack{s\in\lambda\\ n-l'(s) \text{ odd}}} \big(\alpha + a'(s) + (n - l'(s) - 1)r\big) \tag{8.10}$$

and

$$B_\lambda := \frac{\prod\limits_{\substack{1\le i<j\le n\\ j-i \text{ odd}}} [(j-i+1)r \uparrow \lambda_i - \lambda_j]}{\prod\limits_{\substack{1\le i<j\le n\\ j-i \text{ even}}} [(j-i)r \uparrow \lambda_i - \lambda_j]} = \frac{\prod\limits_{\substack{s\in\lambda\\ n-l'(s) \text{ even}}} \big(a'(s)+(n-l'(s))r\big)}{\prod\limits_{\substack{s\in\lambda\\ l_\lambda(s) \text{ odd}}} \big(a_\lambda(s)+(l_\lambda(s)+1)r\big)}. \tag{8.11}$$

Proof. By continuity, we may assume $\alpha \notin -\mathbb{N} - \mathbb{N}\cdot 2r$. We expand both sides of (8.6) as a polynomial in t and compare coefficients. Then the product on the left-hand side becomes $e_d(z_{\text{odd/even}})$ while on the right-hand side the u_I-factor has to be replaced by $e_{d-|I|_o}(z_i | i \notin I \text{ odd/even})$. In particular, only sets I with $|I|_o \le d$ enter the formula. It is easily checked that the set of ε_I with $I \in P_{\text{odd}}$ (resp. $I \in P_{\text{even}}$) with $|I|_o \le d$ has a unique maximum with respect to the order relation $[\mu] \le [\lambda]$, namely $(1^b) = \sum_{i=1}^b \varepsilon_i$ where $b = 2d-1$ (resp. $b = 2d$). Thus, by Theorem 6.1, the monomial $z^{[\mu+(1^b)]}$ appears on the right-hand side exactly once. Comparing its coefficient, we obtain for all $b = 1,\ldots,n$:

$$R_{\mu+(1^b)}(-\varrho_\alpha) = (-1)^{\lceil \frac{b}{2}\rceil} v_{(1^b)}(\varrho_\alpha+\mu) w_{(1^b)}(\varrho_\alpha+\mu;-r) R_\mu(-\varrho_\alpha). \tag{8.12}$$

This is a recursion relation which allows us to compute $R_\mu(-\varrho_\alpha)$ by deleting one column at a time. It follows that $R_\mu(-\varrho_\alpha) = (-1)^{|\mu|_{\text{odd}}} A_\mu B_\mu$ with $A_{\mu+(1^b)} = v_{(1^b)}(\varrho + \underline{\alpha} + \mu) A_\mu$ and $B_{\mu+(1^b)} = w_{(1^b)}(\varrho_\alpha+\mu;-r) B_\mu$.

The first relation implies easily both formulas for A_μ. The expressions for B_μ could be derived in the same way as the analogous formulas [Mac], VI(6.11) and (6.11′) for Macdonald polynomials. Especially the second expression for B_λ in (8.11) is quite tedious to derive which is mainly due to the parity conditions. But there is a trick to derive our formulas directly from Macdonald's formulas (6.11) and (6.11′). For any product of the form

$$P = \prod_i (1 - q^{a_i} t^{b_i}) \tag{8.13}$$

with $a_i, b_i \ge 0$ we define

$$[P]_{\text{even}} := \prod_{\{i | b_i \text{ even}\}} (a_i + b_i r). \tag{8.14}$$

The map $P \mapsto [P]_{\text{even}}$ is multiplicative. The point is now the easily verified formula $[B_{\nu/\mu}]_{\text{even}} = w_{(1^b)}(\varrho_\alpha+\mu;-r)$ with $B_{\nu/\mu}$ as in *loc. cit.* VI(6.4). From *loc. cit.* VI(6.10) we obtain $B_\mu = [u_0(P_\mu)]_{\text{even}}$. Now the formulas for B_μ above are nothing more than $[\cdot]_{\text{even}}$ applied to *loc. cit.* VI(6.11) and (6.11′). □

Remarks. 1. The evaluation formula (8.9) are also valid for $r = 0$ provided one replaces the expressions for B_λ in (8.2) by their limits for $r \to 0$. The same remark holds for all the Pieri formulas in section 9.

2. The polynomials $R_\lambda(z;r)$ have, in general, coefficients in $\mathbb{Q}(r)$. Conjecturally, one obtains an integral form as follows. For $\lambda \in \Lambda$ let

$$[c_\lambda]_{\text{even}} := \prod_{\substack{s\in\lambda\\ l_\lambda(s) \text{ odd}}} \big(a_\lambda(s) + (l_\lambda(s)+1)r\big) \tag{8.15}$$

and $\mathcal{R}_\lambda(z;r) := [c_\lambda]_{\text{even}} R_\lambda(z;r)$.

8.3. Conjecture. *For all* $\lambda \in \Lambda$ *holds* $\mathcal{R}_\lambda(z;r) \in \mathbb{Z}[r,z]$.

The factor $[c_\lambda]_{\rm even}$ is the denominator of the second expression for B_λ in (8.2). For $n \gg 0$ there is no cancellation involving the variable r. Thus, at least up to a rational factor and for big n, the conjectured statement appears to be optimal. The conjecture has been tested for $n \le 6$ and $|\lambda|_{\rm odd} \le 6$. (Shifted) Jack polynomials have also certain positivity properties (see [KS1] and [KS2]) but none of them seem to generalize to semisymmetric polynomials.

Now, we specialize the evaluation and the binomial formula to the homogeneous polynomials $\overline{R}_\lambda$. The evaluation formula (8.9) becomes

8.4. Theorem. *For $\lambda \in \Lambda$ holds*

$$\overline{R}_\lambda(1,\ldots,1) = \begin{cases} B_\lambda & \text{if } n \text{ is odd or } [\lambda]_1 = 0; \\ 0 & \text{otherwise.} \end{cases} \tag{8.16}$$

Proof. This follows from (8.9) by calculating $\lim_{\alpha\to\infty} \alpha^{-|\lambda|_{\rm odd}} R_\lambda(-\varrho_\alpha)$. The parity of n comes in because $\deg_\alpha A_\lambda(\alpha)$ equals $|\lambda|_{\rm odd}$ (resp. $|\lambda|_{\rm even}$) if n is odd (resp. even). $\square$

Now, the binomial formula (7.6) becomes:

8.5. Theorem. *Assume $r \neq 0$. For all $\lambda \in \Lambda$ holds*

$$\frac{\overline{R}_\lambda(\underline{1}+z)}{B_\lambda} = \sum_\mu \frac{R_\mu(\varrho+\lambda)}{R_\mu(\varrho+\mu)} \frac{\overline{R}_\mu(z)}{B_\mu}. \tag{8.17}$$

The sum is over all $\mu \in \Lambda$ with $\mu \sqsubseteq \lambda$ and, in case n is even, additionally with $[\mu]_1 = [\lambda]_1$.

Proof. Using the evaluation formula (8.9), this follows from (7.6) by replacing z by αz, dividing by the appropriate power α^N and taking the limit $\alpha \to \infty$. Here

$$N = \begin{cases} |\lambda|_{\rm odd} - |\lambda|_{\rm odd} = 0 & \text{if } n \text{ is odd;} \\ |\lambda|_{\rm odd} - |\lambda|_{\rm even} = [\lambda]_1 & \text{if } n \text{ is even.} \end{cases} \tag{8.18}$$

$\square$

Remark. In the classical case, Lassalle [La] used the expansion of $\overline{P}_\lambda(\underline{1}+z)$ to *define* "generalized binomial coefficients". Okounkov and Olshanski [OO] proved later the classical analogue of (8.17). This implies in particular that Lassalle's binomial coefficients equal $\frac{P_\mu(\varrho+\lambda)}{P_\mu(\varrho+\mu)}$. Thus, classically it is possible to define shifted polynomials from the theory of the unshifted ones. The theorem above shows that this fails in the semisymmetric case if the number of variables n is even.

9. The Pieri formula

Now, we make the Pieri formula completely explicit. Recall again some notation: For $\lambda = \mu + \varepsilon_I$ let $C_{\lambda/\mu}$ (resp. $R_{\lambda/\mu}$) be the set of boxes of μ which are in the same column (resp. row) as some box of $\lambda \setminus \mu$.

9.1. Theorem. *Assume $r \neq 0$. For every $\mu \in \Lambda$ holds*

$$\prod_{i \text{ odd/even}} (t+z_i)\ R_\mu(z) = \sum_{I \in P_{\rm odd/even}} \prod_{\substack{i \notin I \\ i \text{ odd/even}}} (t+\varrho_i+\mu_i)\ [\psi'_{\lambda/\mu}]_{\rm even}\ R_\lambda(z). \tag{9.1}$$

Here $\lambda = \mu + \varepsilon_I$ and I runs through $P_{\mathrm{odd/even}}$. Moreover,

$$
\begin{aligned}
[\psi'_{\lambda/\mu}]_{\mathrm{even}} &= \frac{\prod\limits_{\substack{i\notin I, j\in I\\ i<j, j-i \text{ odd}}} \big(\mu_i-\mu_j+(j-i-1)r\big)\big(\lambda_i-\lambda_j+(j-i+1)r\big)}{\prod\limits_{\substack{i\notin I, j\in I\\ i<j, j-i \text{ even}}} \big(\mu_i-\mu_j+(j-i)r\big)\big(\lambda_i-\lambda_j+(j-i)r\big)} \\
&= \prod_{s\in C_{\lambda/\mu}-R_{\lambda/\mu}} \frac{[b_\lambda(s)]_{\mathrm{even}}}{[b_\mu(s)]_{\mathrm{even}}}
\end{aligned}
\tag{9.2}
$$

where

$$
[b_\lambda(s)]_{\mathrm{even}} := \begin{cases} a_\lambda(s)+(l_\lambda(s)+1)r & \textit{for } l_\lambda(s) \textit{ odd,} \\ \big((a_\lambda(s)+1)+l_\lambda(s)r\big)^{-1} & \textit{for } l_\lambda(s) \textit{ even.} \end{cases}
\tag{9.3}
$$

Proof. By (8.6) and the evaluation formula (8.9) we have to calculate

$$
\left(v_I(\varrho_\alpha+\mu)\frac{A_\mu(\alpha)}{A_{\mu+\varepsilon_I}(\alpha)}\right)\left(w_I(\varrho_\alpha+\mu;-r)\frac{B_\mu}{B_{\mu+\varepsilon_I}}\right).
\tag{9.4}
$$

One easily verifies that the first factor is 1 (not too surprising, given the fact that the result can not depend on α). For the second factor we apply the same trick as in the proof of Theorem 8.2 and obtain: it is $[\cdot]_{\mathrm{even}}$ applied to the corresponding formulas for Macdonald polynomials. The result follows from [Mac], VI(6.7′), (6.13), and (6.23). □

Remarks. 1. A priori, it might happen that $\lambda = \mu + \varepsilon_I$ is not a partition. But then $[\psi'_{\lambda/\mu}]_{\mathrm{even}} = 0$ and the corresponding summand may be omitted. In fact, in that case there is an index i such that $i \notin I$, $j := i + 1 \in I$ and $\mu_i = \mu_{i+1}$. Thus $[\psi'_{\lambda/\mu}]_{\mathrm{even}}$ contains the factor $\mu_i - \mu_j + (j - i - 1)r = 0$.

2. The first expression for $[\psi'_{\lambda/\mu}]_{\mathrm{even}}$ shows that it is a rational function in μ. The second expression takes the cancellation into account which occurs when the "vertical strip" $\lambda \setminus \mu$ contains boxes in the same column, i.e., if there is an index i such that $i \in I$, $i + 1 \in I$ and $\mu_i = \mu_{i+1}$.

By comparing coefficients of powers of t we easily obtain Pieri formulas involving elementary symmetric functions:

9.2. Corollary.

$$
e_m(z_i \mid i\,{}^{\mathrm{odd}}_{\mathrm{even}})\; R_\mu(z) = \sum_I \; e_{m-s}(\mu_i+\varrho_i \mid i\notin I, i\,{}^{\mathrm{odd}}_{\mathrm{even}})\; [\psi'_{\lambda/\mu}]_{\mathrm{even}}\; R_\lambda(z)
\tag{9.5}
$$

where $\lambda = \mu + \varepsilon_I$ and where I runs through the elements of $P_{\mathrm{odd/even}}$ with $s := |I|_o \leq m$.

Example. For $n = 3$ we obtain:

$$
\begin{aligned}
(z_1+z_3)\cdot R_{\mu_1,\mu_2,\mu_3} &= (\mu_1+\mu_3+2r)R_{\mu_1,\mu_2,\mu_3} \\
&\quad + R_{\mu_1+1,\mu_2,\mu_3} + \frac{(\mu_2-\mu_3)(\mu_2-\mu_3-1+2r)}{(\mu_1-\mu_3+2r)(\mu_1-\mu_3-1+2r)} R_{\mu_1,\mu_2,\mu_3+1} \\
&\quad + R_{\mu_1+1,\mu_2+1,\mu_3} + \frac{(\mu_1-\mu_2)(\mu_1-\mu_2-1+2r)}{(\mu_1-\mu_3+2r)(\mu_1-\mu_3-1+2r)} R_{\mu_1,\mu_2+1,\mu_3+1},
\end{aligned}
\tag{9.6}
$$

$$z_2\cdot R_{\mu_1,\mu_2,\mu_3}=(\mu_2+r)R_{\mu_1,\mu_2,\mu_3}$$
$$+R_{\mu_1+1,\mu_2+1,\mu_3}+\frac{(\mu_1-\mu_2)(\mu_1-\mu_2-1+2r)}{(\mu_1-\mu_3+2r)(\mu_1-\mu_3-1+2r)}R_{\mu_1,\mu_2+1,\mu_3+1},\tag{9.7}$$

(9.8)
$$\begin{aligned}z_1z_3\cdot R_{\mu_1,\mu_2,\mu_3}&=(\mu_1+2r)\mu_3R_{\mu_1,\mu_2,\mu_3}\\&+\mu_3R_{\mu_1+1,\mu_2,\mu_3}+(\mu_1+2r)\frac{(\mu_2-\mu_3)(\mu_2-\mu_3-1+2r)}{(\mu_1-\mu_3+2r)(\mu_1-\mu_3-1+2r)}R_{\mu_1,\mu_2,\mu_3+1}\\&+\mu_3R_{\mu_1+1,\mu_2+1,\mu_3}+(\mu_1+2r)\frac{(\mu_1-\mu_2)(\mu_1-\mu_2-1+2r)}{(\mu_1-\mu_3+2r)(\mu_1-\mu_3-1+2r)}R_{\mu_1,\mu_2+1,\mu_3+1}\\&+R_{\mu_1+1,\mu_2+1,\mu_3+1}.\end{aligned}$$

Formula (9.5) can be used to give a Pieri rule for multiplication with shifted elementary semisymmetric polynomials. For this, we introduce the following notation. Let $I\in P_{\rm odd}$, $s=|I|$ and f a semi-symmetric polynomial in $n-s$ variables. Then we define $f(z|I'):=f(z_{k_{s+1}},\ldots,z_{k_n})$ where $i\mapsto k_i$ is any *parity preserving* bijection from $\{s+1,\ldots,n\}$ to $\{1,\ldots,n\}\setminus I$. For example, if $n=5$, then $f(z|\{2,3,5\}')=f(z_4,z_1)$.

9.3. Theorem. *For $m=0,\ldots,n$ and $\mu\in\Lambda$ holds*

$$R_{(1^m)}(z)R_\mu(z)=\sum_{s=0}^{m}\sum_{|I|=s}R_{(1^{m-s})}(\varrho+\mu|I')\ [\psi'_{\lambda/\mu}]_{\rm even}\ R_\lambda(z).\tag{9.9}$$

Here $\lambda=\mu+\varepsilon_I$ and I runs through $P_{\rm odd/even}$ according to m odd/even.

Proof. From the Triangularity Theorem 6.1 (or Corollary 2.8 and the explicit formulas in [KS1]) it follows that $R_{(1^m)}$ is a linear combination of elementary symmetric functions in the odd or even variables. Thus Corollary 9.2 implies that there are semisymmetric functions $f_p(z_1,\ldots,z_{n-p})$ such that

$$R_{(1^m)}(z)R_\mu(z)=\sum_I f_{|I|}(\varrho+\mu|I')[\psi'_{\lambda/\mu}]_{\rm even}R_\lambda(z).\tag{9.10}$$

Here the sum is over all $I\in P_{\rm odd/even}$ if m is odd/even. Moreover, (with $p=|I|$)

$$\deg f_p\le\deg R_{(1^m)}-|I|_o=\left\lceil\frac{m}{2}\right\rceil-\left\lceil\frac{p}{2}\right\rceil.\tag{9.11}$$

As one easily checks, the formula

$$\left\lceil\frac{m}{2}\right\rceil-\left\lceil\frac{p}{2}\right\rceil=\left\lceil\frac{m-p}{2}\right\rceil\tag{9.12}$$

holds *except* when m is even and p is odd, a case which does not occur. Thus we get $\deg f_p\le\deg R_{(1^{m-p})}$ for all m.

Next, we show that the vanishing conditions hold for f_p. For this, let $I=\{1,\ldots,p\}$. Then $f_{|I|}(\varrho+\mu|I')=f_p(\varrho_{p+1}+\mu_{p+1},\ldots,\varrho_n+\mu_n)$. Put $\lambda=\mu+\varepsilon_I$. Since $[\psi'_{\lambda/\mu}]_{\rm even}\ne0$ it suffices to show that R_λ does not occur in the expansion of $R_{(1^m)}R_\mu$ whenever $\mu_{p+(m-p)}=\mu_m=0$. For this, we put $z_m=\varrho_m,\ldots,z_n=\varrho_n$. Then the left-hand side of (9.10) vanishes while, by Proposition 2.5, on the right-hand side those R_λ's which don't vanish remain linearly independent. Thus, the coefficient in front of $R_{\mu+\varepsilon_I}$ is zero which proves the claim.

We have proved that f_p is a multiple of $R_{(1^{m-p})}$. To show equality we put $z_{m+1}=\varrho_{m+1},\ldots,z_n=\varrho_n$ in (9.10) and then replace z_i by ϱ_m+z_i for $i=1,\ldots,m$.

Then $R_{(1^m)}(z)$ becomes $R_{(1^m)}(z_1,\ldots,z_m)$ which is the last elementary symmetric polynomial in the odd, respectively even, variables. Thus (9.10) becomes simply a special case of Corollary 9.2 which implies $f_p = R_{(1^{m-p})}$. ☐

Example. By (2.22) for $n=3$ holds $R_{(1)}(z) = z_1 - z_2 + z_3 - r$. Thus formulas (9.6) and (9.7) imply

$$\begin{aligned} R_{(1)}\cdot R_{\mu_1,\mu_2,\mu_3} &= (\mu_1{-}\mu_2{+}\mu_3)R_{\mu_1,\mu_2,\mu_3} \\ &\quad + R_{\mu_1+1,\mu_2,\mu_3} + \frac{(\mu_2{-}\mu_3)(\mu_2{-}\mu_3{-}1{+}2r)}{(\mu_1{-}\mu_3{+}2r)(\mu_1{-}\mu_3{-}1{+}2r)}R_{\mu_1,\mu_2,\mu_3+1} \end{aligned} \tag{9.13}$$

in accordance with (9.9), case $m=1$. Observe also the cancellation which occurs when one subtracts (9.7) from (9.6). This is reflected in the fact that in (9.9) for m odd the a priori possible terms with $|I| = m+1$ are missing.

As a consequence of (9.9) we obtain a Pieri rule for the top homogeneous parts:

9.4. Corollary. *For every $\mu\in\Lambda$ and $m=1,\ldots,n$ holds*

$$\mathbf{e}_m(z)\overline{R}_\mu(z) = \sum_I [\psi'_{\lambda/\mu}]_{\mathrm{even}}\overline{R}_\lambda(z). \tag{9.14}$$

Here $\lambda = \mu+\varepsilon_I$ and I runs through all subsets of $\{1,\ldots,n\}$ consisting of $\lceil\frac{m}{2}\rceil$ odd numbers and $\lfloor\frac{m}{2}\rfloor$ even numbers.

Finally, we complete the explicit computation of R_λ, started in Corollary 4.10, where λ is a hook.

9.5. Corollary. *Let $a, m \geq 2$ be integers with m even. Then*

$$\begin{aligned} R_{(a\,1^{m-1})} &= (R_{(1)}-1)(R_{(1)}-2)\ldots \\ &\quad (R_{(1)}-a+2)\big(R_{(1)}R_{(1^m)} - \frac{a-1}{a-1+mr}R_{(1^{m+1})}\big). \end{aligned} \tag{9.15}$$

Proof. From formula (9.9) and some short calculations we get

$$\begin{aligned} R_{(1)}\cdot R_{(a-1\,1^{m-1})} &= (a-2)R_{(a-1\,1^{m-1})} + R_{(a\,1^{m-1})} \\ &\quad + \frac{mr}{(a{-}2{+}mr)(a{-}1{+}mr)}R_{(a-1\,1^m)}. \end{aligned} \tag{9.16}$$

Thus,

$$R_{(a\,1^{m-1})} = (R_{(1)}{-}a{+}2)R_{(a-1\,1^{m-1})} - \frac{mr}{(a{-}2{+}mr)(a{-}1{+}mr)}R_{(a-1\,1^m)}. \tag{9.17}$$

This already implies formula (9.15) for $a=2$. For $a\geq 2$ we are using induction and formula (4.24) for $R_{(a-1\,1^m)}$:

(9.18)

$$\begin{aligned} R_{(a\,1^{m-1})} &= (R_{(1)}-1)\ldots(R_{(1)}-a+2)\big(R_{(1)}R_{(1^m)} - \frac{a-2}{a-2+mr}R_{(1^{m+1})}\big) \\ &\quad - \frac{mr}{(a-2+mr)(a-1+mr)}(R_{(1)}-1)\ldots(R_{(1)}-a+2)R_{(1^{m+1})} \\ &= (R_{(1)}-1)(R_{(1)}-2)\ldots(R_{(1)}-a+2)\big(R_{(1)}R_{(1^m)} - \frac{a{-}1}{a{-}1{+}mr}R_{(1^{m+1})}\big). \end{aligned}$$

☐

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