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# CONSTRUCTIBLE CHARACTERS, LEADING COEFFICIENTS AND LEFT CELLS FOR FINITE COXETER GROUPS WITH UNEQUAL PARAMETERS

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Abstract. Following Lusztig, we investigate constructible characters, leading coefficients and left cells for a finite Coxeter group $W$ in the case of unequal parameters. We obtain explicit results for $W$ of type $F_4$, $B_n$ and $I_2(m)$ ($m$ even) which support Lusztig's conjecture that known results about left cells in the equal parameter case should remain valid in the case of unequal parameters.

## 1. Introduction

Consider a finite Coxeter group $W$ with generating set $S$. Following Lusztig [17], we wish to study the partition of $W$ into *left cells* in the multi-parameter case. This means that we are given a function $\varphi\colon W \to \Gamma$ into an abelian group $\Gamma$ such that $\varphi(w) = \varphi(s_1)\cdots\varphi(s_p)$ whenever $w = s_1\cdots s_p$ with $s_i \in S$ is a reduced expression. Furthermore, we assume that there is a total order on $\Gamma$ which is compatible with the group structure and that $\varphi(s)$ is strictly positive for all $s \in S$. Then the corresponding Iwahori–Hecke algebra $H$ over the group algebra $\mathbb{Z}[\Gamma]$ has a certain distinguished basis and this is used to define a partition of $W$ into *left cells*. In the case where $\varphi$ is constant, one obtains just the original definition of the Kazhdan–Lusztig basis and left cells introduced in [13]. We recall the definitions in the multi-parameter case and establish some basic relations in Section 2.

Using the $a$-function arising from the generic degrees of $H$, one can also define the notions of *constructible characters* and *leading coefficients* in complete analogy to the equal parameter case (which is studied in detail by Lusztig [18] and [21]). This will be worked out in Section 3. In order to determine the leading coefficients, we will introduce in Section 4 the notion of *orthogonal representations* of Iwahori–Hecke algebras. This will allow us to define leading coefficients of matrix representations (and not only of their characters). We show in Theorem 4.10 and its corollaries that these leading coefficients can be used to detect left cells.

The last three sections are concerned with explicit results. First, in Section 5, we show that Hoefsmit's matrix representations in type $B_n$ (or, rather, the variants defined by Ariki–Koike [3]) actually are orthogonal representations satisfying an additional integrality condition. This turns out to be an efficient tool for determining the leading coefficients in type $B_n$; see Theorem 5.4. (In a similar way, one can also recover the known results on leading coefficients in type $A_n$ using orthogonal representations.) In Sections 6 and 7, we present explicit results on constructible

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characters, leading coefficients and left cells for the types $I_2(m)$ and $F_4$. These were obtained by explicit computations using the CHEVIE system; see [9] and [24].

The results we obtain provide some evidence for the conjecture expressed by Lusztig in [17, §7] that known results on left cells in the equal parameter case should continue to hold in the case of unequal parameters. For example, the following results should be true in general:

(1) The constructible characters are precisely the characters afforded by the various left cells of $W$ (see [20] in the equal parameter case).
(2) For $w \in W$ and $\chi \in \text{Irr}(W)$ let $c_{w,\chi}$ be the corresponding leading coefficient; then we have $c_{w,\chi} \neq 0$ for some $\chi \in \text{Irr}(W)$ if and only if $w, w^{-1}$ are in the same left cell (see [21, 3.5] in the equal parameter case).
(3) For any left cell $\mathfrak{C} \subset W$, there exists a unique element $x_0 \in \mathfrak{C}$ such that the character afforded by $\mathfrak{C}$ is given by $\pm \sum_\chi c_{x_0,\chi}\chi$ (see [18, 12.2] in the equal parameter case).

(To be more precise, the above references cover the case of finite Weyl groups. The analogous results for the dihedral groups and the types $H_3$, $H_4$ have been checked using explicit computation by Lusztig [16], Alvis [1] and Alvis–Lusztig [2].)

Note that, according to the classification of finite Coxeter groups, the only types that we have to consider (as far as unequal parameters are concerned) are $I_2(m)$ ($m$ even), $B_n$ ($n \geqslant 2$) and $F_4$. With one exception, we check that the above statements are true in an "extremal" case of unequal parameters. (This means that we have two independent parameters and the ordering is such that one of them dominates any power of the other.) The exception is the statement in (1) for type $B_n$: we show that the constructible characters are precisely the irreducible characters but we cannot show that the characters afforded by left cells are irreducible. (However, this would follow from the explicit description of the left cells by Iancu [12].)

We shall use [10] as a reference for general facts about the characters and representations of finite Coxeter groups and Iwahori–Hecke algebras.

Throughout this paper, we shall use the following notation. Let $\Gamma$ be a free abelian group of finite rank (written multiplicatively) and assume that we are given a total ordering of $\Gamma$. This is specified by a multiplicatively closed subset $\Gamma_+ \subseteq \Gamma \setminus \{1\}$ such that we have $\Gamma = \Gamma_+ \amalg \{1\} \amalg \Gamma_-$, where $\Gamma_- = \{g^{-1} \mid g \in \Gamma_+\}$. We shall write

$$g \leqslant g' \quad \text{if } g'g^{-1} \in \Gamma_+ \cup \{1\}.$$

Let $\{v_s \mid s \in S\} \subseteq \Gamma_+$ be a subset such that $v_s = v_t$ whenever $s, t \in S$ are conjugate in $W$. Then we have a well-defined map $\varphi\colon W \to \Gamma$ such that $\varphi(w) = v_{s_1} \cdots v_{s_p}$ whenever $w = s_1 \cdots s_p$ with $s_i \in S$ is a reduced expression.

Let $R \subseteq \mathbb{C}$ be a subring and let $A = R[\Gamma]$ be the group algebra of $\Gamma$ over $R$. Then the elements of $A$ are Laurent polynomials in the free generators of $\Gamma$. Given any $0 \neq f \in A$, we can write uniquely $f = r_0g_0 + \cdots + r_ng_n$ where $0 \neq r_i \in R$ and $g_i \in \Gamma$, such that $g_ig_0^{-1} \in \Gamma_+$ for all $i > 0$. (Note that any finite subset of $\Gamma$ has a unique maximum and a unique minimum.) For short, we will sometimes just write $f = r_0g_0 + \text{higher terms}$.

Let $H$ be the generic Iwahori–Hecke algebra corresponding to $(W, S)$ with parameters $\{v_s^2 \mid s \in S\}$. Thus, $H$ has an $A$-basis $\{T_w \mid w \in W\}$ and the multiplication is given by the rule

$$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ v_s^2 T_{sw} + (v_s^2 - 1)T_w & \text{if } l(sw) < l(w); \end{cases}$$

here, $l\colon W \to \mathbb{N}_0$ denotes the usual length function on $W$ with respect to $S$. For any $w \in W$ we set $v_w := v_{s_1} \cdots v_{s_p}$ where $w = s_1 \cdots s_p$ with $s_i \in S$ is a reduced expression. It will be convenient to introduce the elements $\tilde{T}_w = v_w^{-1} T_w$ for $w \in W$. Then we have the following relations:

$$\tilde{T}_s\tilde{T}_w = \begin{cases} \tilde{T}_{sw} & \text{if } l(sw) > l(w), \\ \tilde{T}_{sw} + (v_s - v_s^{-1})\tilde{T}_w & \text{if } l(sw) < l(w). \end{cases}$$

The algebra $H$ is *symmetric* with respect to the trace function $\tau\colon H \to A$ defined by $\tau(\tilde{T}_1) = 1$ and $\tau(\tilde{T}_w) = 0$ for $1 \neq w \in W$. We have

$$\tau(\tilde{T}_w\tilde{T}_{w'}) = \begin{cases} 1 & \text{if } w' = w^{-1}, \\ 0 & \text{if } w' \neq w^{-1}; \end{cases}$$

see [10, §8.1].

## 2. On the generalized left cells of $W$

Assume that we have fixed a total order on $\Gamma$ as in Section 1. Following Lusztig [17], we obtain a corresponding Kazhdan–Lusztig basis of $H$ which then gives rise to a partition of $W$ into left cells. (This generalizes the original constructions of Kazhdan–Lusztig [13].) The aim of this section is to establish some results concerning these left cells which generalize similar results due to Kazhdan–Lusztig [13] and Lusztig [18] in the one-parameter case. See also [25, Chap. 1] for further results on left cells in the multi-parameter case.

**2.1. The Kazhdan–Lusztig basis.** Let $a \mapsto \bar{a}$ be the $R$-algebra automorphism of $A$ which takes $g$ to $g^{-1}$ for any $g \in \Gamma$. Let $\leqslant$ denote the Bruhat–Chevalley order on $W$. For each pair of elements $y \leqslant w$ in $W$, we have a corresponding Laurent polynomial $P_{y,w}^* \in A$, as constructed in [17, Prop. 2]; see also [10, 11.1.11] for a recursive formula for the computation of $P_{y,w}^*$. We have $P_{w,w}^* = 1$ and, if $y < w$, then $P_{y,w}^*$ is an integral linear combination of the elements of $\Gamma_-$. For each $w \in W$, the corresponding Kazhdan–Lusztig basis element is given by

$$C_w = \sum_{\substack{y \in W \\ y \leqslant w}} (-1)^{l(w)-l(y)}\, \overline{P}_{y,w}^*\, \tilde{T}_y \in H;$$

see [17, §6]. Since $C_w$ is a linear combination of $\tilde{T}_w$ and basis elements $\tilde{T}_y$ with $y < w$, we see that $\{C_w \mid w \in W\}$ is a basis of $H$. We have the following multiplication rules (see [17, §6]): for $w \in W$ and $s \in S$, we have

$$\tilde{T}_s\, C_w = \begin{cases} C_{sw} + v_sC_w - \displaystyle\sum_{\substack{z<w \\ sz<z}} (-1)^{l(w)-l(z)} M_{z,w}^s C_z & \text{if } sw > w, \\ -v_s^{-1}\, C_w & \text{if } sw < w, \end{cases}$$

where $M_{z,w}^s \in A$ are determined as in [17, §3]. Note that we have $\overline{M}_{z,w}^s = M_{z,w}^s$. Now recall the following definition from [17, §6]. Let $\leqslant_L$ be the preorder relation on $W$ generated by the relation

(L) $\begin{cases} y \leqslant_L w \text{ if there exists some } s \in S \text{ such that } C_y \text{ appears with} \\ \text{nonzero coefficient in } \tilde{T}_sC_w \text{ (expressed in the } C_w\text{-basis).} \end{cases}$

The equivalence relation associated with $\leqslant_L$ will be denoted by $\sim_L$ and the corresponding equivalence classes are called the *left cells* of $W$. The following example shows that these notions depend on the chosen total ordering of $\Gamma$.

**Example 2.2.** Let $(W, S)$ be of type $B_3$, with generators $S = \{t, s_1, s_2\}$ such that $(ts_1)^4 = (s_1s_2)^3 = 1$. Let $v$ be an indeterminate and $\Gamma = \{v^m \mid m \in \mathbb{Z}\}$, $\Gamma_+ = \{v^m \mid m > 0\}$. Let $v_{s_1} = v_{s_2} = v$ and $v_t = v^{m_1}$, where $m_1 \geqslant 1$. Computing the left cells of $W$, we find that it makes a difference if $m_1 = 1$, $m_1 = 2$ or $m_1 > 2$. For example, let $w_1 = s_1s_2s_1$, $w_2 = tw_1$, $w_3 = s_1w_2$, $w_4 = s_2w_3$. Then it can be checked that

- $\{w_1, w_2, w_3\}$ is a left cell if $m_1 = 1$;
- $\{w_1, w_2, w_3, w_4\}$ is a left cell if $m_1 = 2$;
- $\{w_1\}$ and $\{w_2, w_3, w_4\}$ are left cells if $m_1 > 2$.

This example also shows that [25, Conjecture 1.23] does not hold. (I was informed by N. Xi that he knew that this conjecture does not hold.)

**2.3. Left cell representations.** Each left cell $\mathfrak{C}$ gives rise to a representation of $H$. This is constructed as follows (see [17, §7]). Let $V_{\mathfrak{C}}$ be an $A$-module with a free $A$-basis $\{e_w \mid w \in \mathfrak{C}\}$. Then the action of $\tilde{T}_s$ $(s \in S)$ is given by the multiplication formulas in (2.1), i.e., we have

$$\tilde{T}_s\, e_w = \left\{ \begin{array}{ll} e_{sw} + v_s e_w - \displaystyle\sum_{\substack{z<w\\ sz<z}} (-1)^{l(w)-l(z)} M_{z,w}^s e_z & \quad \text{if } sw > w, \\ -v_s^{-1}\, e_w & \quad \text{if } sw < w, \end{array} \right.$$

where we tacitly assume that $e_z = 0$ if $z \notin \mathfrak{C}$.

Upon specialization $v_s \mapsto 1$ $(s \in S)$, we obtain a representation of $W$ which is called the representation carried by $\mathfrak{C}$. We denote by $\chi_{\mathfrak{C}}$ the character of that representation of $W$.

**2.4. The dual basis.** The following constructions are analogous to those in [18, 5.1] (which are concerned with the one-parameter case). For $y \leqslant w$ in $W$ we define a Laurent polynomial $Q_{y,w}^* \in A$ inductively by

$$\sum_{\substack{z\in W\\ y\leqslant z\leqslant w}} (-1)^{l(w)-l(z)} P_{y,z}^*\, Q_{z,w}^* = \left\{ \begin{array}{ll} 1 & \quad \text{if } y = w, \\ 0 & \quad \text{if } y < w. \end{array} \right.$$

We have $Q_{w,w}^* = 1$ and, if $y < w$, then $Q_{y,w}^*$ is an integral linear combination of the elements of $\Gamma_-$. (Proof by induction on $l(w) - l(y)$.) Then we set, for any $y \in W$,

$$D_y := \sum_{\substack{w\in W\\ y\leqslant w}} \overline{Q}_{y,w}^*\, \tilde{T}_w \in H.$$

As in [18, 5.1.10], we see that

$$\tau(C_w D_{y^{-1}}) = \left\{ \begin{array}{ll} 1 & \quad \text{if } y = w, \\ 0 & \quad \text{if } y \neq w, \end{array} \right.$$

where $\tau$ is the symmetrizing trace defined in Section 1. It follows that $\{C_w \mid w \in W\}$ and $\{D_{w^{-1}} \mid w \in W\}$ are dual bases with respect to each other. Consequently, we have the following rules: for $w \in W$ and $s \in S$, we have

$$\tilde{T}_s\, D_y = \begin{cases} D_{sy} - v_s^{-1} D_y - \displaystyle\sum_{\substack{w>y\\ sw>w}} (-1)^{l(w)-l(y)} M_{y,w}^s D_w & \text{if } sy < y, \\ v_s D_y & \text{if } sy > y. \end{cases}$$

(Compare with the formulas [18, 5.1.16] in the one-parameter case.) Indeed, let us write $\tilde{T}_s D_y = \sum_w a_{yw} D_w$ with $a_{yz} \in A$. Then we have $a_{yw} = \tau(\tilde{T}_s D_y C_{w^{-1}}) = \tau(D_y C_{w^{-1}} \tilde{T}_s)$. On the other hand, the map $\tilde{T}_w \mapsto \tilde{T}_{w^{-1}}$ defines an anti-automorphism of $H$. Thus, writing $\tilde{T}_s C_w = \sum_z b_{wz} C_z$ with $b_{wz} \in A$, we have $C_{w^{-1}} \tilde{T}_s = \sum_z b_{wz} C_{z^{-1}}$ and so $a_{yw} = \tau(D_y C_{w^{-1}} \tilde{T}_s) = b_{wy}$. Hence, the multiplication formulas for the $D_y$-basis are obtained by "transposing" those for the $C_w$-basis in (2.1).

**Lemma 2.5** (See Lusztig [18, 5.1.14] in the one-parameter case)**.** *Let $y, w \in W$. If $C_y D_{w^{-1}} \neq 0$, then we have $w \leqslant_L y$.*

*Proof.* This is word for word the same as in [19, Lemma 4.5]. □

Note that Lusztig actually also proves the converse of the above relation but this requires the positivity properties of the $\mu$-constants occurring in the Kazhdan–Lusztig polynomials in the one-parameter case. (This positivity is no longer satisfied in the unequal parameter case, see the examples in [17] and also Example 2.7 below.)

The following result appears in Lusztig's lecture notes [22, §11]. Note that the argument we give for (b) is different from that in [13, 3.2] (in the equal parameter case). In Example 2.7 below we will see that the $M$-polynomials do not behave in exactly the same way as the $\mu$-constants in the equal parameter case.

**Lemma 2.6** (See Kazhdan–Lusztig [13] in the one-parameter case)**.** *Let $w_0 \in W$ be the longest element. Then we have*

(a) $$Q_{y,w}^* = P_{ww_0,yw_0}^* \qquad \textit{for any } y, w \in W \textit{ with } y \leqslant w.$$

*Furthermore, if $y, w \in W$ and $s \in S$ are such that $sy < y < w < sw$, then*

(b) $$M_{ww_0,yw_0}^s = -(-1)^{l(w)-l(y)} M_{y,w}^s.$$

*Proof.* The proof of (a) is similar to that in [13]; we sketch the main ingredients involving the $R$-polynomials, which are defined as follows. As in [17, §1], we write for any $w \in W$:

$$\tilde{T}_{w^{-1}}^{-1} = \sum_{y \in W} \overline{R}_{y,w}^* \, \tilde{T}_y \qquad \text{where } R_{y,w}^* \in A.$$

We have $R_{w,w}^* = 1$ and $R_{y,w}^* = 0$ unless $y \leqslant w$. Furthermore, we have the following recursion formula for the computation of the $R$-polynomials. Let $y, w \in W$ and assume that $y \leqslant w$, $w \neq 1$. If $s \in S$ is such that $sw < w$, then

$$R_{y,w}^* = \begin{cases} R_{sy,sw}^* + (v_s - v_s^{-1}) R_{y,sw}^* & \text{if } sy > y, \\ R_{sy,sw}^* & \text{if } sy < y. \end{cases}$$

Using the above formula and induction on $l(w) - l(y)$, one easily shows the following relations:

$$R_{ww_0,yw_0}^* = R_{y,w}^* \qquad \text{and} \qquad \overline{R}_{y,w}^* = (-1)^{l(w)-l(y)} R_{y,w}^*.$$

(In particular, this shows that $y \leqslant w$ if and only if $ww_0 \leqslant yw_0$.) Combining this with [17, 1.1], we obtain the following identity:

$$(*) \qquad \sum_{\substack{z\in W\\ y\leqslant z\leqslant w}} (-1)^{l(w)-l(z)} R^*_{y,z}\, R^*_{ww_0,zw_0} = \begin{cases} 1 & \text{if } y=w,\\ 0 & \text{if } y\neq w.\end{cases}$$

Now, in order to prove the identity $Q^*_{y,w}=P^*_{ww_0,yw_0}$, we must show that

$$(**) \qquad \sum_{\substack{z\in W\\ y\leqslant z\leqslant w}} (-1)^{l(w)-l(z)} P^*_{y,z}\, P^*_{ww_0,zw_0} = \begin{cases} 1 & \text{if } y=w,\\ 0 & \text{if } y< w.\end{cases}$$

As in [13, 3.1], let $U_{y,w}\in A$ denote the left-hand side of $(**)$. We show by induction on $l(w)-l(y)$ that $U_{y,w}=0$ if $y<w$ and $U_{w,w}=1$. If $y=w$, this is clear. Now assume that $y<w$. By [17, 2.2], we have

$$\overline{P}^*_{y,w} = \sum_{\substack{z\in W\\ y\leqslant z\leqslant w}} R^*_{y,z}\, P^*_{z,w}.$$

We have similar expressions for $\overline{P}^*_{y,z}$ and $\overline{P}^*_{ww_0,zw_0}$. Inserting these expressions into $\overline{U}_{y,w}=\sum_{y\leqslant z\leqslant w}(-1)^{l(w)-l(z)}\overline{P}^*_{z,w}\,\overline{P}^*_{ww_0,zw_0}$, we obtain

$$\overline{U}_{y,w} = \sum_{\substack{u,u'\in W\\ y\leqslant u\leqslant u'\leqslant w}} (-1)^{l(w)-l(u')} R^*_{y,u}\, R^*_{ww_0,u'w_0}\, U_{u,u'}.$$

Using the inductive hypothesis on $U_{u,u'}$, the above expression simplifies to

$$\overline{U}_{y,w} = U_{y,w} + \sum_{y\leqslant u\leqslant w} (-1)^{l(w)-l(u)} R^*_{y,u}\, R^*_{ww_0,uw_0}.$$

By $(*)$, we have $\overline{U}_{y,w}=U_{y,w}$. However, the definition of the polynomials $P^*_{y,w}$ shows that $U_{y,w}$ is a linear combination of elements in $\Gamma_-$. Since $\overline{U}_{y,w}=U_{y,w}$, we conclude that $U_{y,w}=0$, as desired.

To prove (b), we proceed as follows. For any $w\in W$, we set $C'_w:=\sum_{y\leqslant w} P^*_{y,w}\tilde{T}_y$. Then we have $C'_w=(-1)^{l(w)}j(C_w)$, where $j\colon H\to H$ is the ring involution defined by $j(\sum_y a_w\tilde{T}_y)=\sum_y \bar{a}_y(-1)^{l(y)}\tilde{T}_y$; see [17, §6]. We have the following multiplication rules: for any $w\in W$ and $s\in S$, we have

$$\tilde{T}_s\, C'_w = \begin{cases} C'_{sw} - v_s^{-1}C'_w + \displaystyle\sum_{\substack{z<w\\ sz<z}} M^s_{z,w} C'_z & \text{if } sw>w,\\[2ex] v_s\, C'_w & \text{if } sw<w.\end{cases}$$

Now we claim that (see [18, 5.1.8] in the one-parameter case)

$$(\dagger) \qquad D_y = C'_{yw_0}\tilde{T}_{w_0} \qquad \text{for any } y\in W.$$

This is seen as follows. By [17, Prop. 3], we have $C'_w=\sum_{y\leqslant w}\overline{P}^*_{y,w}\,\tilde{T}^{-1}_{y^{-1}}$ for any $w\in W$. This yields

$$C'_{yw_0}\tilde{T}_{w_0} = \Big(\sum_{\substack{w\in W\\ y\leqslant w}} \overline{P}^*_{ww_0,yw_0}\,\tilde{T}^{-1}_{w_0w^{-1}}\Big)\tilde{T}_{w_0}.$$

Now, for any $w \in W$, we have $\tilde{T}_{w_0w^{-1}}\tilde{T}_w = \tilde{T}_{w_0}$. Inserting this into the above equation, we obtain

$$C'_{yw_0}\tilde{T}_{w_0} = \sum_{\substack{w\in W\\ y\leqslant w}} \overline{P}^{\,*}_{ww_0,yw_0}\,\tilde{T}_w = \sum_{\substack{w\in W\\ y\leqslant w}} \overline{Q}^{\,*}_{y,w}\,\tilde{T}_w = D_y,$$

where the second equality holds by (a). Thus, (†) is proved. Now fix $y \in W$ and let $s \in S$. We have $l(syw_0) = l(w_0) - l(sy)$ and so $syw_0 > yw_0$ if and only if $sy < y$. Thus, using the above multiplication rules for the $C'_w$-basis, we obtain

$$\tilde{T}_sD_y = \tilde{T}_sC'_{yw_0}\tilde{T}_{w_0} = \begin{cases} D_{sy} - v_s^{-1}D_y + \sum\limits_{\substack{w>y\\ sw>w}} M^s_{ww_0,yw_0}D_w & \text{if } sy < y,\\ v_sD_y & \text{if } sy > y.\end{cases}$$

A comparison with the multiplication rules in (2.4) yields the desired relation. □

**Example 2.7.** Let $(W, S)$ and $\Gamma_+ \subset \Gamma$ be as in Example 2.2, where $m_1 > 2$. Then we have

$$M^t_{y,w} = v^{m_1-2} + v^{2-m_1} \qquad \text{where} \qquad y = ts_2, \quad w = s_1s_2s_1t.$$

Since $l(w) - l(y) = 2$, the formula in Lemma 2.6 yields $M^t_{ww_0,yw_0} = -M^t_{y,w} \neq 0$. This phenomenon cannot occur in the one-parameter case, where multiplication with $w_0$ does not change the $M$-polynomials (which are constant); see [13, 3.2].

**Corollary 2.8** (See Lusztig [18, 5.1.8 and 5.14] in the equal parameter case)**.**

(a) *For any $y, w \in W$, we have $y \leqslant_L w$ if and only if $ww_0 \leqslant_L yw_0$. In particular, if $\mathfrak{C}$ is a left cell in $W$, then so is $\mathfrak{C}\,w_0 = \{ww_0 \mid w \in \mathfrak{C}\}$.*

(b) *Let $\mathfrak{C}$ be a left cell and let $\chi_{\mathfrak{C}}$ be the character of the representation of $W$ carried by $\mathfrak{C}$ (see 2.3). Then we have $\chi_{\mathfrak{C}w_0} = \varepsilon \otimes \chi_{\mathfrak{C}}$, where $\varepsilon$ is the sign character.*

*Proof.* The statement in (a) is an immediate consequence of Lemma 2.6 and the definition of left cells. The proof of the statement in (b) is essentially the same as that in [4, 2.25]. For this purpose, recall the construction of the $H$-module $V_{\mathfrak{C}}$ from (2.3). Now, since every element of $W$ is conjugate to its inverse, the representation of $H$ on $V_{\mathfrak{C}}$ is equivalent (over the field of fractions of $A$) to its contragredient dual (see [10, 8.2.6]). Thus, "transposing" the formulas in (2.3), we see that we may also regard $V_{\mathfrak{C}}$ as an $H$-module where $\tilde{T}_s$ acts as follows (compare with the formulas in (2.4)):

$$\tilde{T}_s\,e_y = \begin{cases} e_{sy} - v_s^{-1}e_y - \sum\limits_{\substack{w>y\\ sw>w}} (-1)^{l(w)-l(y)}M^s_{y,w}e_w & \text{if } sy < y,\\ v_se_y & \text{if } sy > y,\end{cases}$$

where we assume again that $e_{sy} = 0$ if $sy \notin \mathfrak{C}$. Now compose the above action of $H$ on $V_{\mathfrak{C}}$ with the $A$-algebra automorphism $\gamma\colon H \to H$ given by $\gamma(\tilde{T}_s) = -\tilde{T}_s^{-1}$ $(s \in S)$. Then we obtain a new $H$-module, denoted by $V_{\mathfrak{C}}^{\gamma}$, on which $\tilde{T}_s$ acts via

$$(*) \qquad \tilde{T}_s\,e_y = \begin{cases} -e_{sy} + v_se_y + \sum\limits_{\substack{w>y\\ sw>w}} (-1)^{l(w)-l(y)}M^s_{y,w}e_w & \text{if } sy < y,\\ -v_s^{-1}e_y & \text{if } sy > y.\end{cases}$$

By [10, 9.4.1], the above representation of $H$ on $V_{\mathfrak{C}}^{\gamma}$ yields the character $\varepsilon \otimes \chi_{\mathfrak{C}}$ upon specializing $v_s \mapsto 1$.

On the other hand, consider the left cell $\mathfrak{C}w_0$. Let $V_{\mathfrak{C}w_0}$ be the corresponding $H$-module as in (2.3). Then $V_{\mathfrak{C}w_0}$ has a free $A$-basis $\{e_{ww_0} \mid w \in \mathfrak{C}\}$ and the action of $\tilde{T}_s$ is given by

$$\tilde{T}_s\, e_{ww_0} = \begin{cases} e_{sww_0} + v_s e_{ww_0} + \displaystyle\sum_{\substack{zw_0 < ww_0 \\ szw_0 < z}} M_{w,z}^s e_{zw_0} & \text{if } sww_0 > ww_0, \\ -v_s^{-1}\, e_{ww_0} & \text{if } sww_0 < ww_0, \end{cases}$$

where we used the identity for the $M$-polynomials in Lemma 2.6. Using the fact that multiplication with $w_0$ reverses the Bruhat–Chevalley order, we can rewrite the above formulas as follows:

$$\tilde{T}_s\, e_{ww_0} = \begin{cases} e_{sww_0} + v_s e_{ww_0} + \displaystyle\sum_{\substack{z > w \\ sz > z}} M_{w,z}^s e_{zw_0} & \text{if } sw < w, \\ -v_s^{-1}\, e_{ww_0} & \text{if } sw > w. \end{cases}$$

Now set $e'_w := (-1)^{l(w)} e_{ww_0}$ for $w \in \mathfrak{C}$. Then the action of $\tilde{T}_s$ on the basis $\{e'_w \mid w \in \mathfrak{C}\}$ of $V_{\mathfrak{C}w_0}$ is given by

$$\tilde{T}_s\, e'_w = \begin{cases} -e'_{sw} + v_s e'_w + \displaystyle\sum_{\substack{z > w \\ sz > z}} (-1)^{l(z)-l(w)} M_{w,z}^s e'_z & \text{if } sw < w, \\ -v_s^{-1}\, e'_w & \text{if } sw > w. \end{cases}$$

Hence we obtain the same action as that given by $(*)$. $\square$

## 3. Generalized $a$-invariants and leading coefficients

The aim of this section is to provide a precise definition of the $a$-invariants and the leading coefficients of characters of finite Coxeter groups for the case of unequal parameters. In the one-parameter case, these notions are due to Lusztig [18]; some indications as to the general case are given in [17, §7]. Here, we follow the exposition in [10, §9.4].

We keep all the assumptions of Section 1 but we now make a special choice of the ring $R$: we take $R = \mathbb{R}$. Let $\mathrm{Irr}(W)$ denote the set of (complex) irreducible characters of $W$; each of these characters is afforded by a representation over $\mathbb{R}$. (Note that $\mathbb{R}$ is a splitting field for $W$; see [10, 6.3.8]). Furthermore, let $K$ be the field of fractions of $A$ and denote $KH := K \otimes_A H$. It is known that $KH$ is a split semisimple algebra, which is isomorphic to the group algebra of $W$ over $K$. Let $\mathrm{Irr}(KH)$ be the set of irreducible characters of $KH$. By Tits' Deformation Theorem, each $\chi \in \mathrm{Irr}(W)$ determines a unique irreducible character $\hat{\chi} \in \mathrm{Irr}(KH)$; for all $w \in W$, we have $\hat{\chi}(\tilde{T}_w) \in A$ and applying the $R$-algebra homomorphism $A \to R$, $g \mapsto 1$ ($g \in \Gamma$), yields $\chi(w)$. (For all this, see [10, 8.1.7 and 9.3.5].)

**3.1. Schur elements.** For each $\chi \in \mathrm{Irr}(W)$ there exists a unique nonzero element $c_\chi \in K$ such that

$$\sum_{\chi \in \mathrm{Irr}(W)} \frac{1}{c_\chi}\, \hat{\chi}(\tilde{T}_w) = \begin{cases} 1 & \text{if } w = 1, \\ 0 & \text{if } w \neq 1; \end{cases}$$

see [10, 9.4.4]. We have in fact $c_\chi \in A$ for all $\chi \in \mathrm{Irr}(W)$, by [10, 9.3.5]. The elements $c_\chi$ are called Schur elements; they can be characterized alternatively as follows. Let $d = \chi(1)$ and consider a matrix representation $\mathfrak{X}\colon KH \to M_d(K)$ affording $\hat{\chi}$. Then we have the following Schur relations (see [10, 7.2.2]):

$$\sum_{w\in W} \mathfrak{X}(\tilde{T}_w)_{ij}\, \mathfrak{X}(\tilde{T}_{w^{-1}})_{kl} = \delta_{il}\, \delta_{kj}\, c_\chi \qquad \text{for } 1 \leqslant i,j,k,l \leqslant d.$$

These relations are a consequence of the fact that $H$ is a symmetric algebra with respect to the trace function $\tau\colon H \to A$ defined in Section 1.

The following result shows that the Schur elements also arise in connection with left cell representations.

**Proposition 3.2** (See Lusztig [18, 5.4 and 5.7] in the one-parameter case)**.** *Let $\mathfrak{C}$ be a left cell in $W$. Then the element*

$$z_{\mathfrak{C}} := \sum_{w\in\mathfrak{C}} C_w\, D_{w^{-1}} \in H$$

*lies in the center $Z(KH)$ of $KH$. For $\chi \in \mathrm{Irr}(W)$, the element $z_{\mathfrak{C}}$ acts by the scalar $m(\chi_{\mathfrak{C}}, \chi)\, c_\chi$ on a representation affording $\hat{\chi} \in \mathrm{Irr}(KH)$, where $m(\chi_{\mathfrak{C}}, \chi)$ denotes the multiplicity of $\chi$ in the character $\chi_{\mathfrak{C}}$ afforded by the representation carried by $\mathfrak{C}$.*

*Proof.* Copying word for word the proof of [18, 5.7(ii)], we obtain the identity $z_{\mathfrak{C}} = \sum_{w\in W} \hat{\chi}_{\mathfrak{C}}(C_w) D_{w^{-1}}$. This yields

$$z_{\mathfrak{C}} = \sum_{\chi'\in\mathrm{Irr}(W)} m(\chi_{\mathfrak{C}}, \chi')\Big(\sum_{w\in W} \hat{\chi}'(C_w)\, D_{w^{-1}}\Big).$$

In order to show that $z_{\mathfrak{C}} \in Z(H)$, we use an argument which is different from that in [18]. We fix a character $\chi' \in \mathrm{Irr}(W)$ and let $\mathfrak{X}'\colon KH \to M_{d'}(K)$ be a representation affording $\hat{\chi}'$; we write $\mathfrak{X}'(h) = (x'_{ij}(h))_{1\leqslant i,j\leqslant d'}$ for $h \in KH$. Then we have

$$\sum_{w\in W} \hat{\chi}'(C_w) D_{w^{-1}} = \sum_{i=1}^{d'} \sum_{w\in W} x'_{ii}(C_w) D_{w^{-1}}.$$

Furthermore, let $\chi \in \mathrm{Irr}(W)$ and $\mathfrak{X}\colon KH \to M_d(K)$ be a representation affording $\hat{\chi}$; we write $\mathfrak{X}(h) = (x_{kl}(h))_{1\leqslant k,l\leqslant d}$. If $\chi = \chi'$, we assume that $\mathfrak{X} = \mathfrak{X}'$. Since $\{C_w\}$ and $\{D_{w^{-1}}\}$ form a pair of dual bases with respect to the symmetrizing trace $\tau$, the Schur relations for the matrix coefficients of $\mathfrak{X}$ and $\mathfrak{X}'$ (see (3.1)) imply that

$$x_{kl}\Big(\sum_{w\in W} x'_{ii}(C_w) D_{w^{-1}}\Big) = \sum_{w\in W} x'_{ii}(C_w) x_{kl}(D_{w^{-1}}) = \begin{cases} \delta_{il}\, \delta_{ik}\, c_\chi & \text{if } \chi = \chi', \\ 0 & \text{if } \chi \neq \chi', \end{cases}$$

for any $1 \leqslant k,l \leqslant d$ and any $1 \leqslant i \leqslant d'$. Summing over all $i \in \{1, \ldots, d'\}$ and inserting this into the above expression for $z_{\mathfrak{C}}$, we obtain

$$x_{kl}(z_{\mathfrak{C}}) = m(\chi_{\mathfrak{C}}, \chi)\, \delta_{kl}\, c_\chi.$$

Thus, we see that $\mathfrak{X}(z_{\mathfrak{C}})$ is a scalar matrix, where the scalar is as required. Since this holds for all irreducible characters of $KH$, we conclude that $z_{\mathfrak{C}}$ is central. ☐

**Definition 3.3.** Let $\chi \in \mathrm{Irr}(W)$. Then the set

$$\{\alpha \in \Gamma \mid \alpha\, \hat{\chi}(\tilde{T}_w) \in \mathbb{R}[\Gamma_+ \cup \{1\}] \quad \text{for all } w \in W\}$$

has a unique minimal element, which is denoted by $\alpha_\chi$ and called the *generalized a-invariant* of $\chi$. We have

$$\alpha_\chi \in \Gamma_+ \cup \{1\} \quad \text{and} \quad c_\chi = r_\chi \alpha_\chi^{-2} + \text{higher terms, where } r_\chi \in \mathbb{R},\ r_\chi > 0.$$

(See Lusztig [18, 5.1.21] in the one-parameter case and [10, 9.4.7] in general.) The constant term of $(-1)^{l(w)}\alpha_\chi\hat{\chi}(\tilde{T}_w)$ will be denoted by $c_{w,\chi}$ and called the *leading coefficient* of $\hat{\chi}(T_w)$. We have

$$\sum_{w\in W} c_{w,\chi}\, c_{w,\chi'} = \begin{cases} r_\chi\, \chi(1) & \text{if } \chi=\chi',\\ 0 & \text{if } \chi\neq\chi'.\end{cases}$$

(See [18, 5.2] in the one-parameter case and [10, §9.4] in general.)

If all parameters $v_s$ are equal, we write $v = v_s$ $(s \in S)$. Then the generalized $a$-invariant is of the form $v^{a_\chi}$ for some $a_\chi \in \mathbb{N}_0$; the integer $a_\chi$ is precisely the $a$-invariant introduced by Lusztig [14]. The leading coefficients of the characters of $W$ are essential in understanding the structure of the left cells of $W$ (see [21] for the one-parameter case). They also play a basic role in the representation theory of reductive groups over finite fields; see [18].

**3.4. Induction from parabolic subgroups.** Let $I \subseteq S$ and consider the parabolic subgroup $W_I \subseteq W$ generated by $I$. Then we have a corresponding parabolic subalgebra $H_I \subseteq H$. Denote by $\text{Ind}_I^S$ the induction of characters, either from $W_I$ to $W$ or from $H_I$ to $H$. For any $\psi \in \text{Irr}(W_I)$ and $\chi \in \text{Irr}(W)$ let $m(\chi,\psi)$ be the multiplicity of $\chi$ in $\text{Ind}_I^S(\psi)$. Since $\text{Ind}_I^S$ is compatible with the specialization $v_s \mapsto 1$ (see [10, 9.1.9]), we have

(a) $$\text{Ind}_I^S(\hat{\psi}) = \sum_{\chi\in\text{Irr}(W)} m(\chi,\psi)\,\hat{\chi} \qquad \text{for all } \psi\in\text{Irr}(W_I).$$

Furthermore, by [10, 9.4.6], we have

(b) $$c_\psi^{-1} = \sum_{\chi\in\text{Irr}(W)} m(\chi,\psi)\, c_\chi^{-1} \qquad \text{for all } \psi\in\text{Irr}(W_I),$$

where $c_\psi \in A$ is the Schur element with respect to $KH_I$.

**Lemma 3.5** (See Lusztig [14] in the one-parameter case)**.** *Let $\psi \in \text{Irr}(W_I)$ where $I \subseteq S$. Let $\alpha_\psi$ be the corresponding generalized $a$-invariant. Then, for any $\chi \in \text{Irr}(W)$ with $m(\chi,\psi) \neq 0$, we have $\alpha_\psi \leqslant \alpha_\chi$. Furthermore, we have*

$$\frac{1}{r_\psi} = \sum_{\substack{\chi\in\text{Irr}(W)\\ \alpha_\chi=\alpha_\psi}} m(\chi,\psi)\,\frac{1}{r_\chi}.$$

*In particular, there exists at least one $\chi \in \text{Irr}(W)$ with $m(\chi,\psi) \neq 0$ and $\alpha_\chi = \alpha_\psi$.*

*Proof.* For each $\chi \in \text{Irr}(W)$, we write $c_\chi = r_\chi\alpha_\chi^{-2}(1 + f_\chi)$ where $f_\chi$ is an $\mathbb{R}$-linear combination of elements of $\Gamma_+$. Similarly, we write $c_\psi = r_\psi\alpha_\psi^{-2}(1+g_\psi)$, where $g_\psi$ is an $\mathbb{R}$-linear combination of elements of $\Gamma_+$. Now we also have $\prod_{\chi\in\text{Irr}(W)}(1 + f_\chi) = 1 + f$, where $f$ is an $\mathbb{R}$-linear combination of elements of $\Gamma_+$. Consequently, for each $\chi \in \text{Irr}(W)$, we have $(1 + f)/(1 + f_\chi) = 1 + f'_\chi$, where $f'_\chi$ is again an $\mathbb{R}$-linear combination of elements of $\Gamma_+$. Finally, let $\alpha \in \Gamma$ be the minimum of all $\alpha_\chi$ where $\chi \in \text{Irr}(W)$ is such that $m(\chi,\psi) \neq 0$. Then we have

$$\frac{\alpha^{-2}(1+f)}{c_\chi} = \frac{\alpha^{-2}\alpha_\chi^2(1+f)}{r_\chi(1+f_\chi)} = \frac{1}{r_\chi}(\alpha^{-1}\alpha_\chi)^2(1+f'_\chi).$$

If $m(\chi,\psi) \neq 0$, the above expression is an $\mathbb{R}$-linear combination of elements in $\Gamma_+ \cup \{1\}$, with constant term $1/r_\chi$ if $\alpha = \alpha_\chi$ and 0 otherwise. It follows that

$$\alpha^{-2}(1+f) \sum_{\chi \in \mathrm{Irr}(W)} m(\chi,\psi)\, \frac{1}{c_\chi}$$

is an $\mathbb{R}$-linear combination of elements in $\Gamma_+ \cup \{1\}$, with constant term

$$\sum_{\substack{\chi \in \mathrm{Irr}(W) \\ \alpha_\chi = \alpha}} \frac{1}{r_\chi} m(\chi,\psi).$$

Since all $r_\chi$ are positive, there are no cancellations in the sum and so the constant term is nonzero. On the other hand, using equation (b) in (3.4) we conclude that

$$\alpha^{-2}(1+f)\frac{1}{c_\psi} = \frac{1}{r_\psi}\alpha^{-2}\alpha_\psi^2 \frac{1+f}{1+g_\psi}$$

is also a $\mathbb{R}$-linear combination of elements of $\Gamma_+ \cup \{1\}$. In particular, this means that $1+g_\psi$ divides $1+f$ in $A$; it is easy to see that then $(1+f)/(1+g_\psi)$ is also of form $1+h$ where $h$ is an $\mathbb{R}$-linear combination of elements of $\Gamma_+$. We conclude that $\alpha \leqslant \alpha_\psi$ and

$$\alpha^{-2}(1+f)\frac{1}{c_\psi} = \frac{1}{r_\psi}\alpha^{-2}\alpha_\psi^2 + \text{higher terms}.$$

Thus, the constant term of $\alpha^{-2}(1+f)/c_\psi$ is $1/r_\psi$ if $\alpha = \alpha_\psi$ and 0 otherwise. We have seen above that the constant term must be nonzero. So we conclude that

$$\alpha = \alpha_\psi \qquad \text{and} \qquad \frac{1}{r_\psi} = \sum_{\substack{\chi \in \mathrm{Irr}(W) \\ \alpha_\chi = \alpha}} m(\chi,\psi)\, \frac{1}{r_\chi}.$$

This implies all the desired statements. ☐

**Definition 3.6.** Let $I \subseteq S$ and $\psi \in \mathrm{Irr}(W_I)$. Then we set

$$\mathrm{J}_{W_I}^W(\psi) = \sum_{\substack{\chi \in \mathrm{Irr}(W) \\ \alpha_\chi = \alpha_\psi}} m(\chi,\psi)\, \chi.$$

Extending by linearity, we obtain a map $\mathrm{J}_{W_I}^W : \mathbb{Z}\,\mathrm{Irr}(W_I) \to \mathbb{Z}\,\mathrm{Irr}(W)$, which is called the *generalized truncated induction*. By Lemma 3.5, we have $\mathrm{J}_{W_I}^W(\psi) \neq 0$ for all $\psi \in \mathrm{Irr}(W_I)$. Furthermore, this operation is transitive; we have

$$\mathrm{J}_{W_L}^W = \mathrm{J}_{W_I}^W \circ \mathrm{J}_{W_L}^{W_I} : \mathbb{Z}\,\mathrm{Irr}(W_L) \to \mathbb{Z}\,\mathrm{Irr}(W) \quad \text{for any subsets } L \subseteq I \subseteq S.$$

If there is no danger of confusion, we will also write $\mathrm{J}_I^S(\psi)$ instead of $\mathrm{J}_{W_I}^W(\psi)$ to abbreviate notation.

In the case of equal parameters, the above definition is due to Lusztig [14].

**Definition 3.7** (See Lusztig [16] and [18, 5.29] in the one-parameter case)**.** Let $\rho$ be the character of a representation of $W$ (not necessarily irreducible). We say that $\rho$ is *constructible* (with respect to the chosen total ordering of $\Gamma$) if it satisfies:

(a) If $W = \{1\}$, then $\rho = 1_W$ is the trivial character.
(b) If $W \neq \{1\}$, then there exists a proper subset $I \subsetneqq S$ and a constructible character $\rho'$ of $W_I$ such that $\rho = \mathrm{J}_I^S(\rho')$ or $\varepsilon \otimes \rho = \mathrm{J}_I^S(\rho')$, where $\varepsilon$ denotes the sign character.

(Note that this is an inductive definition.) Furthermore, we say that $\chi, \chi' \in \mathrm{Irr}(W)$ belong to the same *family* if and only if there exist constructible characters $\rho_1, \rho_2, \ldots, \rho_m$ (for some $m \geqslant 1$) such that $\chi$ occurs in $\rho_1$, $\chi'$ occurs in $\rho_m$, and $\rho_i, \rho_{i+1}$ have an irreducible constituent in common for all $i$. Thus, the families form a partition of $\mathrm{Irr}(W)$.

We say that the chosen total ordering of $\Gamma$ is *saturated* if every irreducible character of $W$ occurs with nonzero multiplicity in some constructible character.

There is one case in which the constructible characters are easily seen to be the irreducible characters of $W$. This is given by the following result.

**Proposition 3.8.** *Assume that, for any $I \subseteq S$ and any $\psi \in \mathrm{Irr}(W_I)$, we have $r_\psi = 1$ (with $r_\psi$ as in Definition 3.3). Then all constructible characters of $W$ are irreducible and each $\chi \in \mathrm{Irr}(W)$ forms a family by itself. Moreover, if the chosen total ordering is saturated, then every irreducible character is constructible.*

*Proof.* The statement about the families is clear once the assertions about the constructible characters are shown. To prove that all constructible characters are irreducible, we proceed by induction on the order of $W$. If $W = \{1\}$, then this is clearly true by definition. Now assume that $W \neq \{1\}$ and let $\rho$ be a constructible character of $W$. By definition, there exists a proper subset $I \subsetneqq S$ and a constructible character $\psi$ of $W_I$ such that $\rho = \mathrm{J}_I^S(\psi)$ or $\varepsilon \otimes \rho = \mathrm{J}_I^S(\psi)$. The induction hypothesis applies to $W_I$, so we know that $\psi$ is irreducible. Using now the identity in Lemma 3.5 and the fact that $r_\psi = r_\chi = 1$ for all $\chi \in \mathrm{Irr}(W)$, we obtain

$$1 = \sum_{\substack{\chi \in \mathrm{Irr}(W) \\ \alpha_\chi = \alpha_\psi}} m(\chi, \psi).$$

In other words, we have $\mathrm{J}_I^S(\psi) \in \mathrm{Irr}(W)$ and $\varepsilon \otimes \mathrm{J}_I^S(\psi) \in \mathrm{Irr}(W)$. So $\rho$ must be irreducible. This argument also shows that if the total ordering is saturated, then every irreducible character is constructible. ☐

**Example 3.9.** Assume that $(W, S)$ is of type $A_{n-1}$ such that $W$ is isomorphic to the symmetric group $\mathfrak{S}_n$. In this case, all parameters are equal; let us write $v = v_s$ ($s \in S$). Then we have $\Gamma = \{v^m \mid m \in \mathbb{Z}\}$ and the total ordering is given by $\Gamma_+ = \{v^m \mid m > 0\}$.

The $a$-invariants are simply given by the so-called $b$-invariants and the truncated induction reduces to the Macdonald–Lusztig–Spaltenstein induction; see [10, §5.2 and 9.4.5]. By the construction of the irreducible characters of $W$ in [10, §5.4], the ordering of $\Gamma$ is saturated. Furthermore, the hypothesis of Proposition 3.8 is satisfied (see again [10, 9.4.5]). Thus,

(a) the constructible characters are precisely the irreducible characters of $W$.

Now, Kazhdan and Lusztig have shown in [13, Theorem 1.4] that we have $\chi_{\mathfrak{C}} \in \mathrm{Irr}(W)$ for all left cells $\mathfrak{C}$ in $W$. In particular, this means that

(b) the constructible characters are precisely the characters afforded by the various left cells in $W$.

The leading coefficients of the character values of $H$ are determined by Lusztig in [18, 5.16]. The result is as follows.

(c) Let $\mathfrak{C}$ be a left cell in $W$ and $\chi = \chi_{\mathfrak{C}} \in \mathrm{Irr}(W)$. Then there exists a unique $x_0 \in \mathfrak{C}$ such that $c_{x_0,\chi} = \pm 1$ and we have $c_{x,\chi} = 0$ for all $x \in \mathfrak{C}$, $x_0 \neq x$.

Note, however, that the proof of (c) uses the orthogonality relations for leading coefficients in [18, Cor. 5.8] whose proof in turn is based on a property of left cells (see [19, 6.3(c)]) for which no elementary proof is known and which is only known to hold in the equal parameter case. In Theorem 5.4 below, we obtain an elementary proof of (c).

## 4. Orthogonal representations and leading coefficients

We keep the setting of the previous section. We will now introduce so-called "orthogonal" representations of $KH$; these are analogues of the "orthogonal" representations of finite groups, in the sense of [7, §73A]. We shall use them to define a refinement of the leading coefficients. This refinement will actually allow us to compute the leading coefficients in type $A_{n-1}$ and $B_n$; see Section 5.

**4.1.** Recall that $A = \mathbb{R}[\Gamma]$ and that we have chosen a total ordering of $\Gamma$. We define $\mathfrak{I}_+ \subset A$ to be the set of all $f \in A$ such that

$$f = 1 + \mathbb{R}\text{-linear combination of elements of } \Gamma_+.$$

Note that $\mathfrak{I}_+$ is multiplicatively closed. Furthermore, every element $x \in K$ can be written in the form

(a) $$x = r\gamma f/g \qquad \text{where } r \in \mathbb{R},\ \gamma \in \Gamma \text{ and } f, g \in \mathfrak{I}_+;$$

note that, here, $r$ and $\gamma$ are *uniquely determined*. Furthermore, let

(b) $$K_+ := \{x \in K \mid x = r\gamma f/g \text{ as in (a) with } \gamma = 1 \text{ or } \gamma \in \Gamma_+\}.$$

Note that we have $K_+ \cap \mathbb{R}[\Gamma] = \mathbb{R}[\Gamma_+ \cup \{1\}]$. Now let $x \in K_+$. Expressing $x$ as in (a), we define the *constant term* of $x$ to be 0 if $\gamma \in \Gamma_+$ and $r$ if $\gamma = 1$.

**Definition 4.2.** Let $\chi \in \mathrm{Irr}(W)$ and consider the corresponding irreducible character $\hat{\chi}$ of $KH$. Let $d$ be the degree of $\chi$ and $\mathfrak{X}\colon KH \to M_d(K)$ be a matrix representation affording $\hat{\chi}$. We say that $\mathfrak{X}$ is *orthogonal* if there exists a diagonal matrix $D \in M_d(K)$ satisfying the following conditions:

(a) We have $\mathfrak{X}(\tilde{T}_{w^{-1}}) = D^{-1}\mathfrak{X}(\tilde{T}_w)^{\mathrm{tr}}D$ for all $w \in W$.
(b) The diagonal entries of $D$ lie in $\mathfrak{I}_+$.

(Compare this definition with the similar notion for finite groups in [7, 73.10].) Note that if there is an orthogonal representation affording $\hat{\chi}$, then we have

$$\hat{\chi}(\tilde{T}_{w^{-1}}) = \hat{\chi}(\tilde{T}_w) \qquad \text{for all } w \in W.$$

If $E$ is a $KH$-module affording $\hat{\chi}$, the above conditions mean that there exists a symmetric bilinear form $(\ ,\ )$ on $E$ such that the following two conditions hold:

(a') We have $(\tilde{T}_w.e, e') = (e, \tilde{T}_{w^{-1}}.e')$ for all $w \in W$ and all $e, e' \in E$.
(b') There exists an orthogonal basis $\{v_i\}$ of $E$ such that $(v_i, v_i) \in \mathfrak{I}_+$ for all $i$.

Our first result shows that orthogonal representations always exist. The proof is a refinement of Lusztig's argument in [15, 1.7].

**Proposition 4.3.** *Every $\hat{\chi} \in \mathrm{Irr}(KH)$ is afforded by an orthogonal representation.*

*Proof.* Let $\hat{\chi} \in \mathrm{Irr}(KH)$ and consider a representation $\mathfrak{Y}\colon KH \to M_d(K)$ affording $\hat{\chi}$. Let $E = K^d$ be the corresponding $KH$-module with standard basis $\{e_1, \ldots, e_d\}$. Let $(\ ,\ )_\circ\colon E \times E \to K$ be the symmetric bilinear form for which $\{e_i\}$

is an orthonormal basis. Following the argument in the proof of [15, 1.7], we define a new bilinear form $(\ ,\ )\colon E \times E \to K$ by the formula

$$(e, e') := \sum_{w \in W} (\tilde{T}_w.e, \tilde{T}_w.e')_\circ \qquad \text{for any } e, e' \in E.$$

As in the proof in [15], it is easily checked that $(\tilde{T}_s.e, e') = (e, \tilde{T}_s.e')$ for all $s \in S$ and, hence, $(\tilde{T}_w.e, e') = (e, \tilde{T}_{w^{-1}}.e')$ for all $w \in W$. We now proceed in three steps.

*Step* 1. We claim that $(e, e) \neq 0$ for any $0 \neq e \in E$. More precisely, we claim that

$$\text{for any } 0 \neq e \in E, \text{ there exists some } a \in A \text{ such that } (ae, ae) \in \mathfrak{I}_+.$$

This is seen as follows. Let us write $e = \sum_i x_i e_i$ where $x_i \in K$. Multiplying $e$ by a nonzero common multiple of the denominators occurring in the coefficients $x_i$, we can assume that $x_i \in A$ for all $i$. Taking a further scalar multiple of $e$, we can also assume that $\tilde{T}_w.e$ is an $A$-linear combination of $e_1, \ldots, e_d$ for any $w \in W$. Then the defining formula for $(\ ,\ )$ shows that $(e, e)$ is a sum of squares of elements of $A$, at least one of which is nonzero. Consequently, there exists some $b_0 \in \mathbb{R}$, $b_0 > 0$, and some $\gamma_0 \in \Gamma$ such that

$$(e, e) = b_0\gamma_0^2 + \text{linear combination of } \gamma \in \Gamma \text{ with } \gamma\gamma_0^{-2} \in \Gamma_+.$$

Since we are working in $\mathbb{R}$, we can find a square root of $b_0$. Thus, dividing $e$ by $\sqrt{b_0}\gamma_0 \in A$, the above claim is established.

*Step* 2. We claim that there exists an orthogonal basis $\{v_1, \ldots, v_d\}$ of $E$ with respect to $(\ ,\ )$ such that the following conditions are satisfied for any $1 \leqslant i \leqslant d$:

$$(v_i, v_i) \in \mathfrak{I}_+ \qquad \text{and} \qquad v_i = \sum_{j \leqslant i} t_{ji} e_j \quad \text{with } t_{ij} \in K \text{ and } t_{ii} \neq 0.$$

We proceed by induction on $i$. First, consider $e_1$. By Step 1, there exists some $a_1 \in A$ such that $(a_1e_1, a_1e_1) \in \mathfrak{I}_+$. Thus, setting $v_1 := a_1e_1$, the desired conditions are satisfied for $v_1$. Now let $i > 1$ and assume that $v_1, \ldots, v_{i-1}$ have already been constructed. Then (as in the Gram–Schmidt orthogonalisation procedure) we set

$$v_i' = e_i - \sum_{k=1}^{i-1} \frac{(v_k, e_i)}{(v_k, v_k)}\, v_k.$$

Now, by Step 1, there exists some $a_i \in A$ such that $(a_iv_i', a_iv_i') \in \mathfrak{I}_+$. Thus, setting $v_i := a_iv_i'$, the required conditions are satisfied for $v_i$.

*Step* 3. Let $T = (t_{ij}) \in M_d(K)$ be the transition matrix between the bases $\{e_i\}$ and $\{v_i\}$. We define

$$\mathfrak{X}\colon KH \to M_d(K), \qquad \mathfrak{X}(\tilde{T}_w) = T^{-1}\mathfrak{Y}(\tilde{T}_w)T \in M_d(K)$$

and write $\mathfrak{X}(\tilde{T}_w) = (x_{ij}(\tilde{T}_w))_{1 \leqslant i,j \leqslant d}$ where $x_{ij}(\tilde{T}_w) \in K$. Then the action of $\tilde{T}_w$ on $E$ is given by $\tilde{T}_w.v_i = \sum_j x_{ji}(\tilde{T}_w)v_j$. Now, we have seen in the beginning of the proof that $(\tilde{T}_w.e, e') = (e, \tilde{T}_{w^{-1}}.e')$ for any $w \in W$ and any $e, e' \in E$. Applying this to $e = v_k$, $e' = v_l$ and the element $w^{-1} \in W$ yields

$$(v_l, v_l)x_{lk}(\tilde{T}_{w^{-1}}) = (\tilde{T}_{w^{-1}}.v_k, v_l) = (v_k, \tilde{T}_w.v_l) = (v_k, v_k)x_{kl}(\tilde{T}_w).$$

Thus, if $D \in M_d(K)$ is the diagonal matrix with diagonal entries $(v_i, v_i) \in \mathfrak{I}_+$ $(1 \leqslant i \leqslant d)$, then we have $D\mathfrak{X}(\tilde{T}_{w^{-1}}) = \mathfrak{X}(\tilde{T}_w)^{\mathrm{tr}}D$ for all $w \in W$. Consequently, $\mathfrak{X}$ is an orthogonal representation, and it affords $\hat{\chi}$ since it is equivalent to $\mathfrak{Y}$. $\square$

We can now define a refinement of the leading coefficients. Recall the definitions of $\mathfrak{I}_+$ and $K_+$ from (4.1).

**Theorem 4.4.** *Let $\mathfrak{X}\colon KH \to M_d(K)$ be an orthogonal representation affording $\hat{\chi} \in \mathrm{Irr}(KH)$. Let us write $\mathfrak{X}(h) = (x_{ij}(h))_{1\leqslant i,j\leqslant d}$ for $h \in KH$. Then, for any $1 \leqslant i,j \leqslant d$ and any $w \in W$, we have*

$$(-1)^{l(w)}\,\alpha_\chi\, x_{ij}(\tilde{T}_w) \in K_+, \qquad \text{with } \alpha_\chi \text{ as in Definition 3.3;}$$

*the constant term of this element will be denoted by $c_{w,\mathfrak{X}}^{ij}$. Then the following hold.*

(a) *For each $1 \leqslant i,j \leqslant d$, there exists some $w \in W$ such that $c_{w,\mathfrak{X}}^{ij} \neq 0$.*
(b) *For all $w \in W$ and all $1 \leqslant i,j \leqslant d$, we have $c_{w^{-1},\mathfrak{X}}^{ij} = c_{w,\mathfrak{X}}^{ji}$ and*

$$c_{w,\chi} = \sum_{i=1}^{d} c_{w,\mathfrak{X}}^{ii}, \quad \text{with } c_{w,\chi} \in \mathbb{R} \text{ as in Definition 3.3.}$$

(c) *Let $\mathfrak{Y}\colon KH \to M_e(K)$ be orthogonal affording $\hat{\psi} \in \mathrm{Irr}(KH)$. Then we have the following Schur relations, for any $1 \leqslant i,j \leqslant d$ and $1 \leqslant k,l \leqslant e$:*

$$\sum_{w\in W} c_{w,\mathfrak{X}}^{ij} c_{w,\mathfrak{Y}}^{lk} = \begin{cases} \delta_{il}\,\delta_{jk}\, r_\chi & \text{if } \mathfrak{X} = \mathfrak{Y} \text{ and with } r_\chi \in \mathbb{R} \text{ as in Def. 3.3,} \\ 0 & \text{if } \mathfrak{X} \text{ is not equivalent to } \mathfrak{Y} \text{ (i.e., } \hat{\chi} \neq \hat{\psi}\text{).} \end{cases}$$

We call the elements $c_{w,\mathfrak{X}}^{ij}$ the *leading coefficients* of the matrix entries of $\mathfrak{X}$.

*Proof.* Let us fix $1 \leqslant i,j \leqslant d$. We can write $x_{ij}(\tilde{T}_w) = f_w/g_w$ with $f_w \in A$ and $g_w \in \mathfrak{I}_+$. By the definition of $\mathfrak{I}_+$, we have $\prod_{w\in W} g_w = 1+g$, where $g$ is an $\mathbb{R}$-linear combination of elements of $\Gamma_+$. We also write $1+g = g_w(1+g_w')$ where $g_w'$ is an $\mathbb{R}$-linear combination of elements of $\Gamma_+$. Now the set

$$\{\alpha \in \Gamma \mid \alpha\, f_w \in \mathbb{R}[\Gamma_+ \cup \{1\}] \text{ for all } w \in W\}$$

has a unique minimal element, which we denote by $\alpha_0 \in \Gamma$. Then we can write uniquely

$$(*) \qquad \alpha_0\, f_w = r_w + \mathbb{R}\text{-linear combination of elements of } \Gamma_+,$$

where $r_w \in \mathbb{R}$; furthermore, there exists some $w \in W$ for which $r_w \neq 0$.
*Claim.*

$$\alpha_0 = \alpha_\chi \qquad \text{and} \qquad r_w = (-1)^{l(w)} c_{w,\mathfrak{X}}^{ij}.$$

To prove this, we consider the Schur relations for the matrix coefficients of $\mathfrak{X}$:

$$c_\chi = \sum_{w\in W} x_{ij}(\tilde{T}_w) x_{ji}(\tilde{T}_{w^{-1}}).$$

The condition in Definition 4.2(a) means that $x_{ji}(\tilde{T}_{w^{-1}}) = d_j^{-1} x_{ij}(\tilde{T}_w) d_i$, where $d_i$ and $d_j$ are diagonal entries in $D$. Hence we obtain

$$(1+g)^2 d_j\, c_\chi = (1+g)^2 \sum_{w\in W} d_i (x_{ij}(\tilde{T}_w))^2 = \sum_{w\in W} d_i (1+g_w')^2 f_w^2.$$

Multiplying the above equation with $\alpha_0^2$ and inserting $(*)$ yields (note that we also have $d_i - 1 \in \mathbb{R}[\Gamma_+]$ by Definition 4.2(b)):

$$\begin{aligned}\alpha_0^2(1+g)^2 d_j c_\chi &= \sum_{w\in W} d_i(1+g'_w)^2(r_w + \text{combination of elements of } \Gamma_+)^2\\ &= \sum_{w\in W}(r_w + \text{combination of elements of } \Gamma_+)^2\\ &= \underbrace{\Big(\sum_{w\in W} r_w^2\Big)}_{>0} + \text{combination of elements of } \Gamma_+.\end{aligned}$$

In particular, the right-hand side has a nonzero constant term and does not contain any terms in $\Gamma_-$. Now consider the left-hand side. We can write $\alpha_\chi^2 c_\chi = r_\chi +$ combination of elements of $\Gamma_+$. Noting also that $d_j - 1 \in \mathbb{R}[\Gamma_+]$, we conclude that

$$\alpha_0^2(1+g)^2 d_j\, c_\chi = \Big(\frac{\alpha_0}{\alpha_\chi}\Big)^2\Big(r_\chi + \text{combination of elements of } \Gamma_+\Big).$$

Comparing with the above expression for the right-hand side shows that

$$\alpha_0 = \alpha_\chi \qquad \text{and} \qquad \sum_{w\in W} r_w^2 = r_\chi.$$

Equation $(*)$ and the fact that $r_w \neq 0$ for some $w \in W$ now imply that $r_w = (-1)^{l(w)} c_{w,\mathfrak{X}}^{ij}$. Thus, the above claim is proved. Now we can write

$$(-1)^{l(w)}\alpha_\chi g_{w,\mathfrak{X}}^{ij} x_{ij}(\tilde{T}_w) = c_{w,\mathfrak{X}}^{ij} + \mathbb{R}\text{-linear combination of elements of } \Gamma_+,$$

where $g_{w,\mathfrak{X}}^{ij} \in \mathfrak{I}_+$. Let us consider the assertions in (a), (b) and (c). Since $r_w = (-1)^{l(w)} c_{w,\mathfrak{X}}^{ij}$ and since there exists some $w \in W$ with $r_w \neq 0$, we have (a). To prove (b), we argue as follows. Since $d_i, g_{w,\mathfrak{X}}^{ij} \in \mathfrak{I}_+$, the real number $c_{w,\mathfrak{X}}^{ij}$ is the constant term of

$$d_i(-1)^{l(w)}\alpha_\chi\, g_{w,\mathfrak{X}}^{ij} x_{ij}(\tilde{T}_w) \in \mathbb{R}[\Gamma_+ \cup \{1\}].$$

Now note that $d_i x_{ij}(\tilde{T}_w) = d_j x_{ji}(\tilde{T}_{w^{-1}})$ by Definition 4.2(a). So the above expression equals

$$d_j(-1)^{l(w)}\alpha_\chi\, g_{w,\mathfrak{X}}^{ij} x_{ji}(\tilde{T}_{w^{-1}}) \in \mathbb{R}[\Gamma_+ \cup \{1\}]$$

and this expression has constant term $c_{w^{-1},\mathfrak{X}}^{ji}$. Thus, (b) is proved. Next, we have already seen above that

$$r_\chi = \sum_{w\in W} r_w^2 = \sum_{w\in W}(c_{w,\mathfrak{X}}^{ij})^2$$

which is the desired relation in (c) for $\mathfrak{X} = \mathfrak{Y}$, $i = l$, $j = k$. If $i \neq l$ or $j \neq k$, the identity $\sum_{w\in W} c_{w,\mathfrak{X}}^{ij} c_{w,\mathfrak{X}}^{lk} = 0$ follows from the Schur relation

$$\sum_{w\in W} x_{ij}(\tilde{T}_w) x_{kl}(\tilde{T}_{w^{-1}}) = 0,$$

by an argument completely analogous to that above. Finally, in order to prove (c) in the case where $\mathfrak{X} \not\cong \mathfrak{Y}$, let us write $\mathfrak{Y}(h) = (y_{kl}(h))_{1\leqslant k,l\leqslant e}$ for $h \in KH$. We have already seen above that

$$(-1)^{l(w)}\alpha_\chi g_{w,\mathfrak{X}}^{ij} x_{ij}(\tilde{T}_w) = c_{w,\mathfrak{X}}^{ij} + \mathbb{R}\text{-linear combination of elements of } \Gamma_+,$$
$$(-1)^{l(w)}\alpha_{\chi'} g_{w,\mathfrak{Y}}^{kl} y_{kl}(\tilde{T}_w) = c_{w,\mathfrak{Y}}^{kl} + \mathbb{R}\text{-linear combination of elements of } \Gamma_+,$$

for some $g_{w,\mathfrak{X}}^{ij}, g_{w,\mathfrak{Y}}^{kl} \in \mathfrak{I}_+$. We can now argue by a reasoning entirely analogous to that above, using the Schur relations [10, 7.2.2]:

$$\sum_{w\in W} x_{ij}(\tilde{T}_w) y_{kl}(\tilde{T}_{w^{-1}}) = 0$$

for any $1 \leqslant i,j \leqslant d$ and $1 \leqslant k,l \leqslant e$. □

*Remark* 4.5. The leading coefficients of the matrix entries of $\mathfrak{X}$ also have the following properties with respect to the Kazhdan–Lusztig basis of $H$ and its dual. For any $w \in W$, we have

$$(-1)^{l(w)}\alpha_\chi\, x_{ij}(C_w) \in K_+ \text{ and the constant term is } c_{w,\mathfrak{X}}^{ij},$$
$$(-1)^{l(w)}\alpha_\chi\, x_{ij}(D_w) \in K_+ \text{ and the constant term is } c_{w,\mathfrak{X}}^{ij}.$$

This immediately follows from Definition 3.3 and the expressions for $C_w$ and $D_w$ in (2.4), using that $\overline{P}_{y,w}^*$ and $\overline{Q}_{y,w}^*$ are linear combinations of elements of $\Gamma_+$ if $y < w$ (see also [18, 5.2.1 and 5.2.2]).

*Remark* 4.6. By [10, 9.3.5], the character values $\hat{\chi}(\tilde{T}_w)$ lie in $A = R[v_s^{\pm 1} \mid s \in S]$, where $R$ is the ring of algebraic integers in a splitting field for $W$. Consequently, the leading coefficients $c_{w,\chi}$ in Definition 3.3 are algebraic integers in a splitting field for $W$. In particular, if $W$ is crystallographic, then $\mathbb{Q}$ is a splitting field and the $c_{w,\chi}$ are rational integers. On the other hand, in general, we cannot expect the leading coefficients $c_{w,\mathfrak{X}}^{ij}$ in Theorem 4.4 to be algebraic integers.

We keep the above set-up where we assume that $\hat{\chi} \in \mathrm{Irr}(KH)$ is afforded by an orthogonal representation $\mathfrak{X}\colon KH \to M_d(K)$. Now we consider a left cell $\mathfrak{C}$ in $W$ with respect to the chosen total ordering of $\Gamma$ (see Section 2).

**Proposition 4.7.** *Let $\mathfrak{C}$ be a left cell in $W$ and $1 \leqslant i,j \leqslant d$. Then we have*

$$\sum_{k=1}^{d}\sum_{w\in\mathfrak{C}} c_{w,\mathfrak{X}}^{ik}\, c_{w,\mathfrak{X}}^{jk} = \delta_{ij}\, m(\chi_{\mathfrak{C}},\chi)\, r_\chi.$$

*In particular, if $\chi$ occurs with nonzero multiplicity in $\chi_{\mathfrak{C}}$, then some leading coefficient $c_{w,\mathfrak{X}}^{ik}$ with $w \in \mathfrak{C}$ is nonzero.*

*Proof.* Consider the element $z_{\mathfrak{C}}$ defined in Proposition 3.2. We have $\alpha_\chi^2\, x_{ij}(z_{\mathfrak{C}}) = \delta_{ij}\, m(\chi_{\mathfrak{C}},\chi)\,\alpha_\chi^2\, c_\chi$ and, by Definition 3.3, this expression lies in $K_+$ and has constant term $\delta_{ij}\, m(\chi_{\mathfrak{C}},\chi) r_\chi$. On the other hand, we also have

$$\alpha_\chi^2\, x_{ij}(z_{\mathfrak{C}}) = \sum_{w\in\mathfrak{C}} \alpha_\chi^2\, x_{ij}(C_w D_{w^{-1}}) = \sum_{k=1}^{d}\sum_{w\in\mathfrak{C}} (\alpha_\chi\, x_{ik}(C_w))(\alpha_\chi\, x_{kj}(D_{w^{-1}}))$$

and, using Remark 4.5, this expression lies in $K_+$ and has constant term

$$\sum_{k=1}^{d}\sum_{w\in\mathfrak{C}} c_{w,\mathfrak{X}}^{ik}\, c_{w,\mathfrak{X}}^{jk}.$$

This yields the desired identity. □

Our next aim is to show that orthogonal representations lead to strong results about leading coefficients if an additional integrality condition is satisfied. For this purpose, we need the following definition, which is inspired from Rouquier [23].

**Definition 4.8.** Recall that we have chosen a total ordering of $\Gamma$. Let $\mathfrak{I}_+$ be as in (4.1). Then we set $A_0 = \mathbb{Z}[\Gamma]$ and define

$$\mathcal{O} := \left\{ \frac{f}{g} \,\middle|\, f \in A_0, g \in \mathfrak{I}_+ \cap A_0 \right\} \subseteq K.$$

As a first application we obtain the following result.

**Lemma 4.9.** *Assume that $\hat{\chi} \in \mathrm{Irr}(KH)$ is afforded by an orthogonal representation $\mathfrak{X}\colon KH \to M_d(K)$ such that $\mathfrak{X}(\tilde{T}_w) \in M_d(\mathcal{O})$ for all $w \in W$. Then we have*

$$c_{w,\mathfrak{X}}^{ij} \in \mathbb{Z} \quad \textit{for all } w \in W \textit{ and } 1 \leqslant i, j \leqslant d.$$

*Proof.* Write $\mathfrak{X}(\tilde{T}_w) = (x_{ij}(\tilde{T}_w))_{1 \leqslant i,j \leqslant d}$. By definition, we have

$$(-1)^{l(w)} \alpha_\chi x_{ij}(\tilde{T}_w) \in K_+ \cap \mathcal{O}.$$

It remains to observe that any element in $K_+ \cap \mathcal{O}$ has a constant term in $\mathbb{Z}$. □

**Theorem 4.10.** *Let $\chi \in \mathrm{Irr}(W)$ and $\mathfrak{X}\colon KH \to M_d(K)$ be an orthogonal representation affording the corresponding $\hat{\chi} \in \mathrm{Irr}(KH)$. Assume that $r_\chi = 1$ and $\mathfrak{X}(\tilde{T}_w) \in M_d(\mathcal{O})$ for all $w \in W$. (Recall the definition of $r_\chi$ in 3.3.) Then there exists a unique injective map*

$$\{(i,j) \mid 1 \leqslant i,j \leqslant d\} \to W, \qquad (i,j) \mapsto w(i,j),$$

*such that $c_{w,\mathfrak{X}}^{ij} \neq 0$ for $w = w(i,j)$. Furthermore, the following hold:*

- (a) *For $1 \leqslant i,j \leqslant d$, we have $c_{w(i,j),\mathfrak{X}}^{ij} = \pm 1$ and $w(i,j) = w(j,i)^{-1}$; in particular, the elements $w(i,i)$ are involutions. We have $c_{w(i,i),\chi} = \pm 1$ for $1 \leqslant i \leqslant d$.*
- (b) *For a fixed $j$, the elements $w(i,j)$ $(1 \leqslant i \leqslant d)$ lie in the same left cell and the elements $w(j,i)$ $(1 \leqslant i \leqslant d)$ lie in the same right cell. Consequently, all the elements $w(i,j)$ $(1 \leqslant i,j \leqslant d)$ lie in the same two-sided cell.*
- (c) *If $\mathfrak{C}$ is any left cell such that $\chi \in \mathrm{Irr}(W)$ occurs as a constituent in $\chi_{\mathfrak{C}}$, then $\mathfrak{C}$ contains an involution $w(k,k)$, for some $1 \leqslant k \leqslant d$.*

*Proof.* Let $1 \leqslant i,j \leqslant d$. Then the orthogonality relations in Theorem 4.4(c) yield

$$\sum_{w \in W} (c_{w,\mathfrak{X}}^{ij})^2 = r_\chi = 1.$$

Since the leading coefficients are integers by Lemma 4.9, we conclude that there exists a unique $w = w(i,j) \in W$ such that $c_{w,\mathfrak{X}}^{ij} \neq 0$; in fact, we have $c_{w,\mathfrak{X}}^{ij} = \pm 1$. The uniqueness and Theorem 4.4(b) imply that

$$w(j,i) = w(i,j)^{-1} \qquad \text{for all } 1 \leqslant i,j \leqslant d. \tag{1}$$

Now let $1 \leqslant k,l \leqslant d$ be such that $i \neq k$ or $j \neq l$. Then we claim that $w(k,l) \neq w(i,j)$. Indeed, if this were not the case, the orthogonality relations would yield

$$\pm 1 = c_{w(i,j),\mathfrak{X}}^{ij}\, c_{w(k,l),\mathfrak{X}}^{kl} = \delta_{ik}\,\delta_{jl}\,c_\chi = 0,$$

a contradiction. So we must have $w(i,j) \neq w(k,l)$. Thus, we have established the existence and uniqueness of the required injection. Furthermore, we have seen that $c_{w,\mathfrak{X}}^{ij} = \pm 1$ for $w = w(i,j)$, proving the first statement in (a). Now assume that $i = j$

and set $w(i) = w(i,i)$. Then (1) shows that $w(i) = w(i)^{-1}$ is an involution. Now Theorem 4.4(b) implies that $c_{w(i),\chi} = \sum_{j=1}^{d} c_{w(i),\mathfrak{X}}^{jj} = c_{w(i),\mathfrak{X}}^{ii} = \pm 1$. Conversely, assume that $c_{w,\chi} \neq 0$ for some $w \in W$. Then Theorem 4.4(b) implies that $c_{w,\mathfrak{X}}^{ii} \neq 0$ for some $i$. The uniqueness of $w(i)$ then shows that $w = w(i)$. Thus, we have

$$(2) \qquad \begin{cases} c_{w,\chi} \neq 0 \text{ if and only if } w = w(i) \text{ for some } 1 \leqslant i \leqslant d; \\ c_{w(i),\chi} = \pm 1 \text{ and } w(i)^2 = 1 \text{ for all } 1 \leqslant i \leqslant d, \end{cases}$$

and the remaining statements in (a) are proved. To prove (b), fix $j$ and let $w = w(i,j)$, $y = w(i',j)$ for $1 \leqslant i, i' \leqslant d$. First we check that $w \leqslant_L y$. We apply $\mathfrak{X}$ to the product $C_y D_{w^{-1}}$. The $(i',i)$-coefficient is given by

$$\mathfrak{X}(C_y D_{w^{-1}})_{i'i} = (\mathfrak{X}(C_y)\mathfrak{X}(D_{w^{-1}}))_{i'i} = \sum_{k=1}^{d} x_{i'k}(C_y) x_{ki}(D_{w^{-1}}).$$

Using Remark 4.5, this implies that

$$(-1)^{l(y)+l(w)} \alpha_\chi^2 \, \mathfrak{X}(C_y D_{w^{-1}})_{i'i} \in K_+$$

and the constant term of this element equals

$$\sum_{k=1}^{d} c_{y,\mathfrak{X}}^{i'k} \, c_{w^{-1},\mathfrak{X}}^{ki} = \sum_{k=1}^{d} c_{y,\mathfrak{X}}^{i'k} \, c_{w,\mathfrak{X}}^{ik} = c_{y,\mathfrak{X}}^{i'j} \, c_{w,\mathfrak{X}}^{ij} = \pm 1,$$

where the last two equalities hold since $w = w(i,j)$ and $y = w(i',j)$. In particular, this shows $\mathfrak{X}(C_y D_{w^{-1}})_{i'i} \neq 0$. Hence we also have $C_y D_{w^{-1}} \neq 0$ and so $w \leqslant_L y$, by Lemma 2.5. Applying $\mathfrak{X}$ to the product $C_w D_{y^{-1}}$, we also find $y \leqslant_L w$ and, hence, $y \sim_L w$. On the other hand, we also have $w(j,i)^{-1} = w \sim_L y = w(j,i')^{-1}$. Thus, for fixed $j$, all the elements $w(j,i)$ $(1 \leqslant i \leqslant d)$ lie in the same right cell. Hence, all elements $w(i,j)$ lie in the same two-sided cell.

Finally, to prove (c), let $\mathfrak{C}$ be any left cell such that $\chi$ occurs in $\chi_{\mathfrak{C}}$. Let $1 \leqslant i \leqslant d$. Then Proposition 4.7 shows that

$$\sum_{k=1}^{d} \sum_{w \in \mathfrak{C}} (c_{w,\mathfrak{X}}^{ik})^2 = m(\chi_{\mathfrak{C}}, \chi) r_\chi \neq 0.$$

Hence, there exists some $w \in \mathfrak{C}$ and some $k$ such that $c_{w,\mathfrak{X}}^{ik} \neq 0$. Thus, we must have $w = w(i,k) \in \mathfrak{C}$. On the other hand, by (b), all the elements $w(j,k)$ $(1 \leqslant j \leqslant d)$ lie in the same left cell. In particular, we have $w(k,k) \in \mathfrak{C}$. ☐

In the setting of Theorem 4.10 it may be conjectured that, for each fixed $j$, the set $\{w(i,j) \mid 1 \leqslant j \leqslant d\}$ actually is a left cell.

**Corollary 4.11.** *For any $\chi \in \mathrm{Irr}(W)$, we assume that the conditions in Theorem 4.10 are satisfied. Then the following hold:*

(a) *Set $\mathcal{D}_\chi := \{w \in W \mid c_{w,\chi} \neq 0\}$. Then $\mathcal{D}_\chi \cap \mathcal{D}_{\chi'} = \varnothing$ if $\chi \neq \chi'$ and $\bigcup_\chi \mathcal{D}_\chi$ is the set of all involutions in $W$.*

(b) *For any $w \in W$, we have $c_{w,\chi} = 0$ unless $w^2 = 1$. If $c_{w,\chi} \neq 0$, then $c_{w,\chi} = \pm 1$.*

*Proof.* By Theorem 4.10, we have $\mathcal{D}_\chi = \{w(i,i) \mid 1 \leqslant i \leqslant d\}$. Next, let $\chi' \in \mathrm{Irr}(W)$, $\chi' \neq \chi$, and let $\mathfrak{Y} \colon KH \to M_e(K)$ be an orthogonal representation over $\mathcal{O}$ affording the corresponding $\hat{\chi}' \in \mathrm{Irr}(KH)$. Then, for each $1 \leqslant k, l \leqslant e$, there exists a unique element $w(k,l)' \in W$ such that $c_{w(k,l)',\mathfrak{Y}}^{kl} \neq 0$; furthermore, we have

$$\mathcal{D}_{\chi'} = \{w(k,k)' \mid 1 \leqslant k \leqslant e\}.$$

Now assume that there exists some $i \in \{1, \ldots, d\}$ and some $k \in \{1, \ldots, e\}$ such that $w(i,i) = w(k,k)'$. Then the orthogonality relations in Theorem 4.4(c) yield

$$0 = \sum_{w \in W} c_{w,\mathfrak{X}}^{ii} c_{w,\mathfrak{Y}}^{kk} = c_{w(i,i),\mathfrak{X}}^{ii} c_{w(k,k)',\mathfrak{Y}}^{kk} = \pm 1,$$

a contradiction. Thus, we have $\mathcal{D}_\chi \cap \mathcal{D}_{\chi'} = \varnothing$. It follows that

$$\Big| \bigcup_{\chi \in \mathrm{Irr}(W)} \mathcal{D}_\chi \Big| = \sum_{\chi \in \mathrm{Irr}(W)} |\mathcal{D}_\chi| = \sum_{\chi \in \mathrm{Irr}(W)} \chi(1) \geqslant |\{w \in W \mid w^2 = 1\}|,$$

where the last inequality is a well-known result for finite groups (see, for example, [7, Ex. 73.5]). Since each set $\mathcal{D}_\chi$ consists of involutions, we have, in fact, equality. □

## 5. Constructible characters and leading coefficients in type $B_n$

In this section, let $(W, S)$ be of type $A_{n-1}$ or $B_n$ ($n \geqslant 2$), with generators and relations given by the following diagrams:


We assume that $\Gamma = \{v_t^m v^n \mid m, n \in \mathbb{Z}\}$, where $v_t$ and $v$ are independent indeterminates, and that the total ordering on $\Gamma$ is given by

$$(*) \qquad \Gamma_+ = \{v_t^k v^l \mid k > 0, l \in \mathbb{Z}\} \cup \{v^l \mid l > 0\}.$$

In type $A_{n-1}$, the set $\mathrm{Irr}(W)$ has a natural parametrization by the set of partitions of $n$; in type $B_n$, we have a natural parametrization of $\mathrm{Irr}(W)$ by the set of all pairs of partitions $(\lambda, \mu)$ such that $|\lambda| + |\mu| = n$; see [10, Chap. 5]. The set of irreducible characters in type $A_{n-1}$ will be identified with the set of those irreducible characters in type $B_n$ which are labelled by pairs of partitions $(\lambda, \mu)$ such that $\mu = \varnothing$. (This identification is compatible with the subsequent formulas for Schur elements, $a$-invariants etc.; see also [11, §2.3].)

The first consequence of the above condition $(*)$ is the following result.

*Remark* 5.1. Explicit formulas for the Schur elements $c_\chi$ are due to Steinberg (in type $A$) and Hoefsmit [11] (in type $B$); see also [10, 10.5.3]. Writing $c_\chi = r_\chi \alpha_\chi^{-2} +$ higher terms as in Definition 3.3, these formulas show that we have

$$r_\chi = 1 \qquad \text{for all } \chi \in \mathrm{Irr}(W),$$
$$\alpha_\chi = v^{n(\lambda)+2n(\mu)-n^*(\mu)}\, v_t^{|\mu|} \qquad \text{if } \chi \text{ is labelled by } (\lambda, \mu),$$

where for any partition $\nu = (\nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_r \geqslant 0)$ we define

$$n(\nu) = \sum_{i=1}^{r} (i-1)\nu_i \quad \text{and} \quad n^*(\nu) = \sum_{i=1}^{r} \nu_i(\nu_i - 1)/2.$$

**Proposition 5.2.** *All constructible characters of $W$ are irreducible. Furthermore, the above ordering of $\Gamma$ is saturated and so all irreducible characters are constructible.*

*Proof.* Since $r_\chi = 1$ for all $\chi \in \mathrm{Irr}(W)$, the first statement follows from Proposition 3.8. To prove the second statement, we shall need the following preliminary result about the induction of characters. Let $k, l \geqslant 0$ be such that $n = k + l$. Then we have a natural embedding $W_k \times \mathfrak{S}_l \subseteq W_n$, where $W_k$ is generated by

$t, s_1, \ldots, s_{k-1}$ (if $k \geqslant 1$) and $\mathfrak{S}_l$ is generated by $s_{k+1}, \ldots, s_{n-1}$ (if $l \geqslant 1$); we use the convention that $W_0 = \mathfrak{S}_0 = \{1\}$. Now let $(\lambda', \mu')$ be a pair of partitions such that $|\lambda'| + |\mu'| = k$ and consider the corresponding irreducible character $\chi_{(\lambda',\mu')}$ of $W_k$; let $\varepsilon_l$ be the sign character of $\mathfrak{S}_l$. Then we claim that

$$(\dagger) \qquad \mathrm{J}_{W_k \times \mathfrak{S}_l}^{W_n}\Big(\chi_{(\lambda',\mu')} \boxtimes \varepsilon_l\Big) = \chi_{(\lambda,\mu')},$$

where the partition $\lambda$ is obtained from $\lambda'$ by increasing the $l$ largest parts by 1.

This is seen by an argument which is similar to the proof in the case of equal parameters; see [10, 6.4.7]. By Pieri's rule (see [10, Ex. 6.3]), we have

$$\mathrm{Ind}_{W_k \times \mathfrak{S}_l}^{W_n}\Big(\chi_{(\lambda',\mu')} \boxtimes \varepsilon_l\Big) = \sum_{(\lambda,\mu)} \chi_{(\lambda,\mu)},$$

where the sum is over all pairs $(\lambda, \mu)$ such that, for some $0 \leqslant d \leqslant l$, the Young diagram of $\lambda$ (resp., $\mu$) is obtained from that of $\lambda'$ (resp., $\mu'$) by adding $d$ (resp., $l-d$) boxes, with no two boxes in the same row. Consider a pair $(\lambda, \mu)$ occurring in the above sum. If $d < l$ (i.e., the Young diagram of $\mu$ has more boxes than that of $\mu'$), then $\alpha_{(\lambda',\mu')} < \alpha_{(\lambda,\mu)}$ and so $\chi_{(\lambda,\mu)}$ will not occur in $\mathrm{J}_{W_k \times \mathfrak{S}_l}^{W_n}\Big(\chi_{(\lambda',\mu')} \boxtimes \varepsilon_l\Big)$. Hence, it is enough to consider pairs $(\lambda, \mu)$ with $\mu = \mu'$. Then $\lambda$ is obtained by adding $l$ boxes to $\lambda'$, with no two in the same row. Let $1 \leqslant i_1 < \cdots < i_l$ be the rows where a box is added. Then we have

$$n(\lambda) - n(\lambda') = \sum_{i \geqslant 1} (i-1)(\lambda_i - \lambda_i') = \sum_{j=1}^{l} (i_j - 1) \geqslant \sum_{j=1}^{l} (j-1) = l(l-1)/2,$$

with equality if and only if $i_j = j$ for all $j$. Now note that the generalized $a$-invariant of $\chi_{(\chi',\mu)} \boxtimes \varepsilon_l$ is obtained from that of $\chi_{(\chi',\mu)}$ by multiplication with $v^{l(l-1)/2}$. Thus, we see that the above claim (†) is proved.

Now we can argue as follows to prove that the ordering is saturated. Let $n \geqslant 2$. By an inductive hypothesis, we may assume that the saturation condition is satisfied for $W_k$ with $k < n$ and so all irreducible characters of $W_k$ are constructible. Now let $(\lambda, \mu)$ be a pair of partitions such that $|\lambda| + |\mu| = n$. Then we have two cases.

(a) Assume that $\lambda \neq \varnothing$ and write $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > \lambda_{r+1} = 0)$ where $r \geqslant 1$. Let $k \geqslant 1$ be such that $\lambda_1 = \cdots = \lambda_k > \lambda_{k+1}$ and let $\lambda'$ be the partition obtained from $\lambda$ by decreasing the $k$ first parts by 1. We set $l = n - k$. Now (†) applies and so

$$\mathrm{J}_{W_k \times \mathfrak{S}_l}^{W_n}\Big(\chi_{(\lambda',\mu)} \boxtimes \varepsilon_l\Big) = \chi_{(\lambda,\mu)}.$$

Hence, the saturation condition is satisfied for $\chi_{(\lambda,\mu)}$.

(b) Assume that $\lambda = \varnothing$. Then consider $\varepsilon \otimes \chi_{(\varnothing,\mu)} = \chi_{(\mu^*,\varnothing)}$, where $\mu^*$ denotes the transposed partition (see [10, 5.5.6]). We can now apply the argument in (a) to the pair $(\mu^*, \varnothing)$ and so the saturation condition is satisfied for $\chi_{(\mu^*,\varnothing)}$.

Thus, we have shown that the ordering on $\Gamma$ is saturated and so the constructible characters are precisely the irreducible characters of $W$. ☐

An analogy with the case of equal parameters suggests that the characters carried by the various left cells in type $B_n$ are irreducible. (Further evidence for this is also provided by [5].) Our next result is an integral version of Proposition 4.3.

**5.3. Hoefsmit's matrices.** For any $\hat{\chi} \in \mathrm{Irr}(KH)$, Hoefsmit [11, §2] has constructed a representation $\mathfrak{Y}\colon KH \to M_d(K)$ affording $\hat{\chi}$ such that

$$\mathfrak{Y}(\tilde{T}_w) \in M_d(\mathcal{O}) \qquad \text{for all } w \in W;$$

(see also [10, §10.1]). We shall now show that Hoefsmit's representations, slightly modified as in Ariki–Koike [3], are orthogonal (see also [8, §8] for a related statement). The point about Ariki–Koike's construction is that $\mathfrak{Y}\colon KH \to M_d(K)$ can be chosen such that the following holds.

Consider the action of $\mathfrak{Y}(\tilde{T}_w)$ on $K^d$. Let $\{e_1, \ldots, e_d\}$ be the standard basis of $K^d$. First, $\mathfrak{Y}(\tilde{T}_t)$ acts diagonally on $K^d$; each $e_j$ is an eigenvector with eigenvalue $-v_t^{-1}$ or $v_t$. Now let us consider the action of $\mathfrak{Y}(\tilde{T}_{s_i})$ where $1 \leqslant i \leqslant n-1$ is fixed. For a given basis vector $e_j$, we have two possibilities. Either $e_j$ is an eigenvector with eigenvalue $-v^{-1}$ or $v$. Or there exists a basis vector $e_k$ with $k \neq j$ such that $\langle e_j, e_k\rangle$ is an invariant subspace on which the action of $\tilde{T}_{s_i}$ is given by a certain $2 \times 2$-matrix $M$ depending on $i, j, k$. This matrix $M$ is specified as follows.

(H1) *The diagonal entries of $M$ are* $\dfrac{v - v^{-1}}{1 + v_t^2 v^{2r}}$ *and* $\dfrac{v_t^2 v^{2r}(v - v^{-1})}{1 + v_t^2 v^{2r}}$ *for some $r \in \mathbb{Z}$.*

Now, by [3, Remark 3.15], there is some freedom in the choice of the off-diagonal entries in $M$, as long as their product has a specified value. We make the following choice.

(H2) *The off-diagonal entries of $M$ are* $\dfrac{1 + v_t^2 v^{2(r\pm 1)}}{1 + v_t^2 v^{2r}}$*, with $r \in \mathbb{Z}$ as in (H1).*

First of all, we see that all matrix coefficients lie in $\mathcal{O}$.

**Theorem 5.4.** *Let $\chi \in \mathrm{Irr}(W)$ and $\mathfrak{Y}\colon KH \to M_d(K)$ be a matrix representation affording $\hat{\chi} \in \mathrm{Irr}(KH)$. We assume that $\mathfrak{Y}$ is constructed as in Ariki–Koike* [3]*, where (H1) and (H2) are satisfied. Then $\mathfrak{Y}$ is orthogonal.*

*Thus, all the assertions in Theorem 4.10 and Corollary 4.11 hold for $(W, S)$ of type $A_{n-1}$ or $B_n$. In particular, we have $c_{w,\chi} \in \{0, \pm 1\}$ for all $w \in W$.*

*Proof.* First we show that there exists some invertible diagonal matrix $B \in M_d(K)$ such that $\mathfrak{Y}(\tilde{T}_{w^{-1}}) = B^{-1}\mathfrak{Y}(\tilde{T}_w)^{\mathrm{tr}}B$ for all $w \in W$. This can be seen as follows. By Theorem 4.3, there exists an orthogonal representation $\mathfrak{X}\colon KH \to M_d(K)$ affording $\hat{\chi}$. Since $\mathfrak{X}$ and $\mathfrak{Y}$ afford the same character, there exists an invertible matrix $P \in M_d(K)$ such that $\mathfrak{Y}(\tilde{T}_w) = P^{-1}\mathfrak{X}(\tilde{T}_w)P$ for all $w \in W$. Then the condition in Definition 4.2(a) translates to

$$\mathfrak{Y}(\tilde{T}_{w^{-1}}) = B^{-1}\mathfrak{Y}(\tilde{T}_w)^{\mathrm{tr}}B \qquad \text{for all } w \in W, \text{ where } B = P^{\mathrm{tr}}DP;$$

note that $B$ is symmetric and invertible. Hence, it is enough to show that any symmetric matrix $B$ satisfying $\mathfrak{Y}(\tilde{T}_{w^{-1}}) = B^{-1}\mathfrak{Y}(\tilde{T}_w)^{\mathrm{tr}}B$ $(w \in W)$ must be a diagonal matrix. We proceed by induction on $n$.

If $n = 1$, this is clear since $W$ only has 1-dimensional representations. Now assume that $n \geqslant 2$ and let $W' \subset W$ be the parabolic subgroup of type $A_{n-1}$ or $B_{n-1}$ generated by $s_1, \ldots, s_{n-2}$ or $t, s_1, \ldots, s_{n-2}$, respectively. By the branching rule (see [10, §6.1]), the restriction of $\chi$ to $W'$ is multiplicity-free; let $\chi_1', \ldots, \chi_r' \in \mathrm{Irr}(W')$ be the irreducible constituents of that restriction. By induction, each $\chi_i'$ is afforded by a representation $\mathfrak{Y}_i'$ which is constructed in a similar way as $\mathfrak{Y}$. Then, for a suitable numbering, the restriction of $\mathfrak{Y}$ to $KH'$ is the matrix direct sum of $\mathfrak{Y}_1', \ldots, \mathfrak{Y}_r'$ (see [11, 2.2.8] or [3, p. 236] or [10, 10.1.5]). Now we write $B$ as a block matrix with blocks corresponding to $\mathfrak{Y}'_1, \ldots, \mathfrak{Y}'_r$ and consider the relation $B\mathfrak{Y}(\tilde{T}_{w^{-1}}) = \mathfrak{Y}(\tilde{T}_w)^{\text{tr}}B$ block by block for $w \in W'$. Since the $\mathfrak{Y}'_i$ are pairwise nonequivalent, a standard argument using Schur's Lemma shows that $B$ must be a block diagonal matrix, with diagonal blocks $B_1, \ldots, B_r$ satisfying $\mathfrak{Y}'_i(\tilde{T}_{w^{-1}}) = B_i^{-1}\mathfrak{Y}'_i(\tilde{T}_w)^{\text{tr}}B_i$ for all $w \in W'$. By induction, we know that each $B_i$ must be a diagonal matrix and, hence, so is $B$.

It remains to show that $B$ can be chosen to lie in $\mathfrak{I}_+ \cap A_0$. Let us first choose $B$ such that some diagonal coefficient equals 1. Then it easily follows, using (H2), that each diagonal coefficient of $B$ can be written as a product of the quotients of the two off-diagonal coefficients in matrices $M$, for various values of $r$. Multiplying by a suitable common denominator we see that we can find $B$ such that all diagonal coefficients lie in $\mathfrak{I}_+ \cap A_0$. ☐

**Example 5.5.** Assume that $(W, S)$ is of type $B_2$ with generators $t, s_1$. Then $W$ has a unique irreducible character $\chi$ of degree 2; it is labelled by $(\lambda, \mu)$ where $\lambda = \mu = (1)$. So the formula in Remark 5.1 shows that $\alpha_\chi = v_t$. By [10, Table 10.1] (and taking into account the freedom that we have in choosing off-diagonal entries), $\hat{\chi}$ is afforded by the following representation:

$$\mathfrak{Y}\colon \quad \tilde{T}_t \mapsto \begin{bmatrix} v_t & 0 \\ 0 & -v_t^{-1} \end{bmatrix}, \qquad \tilde{T}_{s_1} \mapsto \frac{1}{v_t^2+1}\begin{bmatrix} v - v^{-1} & 1 + v_t^2 v^{-2} \\ 1 + v_t^2 v^2 & v_t^2(v - v^{-1}) \end{bmatrix};$$

it is readily checked that this is an orthogonal representation; also note that $\mathfrak{Y}(\tilde{T}_w) \in M_2(\mathcal{O})$ for all $w \in W$. We compute the following leading coefficients.

$$c_{t,\mathfrak{Y}}^{2,2} = c_{s_1ts_1,\mathfrak{Y}}^{1,1} = 1 \quad \text{and} \quad c_{ts_1,\mathfrak{Y}}^{2,1} = c_{s_1t,\mathfrak{Y}}^{1,2} = -1.$$

Thus, Theorem 4.10(b) shows that $t \sim_L s_1t$ and $ts_1 \sim_L s_1ts_1$, in accordance with the computation in [17, p. 106]; see also (6.3) below.

*Remark* 5.6. Let $(W, S)$ be of type $B_n$ as above. We now make a different choice of parameters, as follows. Let $u$ be an indeterminate and $\Gamma = \{u^m \mid m \in \mathbb{Z}\}$, with total ordering given by $\Gamma_+ = \{u^m \mid m > 0\}$. Assume that

$$v_t = u^a \quad \text{and} \quad v_{s_i} = u^b \quad \text{where } a, b \geqslant 1 \text{ are integers such that } a \not\equiv 0 \bmod b.$$

Then Hoefsmit's formulas also show that $r_\chi = 1$ for all $\chi \in \text{Irr}(W)$ and so all constructible characters are irreducible. In the case where $b = 2$ and $a \in \{1, 3\}$, Lusztig has shown in [17, Theorem 11] that $\chi_{\mathfrak{C}} \in \text{Irr}(W)$ for all left cells $\mathfrak{C} \subset W$. The proof of the latter result requires some deep positivity properties of the structure constants of $H$ with respect to the $C_w$-basis.

## 6. Cells and leading coefficients in type $I_2(m)$

In this section, we determine the leading coefficients and the characters afforded by the left cells in the unequal parameter case for Coxeter groups of type $I_2(m)$ (with $m$ even). To fix notation, let $W = \langle s, t\rangle$ be the dihedral group of order $2m$, where $S = \{s, t\}$ and $st$ has even order $m \geqslant 4$. Assume that $\Gamma = \{v_s^k v_t^l \mid k, l \in \mathbb{Z}\}$, where $v_s, v_t$ are independent indeterminates, and that the total ordering on $\Gamma$ is given by

$$\Gamma_+ = \{v_s^k v_t^l \mid k > 0, l \in \mathbb{Z}\} \cup \{v_t^l \mid l > 0\}.$$

We shall see that the constructible characters are precisely the characters afforded by the various left cells and we will determine all leading coefficients.

**6.1. Irreducible characters.** Let $1_W$ and $\varepsilon$ denote the trivial and the sign character, respectively. We have two more linear characters $\varepsilon_1$ and $\varepsilon_2$ which are given by $\varepsilon_1(s) = \varepsilon_2(t) = 1$ and $\varepsilon_1(t) = \varepsilon_2(s) = -1$. All the remaining irreducible characters of $W$ have degree 2, and they are afforded by the following representations where $a = st$ (see [10, 5.3.4]):

$$\varphi_j : a \mapsto \begin{bmatrix} \zeta^j & 0 \\ 0 & \zeta^{-j} \end{bmatrix}, \qquad s \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where $\zeta = \exp(2\pi\sqrt{-1}/m) \in \mathbb{C}$ and $1 \leqslant j \leqslant (m-2)/2$. Let $\chi_j$ be the character afforded by $\varphi_j$. Using the formulas for the Schur elements in [10, 8.3.4], we find

$$\begin{array}{rclcrcl} \alpha_{1_W} & = & 1 & \qquad & r_{1_W} & = & 1 \\ \alpha_{\varepsilon_1} & = & v_t & & r_{\varepsilon_1} & = & 1 \\ \alpha_{\varepsilon_2} & = & (v_s v_t^{-1})^{m/2} v_t & & r_{\varepsilon_2} & = & 1 \\ \alpha_{\varepsilon} & = & (v_s v_t)^{m/2} & & r_{\varepsilon} & = & 1 \\ \alpha_{\chi_j} & = & v_s & & r_{\chi_j} & = & m/(2 - \zeta^{2j} - \zeta^{-2j}) \end{array}$$

The corresponding irreducible characters of $KH$ will be denoted by $\tilde{1}_W$, $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$, $\tilde{\varepsilon}$, $\hat{\chi}_j$ $(1 \leqslant j \leqslant (m-2)/2)$, respectively.

**6.2. Constructible characters.** We have the following induced characters (here, 1 and $\varepsilon$ stand for the trivial and the sign character of the relevant parabolic subgroup):

$$\operatorname{Ind}_{\{s\}}^S(1) = 1_W + \varepsilon_1 + \sum_j \chi_j \qquad \text{and} \qquad \operatorname{Ind}_{\{s\}}^S(\varepsilon) = \varepsilon + \varepsilon_2 + \sum_j \chi_j,$$
$$\operatorname{Ind}_{\{t\}}^S(1) = 1_W + \varepsilon_2 + \sum_j \chi_j \qquad \text{and} \qquad \operatorname{Ind}_{\{t\}}^S(\varepsilon) = \varepsilon + \varepsilon_1 + \sum_j \chi_j.$$

It is easily checked that the constructible characters for the parabolic subgroups corresponding to the subsets $\{s\}$ and $\{t\}$ (which are of type $A_1$) are precisely the irreducible characters. It follows that the constructible characters for $W$ are

$$1_W, \qquad \varepsilon_1, \qquad \varepsilon_2, \qquad \varepsilon, \qquad \sum_{j=1}^{(m-2)/2} \chi_j.$$

In particular, every irreducible character of $W$ occurs with nonzero multiplicity in some constructible character so the ordering of $\Gamma$ is saturated.

**6.3. Left cells.** The Kazhdan–Lusztig polynomials and the $M$-polynomials with respect to unequal parameters and the above total ordering are determined in [10, Ex. 11.4]. For any $i$ with $1 \leqslant i < m$, let $x_i, y_i \in W$ be the unique elements of length $i$ such that $l(x_i s) < l(x_i)$ and $l(y_i t) < l(y_i)$, respectively. Then the left cells are

$$\{1\}, \quad \{t\}, \quad \{tw_0\}, \quad \{w_0\}, \quad \{x_i \mid 1 \leqslant i < m-1\}, \quad \{y_i \mid 1 < i < m\}.$$

(Note that $y_1 = t$ and $x_{m-1} = tw_0$.) The characters carried by the left cells are, respectively,

$$1_W, \qquad \varepsilon_1, \qquad \varepsilon_2, \qquad \varepsilon, \qquad \sum_{j=1}^{(m-2)/2} \chi_j, \qquad \sum_{j=1}^{(m-2)/2} \chi_j.$$

Thus, we obtain precisely the constructible characters.

**6.4. Leading coefficients.** Using the matrix representations in [10, §8.3], we obtain (by an explicit computation) the following table of leading coefficients:

| Left cell $\mathfrak{C}$ | $w \in \mathfrak{C}$ $(w^2 = 1)$ | $\sum_\chi c_{w,\chi}\,\chi$ |
|---|---|---|
| $\{1\}$ | $1$ | $1_W$ |
| $\{t\}$ | $t$ | $\varepsilon_1$ |
| $\{tw_0\}$ | $tw_0$ | $-(-1)^{m/2}\varepsilon_2$ |
| $\{w_0\}$ | $w_0$ | $\varepsilon$ |
| $\{x_i\}$ | $s$ | $\sum_j \chi_j$ |
| $\{y_i\}$ | $tst$ | $\sum_j \chi_j$ |

For example, if $w = tst$, we compute the character values as follows. We have $\tilde{1}_W(\tilde{T}_w) = v_s v_t^2$, $\tilde{\varepsilon}_1(\tilde{T}_w) = v_t^{-2} v_s$, $\tilde{\varepsilon}_2(T_w) = -v_t^2 v_s^{-1}$, $\tilde{\varepsilon}(\tilde{T}_w) = -v_s^{-1} v_t^{-2}$ and

$$\hat{\chi}_j(\tilde{T}_w) = \hat{\chi}_j(\tilde{T}_s) + (v_t - v_t^{-1})\hat{\chi}_j(\tilde{T}_{st}) = v_s - v_s^{-1} + 2(v_t - v_t^{-1})\cos(2\pi j/m),$$

where we use [10, 8.2.2 and 8.3.3]. Since the $\alpha_\chi$ are known by (6.1), this immediately yields the corresponding leading coefficients. The computations in the remaining cases are similar and will be omitted.

As a result of these computations, we see that each left cell $\mathfrak{C}$ contains an involution $w \in \mathfrak{C}$ such that $\chi_{\mathfrak{C}} = \pm \sum_\chi c_{w,\chi}\,\chi$.

**Lemma 6.5.** *For any $w \in W$ and $\chi \in \mathrm{Irr}(W)$, we have $c_{w,\chi} = 0$ unless $w^2 = 1$. Furthermore, each left cell $\mathfrak{C}$ of $W$ contains a unique element $w$ such that $\chi_{\mathfrak{C}} = \pm \sum_\chi c_{w,\chi}\,\chi$.*

*Proof.* Let $w \in W$. First note that if $l(w)$ is odd, then $w$ is conjugate to $s$ or to $t$ and so $w$ is an involution. Assume now that $w$ is not an involution; then $l(w)$ is even and so $w$ is conjugate by cyclic shift (in the sense of [10, Def. 3.1.3]) to an element of form $(st)^k$, for some $1 \leqslant k \leqslant (m-2)/2$. Thus, as far as character values are concerned, we can assume that $w = (st)^k$ $(1 \leqslant k \leqslant (m-2)/2)$ and so $v_w = (v_s v_t)^k$. Using the data in (6.1) and [10, 8.3.3], we find that

$$\begin{aligned}
(-1)^{l(w)}\alpha_{1_W}\,\tilde{1}_W(\tilde{T}_w) &= (v_s v_t)^k,\\
(-1)^{l(w)}\alpha_{\varepsilon_1}\tilde{\varepsilon}_1(\tilde{T}_w) &= (-1)^k v_s^k v_t^{1-k},\\
(-1)^{l(w)}\alpha_{\varepsilon_2}\tilde{\varepsilon}_2(\tilde{T}_w) &= (-1)^k v_s^{m/2+k} v_t^{1-m/2-k},\\
(-1)^{l(w)}\alpha_{\varepsilon}\tilde{\varepsilon}(\tilde{T}_w) &= (v_s v_t)^{m/2-k},\\
(-1)^{l(w)}\alpha_{\chi_j}\hat{\chi}_j(\tilde{T}_w) &= 2v_s\cos(2\pi jk/m).
\end{aligned}$$

In any case, we see that the constant terms of the above expressions are zero and so $c_{w,\chi} = 0$. Thus, the first statement is proved. As far as the second statement is concerned, we only need to consider the left cells $\{x_i\}$ and $\{y_i\}$.

Consider first the left cell $\{x_i\}$. If $m = 4$, then $\{x_i\} = \{s, ts\}$ and $ts$ is not an involution. So there is nothing to prove in this case. Now assume that $m > 4$ so that $\{x_i\} = \{s, ts, sts, tsts, \ldots\}$. Let $w_k = (st)^k s$ be an involution in this set, where $1 \leqslant k < (m-2)/2$. We must show that there exist $1 \leqslant j < j' \leqslant (m-2)/2$ such that $c_{w,\chi_j} \neq c_{w,\chi_{j'}}$. As in the proof of [10, 8.3.4], we compute $\chi_j(\tilde{T}_w) = \alpha\zeta^{jk} + \delta\zeta^{-jk}$ where $\alpha, \delta$ are given by the expressions in [10, p. 257] and $\zeta$ is a primitive $m$th root of unity. Using these expressions for $\alpha$ and $\delta$, we find that

$$(-1)^{l(w)}\alpha_{\chi_j}\hat{\chi}_j(\tilde{T}_w) = -(\alpha\zeta^{jk} + \delta\zeta^{-jk})$$

and so $c_{w,\chi_j} = (\zeta^{jk+2} - \zeta^{-jk})/(\zeta^2 - 1)$. Now consider these expressions for $j = 1$, $j = 2$ and suppose they are equal. This would imply that $\zeta^{2k+2} - \zeta^{-k} = 2$. Since $\zeta$ is a root of unity, this implies $\zeta^k = -1$ and so $k \geqslant m/2$, a contradiction.

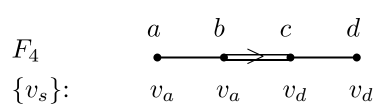
The argument for the left cell $\{y_i\}$ is similar and will be omitted. $\square$

**Example 6.6.** Assume that $m = 6$ so that $(W, S)$ is of type $G_2$. In this case, the above results show that we have the following table of leading coefficients:

| $w \in W$ $(w^2 = 1)$ | $\sum_\chi c_{w,\chi}\,\chi$ |
|---|---|
| $1$ | $1_W$ |
| $t$ | $\varepsilon_1$ |
| $ststs$ | $\varepsilon_2$ |
| $w_0$ | $\varepsilon$ |
| $s$ | $\chi_1 + \chi_2$ |
| $sts$ | $\chi_1 - \chi_2$ |
| $tst$ | $\chi_1 + \chi_2$ |
| $tstst$ | $\chi_1 - \chi_2$ |

## 7. Cells and leading coefficients in type $F_4$

In this section, we determine the leading coefficients and the characters afforded by the left cells in the unequal parameter case for Coxeter groups of type $F_4$. To fix notation, let $(W, S)$ be of type $F_4$, where $S = \{a, b, c, d\}$ and the defining relations are given the following diagram:




We assume that $\Gamma = \{v_a^k v_d^l \mid k, l \in \mathbb{Z}\}$, where $v_a$ and $v_d$ are independent indeterminates, and that the total ordering on $\Gamma$ is given by

$$\Gamma_+ = \{v_a^k v_d^l \mid l > 0, k \in \mathbb{Z}\} \cup \{v_a^k \mid k > 0\}.$$

We shall see that the constructible characters are precisely the characters afforded by the various left cells and we will determine all leading coefficients.

**7.1. Irreducible characters.** We refer to Kondo's labelling of the irreducible characters of $W$; see [6, p. 413] or [10, 5.3.6 and Table C.3]. Using the formulas for the Schur elements in [10, Table 11.1], we find

| $\chi$ | $\alpha_\chi$ | $r_\chi$ |
|---|---|---|
| $1_1$ | $1$ | 1 |
| $1_2$ | $v_d^{12}$ | 1 |
| $1_3$ | $v_a^{12}$ | 1 |
| $1_4$ | $(v_av_d)^{12}$ | 1 |
| $2_1$ | $v_d^3$ | 1 |
| $2_2$ | $v_a^{12}v_d^3$ | 1 |
| $2_3$ | $v_a^3$ | 1 |
| $2_4$ | $v_a^3v_d^{12}$ | 1 |
| $4_1$ | $(v_av_d)^3$ | 2 |

| $\chi$ | $\alpha_\chi$ | $r_\chi$ |
|---|---|---|
| $9_1$ | $(v_av_d)^2$ | 1 |
| $9_2$ | $v_a^2v_d^6$ | 1 |
| $9_3$ | $v_a^6v_d^2$ | 1 |
| $9_4$ | $(v_av_d)^6$ | 1 |
| $6_1$ | $(v_av_d)^3$ | 3 |
| $6_2$ | $(v_av_d)^3$ | 3 |
| $12$ | $(v_av_d)^3$ | 6 |
| $4_2$ | $v_av_d$ | 1 |

| $\chi$ | $\alpha_\chi$ | $r_\chi$ |
|---|---|---|
| $4_3$ | $v_av_d^7$ | 1 |
| $4_4$ | $v_a^7v_d$ | 1 |
| $4_5$ | $(v_av_d)^7$ | 1 |
| $8_1$ | $v_av_d^3$ | 1 |
| $8_2$ | $v_a^7v_d^3$ | 1 |
| $8_3$ | $v_a^3v_d$ | 1 |
| $8_4$ | $v_a^3v_d^7$ | 1 |
| $16$ | $(v_av_d)^3$ | 2 |

**7.2. Constructible characters and left cells.** Let $I \subset S$ be a proper subset. Then the corresponding parabolic subgroup $W_I$ is a direct product of Coxeter groups of type $A_1$, $A_2$, $B_2$ or $B_3$. It is easily checked that $r_\psi = 1$ for all $\psi \in \text{Irr}(W_I)$ and so the constructible characters of $W_I$ are irreducible by Proposition 3.8. Furthermore, the decomposition of the induced characters from $W_I$ to $W$ into irreducibles is easily computed (with the help of the CHEVIE function `InductionTable`, for example). Thus, the constructible characters of $W$ can be determined; they are given as follows.

- All irreducible characters except $4_1$, $6_1$, $6_2$, 12 and 16 are constructible.
- The characters $16 + 4_1$, $16 + 12 + 6_1$ and $16 + 12 + 6_2$ are constructible.

In particular, we see that every irreducible character occurs in some constructible character. On the other hand, the Kazhdan–Lusztig cells with respect to two independent parameters and the above choice of $\Gamma_+ \subset \Gamma$ have been determined explicitly in [10, 11.3.1] and the results of these computations show that the characters carried by the various left cells are precisely the constructible characters.

**7.3. Leading coefficients.** All the character values (and, hence, their leading coefficients) of $KH$ can be computed using the "class polynomials" and the explicit knowledge of the character table of $H$; see the algorithmic description in [10, §8.2]. (In the CHEVIE system [9], this can be done using the function `HeckeCharValues`.) In Table 1 we list the expressions $\sum_\chi c_{w,\chi}\,\chi$ for each involution $w \in W$. (There are 140 involutions in $W$.) In that table, we use the convention that rows which are not separated by bars correspond to elements in the same left cell; furthermore, double bars separate elements in different two-sided cells. Thus, we see that in each left cell $\mathfrak{C}$, there exists a unique involution $w$ such that $\chi_{\mathfrak{C}} = \pm\sum_\chi c_{w,\chi}\,\chi$. Similar computations also show that $c_{w,\chi} = 0$ unless $w^2 = 1$.

TABLE 1. The leading coefficients of all involutions in type $F_4$

| $w \in W$ $(w^2=1)$ | $\sum_\chi c_{w,\chi}\,\chi$ |
|---|---|
| $1$ | $1_1$ |
| $w_0aba$ | $-1_2$ |
| $aba$ | $1_3$ |
| $w_0$ | $1_4$ |
| $cbacbc$ | $-2_1$ |
| $dcbacbcd$ | $-2_1$ |
| $w_0cbacbc$ | $-2_2$ |
| $w_0dcbacbcd$ | $-2_2$ |
| $a$ | $2_3$ |
| $b$ | $2_3$ |
| $w_0a$ | $-2_4$ |
| $w_0b$ | $-2_4$ |
| $c$ | $4_2$ |
| $d$ | $4_2$ |
| $bcb$ | $4_2$ |
| $abcba$ | $4_2$ |
| $w_0abad$ | $-4_3$ |
| $w_0acbadc$ | $-4_3$ |
| $w_0bacbadcb$ | $-4_3$ |
| $w_0abacbadcba$ | $-4_3$ |
| $abad$ | $4_4$ |
| $acbadc$ | $4_4$ |
| $bacbadcb$ | $4_4$ |
| $abacbadcba$ | $4_4$ |
| $w_0c$ | $4_5$ |
| $w_0d$ | $4_5$ |
| $w_0bcb$ | $4_5$ |
| $w_0abcba$ | $4_5$ |
| $cdc$ | $8_1$ |
| $bcdcb$ | $8_1$ |
| $abcdcba$ | $8_1$ |
| $cbcdcbc$ | $8_1$ |
| $acbcdcbac$ | $8_1$ |
| $bacbcdcbacb$ | $8_1$ |

| $w \in W$ $(w^2=1)$ | $\sum_\chi c_{w,\chi}\,\chi$ |
|---|---|
| $w_0abacbacbc$ | $8_1$ |
| $w_0abadcbacbcd$ | $8_1$ |
| $abacbacbc$ | $8_2$ |
| $abadcbacbcd$ | $8_2$ |
| $w_0cdc$ | $8_2$ |
| $w_0bcdcb$ | $8_2$ |
| $w_0cbcdcbc$ | $8_2$ |
| $w_0abcdcba$ | $8_2$ |
| $w_0acbcdcbac$ | $8_2$ |
| $w_0bacbcdcbacb$ | $8_2$ |
| $ac$ | $8_3$ |
| $ad$ | $8_3$ |
| $bd$ | $8_3$ |
| $bacb$ | $8_3$ |
| $cbdc$ | $8_3$ |
| $abacba$ | $8_3$ |
| $bcbdcb$ | $8_3$ |
| $abcbdcba$ | $8_3$ |
| $w_0ac$ | $-8_4$ |
| $w_0ad$ | $-8_4$ |
| $w_0bd$ | $-8_4$ |
| $w_0bacb$ | $-8_4$ |
| $w_0cbdc$ | $-8_4$ |
| $w_0abacba$ | $-8_4$ |
| $w_0bcbdcb$ | $-8_4$ |
| $w_0abcbdcba$ | $-8_4$ |
| $cbc$ | $-9_1$ |
| $acbac$ | $-9_1$ |
| $dcbcd$ | $-9_1$ |
| $adcbacd$ | $-9_1$ |
| $bacbacb$ | $-9_1$ |
| $badcbacbd$ | $-9_1$ |
| $cbadcbacbdc$ | $-9_1$ |
| $w_0bcbcdcbcd$ | $-9_1$ |
| $w_0abcbcdcbacd$ | $-9_1$ |

Table 1. The leading coefficients of all involutions in type $F_4$ (con't)

| $w \in W$ $(w^2 = 1)$ | $\sum_\chi c_{w,\chi}\,\chi$ |
|---|---|
| $cbcdcbcd$ | $9_2$ |
| $acbcdcbacd$ | $9_2$ |
| $bacbcdcbacbd$ | $9_2$ |
| $w_0bcbc$ | $9_2$ |
| $w_0abcbac$ | $9_2$ |
| $w_0bdcbcd$ | $9_2$ |
| $w_0abacbacb$ | $9_2$ |
| $w_0abdcbacd$ | $9_2$ |
| $w_0abadcbacbd$ | $9_2$ |
| $bcbc$ | $9_3$ |
| $abcbac$ | $9_3$ |
| $bdcbcd$ | $9_3$ |
| $abacbacb$ | $9_3$ |
| $abdcbacd$ | $9_3$ |
| $abadcbacbd$ | $9_3$ |
| $acbadcbacbdc$ | $9_3$ |
| $w_0cbcdcbcd$ | $9_3$ |
| $w_0acbcdcbacd$ | $9_3$ |
| $bcbcdcbcd$ | $9_4$ |
| $abcbcdcbacd$ | $9_4$ |
| $w_0cbc$ | $9_4$ |
| $w_0acbac$ | $9_4$ |
| $w_0dcbcd$ | $9_4$ |
| $w_0adcbacd$ | $9_4$ |
| $w_0bacbacb$ | $9_4$ |
| $w_0badcbacbd$ | $9_4$ |
| $w_0cbadcbacbdc$ | $9_4$ |
| $abcbacbc$ | $-4_1 - 16$ |
| $w_0badcbacbcd$ | $-4_1 + 16$ |
| $bacbacbc$ | $-4_1 - 16$ |
| $w_0abdcbacbcd$ | $-4_1 + 16$ |
| $abdcbacbcd$ | $-4_1 - 16$ |
| $w_0bacbacbc$ | $-4_1 + 16$ |
| $badcbacbcd$ | $-4_1 - 16$ |
| $w_0abcbacbc$ | $-4_1 + 16$ |

| $w \in W$ $(w^2 = 1)$ | $\sum_\chi c_{w,\chi}\,\chi$ |
|---|---|
| $acdc$ | $6_1 + 12 + 16$ |
| $acdcbacd$ | $-6_1 + 2 \cdot 12$ |
| $acbdcbacbcdc$ | $6_1 + 12 - 16$ |
| $abacdcba$ | $6_1 + 12 + 16$ |
| $abacbdcbacbd$ | $-6_1 + 2 \cdot 12$ |
| $w_0cbacdcbc$ | $6_1 + 12 - 16$ |
| $bacbacdcbacb$ | $6_1 + 12 + 16$ |
| $w_0acdcbacd$ | $-6_1 + 2 \cdot 12$ |
| $w_0acdc$ | $6_1 + 12 - 16$ |
| $cbacdcbc$ | $6_1 + 12 + 16$ |
| $cbacdcbacbdc$ | $-6_1 + 2 \cdot 12$ |
| $w_0abacdcba$ | $6_1 + 12 - 16$ |
| $acbacdcbac$ | $6_1 + 12 + 16$ |
| $w_0bacdcbacbd$ | $-6_1 + 2 \cdot 12$ |
| $w_0bacdcb$ | $6_1 + 12 - 16$ |
| $bacdcb$ | $6_1 + 12 + 16$ |
| $bacdcbacbd$ | $-6_1 + 2 \cdot 12$ |
| $w_0acbacdcbac$ | $6_1 + 12 - 16$ |
| $abacbcdcbacb$ | $6_2 + 12 + 16$ |
| $w_0cdcbacbcdc$ | $-6_2 + 2 \cdot 12$ |
| $w_0cdcbcd$ | $6_2 + 12 - 16$ |
| $cdcbcd$ | $6_2 + 12 + 16$ |
| $cdcbacbcdc$ | $-6_2 + 2 \cdot 12$ |
| $cbadcbacbcdc$ | $6_2 + 12 - 16$ |
| $bcbcdcbc$ | $6_2 + 12 + 16$ |
| $bcbacbdcbc$ | $-6_2 + 2 \cdot 12$ |
| $w_0abcbdcbacd$ | $6_2 + 12 - 16$ |
| $abcbdcbacd$ | $6_2 + 12 + 16$ |
| $w_0bcbacbdcbc$ | $-6_2 + 2 \cdot 12$ |
| $w_0bcbcdcbc$ | $6_2 + 12 - 16$ |
| $abcbcdcbac$ | $6_2 + 12 + 16$ |
| $abcbacbdcbac$ | $-6_2 + 2 \cdot 12$ |
| $w_0bcbdcbcd$ | $6_2 + 12 - 16$ |
| $bcbdcbcd$ | $6_2 + 12 + 16$ |
| $bcdcbacbcdcb$ | $-6_2 + 2 \cdot 12$ |
| $w_0abcbcdcbac$ | $6_2 + 12 - 16$ |

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