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THE HOM-SPACES BETWEEN PROJECTIVE FUNCTORS

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ABSTRACT. The category of projective functors on a block of the category $\mathcal{O}(\mathfrak{g})$ of Bernstein, Gelfand and Gelfand, over a complex semisimple Lie algebra $\mathfrak{g}$, embeds to a corresponding block of the category $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})$. In this paper we give a nice description of the object V in $\mathcal{O}(\mathfrak{g}\times\mathfrak{g})$ corresponding to the identity functor; we show that V is isomorphic to the module of invariants, under the diagonal action of the center $\mathcal{Z}$ of the universal enveloping algebra of $\mathfrak{g}$, in the so-called anti-dominant projective.

As an application we use Soergel's theory about modules over the coinvariant algebra C, of the Weyl group, to describe the space of homomorphisms of two projective functors T and T'. We show that there exists a natural C-bimodule structure on $\mathrm{Hom}_{\{\mathrm{Functors}\}}(T, T')$ such that this space becomes free as a left (and right) C-module and that evaluation induces a canonical isomorphism $k \otimes_C \mathrm{Hom}_{\{\mathrm{Functors}\}}(T, T') \cong \mathrm{Hom}_{\mathcal{O}(\mathfrak{g})}(T(M_e), T'(M_e))$, where M_e denotes the dominant Verma module in the block and k is the complex numbers.

1. INTRODUCTION

1.1. Beginning around 1970, a number of mathematicians made great progress in understanding the structure of infinite-dimensional representations of a complex semisimple (or reductive) Lie algebra $\mathfrak{g}$ by using the operation of tensor product (over the complex numbers) with a finite-dimensional representation. This operation is an exact functor on the category of representations, preserving many important subcategories such as the category $\mathcal{O}$ (see section 2.2). Bernstein and Gelfand in 1981 (see [BG]) began a systematic abstract study of these functors. They define a projective functor on any category of representations of $\mathfrak{g}$ (which is stable under tensoring with finite dimensional representations) to be a direct summand of a tensor product functor restricted to this category. The term projective functor comes from the fact that such a functor on $\mathcal{O}$ maps projectives to projectives. They were able to to establish many general properties of projective functors, and to apply them to obtain new results about Harish-Chandra modules for complex reductive groups.

A crucial point in the investigation of Bernstein and Gelfand is the determination of the space of homomorphisms between projective functors on the category of representations where the center $\mathcal{Z}$ of the enveloping algebra $\mathcal{U}$ of $\mathfrak{g}$ acts diagonally.

In the present paper we are able to do the same thing for projective functors on the category $\mathcal{O}$ (on which the action of $\mathcal{Z}$ is merely locally finite). Specializing to a true central character then recovers Bernstein and Gelfand's result. In order to

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simplify things we have only considered functors from a block $\mathcal{O}_\lambda$ ($\lambda \in \mathfrak{h}^\star$, $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra) of $\mathcal{O}$ to itself but it shouldn't be very difficult to generalize the results here to functors between different blocks.

By general nonsense we construct a full embedding of categories

$$\{\textit{Projective functors on } \mathcal{O}_\lambda\} \rightsquigarrow \mathcal{O}_{\lambda,\lambda}(\mathfrak{g} \times \mathfrak{g}).$$

Denote by V the object in $\mathcal{O}_{\lambda,\lambda}(\mathfrak{g} \times \mathfrak{g})$ that corresponds in this way to the identity functor, $\mathrm{Id}_{\mathcal{O}_\lambda}$, on $\mathcal{O}_\lambda$. This is in my opinion an interesting object. It turns out (Theorem 3.1) that V is isomorphic to $P^{\Delta}_{w_0,w_0} \overset{def}{=} \{v \in P_{w_0,w_0};\, \Delta v = 0\}$. Here Δ is the ideal in $\mathcal{Z} \otimes \mathcal{Z}$ generated by $z \otimes 1 - 1 \otimes z$, $z \in \mathcal{Z}$, and P_{w_0,w_0} denotes a projective cover of the simple Verma module in $\mathcal{O}_{\lambda,\lambda}(\mathfrak{g} \times \mathfrak{g})$.

Let C be the subalgebra of λ-invariants of the coinvariant algebra of the Weyl group W (see section 2.4). Using Theorem 3.1 and Soergel's theory of C-modules ([S], [S2]) we describe in Theorem 4.9 the Hom-space between two projective functors T and T' on $\mathcal{O}_\lambda$. We show that $\mathrm{Hom}_{\{\mathrm{Functors}\}}(T, T')$ is a C-bimodule which is free as a left (and right) C-module and that evaluation induces a canonical right C-module isomorphism $k \otimes_C \mathrm{Hom}_{\{\mathrm{Functors}\}}(T, T') \cong \mathrm{Hom}_{\mathcal{O}(\mathfrak{g})}(T(M_e), T'(M_e))$, where M_e denotes the dominant Verma module. For the sake of completeness we include a section 4.3 where it is explained how the Kazhdan-Lusztig conjectures can be used to calculate homomorphisms between the type of C-modules that occur here.

We have adopted the philosophy that projective functors are worth studying for their own sake. The most interesting case, however, which was the starting point for these investigations, is to consider projective functors on a parabolic subcategory of $\mathcal{O}$. Because here very little is known and one might also hope for some important applications to representation theory, for instance to describe the homomorphisms between parabolic Verma modules. Two fundamental questions concerning projective functors on parabolic category $\mathcal{O}$ are open:

- Are projective functors determined up to isomorphism by their action on the Grothendieck group?
- Which are the indecomposable projective functors?

This paper contains unfortunately no results in this direction. One problem is that Soergel's Structure Theorem 2.11 is no longer true for parabolic $\mathcal{O}$. I know that the object V in $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})$ corresponding to the identity functor on a parabolic subcategory of $\mathcal{O}(\mathfrak{g})$ cannot be given such a simple description as in the non-parabolic case (in fact, already the statements in Lemma 3.3 fail to hold in general, so V does not embed to any single indecomposable injective). But, on the other hand, V can probably be obtained by glueing nice modules of $\mathcal{Z}$-invariants in some way. I think that giving this sort of description of V would be useful.

Another result of this paper (which is unexpected since it is not compatible with the grading on $\mathcal{O}$) is this. On each projective object in $\mathcal{O}_\lambda$ we consider the maximal increasing filtration whose degree i term is annihilated by the i-th power of the central character. We prove in Proposition 2.12 that the $(i+1)$-th subquotient in this filtration is isomorphic to a direct sum of Vermas with multiplicities corresponding to Weyl group elements of length i. We apply this in Proposition 3.2 to prove that V admits a Verma flag.

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Müller for explaining some facts about parabolic category $\mathcal{O}$ and for showing me her Diplomarbeit [BM]; attempts to understand that paper were the starting point for these investigations. This work was financed by a STINT-postdoc at the Albert-Ludwigs-Universität in Freiburg, Germany.

2. The category of highest weight modules over $\mathfrak{g} \times \mathfrak{g}$

2.1. **Preliminaries.** Let $\mathfrak{g}$ be a semisimple Lie algebra over k, where k denotes the field of complex numbers. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and let $\mathcal{U}$ be the universal enveloping algebra of $\mathfrak{g}$, $\mathcal{Z}$ the center of $\mathcal{U}$. Denote by R_+ the set of positive roots, ρ the half-sum of positive roots, W the Weyl group and S the set of simple reflections. For $x \in W$, let $l(x)$ denote its length relative to S. Denote by w_0 the longest element of W and by e its unit. The dot-action $(\,\cdot\,)$ of W on $\mathfrak{h}^\star$ is given by $x \cdot \chi = x(\chi + \rho) - \rho$.

We fix once and for all a dominant weight λ, which we assume is integral to simplify the exposition. So $\lambda(H_\alpha)$ is an integer ≥ -1 for each positive coroot H_α. (However, all results in this paper remain true for non-integral weights.) Let W_λ denote the stabilizer with respect to the dot-action of λ in W. We let W^λ denote a set of representatives of the cosets W/W_λ and to simplify notations we assume that $e, w_0 \in W^\lambda$. For $x \in W^\lambda$, we simply write x for $x \cdot \lambda$. Let M_x denote the Verma module with highest weight x and let L_x be its simple quotient.

For any ring A we shall use the notation A-mod, (resp. mod-A) for the category of finitely generated left, (resp. right) A-modules. Analogously we define A-mod-A. If $I \subset A$ is a subset and M an (e.g. left) A-module, we define the invariants $M^I = \{m \in M; Im = 0\}$. If M is a $\mathcal{U}$-module and I an ideal in $\mathcal{Z}$, then M^I is a $\mathcal{U}$-submodule of M.

Tensor products. The symbol $\otimes$ denotes $\otimes_k$ unless otherwise specified. If M and N are representations of a Lie-algebra $\mathfrak{a}$, then $M \otimes N$ denotes their tensor product representation, so $a \cdot (m \otimes n) = am \otimes n + m \otimes an$, for $a \in \mathfrak{a}, m \in M$ and $n \in N$.

Let A_1 and A_2 be k-algebras; we define their *external tensor product* $A_1 \boxtimes A_2$ to be the k-algebra whose underlying set is $A_1 \otimes A_2$ and multiplication given by $(a_1 \otimes a_2) \cdot (a_1' \otimes a_2') = a_1 a_1' \otimes a_2 a_2'$.

Assume $M_i \in A_i$-mod, $i = 1, 2$. The external tensor product $M_1 \boxtimes M_2$ is the $A_1 \boxtimes A_2$-module whose underlying set is $M_1 \otimes M_2$ and $(a_1 \otimes a_2) \cdot (m_1 \otimes m_2) = a_1 m_1 \otimes a_2 m_2$.

Denote by $Z(A_i)$ the center of A_i; then $Z(A_1 \boxtimes A_2) = Z(A_1) \boxtimes Z(A_2)$.

Lemma 2.1. *Assume that A_1 and A_2 are noetherian. Let $M_i, N_i \in A_i$-mod, $i = 1, 2$. Then there is a $Z(A_1 \boxtimes A_2)$-module isomorphism*

$$\Lambda : \mathrm{Hom}_{A_1}(M_1, N_1) \boxtimes \mathrm{Hom}_{A_2}(M_2, N_2) \to \mathrm{Hom}_{A_1 \boxtimes A_2}(M_1 \boxtimes M_2, N_1 \boxtimes N_2)$$

given by $\Lambda(\phi_1 \otimes \phi_2)(m_1 \otimes m_2) = \phi_1(m_1) \otimes \phi_2(m_2)$. If $M_i = N_i$, this is a ring isomorphism.

Proof. It is clear that Λ is a $Z(A_1 \boxtimes A_2)$-linear map which is a ring homomorphism when $M_i = N_i$. It is easy to verify that Λ is bijective when the M_i's are free of finite rank; the general case follows from the Five Lemma by taking 2-step finite rank free resolutions. $\square$

2.2. **Category $\mathcal{O}$.** (See [BGG] for details.) Denote by $\mathcal{O} = \mathcal{O}(\mathfrak{g})$ the category of finitely generated left $\mathcal{U}$-modules which are semisimple over $\mathfrak{h}$ and locally finite over $U(\mathfrak{n}_+)$. Associated to the maximal ideal $\mathfrak{m}_\lambda = \mathrm{Ann}_{\mathcal{Z}}(L_e)$ in $\mathcal{Z}$ there is the full subcategory of $\mathcal{O}$ (so-called block)

$$\mathcal{O}_\lambda = \{M \in \mathcal{O};\ M = \bigcup_{k\geq 0} M^{\mathfrak{m}_\lambda^k}\}.$$

Thus the objects of $\mathcal{O}_\lambda$ have composition factors in $\{L_x;\ x \in W^\lambda\}$. From now on we consider only the block $\mathcal{O}_\lambda$. The functor $\mathrm{Hom}_{\mathcal{O}}(\ ,\ )$ will often simply be denoted by $\mathrm{Hom}(\ ,\ )$.

For $M \in \mathcal{O}_\lambda$ we denote by $[M]$ the image of M in the Grothendieck group $K(\mathcal{O}_\lambda)$. The simple modules form a basis of $K(\mathcal{O}_\lambda)$. Thus we define the multiplicity $[M : L_x]$ to be the coefficient of $[L_x]$ in the representation of $[M]$ in this basis. Also, the Verma modules form a basis of $K(\mathcal{O}_\lambda)$ and we define the multiplicity $[M : M_x]$ similarly.

For each $x \in W^\lambda$, let us fix a projective cover P_x of L_x in $\mathcal{O}_\lambda$. Each P_x admits a filtration whose subquotients are Verma modules with parameters in W^λ. Denote by $(P_x : M_y)$ the number of such subquotients isomorphic to M_y. Then $(P_x : M_y) = [P_x : M_y]$ and the Bernstein-Gelfand-Gelfand (BGG) *reciprocity formula* $(P_x : M_y) = [M_y : L_x]$ holds.

We define the dual module $M^\star$ of $M \in \mathcal{O}_\lambda$ to be the direct sum of the duals of the weight spaces of M with the $\mathfrak{g}$-action on $M^\star$ given by the Chevalley involution of $\mathfrak{g}$. Then $M = M^{\star\star}$ and $L \cong L^\star$, when L is simple.

Denote by $I_x = P_x^\star$. This is an injective hull of L_x.

Remark 2.2. We have $L_{w_0} = M_{w_0}$, $P_e = M_e$, $P_{w_0}^\star \cong P_{w_0}$ and P_{w_0} is an injective hull of L_{w_0}.

2.3. **The product category.** The Lie algebra $\mathfrak{g} \times \mathfrak{g}$ has the triangular decomposition

$$\mathfrak{g} \times \mathfrak{g} = (\mathfrak{n}_- \times \mathfrak{n}_-) \oplus (\mathfrak{h} \times \mathfrak{h}) \oplus (\mathfrak{n}_+ \times \mathfrak{n}_+).$$

We denote by $\mathcal{U}^2$ the universal enveloping algebra of $\mathfrak{g} \times \mathfrak{g}$ and by $\mathcal{Z}^2$ its center. We identify $\mathcal{U}^2 = \mathcal{U} \boxtimes \mathcal{U}$ and $\mathcal{Z}^2 = \mathcal{Z} \boxtimes \mathcal{Z}$. We write $\mathcal{O}_{\lambda,\lambda} = \mathcal{O}_{\lambda,\lambda}(\mathfrak{g} \times \mathfrak{g})$, where $(\lambda, \lambda) \in \mathfrak{h}^\star \times \mathfrak{h}^\star = (\mathfrak{h} \times \mathfrak{h})^\star$, and let $pr_{\lambda,\lambda} : \mathcal{O}(\mathfrak{g} \times \mathfrak{g}) \to \mathcal{O}_{\lambda,\lambda}$ be the projection.

Lemma 2.3. i) *The external tensor product defines a map from $\mathcal{O}_\lambda \times \mathcal{O}_\lambda$ to $\mathcal{O}_{\lambda,\lambda}$ and $(\ \boxtimes\ )^\star = (\ )^\star \boxtimes (\ )^\star$.*

ii) *We have canonical isomorphisms $M_{x,y} \cong M_x \boxtimes M_y$, $L_{x,y} \cong L_x \boxtimes L_y$ and $P_{x,y} \cong P_x \boxtimes P_y$.*

Proof. i) is obvious.

The first assertion in ii) is clear. The second assertion holds, since $(L_x \boxtimes L_y)$ is self-dual by i) and a self-dual highest weight module is simple. From BGG reciprocity we now get $[P_{x,y}] = [P_x \boxtimes P_y]$. Since $P_x \boxtimes P_y$ has the unique simple quotient $L_{x,y}$, it follows that $P_{x,y}$ surjects to $P_x \boxtimes P_y$; this is then necessarily an isomorphism. □

Let $P = \bigoplus_{x\in W^\lambda} P_x$ be a minimal projective generator of $\mathcal{O}_\lambda$ and denote by $\pi_x : P \to P_x$ the natural projection. Put $P^2 = P \boxtimes P$. Lemma 2.3 ii) implies that

$P^2 \cong \bigoplus_{x,y \in W^\lambda} P_{x,y}$ is a minimal projective generator of $\mathcal{O}_{\lambda,\lambda}$. Define the basic Artin algebra $R = \mathrm{End}_{\mathcal{O}}(P)$. By abstract reasoning, e.g., [Bass], the functor

$$\mathrm{Hom}_{\mathcal{O}}(P,\ ) : \mathcal{O}_\lambda \rightsquigarrow \text{mod-}R \tag{2.1}$$

is an equivalence of categories. The inverse functor is given by $(\ ) \otimes_R P$.

The ring R has been investigated before; in [BGS] it is proved that R is a Koszul ring and an explicit construction of its Koszul dual is given, so-called parabolic-singular duality. (See [Bac] for the Koszul duality theorem in the case of a singular *and* parabolic block.) Put $R^2 = \mathrm{End}_{\mathcal{O}_{\lambda,\lambda}}(P \boxtimes P)$ and denote by R^{op} the opposite ring of R.

Lemma 2.4. *There exists an involution* ^{op} *of* R *(i.e., an anti-isomorphism of order two); it satisfies* $\mathrm{Hom}_{\mathcal{O}}(P_x, P_y)^{op} = \mathrm{Hom}_{\mathcal{O}}(P_y, P_x)$ *and* $\pi_x^{op} = \pi_x$.

Proof. The duality * on $\mathcal{O}_\lambda$ defines an equivalence between $\mathcal{O}_\lambda$ and the opposite category $\mathcal{O}_\lambda^{op}$. Thus mod-R is equivalent to (mod-R)^{op}. Vector space duality defines an equivalence (mod-R)$^{op} \cong R$-mod. Now R-mod = mod-R^{op}, so mod-R is equivalent to mod-R^{op} and, since R and R^{op} are basic algebras, we conclude that $R \cong R^{op}$. This gives the requested involution $^{op} : R \to R$. The other assertions hold for general reasons. $\square$

As in 2.1, the functor $\mathrm{Hom}_{\mathcal{O}_{\lambda,\lambda}}(P^2,\ )$ defines an equivalence $\mathcal{O}_{\lambda,\lambda} \rightsquigarrow$ mod-R^2 and Lemma 2.1 gives an isomorphism $R \boxtimes R \cong R^2$. (This can be interpreted as $\mathcal{O}_{\lambda,\lambda}$ is the tensor product category in the sense of P. Deligne [D] of the category $\mathcal{O}_\lambda$ with itself.) From now on we fix an involution ^{op} of R. Clearly ^{op} defines an isomorphism $R \cong R^{op}$, which induces the isomorphism $R \boxtimes R^{op} \cong R \boxtimes R$; we conclude

Proposition 2.5. *The categories* mod-$R \boxtimes R^{op}$, mod-$R \boxtimes R$, mod-R^2 *and* $\mathcal{O}_{\lambda,\lambda}$ *are all naturally equivalent.*

2.4. **Projective functors on $\mathcal{O}_\lambda$.** Let E be a finite dimensional $\mathfrak{g}$-module and recall that pr_λ denotes the projection from $\mathcal{O}$ onto the block $\mathcal{O}_\lambda$. We consider $T_E = pr_\lambda \circ E \otimes (\ )$ as a functor from $\mathcal{O}_\lambda$ to $\mathcal{O}_\lambda$.

Definition 2.6. A direct summand T of T_E is called a projective functor. Let $PF(\mathcal{O}_\lambda)$ denote the category of projective functors (morphisms being all natural transformations of functors).

It is immediate that:

- T is exact and commutes with duality on $\mathcal{O}_\lambda$.
- T maps projectives (resp. injectives) to projectives (resp. injectives).

J. Bernstein and S. Gelfand classified projective functors

Theorem 2.7 ([BG], Theorem 3.3 and Theorem 3.5). *If T and T' are projective functors, then $T \cong T'$ iff $T(M_e) \cong T'(M_e)$. The isomorphism classes of indecomposable projective functors are parametrized by W^λ: for each $x \in W^\lambda$ there is a unique projective functor whose value on M_e is P_x.*

We now explain how $PF(\mathcal{O}_\lambda)$ embeds to $\mathcal{O}_{\lambda,\lambda}$ (see [Bass] for details). Let $REF(\mathcal{A})$ denote the category of right exact functors on an abelian category $\mathcal{A}$. When $\mathcal{A} =$ mod-A for an Artin algebra A, we have an equivalence

$$REF(\text{mod-}A) \xrightarrow{\sim} \text{mod-}A \boxtimes A^{op} \text{ given by the assignment } F \rightsquigarrow F(A) \tag{2.2}$$

where the right A-action on $F(A)$ is the natural one (i.e., given by $F(A) \in$ mod-A) and the left A-action is given by the composition

$$A \to \mathrm{Hom}_{\text{mod-}A}(A, A) \to \mathrm{Hom}_{\text{mod-}A}(F(A), F(A))$$

where the first map is left multiplication and the second map is defined by F. The inverse map to (2.2) sends $B \in$ mod-$A \otimes A^{op}$ to the functor $(\ ) \otimes_A B$. Thus, by Proposition 2.5

$$PF(\mathcal{O}_\lambda) \hookrightarrow REF(\mathcal{O}_\lambda) \cong REF(\text{mod-}R) \cong \text{mod-}R \boxtimes R^{op} \cong \mathcal{O}_{\lambda,\lambda}. \tag{2.3}$$

Definition 2.8. We denote by $P(\mathcal{O}_{\lambda,\lambda})$ the (full) subcategory of $\mathcal{O}_{\lambda,\lambda}$ equivalent to $PF(\mathcal{O}_\lambda)$ given by (2.3). Denote by V_T the object in $P(\mathcal{O}_{\lambda,\lambda})$ corresponding to $T \in PF(\mathcal{O}_\lambda)$. When $T = T_E$ we simply write V_E for V_{T_E}. Let R_{bi} denote the ring R considered as a bimodule over itself. Denote by V the object in $\mathcal{O}_{\lambda,\lambda}$ corresponding to R_{bi} in (2.3); thus $V = V_{\mathrm{Id}_{\mathcal{O}_\lambda}}$, where $\mathrm{Id}_{\mathcal{O}_\lambda}$ is the identity functor on $\mathcal{O}_\lambda$.

2.5. **Modules over the coinvariant algebra.** Let $S(\mathfrak{h})$ denote the symmetric algebra of $\mathfrak{h}$ and denote by $S(\mathfrak{h})_+$ its positive part with respect to the $\mathbb{N}$-grading in which $\mathfrak{h}$ has degree 1. The Weyl group W acts naturally on $S(\mathfrak{h})$. Let C denote the algebra $(S(\mathfrak{h})/S(\mathfrak{h})_+^W)^{W_\lambda}$ of W_λ-invariants in the coinvariant algebra $S(\mathfrak{h})/S(\mathfrak{h})_+^W$.

We get an induced grading on C. Denote by C_+ the positive part of C and by $k = C/C_+$ the (unique) simple C-module. (Sometimes k will be considered as a subring or quotient ring of C.) In [B], e.g., an isomorphism of C and the cohomology ring of a partial flag manifold of $\mathfrak{g}$ is constructed. Since the partial flag manifold is a compact manifold, it follows that its highest cohomology group is 1-dimensional. This highest cohomology group corresponds to the socle

$$\operatorname{soc} C \stackrel{def}{=} \{c \in C; C_+ c = 0\}$$

of C under this isomorphism. Thus $\operatorname{soc} C$ is 1-dimensional and we conclude that C is a Gorenstein ring.

On mod-C we have the two functors $\mathrm{Hom}_C(\ , C)$ and $\mathrm{Hom}_k(\ , k)$. The latter functor is obviously a duality, i.e., its square is equivalent to the identity functor.

Choose any k-linear map $f : C \to k$ which is non-zero on $\operatorname{soc} C$. Then

$$f_\star : \mathrm{Hom}_C(M, C) \to \mathrm{Hom}_k(M, k)$$

is a functorial isomorphism in $M \in$ mod-C as is easily deduced from the Gorenstein property. Thus $\mathrm{Hom}_C(\ , C)$ and $\mathrm{Hom}_k(\ , k)$ are (non-canonically) equivalent functors. We denote $\mathrm{Hom}_C(\ , C)$ by *.

Multiplication gives an isomorphism $C \cong C^\star$ in mod-C; since C is projective as a module over itself and we just have shown that * is a duality, it follows that C is injective in mod-C.

We shall need the following theorems of W. Soergel.

Theorem 2.9 ([S], Endomorphism Theorem 7)**.** *Multiplication gives a surjection* $nat : \mathcal{Z} \twoheadrightarrow \mathrm{End}(P_{w_0})$. *Moreover, C is naturally isomorphic to* $Z/\operatorname{Ker} nat \cong \mathrm{End}(P_{w_0})$.

(See also [B] for a simpler proof and [BeilGin] for the $\mathcal{D}$-module approach.) It now follows from BGG reciprocity that $\dim C = \operatorname{card} W^\lambda$.

Definition 2.10. Define the functor $\mathbb{V} = \mathrm{Hom}(P_{w_0},\ ) : \mathcal{O}_\lambda \to$ mod-C, where we have identified C with $\mathrm{End}(P_{w_0})$.

Clearly $\mathbb{V}$ is exact. It is shown in [S] that we have $\mathbb{V} \circ {}^\star \cong {}^\star \circ \mathbb{V}$ and that $\mathbb{V}(P_x)^\star \cong \mathbb{V}(P_x)$ for each $x \in W^\lambda$.

Theorem 2.11 ([S], Theorem 9)**.** *Let $M, N \in \mathcal{O}_\lambda$. The natural map*

$$\mathrm{Hom}_{\mathcal{O}_\lambda}(M, N) \to \mathrm{Hom}_C(\mathbb{V}M, \mathbb{V}N)$$

is bijective when N is a projective or M is injective.

2.6. **Filtrations on projectives.** For each $x \in W^\lambda$ we associate the multiset Λ_x such that y is an element of Λ_x with multiplicity $n_{y,x}$ iff $y \in W^\lambda$ and $n_{y,x} = (P_x : M_y)$.

Let $x_1, \dots, x_t$ be *any* ordering of Λ_x such that $x_i < x_j \implies i > j$; it is then well-known (see, e.g., [BGG]) that P_x admits a filtration $0 \subset P_{1,x} \subset \dots \subset P_{t,x} = P_x$ such that $P_{i,x}/P_{i-1,x} \cong M_{x_i}$.

We now choose an ordering $x_1, \dots, x_t$ of Λ_x in which, in addition, all occurring elements of a given length are adjacent and consider the corresponding filtration as above. We define $G_{i,x} = \bigcup_{l(x_j) \le i} P_{j,x}$ for each i. Thus

$$0 \subset G_{0,x} \subset \dots \subset G_{l(x),x} = P_x \tag{2.4}$$

and $G_{i,x}/G_{i-1,x} \cong \bigoplus_{y \in W^\lambda, l(y)=i} M_y^{n_{y,x}}$, because $\mathrm{Ext}^1_{\mathcal{O}}(M_y, M_z) = 0$ whenever $l(y) = l(z)$. We consider also on P_x the filtration

$$F_{i,x} = P_x^{\mathfrak{m}^{i+1}}. \tag{2.5}$$

Here we simply write $\mathfrak{m}$ for $\mathfrak{m}_\lambda$.

Since any Verma module in $\mathcal{O}_\lambda$ is annihilated by $\mathfrak{m}$ it follows that $G_{i,x} \subset F_{i,x}$. We shall prove

Proposition 2.12. *For all $i = 0, \dots, l(x)$ we have $G_{i,x} = F_{i,x}$.*

Lemma 2.13. *Proposition 2.12 holds when $x = w_0$.*

Proof of Lemma 2.13. **a**) Since P_{w_0} is the projective cover of L_{w_0}, we have $[F_{i,w_0} : L_{w_0}] = \dim \mathrm{Hom}_{\mathcal{O}}(P_{w_0}, F_{i,w_0})$. On the other hand,

$$\mathrm{Hom}_{\mathcal{O}}(P_{w_0}, F_{i,w_0}) \cong \mathrm{Hom}_{\mathcal{O}}(P_{w_0}, P_{w_0})^{\mathfrak{m}^{i+1}} = C^{\mathfrak{m}^{i+1}}.$$

Thus $[F_{i,w_0} : L_{w_0}] = \dim C^{\mathfrak{m}^{i+1}}$.

b) We calculate $\dim C^{\mathfrak{m}^{i+1}}$. We have a graded isomorphism $\mathcal{Z}/J \cong C$ from Theorem 2.9. Denote by C_i the degree i component of C with respect to this grading; thus $C = \bigoplus_{j=0}^{l(w_0)} C_j$ where $1 \in C_0$. It follows that the ideal $C_{\ge k} = \bigoplus_{j \ge k}^{l(w_0)} C_j$ equals $\mathfrak{m}^k \cdot C$ for any $k \in \mathbb{N}$. Put $n_i = \mathrm{card}\{x \in W^\lambda; l(x) \le i\}$. It is known that $\dim C_j = \mathrm{card}\{x \in W^\lambda; l(x) = j\}$ and it follows that $\dim C/\mathfrak{m}^{i+1}C = n_i$. We have

$$C^{\mathfrak{m}^{i+1}} \cong \mathrm{Hom}_C(C,\, C^{\mathfrak{m}^{i+1}}) \cong \mathrm{Hom}_C(C/\mathfrak{m}^{i+1}C, C)$$

and the dimension of the last space equals the dimension of $C/\mathfrak{m}^{i+1}C$, since the functor ${}^\star = \mathrm{Hom}_C(\ , C)$ is a duality and therefore preserves vector space dimension. Thus $\dim C^{\mathfrak{m}^{i+1}} = n_i$.

c) Assume by induction on i (starting with $i = -1$) that $F_{i,w_0} = G_{i,w_0}$. We prove $F_{i+1,w_0} = G_{i+1,w_0}$. We know that $F_{i+1,w_0} \supset G_{i+1,w_0}$ and that $P_{w_0}/G_{i+1,w_0}$

has a Verma flag. Thus, if $F_{i+1,w_0} \neq G_{i+1,w_0}$, then necessarily $F_{i+1,w_0}/G_{i+1,w_0}$ contains a submodule isomorphic to the simple Verma module L_{w_0} and hence

$$[F_{i+1,w_0} : L_{w_0}] > [G_{i+1,w_0} : L_{w_0}].$$

But then $[F_{i+1,w_0}/F_{i,w_0} : L_{w_0}] > [G_{i+1,w_0}/F_{i,w_0} : L_{w_0}]$ and the latter number equals $n_{i+1} - n_i$ since $G_{i+1,w_0}/F_{i,w_0} \cong G_{i+1,w_0}/G_{i,w_0} \cong \bigoplus_{l(x)=i+1, x\in W^\lambda} M_x$. This is a contradiciton since we have shown that $[F_{i+1,w_0}/F_{i,w_0} : L_{w_0}] = n_{i+1} - n_i$. Thus $F_{i+1,w_0} = G_{i+1,w_0}$. □

Now let $x \in W^\lambda$ be arbitrary.

Lemma 2.14. *We have* $[F_{i,x} : L_{w_0}] = \sum_{l(y)\leq i} n_{y,x}$.

Proof of Lemma 2.14. We have $[F_{i,x} : L_{w_0}] = \dim \mathrm{Hom}_{\mathcal{O}}(P_{w_0}, F_{i,x})$, since P_{w_0} is the projective cover of L_{w_0}. Now

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}}(P_{w_0}, F_{i,x}) &\cong \mathrm{Hom}_{\mathcal{O}}(P_{w_0}, P_x)^{\mathfrak{m}^{i+1}} \cong \mathrm{Hom}_{\mathcal{O}}(I_x, P_{w_0})^{\mathfrak{m}^{i+1}} \\ &= \mathrm{Hom}_C(\mathbb{V}I_x, \mathbb{V}P_{w_0})^{\mathfrak{m}^{i+1}} \cong \mathrm{Hom}_C(\mathbb{V}P_x, \mathbb{V}P_{w_0})^{\mathfrak{m}^{i+1}} \\ &\cong \mathrm{Hom}_{\mathcal{O}}(P_x, P_{w_0})^{\mathfrak{m}^{i+1}} \cong \mathrm{Hom}_{\mathcal{O}}(P_x, P_{w_0}^{\mathfrak{m}^{i+1}}).\end{aligned}$$

Here the third and the fifth isomorphisms are given by Theorem 2.11 since I_x is injective and P_{w_0} is projective, respectively. The fourth isomorphism holds since $\mathbb{V}P_x \cong \mathbb{V}I_x$. We have $\dim \mathrm{Hom}_{\mathcal{O}}(P_x, P_{w_0}^{\mathfrak{m}^{i+1}}) = [P_{w_0}^{\mathfrak{m}^{i+1}} : L_x]$, since P_x is projective. But then $[P_{w_0}^{\mathfrak{m}^{i+1}} : L_x] = \sum_{l(y)\leq i, y\in W^\lambda} [M_y : L_x]$ from Lemma 2.13. □

Proof of Proposition 2.12. With Lemma 2.14 in hand this is practically identical to the proof of **c**) in Lemma 2.13 and is left to the reader. □

3. The bimodule R_{bi} as an object in $\mathcal{O}(\mathfrak{g} \times \mathfrak{g})$

3.1. **The object V and statement of the main theorem.** Recall from Definition 2.8 the object V in $\mathcal{O}_{\lambda,\lambda}$ corresponding to the identity functor $\mathrm{Id}_{\mathcal{O}_\lambda}$ on $\mathcal{O}_\lambda$, as well as to $R_{bi} \in$ mod-$R^2 =$ mod-$R \boxtimes R$. We see that V is determined by $\mathrm{Hom}_{\mathcal{O}_{\lambda,\lambda}}(P^2, V) \cong R_{bi}$. The map

$$\Theta : R^2 \twoheadrightarrow R_{bi},\ \Theta(\phi \otimes \psi) = \phi^{op} \circ \psi \tag{3.1}$$

is a surjection in mod-R^2. Since P^2 is a projective generator of $\mathcal{O}_{\lambda,\lambda}$, it follows that V is the (unique) quotient of P^2, such that Θ induces an isomorphism $\overline{\Theta} : \mathrm{Hom}(P^2, V) \to R_{bi}$. In fact, V is isomorphic to P^2 modulo the submodule generated by $\{\mathrm{Im}(\phi^{op} \otimes 1 - 1 \otimes \phi);\, \phi \in R\}$. Let Δ be the ideal in $\mathcal{Z}^2$ generated by $\{z \otimes 1 - 1 \otimes z;\, z \in Z\}$. At the end of this section we shall prove

Theorem 3.1. *There is an isomorphism* $V \cong P^{\Delta}_{w_0,w_0}$.

3.2. **Filtration and the socle of V.** We prove that V admits a Verma flag.

Proposition 3.2. *We have* $V^{\mathfrak{m}^{i+1}_{\lambda,\lambda}}/V^{\mathfrak{m}^{i}_{\lambda,\lambda}} \cong \sum_{x\in W^\lambda, l(x)=i} M_{x,x}$ *for each* $i \in \mathbb{N}$.

Proof. **a**) Fix $i \in \mathbb{N}$. In this proof direct sums are taken over the set $\{x \in W^\lambda, l(x) = i\}$ unless otherwise specified. We must show that

$$R_{bi}^{\mathfrak{m}^{i+1}_{\lambda,\lambda}}/R_{bi}^{\mathfrak{m}^{i}_{\lambda,\lambda}} \cong \mathrm{Hom}(P^2, \bigoplus M_{x,x}). \tag{3.2}$$

Write for simplicity $\mathfrak{m} = \mathfrak{m}_\lambda$. Since the left and right action of $\mathcal{Z}$ on R_{bi} coincide, we have $R_{bi}^{\mathfrak{m}^i_{\lambda,\lambda}} = \mathrm{Hom}(P, P^{\mathfrak{m}^i})$. From Proposition 2.12 we have $P^{\mathfrak{m}^{i+1}}/P^{\mathfrak{m}^i} \cong \bigoplus M_x^{(P:M_x)}$ and therefore

$$R_{bi}^{\mathfrak{m}^{i+1}_{\lambda,\lambda}}/R_{bi}^{\mathfrak{m}^i_{\lambda,\lambda}} \cong \mathrm{Hom}(P, \oplus M_x^{(P:M_x)}). \tag{3.3}$$

We consider from now on the induced right R^2-module structure on the latter module. Write $\tilde{P} = P/P^{\mathfrak{m}^i}$ and $\tilde{P}_y = P_y/P_y^{\mathfrak{m}^i}$, for $y \in W^\lambda$. Since $\mathrm{Hom}(M_y, M_z) = 0$ if $l(y) < l(z)$, we see that $\mathrm{Hom}(P^{\mathfrak{m}^i}, \bigoplus M_x^{(P:M_x)}) = 0$. Thus 3.3 implies

$$R_{bi}^{\mathfrak{m}^{i+1}_{\lambda,\lambda}}/R_{bi}^{\mathfrak{m}^i_{\lambda,\lambda}} \cong \mathrm{Hom}(\tilde{P}, \bigoplus M_x^{(P:M_x)}). \tag{3.4}$$

We get from Proposition 2.12 that $\tilde{P}_x \cong M_x$ whenever $l(x) = i$. Hence

(3.5)
$$\begin{aligned}\mathrm{Hom}(P^2, \bigoplus M_{x,x}) &\cong \mathrm{Hom}(P^2, \bigoplus \tilde{P}_x \boxtimes \tilde{P}_x) \cong \bigoplus(\mathrm{Hom}(P, \tilde{P}_x) \boxtimes \mathrm{Hom}(P, \tilde{P}_x))\\ &\cong \bigoplus(\mathrm{Hom}(\tilde{P}, \tilde{P}_x) \boxtimes \mathrm{Hom}(\tilde{P}, \tilde{P}_x)) \cong \mathrm{Hom}(\tilde{P} \boxtimes \tilde{P}, \bigoplus(\tilde{P}_x \boxtimes \tilde{P}_x)).\end{aligned}$$

b) In order to prove 3.2 it remains to construct an isomorphism

$$\pi : \mathrm{Hom}(\tilde{P} \boxtimes \tilde{P}, \bigoplus(\tilde{P}_x \boxtimes \tilde{P}_x)) \to \mathrm{Hom}(\tilde{P}, \bigoplus M_x^{(P:M_x)}) \tag{3.6}$$

in mod-R^2. We define $\pi(\phi \otimes \psi) = \phi^{op} \circ \psi$, for $\phi, \psi \in \mathrm{Hom}(\tilde{P}, \tilde{P}_x)$.

It is clear that π is a right R^2-linear map, and it follows from BGG-reciprocity that both objects in 3.6 have the same dimension.

c) We prove that π is injective. First, note that it suffices to prove that for each $x_0 \in W^\lambda$ with $l(x_0) = i$ the restriction of π to $\mathrm{Hom}(\tilde{P} \boxtimes \tilde{P}, \tilde{P}_{x_0} \boxtimes \tilde{P}_{x_0})$. Indeed, if $\phi \in \mathrm{Hom}(\tilde{P}, \tilde{P}_{x_0})$ then, since $\tilde{P}_{x_0}^{\mathfrak{m}} = \tilde{P}_{x_0} = M_{x_0}$, we have $\mathrm{Im}\,\phi \subset \tilde{P}^{\mathfrak{m}} \cong \bigoplus M_x^{(P:M_x)}$. But, since $\mathrm{Hom}(M_{x_0}, M_x) = 0$ if $l(x) = l(x_0)$ and $x \neq x_0$, we then must have $\mathrm{Im}\,\phi \subset M_{x_0}^{(P:M_{x_0})}$ and the statement follows.

Now, fix $x = x_0$ as above and let $v = \sum_j \phi_j \otimes \psi_j$ (the sum being taken over some finite index set) be any element of $\mathrm{Hom}(\tilde{P} \boxtimes \tilde{P}, \tilde{P}_x \boxtimes \tilde{P}_x)$ and assume, without loss of generality, that the ψ_j's are linearly independent. We know that $\tilde{P}$ has the submodule (isomorphic to) M_x^n, where $n = (P : M_x)$, and that every morphism from $\tilde{P}_x = M_x$ to $\tilde{P}$ has its image in M_x^n. Thus $\phi_i^{op} \in \mathrm{Hom}(M_x, M_x^n)$ so that $\phi_j^{op} = \sum_j \lambda_{jk} \epsilon_k$, where $\epsilon_1, \ldots, \epsilon_n$ is the standard basis of $\mathrm{Hom}(M_x, M_x^n)$ and $\lambda_{jk} \in k$. Then, if $\pi(v) = 0$, we get

$$\sum_{jk} \lambda_{jk}\epsilon_k \circ \psi_k = 0 \implies \forall\, k : \sum_j \lambda_{jk}\epsilon_k \circ \psi_k = 0 \iff \forall\, k : \sum_j \lambda_{jk}\psi_k = 0,$$

so that $\lambda_{jk} = 0$ for all j, k, since the ψ_j's were linearly independent. Thus $v = 0$. $\square$

Recall that the socle of an object X in an abelian category, denoted $\mathrm{soc}\,X$, is defined to be its maximal semisimple subobject.

Lemma 3.3. *The socle of V is isomorphic to L_{w_0,w_0}.*

Proof of Lemma 3.3. i) We have to show that $\mathrm{soc}\,R_{bi} \cong \mathrm{Hom}(P^2, L_{w_0,w_0})$. Note that $\mathrm{soc}\,R_{bi} = \{f \in R;\ f \circ \phi = \phi \circ f = 0, \forall \phi \in \mathrm{rad}\,R\}$. Here $\mathrm{rad}\,R$ can be characterized as the set of those $\phi \in R$ such that there is no $x \in W^\lambda$ for which $\operatorname{Im}\phi \supset P_x$. It is clear that

$$f \circ \phi = 0, \forall \phi \in \operatorname{rad} R \iff \operatorname{Im} f \subset \operatorname{soc} P. \tag{3.7}$$

Using that P_{w_0} is injective and the above characterisation of $\operatorname{rad} R$ it also follows that

$$\phi \circ f = 0, \forall \phi \in \operatorname{rad} R \implies \operatorname{Im} f \subset P_{w_0}. \tag{3.8}$$

Thus, $\operatorname{soc} R_{bi} \subset \operatorname{Hom}(P, \operatorname{soc} P_{w_0})$. But this inclusion must be an isomorphism, since $\operatorname{soc} P_{w_0} \cong L_{w_0}$ and hence $\dim \operatorname{Hom}(P, \operatorname{soc} P_{w_0}) = 1$. We see that $\operatorname{soc} R_{bi}$ is annihilated by $\operatorname{rad} R^2$ and by $\pi_x \otimes \pi_y$ for all $(x, y) \neq (w_0, w_0)$ and it follows that $\operatorname{soc} R_{bi} \cong \operatorname{Hom}(P^2, L_{w_0,w_0})$. $\square$

3.3. **Category $\mathcal{O}^\Delta_{\lambda,\lambda}$ and the object $P^\Delta_{w_0,w_0}$.** Recall the notations and results of section 2.4 and put $C^2 = C \boxtimes C$. Theorem 2.9 and Lemma 2.1 give an isomorphism $C^2 \cong \operatorname{End}(P_{w_0,w_0})$ and a surjection $\mathcal{Z}^2 \twoheadrightarrow C^2$. We denoted by Δ the ideal in $\mathcal{Z}^2$ generated by $z \otimes 1 - 1 \otimes z$ for $z \in \mathcal{Z}$. Abusing notation we also denote by Δ the image of Δ in C^2.

Definition 3.4. Denote by $\mathcal{O}^\Delta_{\lambda,\lambda}$ the subcategory of $\mathcal{O}_{\lambda,\lambda}$ whose objects are annihilated by Δ.

Since the left and right $\mathcal{Z}$-action on R_{bi} coincide, we see that V belongs to $\mathcal{O}^\Delta_{\lambda,\lambda}$. Clearly all Verma modules are in $\mathcal{O}^\Delta_{\lambda,\lambda}$. The functor $\mathcal{O}_{\lambda,\lambda} \ni M \to M^\Delta \in \mathcal{O}^\Delta_{\lambda,\lambda}$is right adjoint to the inclusion $\mathcal{O}^\Delta_{\lambda,\lambda} \hookrightarrow \mathcal{O}_{\lambda,\lambda}$. The latter functor is exact, hence M^Δ is injective in $\mathcal{O}^\Delta_{\lambda,\lambda}$ whenever M is injective in $\mathcal{O}_{\lambda,\lambda}$. In particular, $P^\Delta_{w_0,w_0}$ is injective in $\mathcal{O}^\Delta_{\lambda,\lambda}$. Since its socle is L_{w_0,w_0}, we conclude that $P^\Delta_{w_0,w_0}$ is the injective hull of L_{w_0,w_0} in this category; in particular, $P^\Delta_{w_0,w_0}$ is indecomposable. We have

Lemma 3.5. *There exists an embedding* $V \hookrightarrow P^\Delta_{w_0,w_0}$.

Proof. By Lemma 3.3, we have $\operatorname{soc} V = L_{w_0,w_0}$, so there is an imbedding $\operatorname{soc} V \hookrightarrow P^\Delta_{w_0,w_0}$. Since $P^\Delta_{w_0,w_0}$ is injective in $\mathcal{O}^\Delta_{\lambda,\lambda}$, it follows that this embedding extends to a morphism $V \to P^\Delta_{w_0,w_0}$, which has to be injective, since its restriction to $\operatorname{soc} V$ is. $\square$

Lemma 3.6. *The multiplicity* $[P^\Delta_{w_0,w_0} : L_{w_0,w_0}]$ *equals* $\operatorname{card} W^\lambda$.

Proof. Since $P^\Delta_{w_0,w_0}$ is the injective hull of L_{w_0,w_0} in $\mathcal{O}^\Delta_{\lambda,\lambda}$, we have $[P^\Delta_{w_0,w_0} : L_{w_0,w_0}] = \dim \operatorname{End}(P^\Delta_{w_0,w_0})$. On the other hand, we know that $\dim C = \operatorname{card} W^\lambda$, and we have the vector space (and also ring) isomorphism

$$C \ni x \to \overline{x \otimes 1} \in C^2/\Delta, \tag{3.9}$$

so that also $\dim C^2/\Delta = \operatorname{card} W^\lambda$. The proof of Lemma 3.6 is completed by

Claim 3.7. $\operatorname{End}(P^\Delta_{w_0,w_0})$ *is isomorphic to* C^2/Δ *in* C^2-mod.

Proof of Claim. Clearly $\operatorname{End}(P^\Delta_{w_0,w_0}) = \operatorname{Hom}(P^\Delta_{w_0,w_0}, P_{w_0,w_0})$. Let $i : P^\Delta_{w_0,w_0} \hookrightarrow P_{w_0,w_0}$ be the inclusion. Since P_{w_0,w_0} is injective (in $\mathcal{O}_{\lambda,\lambda}$), the map

$$i^\star : C^2 \cong \operatorname{End}(P_{w_0,w_0}) \to \operatorname{Hom}(P^\Delta_{w_0,w_0}, P_{w_0,w_0})$$

is a surjection. The kernel of $i^\star$ clearly contains Δ and we get a surjection $C^2/\Delta \twoheadrightarrow \operatorname{End}(P^\Delta_{w_0,w_0})$.

To see this is an isomorphism it suffices to show that $\dim \operatorname{End}(P^{\Delta}_{w_0,w_0}) \geq \operatorname{card} W^{\lambda}$. We know by Lemma 3.5 that $V \hookrightarrow P^{\Delta}_{w_0,w_0}$. Thus, by Proposition 3.2,

$$\dim \operatorname{End}(P^{\Delta}_{w_0,w_0}) = [P^{\Delta}_{w_0,w_0} : L_{w_0,w_0}] \geq [V : L_{w_0,w_0}] = \operatorname{card} W^{\lambda}.$$

□

3.4. **Proof of the main theorem.**

Proof of Theorem 3.1. We know from Lemma 3.5 that V embeds to $P^{\Delta}_{w_0,w_0}$. To see that this embedding is an isomorphism we just need to show that V is injective in $\mathcal{O}^{\Delta}_{\lambda,\lambda}$, because $P^{\Delta}_{w_0,w_0}$ is indecomposable and any non-trivial extension of an injective object must split.

By Lemma 3.3 and Proposition 3.2, we see that Lemma 3.8 below—with $\mathcal{O}_\lambda$ replaced by $\mathcal{O}_{\lambda,\lambda}$ and $\mathcal{A} = \mathcal{O}^{\Delta}_{\lambda,\lambda}$—applies to V. So it suffices to show that any extension $\tau_E : V \hookrightarrow E \twoheadrightarrow M_{xy}$ in $\mathcal{O}^{\Delta}_{w_0,w_0}$ splits.

Assume to get a contradiction that τ_E doesn't split. From Lemma 3.9 below, we then get $\operatorname{soc} E = \operatorname{soc} V = \operatorname{soc} P_{w_0,w_0}$, and this extends by injectivity to an embedding $E \hookrightarrow P^{\Delta}_{w_0,w_0}$. It follows that $[E : L_{w_0,w_0}] \leq [P^{\Delta}_{w_0,w_0} : L_{w_0,w_0}] = \operatorname{card} W^{\lambda}$ by Lemma 3.6. But $[E : L_{w_0,w_0}] = [V : L_{w_0,w_0}] + [M_{x,y} : L_{w_0,w_0}] = \operatorname{card} W^{\lambda} + 1$ by Proposition 3.2. Thus τ_E splits and V is injective. □

Lemma 3.8. *Let $\mathcal{A}$ be a full abelian subcategory of $\mathcal{O}_\lambda$ containing all Verma modules. Let $M \in \mathcal{A}$ and assume that M contains a submodule isomorphic to M_e and that $\operatorname{soc} M \cong L_{w_0}$. Then M is injective iff $\operatorname{Ext}^1_{\mathcal{A}}(M_x, M) = 0$ for all $x \in W^{\lambda}$.*

Proof. The only if part is obvious. Assume now $\operatorname{Ext}^1_{\mathcal{A}}(M_x, M) = 0$ for all $x \in W^{\lambda}$. We must show that $\operatorname{Ext}^1_{\mathcal{A}}(L_x, M) = 0$. If $x = w_0$, there is nothing to prove so assume $x \neq w_0$. Then there is a short exact sequence

$$K \hookrightarrow M_x \twoheadrightarrow L_x \tag{3.10}$$

with $K \neq 0$. The assumptions on M imply that $\operatorname{Hom}_{\mathcal{A}}(L_x, M) = 0$ and $\operatorname{Hom}_{\mathcal{A}}(M_x, M) = \operatorname{Hom}_{\mathcal{A}}(K, M) = k$. The long exact sequence obtained by applying $\operatorname{Hom}_{\mathcal{A}}(\ , M)$ to (3.10) now shows that $\operatorname{Ext}^1_{\mathcal{A}}(L_x, M) = 0$. □

Lemma 3.9. *Let $\tau_E : M \hookrightarrow E \twoheadrightarrow M_x$ be a non-split exact sequence in $\mathcal{O}_\lambda$ for some $x \in W^{\lambda}$. Then $\operatorname{soc} E = \operatorname{soc} M$.*

Proof. We must show that $\operatorname{Hom}_{\mathcal{O}}(L_y, M) = \operatorname{Hom}_{\mathcal{O}}(L_y, E)$ for all $y \in W^{\lambda}$. If $y \neq w_0$, this is clear since then $\operatorname{Hom}_{\mathcal{O}}(L_y, M_x) = 0$. Assume $y = w_0$. Applying $\operatorname{Hom}_{\mathcal{O}}(L_{w_0},\ )$ and $\operatorname{Hom}_{\mathcal{O}}(M_x,\ )$ to τ_E, we get the commutative diagram with exact rows (where the vertical maps are induced by the inclusion $L_{w_0} \hookrightarrow M_x$)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \operatorname{Hom}_{\mathcal{O}}(L_{w_0}, M) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{O}}(L_{w_0}, E) \arrow[r, "\pi"] \arrow[d] & \operatorname{Hom}_{\mathcal{O}}(L_{w_0}, M_x) \arrow[d, equal] \\
0 \arrow[r] & \operatorname{Hom}_{\mathcal{O}}(M_x, M) \arrow[r] & \operatorname{Hom}_{\mathcal{O}}(M_x, E) \arrow[r, "\tilde{\pi}"] & \operatorname{Hom}_{\mathcal{O}}(M_x, M_x).
\end{tikzcd}$$

The image of $\tilde{\pi}$ cannot contain Id_{M_x}, since that would give a split of τ_E. Since $\dim \operatorname{Hom}_{\mathcal{O}}(M_x, M_x) = 1$, we conclude that $\tilde{\pi} = 0$. Hence $\pi = 0$. □

4. The structure of homomorphisms between projective functors

4.1. **Projective functors as objects in $\mathcal{O}_{\lambda,\lambda}$.** In this section we give a description of the space of all homomorphisms between projective functors. Using the theory developed in [S], this turns out to be a straightforward matter with the neat description of V from the previous section in hand. Recall the terminology from section 2.3. To each finite dimensional $\mathfrak{g}$-module E we assigned a projective functor T_E and we have the corresponding object V_E in $\mathcal{O}_{\lambda,\lambda}$ given by Definition 2.8; it follows from the definition that

$$\mathrm{Hom}(P^2, V_E) \cong \mathrm{Hom}(P, E \otimes P)_{bi}$$

where $\mathrm{Hom}(P, E\otimes P)_{bi}$ is the object of mod-R^2 which equals $\mathrm{Hom}(P, E \otimes P)$ as a space and $f \cdot \phi \otimes \psi = (\mathrm{id}_E \otimes \phi^{op}) \circ f \circ \psi$, for $\phi \otimes \psi \in R^2$ and $f \in \mathrm{Hom}(P, E \otimes P)_{bi}$.

Proposition 4.1. *V_E is isomorphic to $pr_{\lambda,\lambda}((k \boxtimes E) \otimes V)$.*

Proof. The map defining this isomorphism corresponds to

$$\mathrm{Hom}(P, E \otimes P)_{bi} \to \mathrm{Hom}(P^2, (k \boxtimes E) \otimes V);\ f \to \{p \boxtimes q \to \overline{p \boxtimes f(q)}\}.$$

Here $p \boxtimes f(q) \in P \boxtimes (E \otimes P) = (k \boxtimes E) \otimes P^2$ and $\overline{p \boxtimes f(q)}$ denotes the image of $p \boxtimes f(q)$ in $(k \boxtimes E) \otimes V$ (given by $P \boxtimes P \twoheadrightarrow V$). The reader may verify that this assignment indeed defines an isomorphism in mod-$R \boxtimes R$. $\square$

4.2. **Projective functors as bimodules over the Coinvariant Algebra.** Recall that $C^2 = C \boxtimes C \cong \mathrm{End}(P_{w_0,w_0})$. We identify

$$\text{mod-}C^2 = C\text{-mod-}C \tag{4.1}$$

by means of $c_1 m c_2 = c_1 \otimes c_2 \cdot m$ for $m \in M \in \text{mod-}C^2$.

We like to think of C^2-modules as C-bimodules, so we denote this category by C-mod-C. We have the bifunctors $\mathrm{Hom}_{C^2}(\ ,\ )$ and $\bigotimes_C$ and the duality functor $\mathrm{Hom}_k(\ , k)$ on C-mod-C. (Of course C^2 is Gorenstein so $\mathrm{Hom}_k(\ , k)$ is noncanonically equivalent to $\mathrm{Hom}_{C^2}(\ , C^2)$.) Let C_{bi} denote C considered as a bimodule over itself.

Lemma 4.2. *C_{bi} is isomorphic to ${C^2}^{\Delta}$ in C-mod-C. Moreover, C_{bi} is self dual in this category.*

Proof. Clearly the map $C_{bi} \ni c \to \bar{c} \in C^2/\Delta$ is an isomorphism in C-mod-C. On the other hand,

$$(C^2/\Delta)^\star = \mathrm{Hom}_{C^2}(C^2/\Delta, C^2) \cong \mathrm{Hom}_{C^2}(C^2, {C^2}^{\Delta}) \cong {C^2}^{\Delta}.$$

Hence it suffices to show that C_{bi} is self dual in C-mod-C. To see this we choose an isomorphism $C \cong \mathrm{Hom}_k(C, k)$ in mod-C. Since the left and right C-module structures on C_{bi} (and hence also on $\mathrm{Hom}_k(C_{bi}, k)$) coincide, this gives actually an isomorphism $C_{bi} \cong \mathrm{Hom}_k(C_{bi}, k)$ in C-mod-C. $\square$

Similarly to the functor $\mathbb{V}$ from Definition 2.10 we define

Definition 4.3. Let $\mathbb{V}^2$ denote the functor $\mathrm{Hom}(P_{w_0,w_0},\ ) : \mathcal{O}_{\lambda,\lambda} \to C\text{-mod-}C$.

Then

$$\mathbb{V}^2(V) \cong \mathrm{Hom}(P_{w_0,w_0}, P^{\Delta}_{w_0,w_0}) \cong \mathrm{End}(P_{w_0,w_0})^{\Delta} \cong {C^2}^{\Delta} \cong C_{bi}. \tag{4.2}$$

Let T and T' be projective functors. We find finite dimensional $\mathfrak{g}$-modules E and F such that T and T' are direct summands in T_E and T_F, respectively. Thus V_T (resp. $V_{T'}$) is a direct summand in V_E (resp. V_F). We now prove

Proposition 4.4. *The functor $\mathbb{V}^2$ restricted to $P(\mathcal{O}_{\lambda,\lambda})$ is fully faithful.*

Remark 4.5. The analogy with the Structure Theorem 2.11 can be made even closer: Let Λ_{w_0} denote the projective functor with $\Lambda_{w_0}(M_e) \cong P_{w_0}$. It can be shown that $\mathrm{End}_{PF(\mathcal{O}_\lambda)}(\Lambda_{w_0})$ is isomorphic to C^2. This way Proposition 4.4 reads that the functor $\mathrm{Hom}_{PF(\mathcal{O}_\lambda)}(\Lambda_{w_0},\ )$ from $PF(\mathcal{O}_\lambda)$ to mod- $\mathrm{End}_{PF(\mathcal{O}_\lambda)}(\Lambda_{w_0})$ is fully faithful.

Proof. i) We have to show that the map

$$\mathbb{V}^2 : \mathrm{Hom}(V_E, V_F) \to \mathrm{Hom}_{C^2}(\mathbb{V}^2(V_E), \mathbb{V}^2(V_F)) \tag{4.3}$$

is bijective (because this map will then restrict to a bijection between direct summands of V_E and V_F).

Injectivity. It is clear that the socle of $(k \boxtimes F) \otimes P_{w_0,w_0}$ consists of a direct sum of copies of L_{w_0,w_0}. Since $V_F = (k \boxtimes F) \otimes P^{\Delta}_{w_0,w_0}$ is a submodule of $(k \boxtimes F) \otimes P_{w_0,w_0}$, the socle of V_F has this property also. Thus, if $0 \neq \phi \in \mathrm{Hom}(V_E, V_F)$, then $\mathrm{Im}\,\phi$ contains L_{w_0,w_0}. Since P_{w_0,w_0} is the projective cover of L_{w_0,w_0}, this assures that $\mathbb{V}^2(\mathrm{Im}\,\phi) \neq 0$. But $\mathbb{V}^2(\mathrm{Im}\,\phi) = \mathrm{Im}\,\mathbb{V}^2(\phi)$, by exactness of $\mathbb{V}^2$. Hence $\mathbb{V}^2(\phi) \neq 0$ so the map (4.3) is injective.

Both sides have the same dimension. In analogy with the argument in [S] (step 4 in the proof of the Theorem 9) we see that it suffices to consider the case $V_F = V$. Then $\mathbb{V}^2(V_F) \cong {C^2}^{\Delta}$. Thus

$$\begin{aligned}
&\mathrm{Hom}_{C^2}(\mathbb{V}^2(V_E), \mathbb{V}^2(V_F)) \cong \mathrm{Hom}_{C^2}(\mathbb{V}^2(V_E), {C^2}^{\Delta}) \cong \mathrm{Hom}_{C^2}(\mathbb{V}^2(V_E), C^2)^{\Delta} \\
&\quad \cong \mathrm{Hom}_{C^2}({C^2}^{\star}, \mathbb{V}^2(V_E)^{\star})^{\Delta} \cong \mathrm{Hom}_{C^2}(C^2, \mathbb{V}^2(V_E^{\star}))^{\Delta} \cong \mathbb{V}^2(V_E^{\star})^{\Delta} \\
&\quad \cong \mathrm{Hom}(V_E, P^{\star}_{w_0,w_0})^{\Delta} \cong \mathrm{Hom}(V_E, P_{w_0,w_0})^{\Delta} \cong \mathrm{Hom}(V_E, P^{\Delta}_{w_0,w_0}) \cong \mathrm{Hom}(V_E, V_F).
\end{aligned}$$

□

Let $s \in S$ be a simple reflection and let C^s denote the subring of s-invariant elements in C.

Definition 4.6. Denote by Θ_s the wall-crossing (through the s-wall) functor (see [Jan]). Θ_s is a projective functor on $\mathcal{O}_\lambda$. Put also $\tilde{\Theta}_s = \mathrm{id}_{\mathcal{O}_\lambda} \boxtimes \Theta_s : \mathcal{O}_{\lambda,\lambda} \rightsquigarrow \mathcal{O}_{\lambda,\lambda}$.

Define a functor Γ_s : mod-$C \rightsquigarrow$ mod-C, by $\Gamma_s(M) = M \otimes_{C^s} C$ and also $\tilde{\Gamma}_s$: C-mod-$C \rightsquigarrow C$-mod-C, by $\tilde{\Gamma}_s(M) = M \otimes_{C^s} C$.

Wall-crossing functors are projective functors which behave particularly well together with the functor $\mathbb{V}$.

Lemma 4.7 ([S], Corollary 1)**.** *For each $s \in S$ there is a natural equivalence $\mathbb{V} \circ \Theta_s \cong \Gamma_s \circ \mathbb{V}$ of functors from $\mathcal{O}_\lambda \to C$-mod.*

Let $\bar{s} = (s_1, \dots, s_n)$ be any sequence in S and put $\Theta_{\bar{s}} = \Theta_{s_n} \cdots \Theta_{s_1}$ and $\Gamma_{\bar{s}} = \Gamma_{s_n} \cdots \Gamma_{s_1}$. Similarly we define $\tilde{\Theta}_{\bar{s}}$ and $\tilde{\Gamma}_{\bar{s}}$. We get from Lemma 4.7

$$\mathbb{V}(\Theta_{\bar{s}}(M_e)) = \Gamma_{\bar{s}}(\mathbb{V}(M_e)) = \Gamma_{\bar{s}}(k). \tag{4.4}$$

From Proposition 4.1 and Lemma 4.7 we similarly get

$$\mathbb{V}^2(V_{\Theta_{\bar{s}}}) = \mathbb{V}^2(\tilde{\Theta}_{\bar{s}}(V)) = \tilde{\Gamma}_{\bar{s}}(C_{bi}). \tag{4.5}$$

Let $\overline{s}' = (s'_1, \ldots, s'_m)$ be another sequence in S. In [S2], Proposition 7 and Proposition 8, the following result is proved when C is replaced by $S(\mathfrak{h})$ in the definition of each involved object. It is straightforward, however, to verify that the case of $S(\mathfrak{h})$ implies the case of C.

Proposition 4.8. $\mathrm{Hom}_{C^2}(\tilde{\Gamma}_{\bar{s}}(C_{bi}), \tilde{\Gamma}_{\bar{s}'}(C_{bi}))$ *is a graded C-bimodule which is free as a left (and as a right) C-module. The specialization map*

$$k \otimes_C \mathrm{Hom}_{C^2}(\tilde{\Gamma}_{\bar{s}}(C_{bi}), \tilde{\Gamma}_{\bar{s}'}(C_{bi})) \to \mathrm{Hom}_C(k \otimes_C \tilde{\Gamma}_{\bar{s}}(C_{bi}), k \otimes_C \tilde{\Gamma}_{\bar{s}'}(C_{bi}))$$

is an isomorphism of right C-modules.

Since, clearly, $k \otimes_C \tilde{\Gamma}_{\bar{s}}(C_{bi}) \cong \Gamma_{\bar{s}}(k)$ and $k \otimes_C \tilde{\Gamma}_{\bar{s}'}(C_{bi}) \cong \Gamma_{\bar{s}'}(k)$ in mod-C, we get a canonical isomorphism

$$k \otimes_C \mathrm{Hom}_{C^2}(\tilde{\Gamma}_{\bar{s}}(C_{bi}), \tilde{\Gamma}_{\bar{s}'}(C_{bi})) \to \mathrm{Hom}_C(\Gamma_{\bar{s}}(k), \Gamma_{\bar{s}'}(k)) \tag{4.6}$$

of right graded C-modules. By Theorems 2.11 and 4.4 we have

$$\mathrm{Hom}_{\mathcal{O}_\lambda}(\Theta_{\bar{s}}(M_e), \Theta_{\bar{s}'}(M_e)) \cong \mathrm{Hom}_C(\Gamma_{\bar{s}}(k), \Gamma_{\bar{s}'}(k)) \tag{4.7}$$

and by the full embedding $PF(\mathcal{O}_\lambda) \hookrightarrow \mathcal{O}_{\lambda,\lambda}$, respectively, by (4.5) and Proposition 4.4, we have the two isomorphisms

$$\mathrm{Hom}_{PF(\mathcal{O}_\lambda)}(\Theta_{\bar{s}}, \Theta_{\bar{s}'}) \cong \mathrm{Hom}_{\mathcal{O}_{\lambda,\lambda}}(V_{\Theta_{\bar{s}}}, V_{\Theta_{\bar{s}'}}) \cong \mathrm{Hom}_{C^2}(\tilde{\Gamma}_{\bar{s}}(C_{bi}), \tilde{\Gamma}_{\bar{s}'}(C_{bi})). \tag{4.8}$$

We have the evaluation map

$$ev : \mathrm{Hom}_{PF(\mathcal{O}_\lambda)}(\Theta_{\bar{s}}, \Theta_{\bar{s}'}) \to \mathrm{Hom}_{\mathcal{O}_\lambda}(\Theta_{\bar{s}}(M_e), \Theta_{\bar{s}'}(M_e)).$$

Note that the map (4.6) via (4.7) and (4.8) then corresponds to the canonical morphism

$$\overline{ev} : k \otimes_C \mathrm{Hom}_{PF(\mathcal{O}_\lambda)}(\Theta_{\bar{s}}, \Theta_{\bar{s}'}) \to \mathrm{Hom}_{\mathcal{O}_\lambda}(\Theta_{\bar{s}}(M_e), \Theta_{\bar{s}'}(M_e)). \tag{4.9}$$

Thus $\overline{ev}$ is an isomorphism.

Denote by Λ_x the (unique up to isomorphism) projective functor of Theorem 2.7 such that $\Lambda_x(M_e) \cong P_x$. In the beginning of section 4.3 we show that if $\bar{s}$ is a reduced S-sequence for x, then Λ_x is a direct summand in $\Theta_{\bar{s}}$. Moreover, all other indecomposable direct summands in $\Theta_{\bar{s}}$ are isomorphic to Λ_y for some y with $l(y) < l(x)$. The fact that (4.9) is an isomorphism for all $\bar{s}$ now readily implies that $\overline{ev}$ must be an isomorphism when $\Theta_{\bar{s}}$, $\Theta_{\bar{s}'}$ are replaced by any Λ_x, Λ_y and hence when replaced by arbitrary projective functors.

Summing up, we have proved

Theorem 4.9. *For any projective functors T and T' there is a natural graded C-bimodule structure on $\mathrm{Hom}_{PF(\mathcal{O}_\lambda)}(T, T')$ making it a free left (and right) C-module. The canonical map*

$$\overline{ev} : k \otimes_C \mathrm{Hom}_{PF(\mathcal{O}_\lambda)}(T, T') \to \mathrm{Hom}_{\mathcal{O}_\lambda}(T(M_e), T'(M_e))$$

is an isomorphism of right C-modules.

Let T and T' be projective functors and choose a basis $\{\epsilon_i\}$ for the free left C-module $\mathrm{Hom}_{PF(\mathcal{O}_\lambda)}(T,T')$. We get an isomorphism of vector spaces

$$\mathrm{Hom}_{PF(\mathcal{O}_\lambda)}(T,T') \ni x \to \sum x_i \otimes 1 \otimes \epsilon_i \in C \otimes_k k \otimes_C \mathrm{Hom}_{PF(\mathcal{O}_\lambda)}(T,T') \tag{4.10}$$

where $x = \sum x_i \epsilon_i$, $x_i \in C$. Then Theorem 4.9 gives

$$\mathrm{Hom}_{PF(\mathcal{O}_\lambda)}(T,T') \cong C \otimes_k \mathrm{Hom}_{\mathcal{O}_\lambda}(T(M_e), T'(M_e)) \tag{4.11}$$

as vector spaces.

Conjecture 4.10. *For any projective functor T there exists a non-canonical ring isomorphism* $\mathrm{End}_{PF(\mathcal{O}_\lambda)}(T) \cong C \otimes_k \mathrm{End}_{\mathcal{O}_\lambda}(T(M_e))$.

4.3. **Kazhdan-Lusztig theory.** One can give an inductive description of the indecomposable projectives as follows. Fix $x \in W^\lambda$ and let $x = s_1 \cdots s_n$ be a reduced decomposition of x, $s_i \in S$. Then P_x is the uniquely determined indecomposable direct summand in $\Theta_{\bar{s}}(M_e)$, where $\bar{s} = (s_1, \dots, s_n)$, which is *not* isomorphic to P_y for $l(y) < l(x)$. Analogously, we find $\mathbb{V}(P_x)$ as a direct summand in $\Gamma_{\bar{s}}(k)$.

Moreover, the Kazhdan-Lusztig conjectures, (conjectured in [KL], proved in [BB]) enable us to calculate the multiplicities n_y such that

$$\Theta_{\bar{s}}(M_e) = \bigoplus_{y \in W^\lambda} P_y^{n_y}. \tag{4.12}$$

In more detail this goes as follows: Let $\mathcal{H}$ be the Hecke algebra over $\mathcal{L} = \mathbb{Z}[v, v^{-1}]$ associated to the Coxeter group (W,S). Let $\{T_y;\, y \in W\}$ denote the standard basis of $\mathcal{H}$; thus $\mathcal{H} = \bigoplus_{y \in W} \mathcal{L} T_y$ and

$$T_y T_z = T_{yz}, \quad \text{if } l(yz) = l(y) + l(z),$$

$$(T_s + 1)(T_s - v) = 0, \quad \text{if } s \in S.$$

Define the involution $h \to \overline{h}$ of $\mathcal{H}$ by $\overline{v} = v^{-1}$ and $\overline{T_y} = T_{y^{-1}}^{-1}$. Put $H_y = v^{l(y)} T_y$ and let $\{\underline{H}_y;\, y \in W\}$ be the Kazhdan-Lusztig self dual basis of $\mathcal{H}$ inductively determined by $\overline{\underline{H}}_y = \underline{H}_y$ and $\underline{H}_y \in H_y + \sum_{z<y} v\mathbb{Z}[v] H_z$. Let $\tilde{h}_{y,z} \in \mathcal{L}$ be the inverse Kazhdan-Lusztig-polynomials, which are inductively defined by $H_y = \sum_z \tilde{h}_{y,z} \underline{H}_z$. Put $C_s = H_s + v$, for $s \in S$.

Expand $\underline{H}_e C_{s_1} \cdots C_{s_n}$ as a sum $\sum_{y<x} p_y H_y$ for some $p_y \in \mathcal{L}$. Then $n_y = \sum_z p_z(1) \tilde{h}_{y,z}(1)$.

With these multiplicities determined, we conclude from Theorem 2.11 that Hom's between indecomposable projectives in $\mathcal{O}_\lambda$ are completely described by the Hom's between the various $\Gamma_{\bar{s}}(k)$'s in C-mod. This is indeed the best description one might hope for.

We would like to do the same thing for projective functors on $\mathcal{O}$. Recall that Λ_y denotes the projective functor of Theorem 2.7 such that $\Lambda_y(M_e) \cong P_y$. By Theorem 2.7 we have

$$\Theta_{\bar{s}} = \bigoplus_{y \in W^\lambda} \Lambda_y^{n_y} \tag{4.13}$$

where the n_y's are defined by (4.12).

Summing up we get from Theorem 4.9

Proposition 4.11. *The Kazhdan-Lusztig conjectures give us an algorithm that describes the space* $\mathrm{Hom}_{PF(\mathcal{O}_\lambda)}(\Lambda_x, \Lambda_y)$ *in terms of homomorphisms between* C^2*-modules of the type* $\Gamma_{\bar{s}}(k)$.

Example 4.12. Let $\mathfrak{g} = \mathfrak{sl}_2$, $W = \{e, s\}$. Then $C \cong k[x]/(x^2)$ and $C^s = k$. The two indecomposable projective functors on the trivial block $\mathcal{O}_0$ are $\mathrm{Id}_{\mathcal{O}_0}$ and $\Lambda_s = \Theta_s$. Denote by V_y the corresponding object of $\mathcal{O}_{\lambda,\lambda}$ corresponding to Λ_y. Then $\mathbb{V}^2(V_e) = C^{2^\Delta}$ and $\mathbb{V}^2(V_s) = C^2 \otimes_{C\otimes k} C^{2^\Delta} \cong C^2$ (in C^2-mod). Thus

$$\mathrm{End}_{PF(\mathcal{O}_0)}(\mathrm{Id}_{\mathcal{O}_0}) \cong \mathrm{End}_{C^2}(C^{2^\Delta}) \cong \mathrm{End}_C(C) = C,$$
$$\mathrm{End}_{PF(\mathcal{O}_0)}(\Theta_s) \cong \mathrm{End}_{C^2}(C^2) = C^2,$$
$$\mathrm{Hom}_{PF(\mathcal{O}_0)}(\Theta_s, \mathrm{Id}_{\mathcal{O}_0}) \cong \mathrm{Hom}_{C^2}(C^2, C^{2^\Delta}) \cong C^{2^\Delta} \cong C.$$

5. Open questions

Here are some open questions connected to the material in this paper.

Action of the Hecke Algebra. The Hecke algebra $\mathcal{H}$ associated to the Weyl group W of $\mathfrak{g}$ acts on (a graded version of) the Grothendieck group $K(\mathcal{O})$ via (graded) projective functors. In fact, Lusztig's self dual element $\underline{\overline{H}}_s$ acts from the right on $K(\mathcal{O})$ by the wall-crossing functor Θ_s for any simple reflection $s \in W$. (See section 4 for the definitions of $\underline{\overline{H}}_s$ and Θ_s.) One would like to lift this to an action of $\mathcal{H}$ on $\mathcal{O}$. To do this, it must be verified that the composition of certain homomorphisms of projective functors are compatible with the defining relations of $\mathcal{H}$.

Hochschild cohomology. Another possible application of Theorem 3.1 would concern the Hochschild cohomology, $HH^\bullet(\mathcal{O}_\lambda)$, of $\mathcal{O}_\lambda$. Here we define the Hochschild cohomology $HH^\bullet(\mathcal{O}_\lambda)$ to be the algebra $\mathrm{Ext}_{R\otimes R^{op}}(R, R)$ where R is the endomorphism ring of a projective generator of $\mathcal{O}_\lambda$. It follows that this algebra is isomorphic to $\mathrm{Ext}_{\mathcal{O}_{\lambda,\lambda}(\mathfrak{g}\times\mathfrak{g})}(P^\Delta_{w_0,w_0}, P^\Delta_{w_0,w_0})$. The good thing here is that P_{w_0,w_0} is a projective and injective object of $\mathcal{O}_{\lambda,\lambda}(\mathfrak{g}\times\mathfrak{g})$; the bad thing is that P_{w_0,w_0} has a very complicated structure as a module over $\mathcal{Z}$.

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