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BOUNDS FOR FOURIER TRANSFORMS OF REGULAR ORBITAL INTEGRALS ON p-ADIC LIE ALGEBRAS

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Abstract. Let G be a connected reductive p-adic group and let $\mathfrak{g}$ be its Lie algebra. Let $\mathcal{O}$ be a G-orbit in $\mathfrak{g}$. Then the orbital integral $\mu_{\mathcal{O}}$ corresponding to $\mathcal{O}$ is an invariant distribution on $\mathfrak{g}$, and Harish-Chandra proved that its Fourier transform $\hat{\mu}_{\mathcal{O}}$ is a locally constant function on the set $\mathfrak{g}'$ of regular semisimple elements of $\mathfrak{g}$. Furthermore, he showed that a normalized version of the Fourier transform is locally bounded on $\mathfrak{g}$. Suppose that $\mathcal{O}$ is a regular semisimple orbit. Let γ be any semisimple element of $\mathfrak{g}$, and let $\mathfrak{m}$ be the centralizer of γ. We give a formula for $\hat{\mu}_{\mathcal{O}}(tH)$ (in terms of Fourier transforms of orbital integrals on $\mathfrak{m}$), for regular semisimple elements H in a small neighborhood of γ in $\mathfrak{m}$ and $t \in F^{\times}$ sufficiently large. We use this result to prove that Harish-Chandra's normalized Fourier transform is globally bounded on $\mathfrak{g}$ in the case that $\mathcal{O}$ is a regular semisimple orbit.

1. Introduction

Let F be a p-adic field of characteristic zero. Let G be the set of F-rational points of a connected reductive group defined over F, and let $\mathfrak{g}$ be its Lie algebra. For $X \in \mathfrak{g}$, let $\mathcal{O} = \mathcal{O}_X$ denote the G-orbit of X, and let $\mu_{\mathcal{O}}$ denote the orbital integral corresponding to $\mathcal{O}$, so that

$$\mu_{\mathcal{O}}(f) = \int_{G/G_X} f(xX)\ dx^*, f \in C_c^{\infty}(\mathfrak{g}). \tag{1.1}$$

Here G_X denotes the centralizer of X in G and dx^* is an invariant measure on G/G_X. Let B denote a symmetric, nondegenerate, G-invariant bilinear form on $\mathfrak{g}$, and fix a nontrivial additive character ψ of F. Then we have the Fourier transform

$$\hat{f}(X) = \int_{\mathfrak{g}} f(Y)\ \psi(B(X,Y))\ dY,\ X \in \mathfrak{g},\ f \in C_c^{\infty}(\mathfrak{g}). \tag{1.2}$$

The distribution $\hat{\mu}_{\mathcal{O}}(f) = \mu_{\mathcal{O}}(\hat{f}),\ f \in C_c^{\infty}(\mathfrak{g})$, is the Fourier transform of the orbital integral. Harish-Chandra [1] proved that it is a locally constant function on $\mathfrak{g}'$, the set of regular semisimple elements of $\mathfrak{g}$.

For $X \in \mathfrak{g}$, let $\eta_{\mathfrak{g}}(X)$ denote the coefficient of t^l in the polynomial $\det(t - \operatorname{ad} X)$, where t is an indeterminate and l is the rank of $\mathfrak{g}$. Then $\mathfrak{g}' = \{X \in \mathfrak{g} : \eta_{\mathfrak{g}}(X) \neq 0\}$. For any G-orbit $\mathcal{O}$ in $\mathfrak{g}$, we normalize $\hat{\mu}_{\mathcal{O}}$ by defining

$$\Phi(\mathfrak{g}, \mathcal{O}, X) = |\eta_{\mathfrak{g}}(X)|^{1/2}\hat{\mu}_{\mathcal{O}}(X), X \in \mathfrak{g}'. \tag{1.3}$$

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Harish-Chandra [1] proved that the normalized Fourier transform $\Phi(\mathfrak{g}, \mathcal{O})$ is locally bounded on $\mathfrak{g}$. In this paper we will prove the following theorem.

Theorem 1.1. *Let $\mathcal{O}$ be a regular semisimple G-orbit in $\mathfrak{g}$. Then*

$$\sup_{X \in \mathfrak{g}'} |\Phi(\mathfrak{g}, \mathcal{O}, X)| < \infty.$$

It is not true that $\Phi(\mathfrak{g}, \mathcal{O})$ is uniformly bounded on $\mathfrak{g}$ for arbitrary orbits $\mathcal{O}$. Let $\mathcal{O}$ be any orbit, and define

$$d_0(\mathcal{O}) = \dim \mathfrak{g} - \dim \mathcal{O} - \operatorname{rank} \mathfrak{g} = \dim \mathfrak{g}_X - \operatorname{rank} \mathfrak{g} \tag{1.4}$$

where $\mathfrak{g}_X$ denotes the centralizer in $\mathfrak{g}$ of a representative $X \in \mathcal{O}$. Then $d_0(\mathcal{O}) \geq 0$ and $d_0(\mathcal{O}) = 0$ when $\mathcal{O}$ is regular semisimple.

When $\mathcal{O}$ is a nilpotent orbit, it follows from the homogeneity property of nilpotent orbital integrals (section 3.1 of [1]) that

$$\Phi(\mathfrak{g}, \mathcal{O}, t^2 X) = |t|^{d_0(\mathcal{O})} \Phi(\mathfrak{g}, \mathcal{O}, X), \quad X \in \mathfrak{g}', t \in F^{\times}. \tag{1.5}$$

The results of [2] show that for general orbits $\mathcal{O}$, $\Phi(\mathfrak{g}, \mathcal{O}, t^2 X)$ also grows at infinity like $|t|^{d_0(\mathcal{O})}$. Thus $\Phi(\mathfrak{g}, \mathcal{O})$ is not uniformly bounded on $\mathfrak{g}'$ when $d_0(\mathcal{O}) > 0$.

The normalized Fourier transforms of regular semisimple orbital integrals are given by the following formula. Let $\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}$, and let A denote the split component of the Cartan subgroup B of G corresponding to $\mathfrak{b}$. Let K be a compact open subgroup of G. Then for all $X \in \mathfrak{b}', Y \in \mathfrak{g}'$, we define

$$\Phi(\mathfrak{g}, dx^*, X, Y) = |\eta_{\mathfrak{g}}(X)|^{1/2} |\eta_{\mathfrak{g}}(Y)|^{1/2} \int_{G/A} \int_K \psi(B(kY, xX)) dk dx^*, \tag{1.6}$$

where dx^* is an invariant measure on G/A and dk is normalized Haar measure on K. It is independent of the choice K of compact open subgroup. When the choice of invariant measure dx^* is not important, we will drop it from the notation and write $\Phi(\mathfrak{g}, X, Y)$. Harish-Chandra [1] proved that this integral is convergent, and that if $\mathcal{O}_X$ denotes the G-orbit of $X \in \mathfrak{b}'$, then we can normalize the Haar measure on G/G_X in (1.1) so that for all $Y \in \mathfrak{g}'$,

$$\Phi(\mathfrak{g}, dx^*, X, Y) = |\eta_{\mathfrak{g}}(X)|^{1/2} |\eta_{\mathfrak{g}}(Y)|^{1/2} \hat{\mu}_{\mathcal{O}_X}(Y) = |\eta_{\mathfrak{g}}(X)|^{1/2} \Phi(\mathfrak{g}, \mathcal{O}_X, Y). \tag{1.7}$$

Theorem 1.1 is a consequence of the following expansion at infinity. Fix a semisimple element $\gamma \in \mathfrak{g}$ and write $\mathfrak{m} = C_{\mathfrak{g}}(\gamma), M = C_G(\gamma)$. Define $N_G(\mathfrak{b}, \mathfrak{m}) = \{y \in G : y^{-1}\mathfrak{b} \subset \mathfrak{m}\}$. Then as in [2], $y \in N_G(\mathfrak{b}, \mathfrak{m})$ if and only if $yM \subset N_G(\mathfrak{b}, \mathfrak{m})$, and $W = W_G(\mathfrak{b}, \mathfrak{m}) = N_G(\mathfrak{b}, \mathfrak{m})/M$ is a finite set. Let $w \in W$, and let $y_w \in N_G(\mathfrak{b}, \mathfrak{m})$ be a representative for w. Then $y_w^{-1}\mathfrak{b}$ is a Cartan subalgebra of the reductive Lie algebra $\mathfrak{m}$, so that given a normalization of invariant measure dm_w^* on $M/y_w^{-1} A y_w$ we can define $\Phi(\mathfrak{m}, dm_w^*, y_w^{-1}X, Y), X \in \mathfrak{b}', Y \in \mathfrak{m}'$, as in (1.6). For each $w \in W$, there is a locally constant function $c_w(dx^*/dm_w^*, \gamma, \cdot) : \mathfrak{b}' \to \mathbf{C}$ defined in (4.5). It has the property that $|c_w(dx^*/dm_w^*, \gamma, X)|$ is a nonzero constant independent of $X \in \mathfrak{b}'$.

Theorem 1.2. *Let ω be a compact subset of $\mathfrak{b}'$. Then there exist a neighborhood $U(\gamma)$ of γ in $\mathfrak{m}$ and $T(\gamma) > 0$ so that for all $X \in \omega, H \in U(\gamma) \cap \mathfrak{g}'$, and $t \in F, |t| \geq T(\gamma)$,*

$$\Phi(\mathfrak{g}, dx^*, X, tH) = \sum_{w \in W} c_w(dx^*/dm_w^*, \gamma, tX) \; \Phi(\mathfrak{m}, dm_w^*, y_w^{-1}X, tH).$$

In the case that $\gamma \in \mathfrak{g}'$, Theorem 1.2 follows from Theorem 2.2 of [2] or from results of Waldspurger in [3]. The proof in the general case uses techniques from [2].

The following stronger version of Theorem 1.1 is an easy consequence of Theorem 1.2 and induction on the dimension of $\mathfrak{g}$.

Theorem 1.3. *Let $\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}$, and let ω be a compact subset of $\mathfrak{b}'$. Then*

$$\sup_{X\in\omega, Y\in\mathfrak{g}'} |\Phi(\mathfrak{g}, X, Y)| < \infty.$$

This paper is organized as follows. In §2 we show how Theorem 1.2 can be used to prove Theorem 1.3. In §3 we prove technical results which are needed for the proof of Theorem 1.2. Finally, Theorem 1.2 is proven in §4. This is done first in the case that $\mathfrak{g}$ is semisimple and $\mathfrak{b}$ is elliptic. The general case follows from this case using parabolic induction.

2. Proof of Theorem 1.3

The proof of Theorem 1.3 requires only the following simpler version of Theorem 1.2 which is proved in the first part of §4 as the first step in the proof of Theorem 1.2. Assume that $\mathfrak{g}$ is semisimple and $\mathfrak{b}$ is an elliptic Cartan subalgebra of $\mathfrak{g}$. Then the split component of B is trivial. Fix Haar measures dx and dm on G and M respectively, and define $c(\mathfrak{g}, \mathfrak{m}, dx/dm, \gamma, X), X \in \mathfrak{b}'$, as in (3.9).

Proposition 2.1. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with $\gamma \in \mathfrak{h}$, and let ω be a compact subset of $\mathfrak{b}'$. Then there exist a neighborhood $\omega(\gamma)$ of γ in $\mathfrak{h}$ and $T(\gamma) > 0$ so that for all $X \in \omega, H \in \omega(\gamma) \cap \mathfrak{h}'$, and $t \in F, |t| \geq T(\gamma)$,*

$$\Phi(\mathfrak{g}, dx, X, tH) = \sum_{w\in W_G(\mathfrak{b},\mathfrak{m})} c(\mathfrak{g}, \mathfrak{m}, dx/dm, \gamma, t y_w^{-1} X) \quad \Phi(\mathfrak{m}, dm, y_w^{-1} X, tH).$$

The proof of Theorem 1.3 from Proposition 2.1 is by induction on the dimension of $\mathfrak{g}$. Since the normalizations of Haar measures are not important for Theorem 1.3 we drop them from the notation. If $\dim \mathfrak{g} < 3$, then $\mathfrak{g}$ is abelian and $|\Phi(\mathfrak{g}, X, H)| = |\psi(B(X, H))| = 1$ for all $H, X \in \mathfrak{g}$. Assume that $\dim \mathfrak{g} \geq 3$ and that the theorem is true for all reductive Lie algebras of smaller dimension.

Suppose that $\mathfrak{g}$ is not semisimple. Then we can write $\mathfrak{g} = \mathfrak{z} + \mathfrak{g}_1$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$, $\mathfrak{g}_1$ is the derived subalgebra, and $\dim \mathfrak{g}_1 < \dim \mathfrak{g}$. Then $\mathfrak{b} = \mathfrak{z} + \mathfrak{b}_1$ where $\mathfrak{b}_1$ is a Cartan subalgebra of $\mathfrak{g}_1$. Further, $\mathfrak{g}' = \mathfrak{z} + \mathfrak{g}_1'$ and $\mathfrak{b}' = \mathfrak{z} + \mathfrak{b}_1'$. Let $G_1 = G/Z$. Then $A_1 = A/Z$ is the split component of $B_1 = B/Z$, and we can identify G/A and G_1/A_1. Now if we use the same invariant measure to define $\Phi(\mathfrak{g}, X, Y), X \in \mathfrak{b}', Y \in \mathfrak{g}'$, and $\Phi(\mathfrak{g}_1, X_1, Y_1), X_1 \in \mathfrak{b}_1', Y_1 \in \mathfrak{g}_1'$, we have

$$\Phi(\mathfrak{g}, Z_1 + X_1, Z_2 + Y_1) = \psi(B(Z_1, Z_2))\Phi(\mathfrak{g}_1, X_1, Y_1), Z_1, Z_2 \in \mathfrak{z}, X_1 \in \mathfrak{b}_1', Y_1 \in \mathfrak{g}'.$$

Let ω be a compact subset of $\mathfrak{b}'$. Then there is a compact subset ω_1 of $\mathfrak{b}_1'$ so that $\omega \subset \mathfrak{z} + \omega_1$. By the induction hypothesis there is $C > 0$ so that $|\Phi(\mathfrak{g}_1, X_1, Y_1)| \leq C$ for all $X_1 \in \omega_1, Y_1 \in \mathfrak{g}_1'$. Thus for all $Z_1, Z_2 \in \mathfrak{z}, X_1 \in \omega_1, Y_1 \in \mathfrak{g}_1'$,

$$|\Phi(\mathfrak{g}, Z_1 + X_1, Z_2 + Y_1)| = |\psi(B(Z_1, Z_2))\Phi(\mathfrak{g}_1, X_1, Y_1)| \leq C.$$

Thus we may as well assume that $\mathfrak{g}$ is semisimple.

Since $\Phi(\mathfrak{g}, X)$ is a class function on $\mathfrak{g}$, and $\mathfrak{g}$ has a finite number of conjugacy classes of Cartan subalgebras, it suffices to show that for each Cartan subalgebra $\mathfrak{h}$

of $\mathfrak{g}$, $|\Phi(\mathfrak{g}, X, H)|$ is uniformly bounded for $X \in \omega, H \in \mathfrak{h}'$. Fix an arbitrary Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Let A be the split component of B and let $G_{\mathfrak{b}}$ denote the centralizer in G of A. Define $W_G(\mathfrak{h}, \mathfrak{g}_{\mathfrak{b}}) = N_G(\mathfrak{h}, \mathfrak{g}_{\mathfrak{b}})/G_{\mathfrak{b}}$ where $N_G(\mathfrak{h}, \mathfrak{g}_{\mathfrak{b}}) = \{y \in G : y^{-1}\mathfrak{h} \subset \mathfrak{g}_{\mathfrak{b}}\}$. For each $s \in W_G(\mathfrak{h}, \mathfrak{g}_{\mathfrak{b}})$, fix a representative $y_s \in N_G(\mathfrak{h}, \mathfrak{g}_{\mathfrak{b}})$ for s. The following lemma follows from combining Lemmas 1.7 and 1.13 of [1].

Lemma 2.2. *Given a normalization dx^* of invariant measure on G/A, there is a normalization $dx^*_{\mathfrak{b}}$ of invariant measure on $G_{\mathfrak{b}}/A$ (independent of $\mathfrak{h}$) so that for all $X \in \mathfrak{b}', H \in \mathfrak{h}'$,*

$$\Phi(\mathfrak{g}, dx^*, X, H) = \sum_{s \in W_G(\mathfrak{h}, \mathfrak{g}_{\mathfrak{b}})} \Phi(\mathfrak{g}_{\mathfrak{b}}, dx^*_{\mathfrak{b}}, X, y_s^{-1}H).$$

Now if $\mathfrak{b}$ is not elliptic, $\dim \mathfrak{g}_{\mathfrak{b}} < \dim \mathfrak{g}$, so that for each s in the finite set $W_G(\mathfrak{h}, \mathfrak{g}_{\mathfrak{b}})$, $|\Phi(\mathfrak{g}_{\mathfrak{b}}, dx^*_{\mathfrak{b}}, X, y_s^{-1}H)|$ is uniformly bounded for $X \in \omega$ and $H \in \mathfrak{h}'$. Thus we may as well assume that $\mathfrak{b}$ is elliptic.

Let $\|\cdot\|$ denote a norm on $\mathfrak{g}$, and let $\mathfrak{h}^1 = \{H \in \mathfrak{h} : \|H\| = 1\}$. For each $\gamma \in \mathfrak{h}^1$, let $\omega(\gamma) \subset \mathfrak{h}$ and $T(\gamma) > 0$ satisfy the conditions of Proposition 2.1. Since $\mathfrak{h}^1$ is compact, there are $\gamma_1, ..., \gamma_k \in \mathfrak{h}^1$ such that $\mathfrak{h}^1 \subset \bigcup_{1 \le i \le k} \omega(\gamma_i)$. Let $T = \max\{T(\gamma_i) : 1 \le i \le k\}$. Then $\mathfrak{h}_T = \{H \in \mathfrak{h} : \|H\| \le T\}$ is compact so that by Theorem 7.7 of [1], there is C_1 so that $|\Phi(\mathfrak{g}, X, H)| \le C_1$ for all $X \in \omega, H \in \mathfrak{h}_T \cap \mathfrak{g}'$. Further,

$$\{H \in \mathfrak{h} : \|H\| > T\} \subset \bigcup_{1 \le i \le k} \{tH : H \in \omega(\gamma_i), t \in F^\times, |t| > T\}.$$

Thus it suffices to bound $|\Phi(\mathfrak{g}, X, tH)|$, $X \in \omega, H \in \omega(\gamma_i) \cap \mathfrak{g}', t \in F^\times, |t| > T$, for each $1 \le i \le k$.

Fix $1 \le i \le k$, and let $\mathfrak{m}_i = C_{\mathfrak{g}}(\gamma_i)$. For each $w \in W_G(\mathfrak{b}, \mathfrak{m}_i)$, let $y_w \in N_G(\mathfrak{b}, \mathfrak{m}_i)$ be a representative for w. Then by Proposition 2.1, for all $X \in \omega, H \in \omega(\gamma_i) \cap \mathfrak{g}', |t| > T \ge T(\gamma_i)$,

$$\Phi(\mathfrak{g}, X, tH) = \sum_{w \in W_G(\mathfrak{b}, \mathfrak{m}_i)} c(\mathfrak{g}, \mathfrak{m}_i, \gamma_i, ty_w^{-1}X) \ \Phi(\mathfrak{m}_i, y_w^{-1}X, tH).$$

Since $\mathfrak{g}$ is semisimple and $\gamma_i \neq 0$, $\dim \mathfrak{m}_i < \dim \mathfrak{g}$. Fix $w \in W_G(\mathfrak{b}, \mathfrak{m}_i)$. Then $\omega(w) = y_w^{-1}\omega$ is a compact subset of the regular set of the Cartan subalgebra $y_w^{-1}\mathfrak{b}$ of $\mathfrak{m}_i$. Thus by the induction hypothesis there is $C_w > 0$ so that $|\Phi(\mathfrak{m}_i, y_w^{-1}X, Y)| \le C_w$ for all $X \in \omega, Y \in \mathfrak{m}_i'$. Further, by Lemma 3.4, $|c(\mathfrak{g}, \mathfrak{m}_i, \gamma_i, ty_w^{-1}X)| = C'_w$ is a nonzero constant independent of $X \in \mathfrak{b}', t \in F$. Thus for all $X \in \omega, H \in \omega(\gamma_i) \cap \mathfrak{g}', |t| > T$,

$$|\Phi(\mathfrak{g}, X, tH)| \le \sum_{w \in W_G(\mathfrak{b}, \mathfrak{m}_i)} C'_w C_w.$$

This concludes the proof of Theorem 1.3.

3. Evaluation of an Integral

In this section we prove Lemma 3.3 which is a slight generalization of Lemma 4.4 of [2]. This Lemma will be needed in §4 to prove Theorem 1.2.

Let $\mathcal{R}$ denote the ring of integers of F, $\mathcal{P}$ the maximal ideal in $\mathcal{R}$, and ϖ a uniformizing parameter so that $\mathcal{P} = \varpi\mathcal{R}$. Let $|\cdot|$ denote the absolute value on F such that $|\varpi| = q^{-1}$ where $q = [\mathcal{R}/\mathcal{P}]$. We assume that the character ψ of F used to define the Fourier transform in (1.2) has conductor $\mathcal{R}$.

There is $n \geq 1$ so that $\mathfrak{g}$ and G are subsets of $M_n(F)$. We have the usual norm $\|\cdot\|$ on $\mathfrak{g} \subset M_n(F)$ given by

$$\|X\| = \max_{i,j} |X_{ij}|,\ X = [X_{ij}] \in M_n(F). \tag{3.1}$$

Let B denote the symmetric, nondegenerate, bilinear form on $\mathfrak{g}$ given by

$$B(X,Y) = \operatorname{tr} XY,\ X, Y \in \mathfrak{g} \subset M_n(F). \tag{3.2}$$

Fix a reductive subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{m} = C_{\mathfrak{g}}(\gamma)$ for some semisimple element γ of $\mathfrak{g}$. Since $\mathfrak{m}$ is reductive, the restriction of B to $\mathfrak{m}$ is nondegenerate, and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{m}^{\perp}$ where $\mathfrak{m}^{\perp} = \{X \in \mathfrak{g} : B(X,Y) = 0\ \forall\ Y \in \mathfrak{m}\}$. For $X \in \mathfrak{g}$, write $X = X_0 + X_1$ where $X_0 \in \mathfrak{m}, X_1 \in \mathfrak{m}^{\perp}$. Then as in [2] we define new norms on $\mathfrak{g}$ as follows. For $X = X_0 + X_1 \in \mathfrak{g}$, define

$$\|X\|' = \max\{\|X_0\|, \|X_1\|\}, \quad \|X\|'' = \sup_{Z \in \mathfrak{g}, \|Z\|' \leq 1} |B(Z,X)|. \tag{3.3}$$

As in [2] $\|X\|'' = \max\{\|X_0\|'', \|X_1\|''\}$ and there is a constant $0 < C_0 \leq 1$ so that

$$C_0\|X\|' \leq \|X\|'' \leq \|X\| \leq \|X\|', \quad X \in \mathfrak{g}. \tag{3.4}$$

For any integer $c \geq 0$, define

$$\mathfrak{k}_c = \{X \in \mathfrak{g} : \|X\|' \leq q^{-c}\}.$$

It is a lattice in $\mathfrak{g}$. Define $c_0 > 0$ as in Lemma 4.1 of [2]. Then in particular, for any $c \geq c_0$, $\exp : \mathfrak{k}_c \to G$ is well defined and we let $K_c = \exp(\mathfrak{k}_c)$. It is a compact open subgroup of G contained in $GL(n, \mathcal{R})$. For $c \geq c_0$, write

$$\phi_c(X,Y) = \int_{K_c} \psi(B(kX,Y))dk, \quad X, Y \in \mathfrak{g},$$

where dk is normalized Haar measure on K_c.

Let $X \in \mathfrak{m}$. Then the restriction of $\operatorname{ad} X$ to $\mathfrak{m}^{\perp}$ is a linear transformation $T_X : \mathfrak{m}^{\perp} \to \mathfrak{m}^{\perp}$. Define $\mathfrak{m}^{\mathrm{reg}}$ to be the set of all $X \in \mathfrak{m}$ such that X is semisimple and $C_{\mathfrak{g}}(X) \subset \mathfrak{m}$. Then for all $X \in \mathfrak{m}^{\mathrm{reg}}$, T_X is invertible. For any integer $s > 0$, we let

$$\mathfrak{m}^{\mathrm{reg}}_s = \{X \in \mathfrak{m}^{\mathrm{reg}} : \|X\| \leq |2|^{1/2}, \|T_X^{-1}\| \leq q^s\},$$

where $\|T_X^{-1}\|$ is the operator norm of T_X^{-1}. Then for all $X \in \mathfrak{m}^{\mathrm{reg}}_s, Z_1 \in \mathfrak{m}^{\perp}$,

$$q^{-s}\|Z_1\| \leq \|\operatorname{ad} X Z_1\| \leq |2|^{1/2}\|Z_1\|. \tag{3.5}$$

Define C_0 as in (3.4)

Lemma 3.1. *Let $H, Y \in \mathfrak{m}^{\mathrm{reg}}_s$, and let $Z_0 \in \mathfrak{m}, Z_1 \in \mathfrak{m}^{\perp}$. Then*

$$\|Z_1\| \leq q^{2s} C_0^{-1} \|\operatorname{ad} H \operatorname{ad} Y Z_1\|'',$$

and if $Z_0 + Z_1 \in \mathfrak{k}_c$ where c is large enough such that $q^{-c} < q^{-2s}C_0$, then

$$\|[H, \exp(-Z_0 - Z_1)Y]\|'' = \max\{\|\operatorname{ad} H \operatorname{ad} Y Z_1\|'', \|[H, \exp(-Z_0)Y]\|''\}.$$

Proof. Let $Z_0 \in \mathfrak{m}, Z_1 \in \mathfrak{m}^{\perp}$. Then, since $H, Y \in \mathfrak{m}^{\mathrm{reg}}_s$, using (3.5)

$$\|\operatorname{ad} H \operatorname{ad} Y Z_1\| \geq q^{-2s}\|Z_1\|.$$

Thus by (3.4)

$$\|Z_1\| \leq q^{2s}\|\operatorname{ad} H \operatorname{ad} Y Z_1\| \leq q^{2s} C_0^{-1} \|\operatorname{ad} H \operatorname{ad} Y Z_1\|''.$$

Now suppose that $Z_0+Z_1 \in \mathfrak{k}_c$. Then $\|Z_0+Z_1\|' = \max\{\|Z_0\|, \|Z_1\|\} \leq q^{-c}$, so that $\|Z_i\| \leq q^{-c}, i=0,1$. Now

$$[H, \exp(-Z_0-Z_1)Y] = \sum_{k\geq 0} \frac{1}{k!}[H, (-\mathrm{ad}\, Z_0 - \mathrm{ad}\, Z_1)^k Y] = W_0+W_1+V.$$

Here $W_0 = [H, \exp(-Z_0)Y] \in \mathfrak{m}, W_1 = [H, [-Z_1, Y]] = \mathrm{ad}\, H \mathrm{ad}\, Y\, Z_1 \in \mathfrak{m}^{\perp}$, and

$$V = \sum_{k\geq 2}\sum_{\epsilon} \frac{1}{k!}(-1)^k [H, \mathrm{ad}\, Z_{\epsilon_1} \mathrm{ad}\, Z_{\epsilon_2} ... \mathrm{ad}\, Z_{\epsilon_k} Y]$$

where for each $k \geq 2$, the sum is over multi-indices $\epsilon = \{\epsilon_i\}_{i=1}^k$, $\epsilon_i \in \{0,1\}, 1 \leq i \leq k$, for which at least one $\epsilon_i = 1$.

Using Lemma 4.1 of [2], for each $k \geq 2$ and multi-index ϵ as above,

$$\|\frac{1}{k!}[H, \mathrm{ad}\, Z_{\epsilon_1} \mathrm{ad}\, Z_{\epsilon_2} ... \mathrm{ad}\, Z_{\epsilon_k} Y]\| \leq |\frac{1}{k!}|\ \|H\|\ \|Y\| q^{-c(k-1)} \|Z_1\| \leq q^{-c}\|Z_1\|.$$

But by the first part of the lemma,

$$q^{-c}\|Z_1\| \leq q^{-c} q^{2s} C_0^{-1} \|\mathrm{ad}\, H \mathrm{ad}\, Y\, Z_1\|'' < \|\mathrm{ad}\, H \mathrm{ad}\, Y\, Z_1\|'' = \|W_1\|'' \leq \|W_0+W_1\|''$$

when $q^{-c} < q^{-2s}C_0$. Thus for such c we have $\|V\|'' \leq \|V\| < \|W_0+W_1\|''$.

Thus

$$\begin{aligned}\|[H, \exp(-Z_0-Z_1)Y]\|'' &= \|W_0+W_1+V\|'' \\ &= \|W_0+W_1\|'' = \max\{\|W_0\|'', \|W_1\|''\}.\end{aligned}$$

□

For $c \geq c_0$, let

$$\mathfrak{k}_c^M = \{X \in \mathfrak{m} : \|X\| \leq q^{-c}\}, \quad K_c^M = \exp(\mathfrak{k}_c^M), \tag{3.6}$$

Lemma 3.2. *There is $c' \geq c_0$ so that for all $c \geq c'$, $K_c \cap M = K_c^M$.*

Proof. Let γ be a semisimple element of $\mathfrak{g}$ such that $\mathfrak{m} = C_{\mathfrak{g}}(\gamma)$. We may as well assume that $\|\gamma\| \leq |2|$. Then since $\gamma \in \mathfrak{m}^{\mathrm{reg}}$, there is $s>0$ so that for all $Z_1 \in \mathfrak{m}^{\perp}$,

$$q^{-s}\|Z_1\| \leq \|\ [Z_1,\gamma]\ \| \leq |2|\ \|Z_1\|.$$

Let $c \geq c_0$ such that $q^{-c} < q^{-s}$. Then clearly $K_c^M \subset K_c \cap M$. Let $k \in K_c \cap M$. Then we can write $k = \exp(Z_0+Z_1)$ where $Z_0 \in \mathfrak{m}, Z_1 \in \mathfrak{m}^{\perp}$ with $\|Z_0+Z_1\|' \leq q^{-c}$. Thus $\|Z_0\| \leq q^{-c}$ and $\|Z_1\| \leq q^{-c}$. Now since $k \in M, k\gamma = \gamma$. But $k\gamma = \exp(Z_0+Z_1)\gamma = \gamma + [Z_1,\gamma] + W$ where

$$W = \sum_{k\geq 2} \frac{1}{k!} \mathrm{ad}\,(Z_0+Z_1)^{k-1}[Z_1,\gamma].$$

Thus $[Z_1,\gamma] = -W$.

But for each $k \geq 2$, using Lemma 4.1 of [2]

$$\|\frac{1}{k!}\mathrm{ad}\,(Z_0+Z_1)^{k-1}[Z_1,\gamma]\| \leq q^{-c}\|Z_1\|.$$

Thus $\|W\| \leq q^{-c}\|Z_1\| < q^{-s}\|Z_1\|$. But $\|[Z_1,\gamma]\| \geq q^{-s}\|Z_1\|$. Thus $[Z_1,\gamma] = -W$ implies that $Z_1 = 0$. Thus $k = \exp Z_0 \in K_c^M$. □

For $H, Y \in \mathfrak{m}_s^{\text{reg}}$, define

$$\mathfrak{m}^\perp(H,Y) = \{Z_1 \in \mathfrak{m}^\perp : \|\text{ad}\, H \text{ad}\, Y Z_1\|'' \leq 1\}$$

and

$$I(\mathfrak{m}^\perp, H, Y) = \int_{\mathfrak{m}^\perp(H,Y)} \psi(1/2B(Z_1, \text{ad}\, H \text{ad}\, Y Z_1))\, dZ_1 \tag{3.7}$$

where dZ_1 is Haar measure on $\mathfrak{m}^\perp$ normalized so that $\{Z_1 \in \mathfrak{m}^\perp : \|Z_1\| \leq 1\}$ has volume one. Let $d(\mathfrak{m}^\perp)$ denote the dimension of $\mathfrak{m}^\perp$, and for $X, Y \in \mathfrak{m}$, define

$$\phi_c^M(X,Y) = \int_{K_c^M} \psi(B(kX,Y))dk \tag{3.8}$$

where dk is normalized Haar measure on K_c^M.

Lemma 3.3. *$I(\mathfrak{m}^\perp)$ is a locally constant function on $\mathfrak{m}_s^{\text{reg}} \times \mathfrak{m}_s^{\text{reg}}$. Further, let c be large enough so that $q^{-c} < q^{-4s-c_0}C_0^2$. Then for all $H, Y \in \mathfrak{m}_s^{\text{reg}}, |t| \geq q^{2s+c}C_0^{-1}$,*

$$\phi_c(t^2H, Y) = q^{cd(\mathfrak{m}^\perp)}|t|^{-d(\mathfrak{m}^\perp)}\ \phi_c^M(t^2H,Y)\ I(\mathfrak{m}^\perp, H, Y).$$

Proof. The first part is clear from the definition.

Fix $c > 0$ such that $q^{-c} < q^{-4s-c_0}C_0^2$, $H, Y \in \mathfrak{m}_s^{\text{reg}}$, and $t \in F^\times$ such that $|t| \geq q^{2s+c}C_0^{-1}$. By Proposition 4.2 of [2], since $|t| \geq q^c$, we have

$$\phi_c(t^2H,Y) = \int_{K_c(H,Y,t)} \psi(t^2B(kH,Y))\, dk,$$

where $K_c(H,Y,t) = \{k \in K_c : \|[H, k^{-1}Y]\|'' \leq |t|^{-1}\}$. Define

$$\mathfrak{k}_c^M(H,Y,t) = \{Z_0 \in \mathfrak{k}_c^M : \|[H, \exp(-Z_0)Y]\|'' \leq |t|^{-1}\},$$

$$\mathfrak{k}_c^1(H,Y,t) = \{Z_1 \in \mathfrak{m}^\perp \cap \mathfrak{k}_c : \|\text{ad}\, H \text{ad}\, Y Z_1\|'' \leq |t|^{-1}\}.$$

Now $K_c = \{\exp(Z_0 + Z_1) : Z_0 \in \mathfrak{k}_c^M, Z_1 \in \mathfrak{m}^\perp \cap \mathfrak{k}_c\}$, and by Lemma 3.1, since $q^{-c} < q^{-2s}C_0$, for all $Z_0 \in \mathfrak{m}, Z_1 \in \mathfrak{m}^\perp$,

$$\|[H, \exp(-Z_0 - Z_1)Y]\|'' = \max\{\|\text{ad}\, H \text{ad}\, Y Z_1\|'', \|[H, \exp(-Z_0)Y]\|''\}.$$

Thus $K_c(H,Y,t) = \{\exp(Z_0 + Z_1) : Z_0 \in \mathfrak{k}_c^M(H,Y,t), Z_1 \in \mathfrak{k}_c^1(H,Y,t)\}$.

Let dZ denote the Haar measure on $\mathfrak{g}$ for which $\mathfrak{k}_c$ has volume one and let dZ_0 denote the Haar measure on $\mathfrak{m}$ for which $\mathfrak{k}_c^M$ has volume one. Then if $Z = Z_0 + Z_1,\ Z_0 \in \mathfrak{m}, Z_1 \in \mathfrak{m}^\perp$, we have $dZ = q^{cd(\mathfrak{m}^\perp)}dZ_0\ dZ_1$. Thus we have

$$\phi_c(t^2H,Y) = q^{cd(\mathfrak{m}^\perp)} \int_{\mathfrak{k}_c^M(H,Y,t)} \int_{\mathfrak{k}_c^1(H,Y,t)} \psi(t^2B(\exp(Z_0+Z_1)H,Y))\, dZ_0 dZ_1.$$

Let $Z_0 \in \mathfrak{k}_c^M$ and $Z_1 \in \mathfrak{k}_c^1(H,Y,t)$. Then

$$B(\exp(Z_0+Z_1)H,Y) = B(\exp(Z_0)H,Y) + \sum_{k\geq 1} b_k,$$

where for $k \geq 1$,

$$b_k = \sum_\epsilon \frac{1}{k!} B(\text{ad}\, Z_{\epsilon_1} \text{ad}\, Z_{\epsilon_2} ... \text{ad}\, Z_{\epsilon_k} H, Y).$$

Here, as in the proof of Lemma 3.1, the sum is over multi-indices ϵ for which at least one $\epsilon_i = 1$. Suppose that exactly one $\epsilon_i = 1$. Then $\text{ad}\, Z_{\epsilon_1} \text{ad}\, Z_{\epsilon_2} ... \text{ad}\, Z_{\epsilon_k} H \in \mathfrak{m}^\perp$ and $Y \in \mathfrak{m}$, so that $B(\text{ad}\, Z_{\epsilon_1} \text{ad}\, Z_{\epsilon_2} ... \text{ad}\, Z_{\epsilon_k} H, Y) = 0$. Thus $b_1 = 0$ and $b_2 =$

$1/2B((\mathrm{ad}\, Z_1)^2 H, Y) = 1/2B(Z_1, \mathrm{ad}\, H \mathrm{ad}\, Y Z_1)$. Suppose that $k \geq 3$ and at least two of the $\epsilon_i = 1$. Then

$$|\frac{1}{k!} B(\mathrm{ad}\, Z_{\epsilon_1} \mathrm{ad}\, Z_{\epsilon_2} ... \mathrm{ad}\, Z_{\epsilon_k} H, Y)| \\ \leq |\frac{1}{k!}| q^{-(k-2)c} \|H\| \, \|Y\| \, \|Z_1\|^2 \leq q^{-(c-c_0)(k-2)} \|Z_1\|^2.$$

But by Lemma 3.1, for $k \geq 3$,

$$q^{-(c-c_0)(k-2)} \|Z_1\|^2 \leq q^{-(c-c_0)} q^{4s} C_0^{-2} (\|\mathrm{ad}\, H \mathrm{ad}\, Y Z_1\|'')^2 \leq |t|^{-2}$$

for $Z_1 \in \mathfrak{k}_c^1(H, Y, t)$ since $q^{-c} \leq q^{-4s-c_0} C_0^2$. Thus

$$\psi(t^2 B(\exp(Z_0 + Z_1) H, Y)) = \psi(t^2 B(\exp(Z_0) H, Y)) \psi(t^2 1/2B(Z_1, \mathrm{ad}\, H \mathrm{ad}\, Y Z_1)),$$

so that

$$\phi_c(t^2 H, Y) = q^{cd(\mathfrak{m}^\perp)} \\ \times \int_{\mathfrak{k}_c^M(H,Y,t)} \psi(t^2 B(\exp(Z_0) H, Y)) dZ_0 \\ \times \int_{\mathfrak{k}_c^1(H,Y,t)} \psi(t^2 1/2B(Z_1, \mathrm{ad}\, H \mathrm{ad}\, Y Z_1)) \; dZ_1.$$

But applying Lemma 4.2 of [2] to $\mathfrak{m}$ in place of $\mathfrak{g}$, since $|t| \geq q^c$,

$$\int_{\mathfrak{k}_c^M(H,Y,t)} \psi(t^2 B(\exp(Z_0) H, Y)) dZ_0 = \phi_c^M(t^2 H, Y).$$

Further, using the proof of Lemma 4.4 of [2], since $|t| \geq q^{2s+c} C_0^{-1}$,

$$\int_{\mathfrak{k}_c^1(H,Y,t)} \psi(t^2 1/2B(Z_1, \mathrm{ad}\, H \mathrm{ad}\, Y Z_1)) \; dZ_1 = |t|^{-d(\mathfrak{m}^\perp)} I(\mathfrak{m}^\perp, H, Y).$$

□

For $H \in \mathfrak{m}^{\mathrm{reg}}$, define $\eta_{\mathfrak{g}/\mathfrak{m}}(H) = \det \mathrm{ad}\, H|_{\mathfrak{m}^\perp} = \det T_H$. Let

$$\mathfrak{g}(\mathfrak{m}) = \{\gamma \in \mathfrak{g} : C_{\mathfrak{g}}(\gamma) = \mathfrak{m}\}.$$

Then $\mathfrak{g}(\mathfrak{m}) \subset \mathfrak{m}^{\mathrm{reg}}$. The following was proven in Lemma 4.5 and Theorem 2.2 of [2].

Lemma 3.4. *There is a unique locally constant function $c_0(\mathfrak{g}, \mathfrak{m})$ on $\mathfrak{g}(\mathfrak{m}) \times \mathfrak{m}^{\mathrm{reg}}$ with the following properties. First, suppose that $Y \in \mathfrak{g}(\mathfrak{m}) \cap \mathfrak{m}_s^{\mathrm{reg}}$ and $H \in \mathfrak{m}_s^{\mathrm{reg}}$ for some $s > 0$. Then*

$$c_0(\mathfrak{g}, \mathfrak{m}, Y, H) = |\eta_{\mathfrak{g}/\mathfrak{m}}(H)|^{1/2} |\eta_{\mathfrak{g}/\mathfrak{m}}(Y)|^{1/2} I(\mathfrak{m}^\perp, Y, H).$$

In addition, for all $Y \in \mathfrak{g}(\mathfrak{m}), H \in \mathfrak{m}^{\mathrm{reg}}$,

(i) $c_0(\mathfrak{g}, \mathfrak{m}, tY, H) = c_0(\mathfrak{g}, \mathfrak{m}, Y, tH)$ *for all $t \in F^\times$;*
(ii) $c_0(\mathfrak{g}, \mathfrak{m}, Y, t^2 H) = c_0(\mathfrak{g}, \mathfrak{m}, Y, H)$ *for all $t \in F^\times$;*
(iii) $|c_0(\mathfrak{g}, \mathfrak{m}, Y, H)|$ *is nonzero and independent of Y, H;*
(iv) $c_0(\mathfrak{g}, \mathfrak{m}, Y, mH) = c_0(\mathfrak{g}, \mathfrak{m}, Y, H)$ *for all $m \in M$.*

Let dx and dm denote Haar measures on G and M respectively. For $c \geq c_0$, let $V(K_c, dx)$ denote the volume of K_c with respect to dx and let $V(K_c^M, dm)$ denote the volume of K_c^M with respect to dm. Then $q^{cd(\mathfrak{m}^\perp)} V(K_c, dx) V(K_c^M, dm)^{-1}$ is independent of c. For $Y \in \mathfrak{g}(\mathfrak{m}), X \in \mathfrak{m}^{\text{reg}}, c \geq c_0$, define

$$(3.9) \qquad c(\mathfrak{g}, \mathfrak{m}, dx/dm, Y, X) = q^{cd(\mathfrak{m}^\perp)} V(K_c, dx) V(K_c^M, dm)^{-1} c_0(\mathfrak{g}, \mathfrak{m}, Y, X).$$

Suppose that $\mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{m}$ and let A denote the split component of the Cartan subgroup of G corresponding to $\mathfrak{b}$. Fix an invariant measure dx^* on G/A and an invariant measure dm^* on M/A. Then if da is a choice of Haar measure on A, we can normalize Haar measures dx and G and dm on M so that $dx = dx^* da, dm = dm^* da$. In this case we write

$$(3.10) \qquad c(\mathfrak{g}, \mathfrak{m}, dx^*/dm^*, Y, X) = c(\mathfrak{g}, \mathfrak{m}, dx/dm, Y, X), Y \in \mathfrak{g}(\mathfrak{m}), X \in \mathfrak{b}'.$$

Lemma 3.5. *Let $Y \in \mathfrak{g}(\mathfrak{m}), X \in \mathfrak{b}', H \in \mathfrak{m} \cap \mathfrak{g}'$. Then*

(i) *$c(\mathfrak{g}, \mathfrak{m}, dx^*/dm^*, Y, X)\Phi(\mathfrak{m}, dm^*, X, H)$ is independent of the choice of dm^*.*

(ii) *Let $u \in G$ and fix any invariant measure dm_u^* on uMu^{-1}/uAu^{-1}. Then*

$$c(\mathfrak{g}, u\mathfrak{m}, dx^*/dm_u^*, uY, uX)\Phi(u\mathfrak{m}, dm_u^*, uX, uH)$$

$$= c(\mathfrak{g}, \mathfrak{m}, dx^*/dm^*, Y, X)\Phi(\mathfrak{m}, dm^*, X, H).$$

Proof. Part (i) is clear from the definitions. Thus in (ii) we may as well assume that dm_u^* is chosen so that dm^* corresponds to dm_u^* under the map $m \mapsto umu^{-1}$. Then we have

$$\Phi(u\mathfrak{m}, dm_u^*, uX, uH) = \Phi(\mathfrak{m}, dm^*, X, H), X \in \mathfrak{b}', H \in \mathfrak{m}'.$$

Fix $H \in \mathfrak{m} \cap \mathfrak{g}'$, and let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ containing uH. Then $u \in N(\mathfrak{h}, \mathfrak{m})$. It is shown in the last part of the proof of Theorem 2.2 of [2] that for this choice of dm_u^*, $c(\mathfrak{g}, u\mathfrak{m}, dx^*/dm_u^*, uY, uH) = c(\mathfrak{g}, \mathfrak{m}, dx^*/dm^*, Y, H)$ for all $Y \in \mathfrak{g}(m)$. ☐

4. An Expansion at Infinity

In this section we will give the proof of Theorem 1.2. The first step is to prove Proposition 2.1. This will be done in a series of lemmas. Thus we assume through Lemma 4.4 that $\mathfrak{g}$ is semisimple and $\mathfrak{b}$ is elliptic. Let γ be an arbitrary semisimple element of $\mathfrak{g}$, and let $\mathfrak{m} = C_{\mathfrak{g}}(\gamma)$. Then the Cartan subgroup B corresponding to $\mathfrak{b}$ has trivial split component. Fix a normalization dx of invariant measure on G and define $\Phi(\mathfrak{g}, dx, X, H), X \in \mathfrak{b}', H \in \mathfrak{g}'$ as in (1.1). Fix $w \in W_G(\mathfrak{b}, \mathfrak{m})$ and a representative $y_w \in N_G(\mathfrak{b}, \mathfrak{m})$. Then the Cartan subgroup $B_w = y_w^{-1} B y_w$ of M corresponding to $y_w^{-1}\mathfrak{b}$ must also have trivial split component. Fix a normalization dm of invariant measure on M. Then we also have $\Phi(\mathfrak{m}, dm, y_w^{-1}X, H), X \in \mathfrak{b}', H \in \mathfrak{m}'$, as in (1.1). Define $c(\mathfrak{g}, \mathfrak{m}, dx/dm)$ as in (3.9). Since dx and dm are fixed throughout the proof of Proposition 2.1, we drop them from the notation.

Suppose that $\gamma = 0$. Then $\mathfrak{m} = \mathfrak{g}, W_G(\mathfrak{b}, \mathfrak{m}) = \{1\}, c(\mathfrak{g}, \mathfrak{g}, \gamma) \equiv 1$, and Proposition 2.1 is trivial. Thus we may as well assume that $\gamma \neq 0$.

Let ω be a compact subset of $\mathfrak{b}'$, and let $X_0 \in \omega$. Then $C_{\mathfrak{g}}(X_0) = \mathfrak{b}$ is abelian, and so there is an open closed subset ω_0 of $\mathfrak{b}$ with $X_0 \in \omega_0 \subset \omega^B = \omega$ which satisfies the conditions of Corollary 2.3 of [1]. Since ω can be covered by a finite number of sets ω_0, we may as well assume that $\omega = \omega_0$ for some $X_0 \in \omega$. Then $V_0 = \omega_0^G$ is a G-domain (open, closed G-invariant set) in $\mathfrak{g}$ by Corollary 2.4 of [1].

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing γ. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{m}$. Let $\mathfrak{h} = \mathfrak{h}_1, ..., \mathfrak{h}_k$ denote a complete set of representatives for the M-conjugacy classes of Cartan subalgebras of $\mathfrak{m}$. Then we can choose representatives $y_v \in N_G(\mathfrak{h}, \mathfrak{m}) = \{y \in G : y^{-1}\mathfrak{h} \subset \mathfrak{m}\}$ for $W_G(\mathfrak{h}, \mathfrak{m}) = N_G(\mathfrak{h}, \mathfrak{m})/M$ so that for each $v \in W_G(\mathfrak{h}, \mathfrak{m})$, $y_v^{-1}\mathfrak{h} = \mathfrak{h}_j$ for some $1 \leq j \leq k$. We may as well take $y_1 = 1$ as the representative of $1 \in W_G(\mathfrak{h}, \mathfrak{m})$. Also, we can choose representatives $y_w \in N_G(\mathfrak{b}, \mathfrak{m})$ for $W_G(\mathfrak{b}, \mathfrak{m})$ so that for each $w \in W_G(\mathfrak{b}, \mathfrak{m})$, $y_w^{-1}\mathfrak{b} = \mathfrak{h}_j$ for some $1 \leq j \leq k$. These representatives will be fixed throughout the proof of Proposition 2.1.

Since ω_0 is compact and $W_G(\mathfrak{b}, \mathfrak{m})$ is finite, there is $r_0 > 0$ so that $\|\gamma\| \leq q^{r_0}|2|^{1/2}$ and $\|y_w^{-1}X\| \leq q^{r_0}|2|^{1/2}$ for all $w \in W_G(\mathfrak{b}, \mathfrak{m}), X \in \omega_0$. Let $t_0 = \varpi^{r_0}$. Then $\|t_0\gamma\| \leq |2|^{1/2}$ and $\|y_w^{-1}t_0X\| \leq |2|^{1/2}$ for all $w \in W_G(\mathfrak{b}, \mathfrak{m}), X \in \omega_0$. Assume that Proposition 2.1 holds for $\gamma' = t_0\gamma$ and $\omega_0' = \{t_0X : X \in \omega_0\}$. Define $T(\gamma) = q^{-2r_0}T(\gamma')$ and $\omega(\gamma) = t_0^{-1}\omega(\gamma')$. Let $t \in F^\times$ such that $|t| \geq T(\gamma)$, $X \in \omega_0$, and $H \in \omega(\gamma) \cap \mathfrak{g}'$. Then $|tt_0^{-2}| \geq T(\gamma)q^{2r_0} = T(\gamma')$, $t_0X \in \omega_0'$, and $t_0H \in \omega(\gamma')$, so that

$$\Phi(\mathfrak{g}, t_0X, (tt_0^{-2})t_0H) = \sum_{w \in W_G(\mathfrak{b}, \mathfrak{m})} c(\mathfrak{g}, \mathfrak{m}, t_0\gamma, tt_0^{-2}y_w^{-1}t_0X)\ \Phi(\mathfrak{m}, y_w^{-1}t_0X, (tt_0^{-2})t_0H).$$

But it is clear from (1.6) that

$$\Phi(\mathfrak{g}, X, tH) = \Phi(\mathfrak{g}, t_0X, (tt_0^{-2})t_0H),$$
$$\Phi(\mathfrak{m}, y_w^{-1}t_0X, (tt_0^{-2})t_0H) = \Phi(\mathfrak{m}, y_w^{-1}X, tH),$$

and from Lemma 3.4(i) and (3.9) that

$$c(\mathfrak{g}, \mathfrak{m}, t_0\gamma, tt_0^{-2}y_w^{-1}t_0X) = c(\mathfrak{g}, \mathfrak{m}, \gamma, ty_w^{-1}X).$$

Thus we may as well assume that $\|\gamma\| \leq |2|^{1/2}$ and $\|y_w^{-1}X\| \leq |2|^{1/2}$ for all $w \in W_G(\mathfrak{b}, \mathfrak{m}), X \in \omega_0$.

Define

$$\Omega_0 = \{ty_w^{-1}X : X \in \omega_0, w \in W_G(\mathfrak{b}, \mathfrak{m}), t \in F^\times, q^{-1} \leq |t| \leq 1\}. \tag{4.1}$$

Then $\Omega_0 \subset \mathfrak{m} \cap \mathfrak{g}' \subset \mathfrak{m}^{\text{reg}}$ and $\gamma \in \mathfrak{g}(\mathfrak{m}) \subset \mathfrak{m}^{\text{reg}}$, and Ω_0 is compact, so there is $s > 0$ so that $\gamma \in \mathfrak{m}_s^{\text{reg}}$ and $\Omega_0 \subset \mathfrak{m}_s^{\text{reg}}$.

Let ω_γ be a compact open neighborhood of γ in $\mathfrak{m}$ which is small enough such that the following conditions are satisfied. First, since $\mathfrak{m}_s^{\text{reg}}$ is open, we can assume that $\omega_\gamma \subset \mathfrak{m}_s^{\text{reg}}$. Next, since $I(\mathfrak{m}^\perp)$ is a locally constant function on $\mathfrak{m}_s^{\text{reg}} \times \mathfrak{m}_s^{\text{reg}}$ and Ω_0 is compact, and $|\eta_{\mathfrak{g}/\mathfrak{m}}|$ is a locally constant functions on $\mathfrak{m}_s^{\text{reg}}$, we can assume that

$$|\eta_{\mathfrak{g}/\mathfrak{m}}(H)| = |\eta_{\mathfrak{g}/\mathfrak{m}}(\gamma)| \ \text{ and } \ I(\mathfrak{m}^\perp, H, X) = I(\mathfrak{m}^\perp, \gamma, X), \quad H \in \omega_\gamma, X \in \Omega_0. \tag{4.2}$$

Next, since $M = C_G(\gamma)$, the $y_v\gamma$, $v \in W_G(\mathfrak{h}, \mathfrak{m})$, are distinct. Similarly the $y_w\gamma, w \in W_G(\mathfrak{b}, \mathfrak{m})$, are distinct. Thus we can choose ω_γ so that $y_v\omega_\gamma \cap y_{v'}\omega_\gamma \neq \emptyset$ for $v, v' \in W_G(\mathfrak{h}, \mathfrak{m})$ implies that $v = v'$ and $y_w\omega_\gamma \cap y_{w'}\omega_\gamma \neq \emptyset$ for $w, w' \in W_G(\mathfrak{b}, \mathfrak{m})$ implies that $w = w'$.

Fix ω_γ satisfying the above conditions. Since ω_γ and Ω_0 are compact open subsets of $\mathfrak{m}$, there is a compact open subgroup K_M^0 of M small enough such that $K_M^0\omega_\gamma = \omega_\gamma$ and $K_M^0\Omega_0 = \Omega_0$. Now since the sets $y_w\omega_\gamma$ are disjoint and compact, $w \in W_G(\mathfrak{b}, \mathfrak{m})$, we can choose a compact open subgroup K of G which is small enough such that the sets $Ky_w\omega_\gamma$ are disjoint and $y_w^{-1}Ky_w \cap M \subset K_M^0$ for all

$w \in W_G(\mathfrak{b}, \mathfrak{m})$. Fix such a compact open subgroup K, and for $w \in W_G(\mathfrak{b}, \mathfrak{m})$, write $K(w) = y_w^{-1} K y_w$ and $K_M(w) = K(w) \cap M$.

Let U be an M-domain (open, closed, M-invariant set) in $\mathfrak{m}$ such that $\gamma \in U \subset \omega_\gamma^M$ which satisfies the conditions of Corollary 2.3 of [1]. In particular, we can assume that $U \cap \mathfrak{h}_i \subset \omega_\gamma, 1 \leq i \leq k$, $C_\mathfrak{g}(X) \subset \mathfrak{m}$ for all $X \in U$, and for every compact subset Q of $\mathfrak{g}$ there is a compact subset Ω of G such that $xU \cap Q \neq \emptyset$ implies that $x \in \Omega M$. Define $V = U^G$. By Corollary 2.4 of [1], $V \subset \omega_\gamma^G$ and is a G-domain in $\mathfrak{g}$. We will show that $\omega(\gamma) = U \cap \mathfrak{h}$ satisfies the condition of Proposition 2.1.

Define $V(K) = \{kyY : k \in K, y \in N_G(\mathfrak{b}, \mathfrak{m}), Y \in U\}$.

Lemma 4.1. (i) *The double cosets $Ky_wM, w \in W_G(\mathfrak{b}, \mathfrak{m})$, are disjoint.*
(ii) *For all $w \in W_G(\mathfrak{b}, \mathfrak{m}), k \in K(w)$, $U \cap ky_w^{-1}\mathfrak{b} \subset \omega_\gamma$.*
(iii) *For all $w \in W_G(\mathfrak{b}, \mathfrak{m}), k \in K(w)$, $ky_w^{-1}\mathfrak{b}' \cap \mathfrak{m} \neq \emptyset$ if and only if $k \in K_M(w)$.*
(iv) *Let $x \in G$ such that $xH \in V(K)$ for some $H \in \omega(\gamma) \cap \mathfrak{h}'$. Then $x \in KN_G(\mathfrak{b}, \mathfrak{m})$.*

Proof. (i) Suppose that $x \in Ky_wM \cap Ky_{w'}M, w, w' \in W_G(\mathfrak{b}, \mathfrak{m})$. Then there are $k, k' \in K, m, m' \in M$ such that $x = ky_wm = k'y_{w'}m'$. Now $x\gamma = ky_w\gamma = k'y_{w'}\gamma \in Ky_w\omega_\gamma \cap Ky_{w'}\omega_\gamma$. Thus by assumption on K, $w = w'$.

(ii) Let $w \in W_G(\mathfrak{b}, \mathfrak{m}), k \in K(w)$ and $Y \in U \cap ky_w^{-1}\mathfrak{b}$. Then $ky_w^{-1}\mathfrak{b} \subset C_\mathfrak{g}(Y) \subset \mathfrak{m}$ so that $y_wk^{-1} \in N_G(\mathfrak{b}, \mathfrak{m})$. Thus there are $w' \in W, m \in M$, such that $y_wk^{-1} = k_1y_w = y_{w'}m$ where $k_1 = y_wk^{-1}y_w^{-1} \in K$. Thus $y_w \in Ky_{w'}M \cap Ky_wM$. Now by (i), $w = w'$ so that $k = m^{-1} \in M \cap K(w) \subset K_M^0$. Now $k^{-1}Y \in U \cap y_w^{-1}\mathfrak{b} \subset \omega_\gamma$ so that $Y \in k\omega_\gamma = \omega_\gamma$ since $k \in K_M^0$.

(iii) Let $w \in W_G(\mathfrak{b}, \mathfrak{m}), k \in K(w)$. Then $ky_w^{-1}\mathfrak{b}' \cap \mathfrak{m} \neq \emptyset$ if and only if there is $X \in \mathfrak{b}'$ such that $ky_w^{-1}X \in \mathfrak{m}$ if and only if $ky_w^{-1}\mathfrak{b} \subset \mathfrak{m}$ if and only if $y_wk^{-1} \in N_G(\mathfrak{b}, \mathfrak{m})$. But as in the proof of (ii), $y_wk^{-1} \in N_G(\mathfrak{b}, \mathfrak{m})$ implies that $k \in M \cap K(w) = K_M(w)$. Conversely, if $k \in K_M(w)$, then $y_wk^{-1} \in y_wM \subset N_G(\mathfrak{b}, \mathfrak{m})$.

(iv) Let $x \in G$ and $H \in \omega(\gamma) \cap \mathfrak{h}'$ such that $xH \in V(K)$. Then there are $k \in K$ and $w \in W_G(\mathfrak{b}, \mathfrak{m})$ such that $xH \in ky_wU$, so that $y_w^{-1}k^{-1}xH \subset U \subset \mathfrak{m}$. Since $H \in \mathfrak{h}'$, this implies that $y_w^{-1}k^{-1}x\mathfrak{h} \subset \mathfrak{m}$, so that $x^{-1}ky_w \in N_G(\mathfrak{h}, \mathfrak{m})$. Thus there are $v \in W_G(\mathfrak{h}, \mathfrak{m})$ and $m \in M$ such that $x^{-1}ky_w = y_vm$. Now $y_w^{-1}k^{-1}xH = m^{-1}y_v^{-1}H \in U$ implies that $y_v^{-1}H \in mU = U$. But there is $\mathfrak{h}_i, 1 \leq i \leq k$, so that $y_v^{-1}\mathfrak{h} = \mathfrak{h}_i$. Thus $y_v^{-1}H \in U \cap \mathfrak{h}_i \subset \omega_\gamma$. But $\omega(\gamma) = U \cap \mathfrak{h} = U \cap \mathfrak{h}_1 \subset \omega_\gamma$. Thus $H \in \omega_\gamma \cap y_v\omega_\gamma$. Now since $\omega_\gamma \cap y_v\omega_\gamma \neq \emptyset$, we have $y_v = 1$. Thus $x^{-1}ky_w = m$ so that $x = ky_wm^{-1} \in KN_G(\mathfrak{b}, \mathfrak{m})$. □

From now on we write $W = W_G(\mathfrak{b}, \mathfrak{m})$. Define $\eta_{\mathfrak{g}/\mathfrak{m}}$ and $c(\mathfrak{g}, \mathfrak{m})$ as in Lemma 3.4 and (3.9).

Lemma 4.2. *There is $T \geq 1$ with the following properties.*

(i) *For all $X \in \omega_0, Y \in V, |t| \geq T$,*

$$\int_K \psi(B(tY, kX))dk = 0$$

unless $Y \in V(K)$.

(ii) *For all* $X \in \omega_0, w \in W, Y \in U, |t| \geq T$,

$$|\eta_{\mathfrak{g}/\mathfrak{m}}(tY)|^{1/2}|\eta_{\mathfrak{g}/\mathfrak{m}}(y_w^{-1}X))|^{1/2}\int_K \psi(B(ty_wY, kX))dk$$
$$= V(K,dx)^{-1}V(K_M(w),dm)c(\mathfrak{g},\mathfrak{m},\gamma,ty_w^{-1}X)\int_{K_M(w)}\psi(tB(Y,k_1y_w^{-1}X))dk_1,$$

where $K_M(w) = M \cap y_w^{-1}Ky_w$, dk_1 *is normalized Haar measure on* $K_M(w)$, $V(K,dx)$, *is the volume of* K *with respect to* dx, *and* $V(K_M(w),dm)$ *is the volume of* $K_M(w)$ *with respect to* dm.

Proof. By Lemma 5.4 of [2] there is $T_1 \geq 1$ so that for all $X \in \omega_0, Y \in V, |t| \geq T_1$,

$$\int_K \psi(B(tY,kX))dk = 0$$

unless $Y \in V(K)$. Thus (i) will hold for any $T \geq T_1$.

Fix $w \in W$. Then for all $X \in \omega_0, Y \in U, t \in F$,

$$\int_K \psi(B(ty_wY,kX))dk = \int_{K(w)}\psi(tB(Y,k'y_w^{-1}X))dk'$$

where dk' is normalized Haar measure on $K(w) = y_w^{-1}Ky_w$.

For $X \neq 0 \in \mathfrak{g}$, define the integer $\nu(X)$ so that $\|X\| = |\varpi^{\nu(X)}|$. Let $S = \{X \in \mathfrak{g} : \|X\| = 1\}$. Then for all $X \neq 0 \in \mathfrak{g}, \varpi^{-\nu(X)}X \in S$.

Let $U_1 = \{Y \in U : Y \notin \omega_\gamma\}$, and let S_1 denote the closure in S of

$$\{\varpi^{-\nu(Y)}Y : Y \in U_1\}.$$

It is a compact set. Now $U \subset \omega_\gamma^M$ where ω_γ is compact, so the eigenvalues of $\operatorname{ad} X, X \in U$ are bounded. Since U_1 is a closed subset of $\mathfrak{m}$, as in Lemma 7.4 of [1], every element of S_1 is either nilpotent or is of the form $\varpi^{-\nu(Y)}Y$ for some $Y \in U_1$.

Let $Y' \in S_1, X \in \omega_0$, and suppose that $[ky_w^{-1}X, Y'] = 0$ for some $k \in K(w)$. Then $k^{-1}Y' \in y_w^{-1}\mathfrak{b}$, so that Y' is semisimple, and hence of the form $Y' = \varpi^{-\nu(Y)}Y$ for some $Y \in U_1$. But then $k^{-1}Y \in y_w^{-1}\mathfrak{b}$ so that $Y \in U \cap ky_w^{-1}\mathfrak{b}$. By Lemma 4.1 (ii), this implies that $Y \in \omega_\gamma$. This contradicts the assumption that $Y \in U_1$. Thus $[ky_w^{-1}X, Y'] \neq 0$ for all $X \in \omega_0, Y' \in S_1, k \in K(w)$, so by Lemma 3.1 of [2] there is T_2' such that

$$\int_{K(w)}\psi(tB(Y',ky_w^{-1}X))dk = 0$$

for all $X \in \omega_0, Y' \in S_1, |t| \geq T_2'$.

Since $\gamma \neq 0$ and $\mathfrak{g}$ is semisimple, $\mathfrak{m} \neq \mathfrak{g}$. Now since for all $Y \in U, C_{\mathfrak{g}}(Y) \subset \mathfrak{m}$, we have $0 \notin U$. Since U is closed, there is $\delta > 0$ so that $\|Y\| \geq \delta$ for all $Y \in U$. Define $T_2 = T_2'\delta^{-1}$. Then for all $|t| \geq T_2, Y \in U_1, X \in \omega_0$,

$$\int_{K(w)}\psi(tB(Y,ky_w^{-1}X))dk = \int_{K(w)}\psi(t\varpi^{\nu(Y)}B(\varpi^{-\nu(Y)}Y,ky_w^{-1}X))dk = 0$$

since $|t\varpi^{\nu(Y)}| = |t|\ \|Y\| \geq T_2'$ and $\varpi^{-\nu(Y)}Y \in S_1$.

Using the same argument as above with $K_M(w)$ in place of $K(w)$, we can also prove that there is $T_3 > 0$ so that for all $|t| \geq T_3, Y \in U_1, X \in \omega_0$,

$$\int_{K_M(w)}\psi(tB(Y,k_1y_w^{-1}X))dk_1 = 0.$$

Thus as long as $T \geq T_1(w) = \max\{T_2, T_3\}$ and $Y \in U_1$,

$$|\eta_{\mathfrak{g}/\mathfrak{m}}(tY)|^{1/2}|\eta_{\mathfrak{g}/\mathfrak{m}}(y_w^{-1}X)|^{1/2}\int_K \psi(B(ty_wY, kX))dk = 0$$
$$= V(G,dx)^{-1}V(K_M(w),dm)c(\mathfrak{g},\mathfrak{m},\gamma,ty_w^{-1}X)\int_{K_M(w)}\psi(tB(Y,k_1y_w^{-1}X))dk_1$$

for all $|t| \geq T, X \in \omega_0$.

Define c' as in Lemma 3.2, and pick $c \geq c'$ large enough such that $K_c \subset K(w)$ and $q^{-c} < q^{-4s-c_0}C_0^2$. Let $k_1, ..., k_d \in K(w)$ denote a complete set of coset representatives for $K_c\backslash K(w)$. Since $V(K(w), dx) = V(K, dx)$, the volume of K_c with respect to normalized Haar measure on $K(w)$ is $V_1 = V(K, dx)^{-1}V(K_c, dx)$. Thus for all $Y \in \omega_\gamma, X \in \omega_0$, and $t \in F^\times$,

$$\int_{K(w)}\psi(tB(Y,ky_w^{-1}X))dk = V_1\sum_{i=1}^d \phi_c(tY, k_iy_w^{-1}X).$$

Define $I_M = \{1 \leq i \leq d : K_ck_i \cap K_M(w) \neq \emptyset\}, I'_M = \{1 \leq i \leq d : i \notin I_M\}$. For $i \in I_M$, we may as well assume that the coset representative k_i is chosen so that $k_i \in K_M(w)$. Now since $c \geq c'$, by Lemma 3.2 $K_c \cap M = K_c^M$. Thus $K_M(w) = \bigcup_{i\in I_M} K_c^M k_i$, so that for all $Y \in \omega_\gamma, X \in \omega_0$, and $t \in F^\times$,

$$\int_{K_M(w)}\psi(tB(Y,k_1y_w^{-1}X)dk_1 = V_2\sum_{i\in I_M}\phi_c^M(tY, k_iy_w^{-1}X),$$

where $V_2 = V(K_M(w), dm)^{-1}V(K_c^M, dm)$. Further, by Lemma 4.1 (iii),

$$K_ck_iy_w^{-1}\mathfrak{b}' \cap \mathfrak{m} \neq \emptyset$$

if and only if there is $k \in K_c$ such that $kk_i \in K_M(w)$. Thus

$$I_M = \{1 \leq i \leq d : K_ck_iy_w^{-1}\mathfrak{b}' \cap \mathfrak{m} \neq \emptyset\}.$$

Let $1 \leq i \leq d$ and suppose there are $Y \in \omega_\gamma, X \in \omega_0$, and $k \in K_c$ such that $[Y, kk_iy_w^{-1}X] = 0$. Then $kk_iy_w^{-1}X \in \mathfrak{m}\cap K_ck_iy_w^{-1}\mathfrak{b}'$ so that $i \in I_M$. Thus for $i \in I'_M$, for all $Y \in \omega_\gamma, X \in \omega_0$, and $k \in K_c$, $[Y, kk_iy_w^{-1}X] \neq 0$, so that by Lemma 3.1 of [2] there is $T(i) > 0$ so that $\phi_c(tY, k_iy_w^{-1}X) = 0$ for all $Y \in \omega_\gamma, X \in \omega_0, |t| \geq T(i)$. Pick $T'_w = \max\{T(i) : i \in I'_M\}$.

Now suppose that $i \in I_M$, so that $k_i \in K_M(w) \subset K_M^0$. Let $X_0 \in \omega_0, Y \in \omega_\gamma$, and $t \in F^\times, |t| \geq q^{4s+2c}C_0^{-2}$. Then $t = t_1t_0^2$ for some $t_1, t_0 \in F^\times$ such that $q^{-1} \leq |t_1| \leq 1$ and $|t_0| \geq q^{2s+c}C_0^{-1}$. Now $Y \in \omega_\gamma \subset \mathfrak{m}_s^{\rm reg}$ and $t_1k_iy_w^{-1}X \in K_M^0\Omega_0 = \Omega_0 \subset \mathfrak{m}_s^{\rm reg}$. Thus by Lemma 3.3 and (4.2),

$$\begin{aligned}\phi_c(tY, k_iy_w^{-1}X) &= \phi_c(t_0^2Y, t_1k_iy_w^{-1}X)\\ &= q^{cd(\mathfrak{m}^\perp)}|t_0|^{-d(\mathfrak{m}^\perp)}\ \phi_c^M(t_0^2Y, t_1k_iy_w^{-1}X)\ I(\mathfrak{m}^\perp, Y, t_1k_iy_w^{-1}X)\\ &= q^{cd(\mathfrak{m}^\perp)}|t_0|^{-d(\mathfrak{m}^\perp)}\ \phi_c^M(tY, k_iy_w^{-1}X)\ I(\mathfrak{m}^\perp, \gamma, t_1k_iy_w^{-1}X).\end{aligned}$$

But for all $X \in \mathfrak{m}^{\rm reg}, t \in F^\times$,

$$|\eta_{\mathfrak{g}/\mathfrak{m}}(tX)| = |t|^{d(\mathfrak{m}^\perp)}|\eta_{\mathfrak{g}/\mathfrak{m}}(X)|.$$

Thus using Lemma 3.4 and (4.2),

$$\begin{aligned}&|t_0|^{-d(\mathfrak{m}^\perp)} I(\mathfrak{m}^\perp, \gamma, t_1 k_i y_w^{-1} X)\\ &\quad = |t_0|^{-d(\mathfrak{m}^\perp)} |\eta_{\mathfrak{g}/\mathfrak{m}}(\gamma)|^{-1/2} |\eta_{\mathfrak{g}/\mathfrak{m}}(t_1 k_i y_w^{-1} X)|^{-1/2} c_0(\mathfrak{g}, \mathfrak{m}, \gamma, t_1 k_i y_w^{-1} X)\\ &\quad = |t_0|^{-d(\mathfrak{m}^\perp)} |\eta_{\mathfrak{g}/\mathfrak{m}}(Y)|^{-1/2} |\eta_{\mathfrak{g}/\mathfrak{m}}(t_1 y_w^{-1} X)|^{-1/2} c_0(\mathfrak{g}, \mathfrak{m}, \gamma, t_1 y_w^{-1} X)\\ &\quad = |\eta_{\mathfrak{g}/\mathfrak{m}}(tY)|^{-1/2} |\eta_{\mathfrak{g}/\mathfrak{m}}(y_w^{-1} X)|^{-1/2} c_0(\mathfrak{g}, \mathfrak{m}, \gamma, t y_w^{-1} X).\end{aligned}$$

Thus

$$\begin{aligned}&\phi_c(tY, k_i y_w^{-1} X)\\ &\quad = q^{cd(\mathfrak{m}^\perp)} \phi_c^M(tY, k_i y_w^{-1} X) |\eta_{\mathfrak{g}/\mathfrak{m}}(tY)|^{-1/2} |\eta_{\mathfrak{g}/\mathfrak{m}}(y_w^{-1} X)|^{-1/2} c_0(\mathfrak{g}, \mathfrak{m}, \gamma, t y_w^{-1} X).\end{aligned}$$

Let $T_2(w) = \max\{T'_w, q^{4s+2c} C_0^{-2}\}$, and let $Y \in \omega_\gamma, X \in \omega_0, t \in F^\times, |t| \geq T_2(w)$. Then

$$\begin{aligned}&|\eta_{\mathfrak{g}/\mathfrak{m}}(tY)|^{1/2} |\eta_{\mathfrak{g}/\mathfrak{m}}(y_w^{-1} X)|^{1/2} \int_{K(w)} \psi(tB(Y, k y_w^{-1} X)) dk\\ &\quad = V_1 \sum_{i \in I_M} |\eta_{\mathfrak{g}/\mathfrak{m}}(tY)|^{1/2} |\eta_{\mathfrak{g}/\mathfrak{m}}(y_w^{-1} X)|^{1/2} \phi_c(tY, k_i y_w^{-1} X)\\ &\quad = V_1 q^{cd(\mathfrak{m}^\perp)} c_0(\mathfrak{g}, \mathfrak{m}, \gamma, t y_w^{-1} X) \sum_{i \in I_M} \phi_c^M(tY, k_i y_w^{-1} X)\\ &\quad = q^{cd(\mathfrak{m}^\perp)} V_1 V_2^{-1} c_0(\mathfrak{g}, \mathfrak{m}, \gamma, t y_w^{-1} X) \int_{K_M(w)} \psi(tB(Y, k_1 y_w^{-1} X)) dk_1.\end{aligned}$$

But using (3.9),

$$q^{cd(\mathfrak{m}^\perp)} V_1 V_2^{-1} c_0(\mathfrak{g}, \mathfrak{m}, \gamma, t y_w^{-1} X) = V(K, dx)^{-1} V(K_M(w), dm) c(\mathfrak{g}, \mathfrak{m}, \gamma, t y_w^{-1} X).$$

Thus the lemma is valid for $T = \max\{T_1, T_1(w), T_2(w) : w \in W\}$. □

Lemma 4.3. *Fix $H \in \omega(\gamma) \cap \mathfrak{g}'$. Then there is a compact open subset G_H of G satisfying the following conditions.*

(i) *For all $X \in \omega_0, |t| \geq 1$,*

$$\Phi(\mathfrak{g}, X, tH) = |\eta_{\mathfrak{g}}(X)|^{1/2} |\eta_{\mathfrak{g}}(tH)|^{1/2} \int_{G_H} \int_K \psi(tB(x^{-1}H, kX)) dk dx.$$

(ii) *For each $w \in W_G(\mathfrak{b}, \mathfrak{m})$, define $M_H(w) = M \cap G_H y_w$. Then for all $X \in \omega_0$, $|t| \geq 1$,*

$$\begin{aligned}&\Phi(\mathfrak{m}, y_w^{-1} X, tH)\\ &\quad = |\eta_{\mathfrak{m}}(y_w^{-1} X)|^{1/2} |\eta_{\mathfrak{m}}(tH)|^{1/2} \int_{M_H(w)} \int_{K_M(w)} \psi(tB(m^{-1}H, k_1 y_w^{-1} X)) dk_1 dm.\end{aligned}$$

Proof. Let $V_0 = \omega_0^G$. Since $\{H\}$ is a compact subset of $\mathfrak{h}'$, by Lemma 5.4 of [2] there is $C > 0$ so that

$$\int_K \psi(tB(kH, Y)) dk = 0$$

for all $Y \in V_0, |t| \geq 1$ unless $\|Y\| \leq C$. Fix $w \in W_G(\mathfrak{b}, \mathfrak{m})$ and let $V_M = (y_w^{-1} \omega_0)^M$. Applying Lemma 5.4 of [2] to $\mathfrak{m}$ and $K_M = K \cap M$ there is $C_w > 0$ so that

$$\int_{K_M} \psi(tB(kH, Y)) dk = 0$$

for all $Y \in V_M, |t| \geq 1$ unless $\|Y\| \leq C_w$. Let $C_H = \max\{C, C_w : w \in W_G(\mathfrak{b}, \mathfrak{m})\}$.

Let $Q = \{Y \in V_0 : \|Y\| \leq C_H\}$. It is a compact subset of G, so that there is a compact subset Ω of G such that $x\omega_0 \cap Q \neq \emptyset$ implies that $x \in \Omega$. Let $G_H = K\Omega K$. It is a compact open subset of G satisfying $G_H = KG_HK$.

Let $X \in \omega_0, |t| \geq 1$. Then

$$\Phi(\mathfrak{g}, X, tH) = |\eta_{\mathfrak{g}}(X)|^{1/2} |\eta_{\mathfrak{g}}(tH)|^{1/2} \int_G \int_K \psi(tB(kH, xX)) dk dx.$$

Let $x \in G$ and suppose $\|xX\| \leq C_H$. Then $xX \in V_0 \cap Q$ so that $x \in \Omega \subset G_H$. Thus for $x \notin G_H, \|xX\| > C_H$, so that $\int_K \psi(tB(kH, xX))dk = 0$. Thus

$$\int_G \int_K \psi(tB(kH, xX)) dk dx = \int_{G_H} \int_K \psi(tB(kH, xX)) dk dx.$$

But since G_H is compact and K bi-invariant, we have

$$\begin{aligned} &\int_{G_H} \int_K \psi(tB(kH, xX)) dk dx \\ &\quad = \int_K \int_{G_H} \int_K \psi(tB(kH, xk_1X)) dk dx dk_1 \\ &\quad = \int_K \int_{G_H} \int_K \psi(tB(x^{-1}kH, k_1X)) dk_1 dx dk \\ &\quad = \int_{G_H} \int_K \psi(tB(x^{-1}H, k_1X)) dk_1 dx. \end{aligned}$$

Fix $w \in W_G(\mathfrak{b}, \mathfrak{m})$. Let $X \in \omega_0, |t| \geq 1$. Then

$$\Phi(\mathfrak{m}, y_w^{-1}X, tH) = |\eta_{\mathfrak{m}}(y_w^{-1}X)|^{1/2} |\eta_{\mathfrak{m}}(tH)|^{1/2} \int_M \int_{K_M} \psi(tB(kH, my_w^{-1}X)) dk dm.$$

Let $m \in M$ and suppose $\|my_w^{-1}X\| \leq C_H$. Then $my_w^{-1}X \in V_0 \cap Q$ so that $my_w^{-1} \in G_H$. Thus $m \in M \cap G_H y_w = M_H(w)$. Thus we have

$$\int_M \int_{K_M} \psi(tB(kH, my_w^{-1}X)) dk dm = \int_{M_H(w)} \int_{K_M} \psi(tB(kH, my_w^{-1}X)) dk dm.$$

Let $m \in M, k \in K_M = K \cap M, k_1 \in K_M(w) = M \cap y_w^{-1}Ky_w$. Then

$$k^{-1}G_H y_w k_1^{-1} = k^{-1}G_H(y_w k_1 y_w^{-1})^{-1} y_w = G_H y_w$$

since $k,\ y_w k_1 y_w^{-1} \in K$. Thus $kmk_1 \in M_H(w)$ if and only if $kmk_1 \in G_H y_w$ if and only if $m \in k^{-1}G_H y_w k_1^{-1} = G_H y_w$ if and only if $m \in M_H(w)$. Thus as above we can write

$$\begin{aligned} &\int_{M_H(w)} \int_{K_M} \psi(tB(kH, my_w^{-1}X)) dk dm \\ &\quad = \int_{K_M(w)} \int_{M_H(w)} \int_{K_M} \psi(tB(kH, mk_1y_w^{-1}X)) dk dm dk_1 \\ &\quad = \int_{K_M} \int_{M_H(w)} \int_{K_M(w)} \psi(tB(m^{-1}kH, k_1y_w^{-1}X)) dk_1 dm dk \\ &\quad = \int_{M_H(w)} \int_{K_M(w)} \psi(tB(m^{-1}H, k_1y_w^{-1}X)) dk_1 dm. \end{aligned}$$

□

The following lemma completes the proof of Proposition 2.1. Define $T(\gamma) = T$ as in Lemma 4.2.

Lemma 4.4. *For all* $X \in \omega_0, H \in \omega(\gamma) \cap \mathfrak{h}', |t| \geq T$,

$$\Phi(\mathfrak{g}, X, tH) = \sum_{w \in W_G(\mathfrak{b},\mathfrak{m})} c(\mathfrak{g}, \mathfrak{m}, \gamma, ty_w^{-1}X)\Phi(\mathfrak{m}, y_w^{-1}X, tH).$$

Proof. Fix $X \in \omega_0, H \in \omega(\gamma) \cap \mathfrak{h}', |t| \geq T$. Then by Lemma 4.3, since $|t| \geq 1$,

$$\Phi(\mathfrak{g}, X, tH) = |\eta_{\mathfrak{g}}(X)|^{1/2}|\eta_{\mathfrak{g}}(tH)|^{1/2} \int_{G_H} \int_K \psi(tB(x^{-1}H, kX))dkdx.$$

Let $x \in G$. Then by Lemma 4.2, since $|t| \geq T$ and $x^{-1}H \in V$,

$$\int_K \psi(tB(x^{-1}H, kX))dk = 0$$

unless $x^{-1}H \in V(K)$. Now by Lemma 4.1 (iv), this implies that $x^{-1} \in Ky_wM$ for some $w \in W = W_G(\mathfrak{b}, \mathfrak{m})$. Write $x = my_w^{-1}k$ for $m \in M, k \in K$. Then $x \in G_H$ if and only if $my_w^{-1} \in G_H$ if and only if $m \in G_Hy_w \cap M = M_H(w)$. Finally, by Lemma 4.1 (i) the cosets $Ky_wM, w \in W$, are disjoint, so that

$$\begin{aligned}
&\int_{G_H} \int_K \psi(tB(x^{-1}H, kX))dkdx \\
&= \sum_{w \in W} V(K, dx)V(K_M(w), dm)^{-1} \int_K \int_{M_H(w)} \int_K \psi(tB(k_1^{-1}y_wm^{-1}H, kX))dkdmdk_1 \\
&= \sum_{w \in W} V(K, dx)V(K_M(w), dm)^{-1} \int_{M_H(w)} \int_K \psi(tB(y_wm^{-1}H, kX))dkdm.
\end{aligned}$$

Fix $w \in W, m \in M_H(w)$. Then since $m^{-1}H \in U$ and $|t| \geq T$, using Lemma 4.2,

$$\begin{aligned}
V(K, dx)V(K_M(w), dm)^{-1} \int_K \psi(tB(y_wm^{-1}H, kX))dk &= |\eta_{\mathfrak{g}/\mathfrak{m}}(tm^{-1}H)|^{-1/2} \\
\times |\eta_{\mathfrak{g}/\mathfrak{m}}(y_w^{-1}X)|^{-1/2}c(\mathfrak{g}, \mathfrak{m}, \gamma, ty_w^{-1}X) &\int_{K_M(w)} \psi(tB(m^{-1}H, k_1y_w^{-1}X))dk_1.
\end{aligned}$$

But

$$\begin{aligned}
&|\eta_{\mathfrak{g}}(X)|^{1/2}|\eta_{\mathfrak{g}}(tH)|^{1/2}|\eta_{\mathfrak{g}/\mathfrak{m}}(tm^{-1}H)|^{-1/2}|\eta_{\mathfrak{g}/\mathfrak{m}}(y_w^{-1}X)|^{-1/2} \\
&\quad = |\eta_{\mathfrak{m}}(y_w^{-1}X)|^{1/2}|\eta_{\mathfrak{m}}(tH)|^{1/2}.
\end{aligned}$$

Thus using Lemma 4.3,

$$\begin{aligned}
\Phi(\mathfrak{g}, X, tH) &= \sum_{w \in W} c(\mathfrak{g}, \mathfrak{m}, \gamma, ty_w^{-1}X) \\
&\times |\eta_{\mathfrak{m}}(y_w^{-1}X)|^{1/2}|\eta_{\mathfrak{m}}(tH)|^{1/2} \int_{M_H(w)} \int_{K_M(w)} \psi(tB(m^{-1}H, k_1y_w^{-1}X))dk_1dm \\
&= \sum_{w \in W} c(\mathfrak{g}, \mathfrak{m}, \gamma, ty_w^{-1}X)\Phi(\mathfrak{m}, y_w^{-1}X, tH).
\end{aligned}$$

□

We now keep the assumption that $\mathfrak{b}$ is elliptic, but remove the assumption that $\mathfrak{g}$ is semisimple. Let Z denote the split component of the center of G. It is also the split component of the Cartan subgroup of G corresponding to $\mathfrak{b}$. Let dx^* and dm^* be choices of Haar measures on G/Z and M/Z respectively, and define $c(\mathfrak{g}, \mathfrak{m}, dx^*/dm^*)$ as in (3.10).

Lemma 4.5. *Let ω be a compact subset of $\mathfrak{b}'$, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with $\gamma \in \mathfrak{h}$. Then there exist a neighborhood $\omega(\gamma)$ of γ in $\mathfrak{h}$ and $T(\gamma) > 0$ so that for all $X \in \omega, H \in \omega(\gamma) \cap \mathfrak{h}'$, and $t \in F, |t| \geq T(\gamma)$,*

$$\Phi(\mathfrak{g}, dx^*, X, tH) = \sum_{w \in W_G(\mathfrak{b},\mathfrak{m})} c(\mathfrak{g}, \mathfrak{m}, dx^*/dm^*, \gamma, ty_w^{-1}X)\ \Phi(\mathfrak{m}, dm^*, y_w^{-1}X, tH).$$

Proof. Write $\mathfrak{g} = \mathfrak{z}+\mathfrak{g}_1, \mathfrak{b} = \mathfrak{b}_1+\mathfrak{z}, \mathfrak{h} = \mathfrak{h}_1+\mathfrak{z}$, where $\mathfrak{g}_1$ is semisimple, $\mathfrak{b}_1$ is an elliptic Cartan subalgebra of $\mathfrak{g}_1$, and $\mathfrak{h}_1$ is an arbitrary Cartan subalgebra of $\mathfrak{g}_1$. Write $\gamma = Z_0+\gamma_1, Z_0 \in \mathfrak{z}, \gamma_1 \in \mathfrak{h}_1$. Then $\mathfrak{m} = C_{\mathfrak{g}}(\gamma) = \mathfrak{z}+\mathfrak{m}_1$ where $\mathfrak{m}_1 = C_{\mathfrak{g}_1}(\gamma_1)$. We can identify $G_1 = G/Z$ and $M_1 = M/Z$. Let dx_1 and dm_1 be the Haar measures on G_1 and M_1 corresponding to dx^* and dm^* respectively with these identifications. Then for all $Z_1, Z_2 \in \mathfrak{z}, X_1 \in \mathfrak{b}_1', H_1 \in \mathfrak{h}_1', w \in W = W_G(\mathfrak{b},\mathfrak{m}) = W_1 = W_{G_1}(\mathfrak{b}_1, \mathfrak{m}_1)$,

$$\Phi(\mathfrak{g}, dx^*, Z_1 + X_1, Z_2 + H_1) = \psi(B(Z_1, Z_2))\Phi(\mathfrak{g}_1, dx_1, X_1, H_1),$$
$$\Phi(\mathfrak{m}, dm^*, y_w^{-1}(Z_1 + X_1), Z_2 + H_1) = \psi(B(Z_1, Z_2))\Phi(\mathfrak{m}_1, dm_1, y_w^{-1}X_1, H_1),$$
$$c(\mathfrak{g}, \mathfrak{m}, dx^*/dm^*, \gamma, y_w^{-1}(Z_1 + X_1)) = c(\mathfrak{g}_1, \mathfrak{m}_1, dx_1/dm_1, \gamma_1, y_w^{-1}X_1).$$

By Proposition 2.1 there are a neighborhood $\omega_1(\gamma_1)$ in $\mathfrak{h}_1$ and $T(\gamma_1) > 0$ so that for all $X_1 \in \omega_1, H_1 \in \omega_1(\gamma_1) \cap \mathfrak{h}_1', |t| \geq T(\gamma_1)$,

$$\Phi(\mathfrak{g}_1, dx_1, X_1, tH_1) = \sum_{w \in W_1} c(\mathfrak{g}_1, \mathfrak{m}_1, dx_1/dm_1, \gamma_1, ty_w^{-1}X_1)\Phi(\mathfrak{m}_1, dm_1, y_w^{-1}X_1, tH_1).$$

Then for all $Z_1, Z_2 \in \mathfrak{z}, X_1 \in \omega_1, H_1 \in \omega_1(\gamma_1) \cap \mathfrak{h}_1', |t| \geq T(\gamma_1)$,

$$\begin{aligned}&\Phi(\mathfrak{g}, dx^*, Z_1 + X_1, t(Z_2 + H_1)) = \psi(B(Z_1, tZ_2))\Phi(\mathfrak{g}_1, dx_1, X_1, tH_1)\\ &= \psi(B(Z_1, tZ_2)) \sum_{w \in W_1} c(\mathfrak{g}_1, \mathfrak{m}_1, dx_1/dm_1, \gamma_1, ty_w^{-1}X_1)\Phi(\mathfrak{m}_1, dx_1, y_w^{-1}X_1, tH_1)\\ &= \sum_{w \in W} c(\mathfrak{g}, \mathfrak{m}, dx^*/dm^*, \gamma, ty_w^{-1}(Z_1 + X_1))\Phi(\mathfrak{m}, dm^*, y_w^{-1}(Z_1 + X_1), t(Z_2 + H_1)).\end{aligned}$$

Thus we can take $\omega(\gamma) = \mathfrak{z} + \omega_1(\gamma_1)$ and $T(\gamma) = T(\gamma_1)$. □

Suppose now that $\mathfrak{b}$ is an arbitrary Cartan subalgebra of $\mathfrak{g}$. Let A be the split component of B, and fix an invariant measure dx^* on G/A. Let $G_{\mathfrak{b}}$ denote the centralizer in G of A. Normalize the invariant measure $dx^*_{\mathfrak{b}}$ on $G_{\mathfrak{b}}/A$ so that in the notation of Lemma 2.2 we have

$$\Phi(\mathfrak{g}, dx^*, X, H) = \sum_{s \in W_G(\mathfrak{h},\mathfrak{g}_{\mathfrak{b}})} \Phi(\mathfrak{g}_{\mathfrak{b}}, dx^*_{\mathfrak{b}}, X, y_s^{-1}H), X \in \mathfrak{b}', H \in \mathfrak{h}'. \tag{4.3}$$

Fix $w \in W_G(\mathfrak{b},\mathfrak{m})$ and a representative $y_w \in N_G(\mathfrak{b},\mathfrak{m})$. Then $A_w = y_w^{-1}Ay_w$ is the split component of the Cartan subgroup $y_w^{-1}By_w$ of M. Fix an invariant measure dm^*_w on M/A_w. Now the centralizer in M of A_w is $M_{w,\mathfrak{b}} = M \cap y_w^{-1}G_{\mathfrak{b}}y_w$. For each $u \in W_M(\mathfrak{h}, \mathfrak{m}_{w,\mathfrak{b}})$, let $y_u \in N_M(\mathfrak{h}, \mathfrak{m}_{w,\mathfrak{b}})$ be a representative for u. Then by

Lemma 2.2 applied to $\mathfrak{m}$ and $\mathfrak{m}_{w,\mathfrak{b}}$, we can normalize the invariant measure $dm^*_{w,\mathfrak{b}}$ on $(M_{w,\mathfrak{b}})/A_w$ so that for all $X \in \mathfrak{b}', H \in \mathfrak{h}'$ we have

$$(4.4)\qquad \Phi(\mathfrak{m}, dm^*_w, y_w^{-1}X, H) = \sum_{u\in W_M(\mathfrak{h},\mathfrak{m}_{w,\mathfrak{b}})} \Phi(\mathfrak{m}_{w,\mathfrak{b}}, dm^*_{w,\mathfrak{b}}, y_w^{-1}X, y_u^{-1}H).$$

Now $y_w\gamma \in \mathfrak{g}_\mathfrak{b}$ and $C_{\mathfrak{g}_\mathfrak{b}}(y_w\gamma) = \mathfrak{g}_\mathfrak{b} \cap y_w\mathfrak{m} = y_w\mathfrak{m}_{w,\mathfrak{b}}$. Define

$$(4.5)\qquad c_w(dx^*/dm^*_w, \gamma, X) = c(\mathfrak{g}_\mathfrak{b}, y_w\mathfrak{m}_{w,\mathfrak{b}}, dx^*_\mathfrak{b}/(dm^*_{w,\mathfrak{b}})^w, y_w\gamma, X)\ , X \in \mathfrak{b}',$$

where $c(\mathfrak{g}_\mathfrak{b}, y_w\mathfrak{m}_{w,\mathfrak{b}}, dx^*_\mathfrak{b}/(dm^*_{w,\mathfrak{b}})^w, y_w\gamma, X)$ is defined as in (3.10) with $\mathfrak{g}_\mathfrak{b}$ instead of $\mathfrak{g}$ and $y_w\mathfrak{m}_{w,\mathfrak{b}}$ instead of $\mathfrak{m}$, and the invariant measure $(dm^*_{w,\mathfrak{b}})^w$ on $y_wM_{w,\mathfrak{b}}y_w^{-1}/A$ is normalized by transferring the invariant measure $dm^*_{w,\mathfrak{b}}$ on $M_{w,\mathfrak{b}}/A_w$ used in (4.4) via the map $m \to y_wmy_w^{-1}$.

Fix $s \in W_G(\mathfrak{h}, \mathfrak{g}_\mathfrak{b})$ and a representative $y_s \in N_G(\mathfrak{h}, \mathfrak{g}_\mathfrak{b})$. Then $y_s^{-1}\gamma \in y_s^{-1}\mathfrak{h} \subset \mathfrak{g}_\mathfrak{b}$, and we define $\mathfrak{m}_{\mathfrak{b},s} = C_{\mathfrak{g}_\mathfrak{b}}(y_s^{-1}\gamma) = \mathfrak{g}_\mathfrak{b} \cap y_s^{-1}\mathfrak{m}$.

Lemma 4.6. *There is a bijection $(s, v) \leftrightarrow (w, u)$ between*

$$\{(s, v) : s \in W_G(\mathfrak{h}, \mathfrak{g}_\mathfrak{b}), v \in W_{G_\mathfrak{b}}(\mathfrak{b}, \mathfrak{m}_{\mathfrak{b},s})\}$$

and

$$\{(w, u) : w \in W_G(\mathfrak{b}, \mathfrak{m}), u \in W_M(\mathfrak{h}, \mathfrak{m}_{w,\mathfrak{b}})\}$$

such that if $y_s \in N_G(\mathfrak{h}, \mathfrak{g}_\mathfrak{b})$ is a representative for s, $y_v \in N_{G_\mathfrak{b}}(\mathfrak{b}, \mathfrak{m}_{\mathfrak{b},s})$ is a representative for v, and $y_w \in N_G(\mathfrak{b}, \mathfrak{m})$ is a representative for w, then $y_sy_v^{-1}y_w \in N_M(\mathfrak{h}, \mathfrak{m}_{w,\mathfrak{b}})$ is a representative of u.

Proof. Let $s \in W_G(\mathfrak{h}, \mathfrak{g}_\mathfrak{b}), v \in W_{G_\mathfrak{b}}(\mathfrak{b}, \mathfrak{m}_{\mathfrak{b},s})$. Then $y_v^{-1}\mathfrak{b} \subset \mathfrak{m}_{\mathfrak{b},s} \subset y_s^{-1}\mathfrak{m}$ so that $y_sy_v^{-1}\mathfrak{b} \subset \mathfrak{m}$. Thus $y_vy_s^{-1} \in N_G(\mathfrak{b}, \mathfrak{m})$. Thus there are unique $w \in W_G(\mathfrak{b}, \mathfrak{m})$ and $m \in M$ such that $y_vy_s^{-1} = y_wm^{-1}$. Now $y_sy_v^{-1}y_w = m \in M$ and $\mathfrak{h} \subset \mathfrak{m}$, so that $m^{-1}\mathfrak{h} \subset \mathfrak{m}$. Further, $m^{-1}\mathfrak{h} = y_w^{-1}y_vy_s^{-1}\mathfrak{h} \subset y_w^{-1}\mathfrak{g}_\mathfrak{b}$ since $y_vy_s^{-1}\mathfrak{h} \subset y_v\mathfrak{g}_\mathfrak{b} = \mathfrak{g}_\mathfrak{b}$. Thus $m^{-1}\mathfrak{h} \subset \mathfrak{m} \cap y_w^{-1}\mathfrak{g}_\mathfrak{b} = \mathfrak{m}_{w,\mathfrak{b}}$ so that $m \in N_M(\mathfrak{h}, \mathfrak{m}_{w,\mathfrak{b}})$, and so represents a unique class $u \in W_M(\mathfrak{h}, \mathfrak{m}_{w,\mathfrak{b}})$. Now we map $(s, v) \to (w, u)$.

Now let $w \in W_G(\mathfrak{b}, \mathfrak{m}), u \in W_M(\mathfrak{h}, \mathfrak{m}_{w,\mathfrak{b}})$. Then for any representative y_u for u, $y_wy_u^{-1}\mathfrak{h} \subset \mathfrak{g}_\mathfrak{b}$ so there are unique $s \in W_G(\mathfrak{h}, \mathfrak{g}_\mathfrak{b})$ and $x \in G_\mathfrak{b}$ such that $y_uy_w^{-1} = y_sx^{-1}$. But as above, $x^{-1}\mathfrak{b} \subset \mathfrak{m}_{\mathfrak{b},s}$. Thus $x \in N_{G_\mathfrak{b}}(\mathfrak{b}, \mathfrak{m}_{\mathfrak{b},s})$ represents a unique $v \in W_{G_\mathfrak{b}}(\mathfrak{b}, \mathfrak{m}_{\mathfrak{b},s})$. Now if y_v is any representative for v, there is $m \in M_{\mathfrak{b},s}$ such that $x = y_vm$. Now $y_sy_v^{-1}y_w = y_um_1$ where $m_1 = y_w^{-1}y_vmy_v^{-1}y_w \in y_w^{-1}y_v(M_{\mathfrak{b},s})y_v^{-1}y_w = y_w^{-1}G_\mathfrak{b}y_w \cap m_1^{-1}Mm_1$. Thus $m_1 \in y_w^{-1}G_\mathfrak{b}y_w \cap M = M_{w,\mathfrak{b}}$ so that y_um_1 is also a representative of u. Thus the map $(w, u) \to (s, v)$ gives an inverse mapping. □

Lemma 4.7. *Let ω be a compact subset of $\mathfrak{b}'$. Then there exist a neighborhood $\omega(\gamma)$ of γ in $\mathfrak{h}$ and $T(\gamma) > 0$ so that for all $X \in \omega, H \in \omega(\gamma) \cap \mathfrak{h}'$, and $t \in F, |t| \geq T(\gamma)$,*

$$\Phi(\mathfrak{g}, dx^*, X, tH) = \sum_{w\in W_G(\mathfrak{b},\mathfrak{m})} c_w(dx^*/dm^*_w, \gamma, tX)\ \Phi(\mathfrak{m}, dm^*_w, y_w^{-1}X, tH).$$

Proof. By (4.3), for all $X \in \mathfrak{b}', H \in \mathfrak{h}'$,

$$\Phi(\mathfrak{g}, dx^*, X, H) = \sum_{s\in W_G(\mathfrak{h},\mathfrak{g}_\mathfrak{b})} \Phi(\mathfrak{g}_\mathfrak{b}, dx^*_\mathfrak{b}, X, y_s^{-1}H).$$

Fix $s \in W_G(\mathfrak{h}, \mathfrak{m}_\mathfrak{b})$. Then $y_s^{-1}\gamma \in \mathfrak{g}_\mathfrak{b}$ and $C_{\mathfrak{g}_\mathfrak{b}}(y_s^{-1}\gamma) = \mathfrak{g}_\mathfrak{b} \cap y_s^{-1}\mathfrak{m} = \mathfrak{m}_{\mathfrak{b},s}$. Since $\mathfrak{b}$ is an elliptic Cartan subalgebra of $\mathfrak{g}_\mathfrak{b}$ and $\omega \subset \mathfrak{b} \cap \mathfrak{g}' \subset \mathfrak{b} \cap \mathfrak{g}'_\mathfrak{b}$, we can apply Lemma

4.5 to $y_s^{-1}\gamma$ to obtain a neighborhood $\omega'(y_s^{-1}\gamma)$ of $y_s^{-1}\gamma$ in $y_s^{-1}\mathfrak{h}$ and $T'(y_s^{-1}\gamma) > 0$ so that for all $X \in \omega, H \in \omega'(y_s^{-1}\gamma) \cap \mathfrak{h}', |t| \geq T'(y_s^{-1}\gamma)$,

$$\Phi(\mathfrak{g}_\mathfrak{b}, dx_\mathfrak{b}^*, X, tH) = \sum_{v \in W_{G_\mathfrak{b}}(\mathfrak{b}, \mathfrak{m}_{\mathfrak{b},s})} c(\mathfrak{g}_\mathfrak{b}, \mathfrak{m}_{\mathfrak{b},s}, dx_\mathfrak{b}^*, y_s^{-1}\gamma, ty_v^{-1}X)\Phi(\mathfrak{m}_{\mathfrak{b},s}, y_v^{-1}X, tH).$$

Here, since by Lemma 4.5, $c(\mathfrak{g}_\mathfrak{b}, \mathfrak{m}_{\mathfrak{b},s}, dx_\mathfrak{b}^*/dm_s^*)\Phi(\mathfrak{m}_{\mathfrak{b},s}, dm_s^*)$ is independent of the choice dm_s^* of invariant measure on $M_{\mathfrak{b},s}/A$, we drop it from the notation.

Define $T(\gamma) = \max_s T'(y_s^{-1}\gamma)$ and $\omega(\gamma) = \bigcap_s y_s\omega'(y_s^{-1}\gamma)$. Then for all $X \in \omega, H \in \omega(\gamma) \cap \mathfrak{h}', |t| \geq T(\gamma)$, we have

$$\Phi(\mathfrak{g}, dx^*, X, tH) = \sum_{s \in W_G(\mathfrak{h}, \mathfrak{g}_\mathfrak{b})} \sum_{v \in W_{G_\mathfrak{b}}(\mathfrak{b}, \mathfrak{m}_{\mathfrak{b},s})} c(\mathfrak{g}_\mathfrak{b}, \mathfrak{m}_{\mathfrak{b},s}, dx_\mathfrak{b}^*, y_s^{-1}\gamma, ty_v^{-1}X)\Phi(\mathfrak{m}_{\mathfrak{b},s}, y_v^{-1}X, ty_s^{-1}H).$$

Fix a pair (s, v) and let (w, u) be the pair that corresponds to it by Lemma 4.6, so that $y_v \in G_\mathfrak{b}$, $y_u \in M$, and $y_v y_s^{-1} = y_w y_u^{-1}$. Then

$$\begin{aligned} y_v\mathfrak{m}_{b,s} = y_v(\mathfrak{g}_\mathfrak{b} \cap y_s^{-1}\mathfrak{m}) &= \mathfrak{g}_\mathfrak{b} \cap y_v y_s^{-1}\mathfrak{m} = \mathfrak{g}_\mathfrak{b} \cap y_w y_u^{-1}\mathfrak{m} \\ &= y_w(y_w^{-1}\mathfrak{g}_\mathfrak{b} \cap \mathfrak{m}) = y_w\mathfrak{m}_{w,\mathfrak{b}}. \end{aligned}$$

Thus using Lemma 3.5 and (4.5), for all $X \in \mathfrak{b}', H \in \mathfrak{h}'$,

$$\begin{aligned} &c(\mathfrak{g}_\mathfrak{b}, \mathfrak{m}_{\mathfrak{b},s}, dx_\mathfrak{b}^*, y_s^{-1}\gamma, y_v^{-1}X)\Phi(\mathfrak{m}_{\mathfrak{b},s}, y_v^{-1}X, y_s^{-1}H) \\ &= c(\mathfrak{g}_\mathfrak{b}, y_v\mathfrak{m}_{\mathfrak{b},s}, dx_\mathfrak{b}^*, y_v y_s^{-1}\gamma, X)\Phi(y_v\mathfrak{m}_{\mathfrak{b},s}, X, y_v y_s^{-1}H) \\ &= c(\mathfrak{g}_\mathfrak{b}, y_w\mathfrak{m}_{w,\mathfrak{b}}, dx_\mathfrak{b}^*/(dm_{w,\mathfrak{b}}^*)^w, y_w\gamma, X)\Phi(y_w\mathfrak{m}_{w,\mathfrak{b}}, (dm_{w,\mathfrak{b}}^*)^w, X, y_w y_u^{-1}H) \\ &= c_w(dx^*/dm_w^*, \gamma, X)\Phi(\mathfrak{m}_{w,\mathfrak{b}}, dm_{w,\mathfrak{b}}^*, y_w^{-1}X, y_u^{-1}H). \end{aligned}$$

Finally, using (4.4) and Lemma 4.6, for all $X \in \omega, H \in \omega(\gamma) \cap \mathfrak{h}', |t| \geq T(\gamma)$, we have

$$\begin{aligned} &\Phi(\mathfrak{g}, dx^*, X, tH) \\ &= \sum_{w \in W_G(\mathfrak{b}, \mathfrak{m})} c_w(dx^*/dm_w^*, \gamma, tX) \sum_{u \in W_M(\mathfrak{h}, \mathfrak{m}_{w,\mathfrak{b}})} \Phi(\mathfrak{m}_{w,\mathfrak{b}}, dm_{\mathfrak{b},w}^*, y_w^{-1}X, ty_u^{-1}H) \\ &= \sum_{w \in W_G(\mathfrak{b}, \mathfrak{m})} c_w(dx^*/dm_w^*, \gamma, tX)\Phi(\mathfrak{m}, dm_w^*, y_w^{-1}X, tH). \end{aligned}$$

□

The following proposition completes the proof of Theorem 1.2.

Proposition 4.8. *Let ω be a compact subset of $\mathfrak{b}'$. Then there exist a neighborhood $U(\gamma)$ of γ in $\mathfrak{m}$ and $T(\gamma) > 0$ so that for all $X \in \omega, H \in U(\gamma) \cap \mathfrak{g}'$, and $t \in F, |t| \geq T(\gamma)$,*

$$\Phi(\mathfrak{g}, dx^*, X, tH) = \sum_{w \in W_G(\mathfrak{b}, \mathfrak{m})} c_w(dx^*/dm_w^*, \gamma, tX)\ \Phi(\mathfrak{m}, dm_w^*, y_w^{-1}X, tH).$$

Proof. Since the measures dx^* and $dm_w^*, w \in W = W_G(\mathfrak{b}, \mathfrak{m})$ are fixed, we drop them from the notation. Let $\mathfrak{h}_1, ..., \mathfrak{h}_k$ denote a complete set of representatives for the M-conjugacy classes of Cartan subalgebras of $\mathfrak{m}$, and fix $1 \leq i \leq k$. By

Lemma 4.7 there are a neighborhood $\omega_i(\gamma)$ of γ in $\mathfrak{h}_i$ and $T_i(\gamma) > 0$ so that for all $X \in \omega, H \in \omega_i(\gamma) \cap \mathfrak{g}'$, and $t \in F, |t| \geq T_i(\gamma)$,

$$\Phi(\mathfrak{g}, X, tH) = \sum_{w \in W} c_w(\gamma, tX)\ \Phi(\mathfrak{m}, y_w^{-1}X, tH).$$

Let $T(\gamma) = \max_{1 \leq i \leq k} T_i(\gamma)$, and let $\omega(\gamma)$ be a neighborhood of γ in $\mathfrak{m}$ small enough such that $\omega(\gamma) \cap \mathfrak{h}_i \subset \omega_i(\gamma)$ for $1 \leq i \leq k$. Now by Corollary 2.3 of [1] there is an open, closed, M-invariant neighborhood $U(\gamma)$ of γ in $\mathfrak{m}$ such that $U(\gamma) \cap \mathfrak{h}_i \subset \omega(\gamma) \cap \mathfrak{h}_i \subset \omega_i(\gamma), 1 \leq i \leq k$. Now let $X \in \omega, H \in U(\gamma) \cap \mathfrak{g}', |t| \geq T(\gamma)$. Then there are $m \in M, 1 \leq i \leq k, H_i \in \mathfrak{h}_i$, so that $H = mH_i$. But $H_i = m^{-1}H \in U(\gamma) \cap \mathfrak{g}' \cap \mathfrak{h}_i \subset \omega_i(\gamma) \cap \mathfrak{g}'$. Thus

$$\begin{aligned}\Phi(\mathfrak{g}, X, tH) = \Phi(\mathfrak{g}, X, tH_i) &= \sum_{w \in W} c_w(\gamma, tX)\ \Phi(\mathfrak{m}, y_w^{-1}X, tH_i) \\ &= \sum_{w \in W} c_w(\gamma, tX)\ \Phi(\mathfrak{m}, y_w^{-1}X, tH).\end{aligned}$$

□

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