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THE BERNSTEIN CENTER IN TERMS OF INVARIANT LOCALLY INTEGRABLE FUNCTIONS

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ABSTRACT. We give a description of the Bernstein center of a reductive p-adic group G in terms of invariant locally integrable functions and compute a basis of these functions for the group $SL(2)$.

1. INTRODUCTION

1.1. The Lie algebra $\mathfrak{g}$ and associated enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of a connected Lie group G are fundamental tools in the study of the group's representations. The center $\mathfrak{Z}(\mathfrak{U}(\mathfrak{g}))$ of the enveloping algebra is particularly useful for a number of purposes. There are three useful descriptions of the center: (i) an algebraic description of the center in terms of the Lie algebra, (ii) the Harish-Chandra homomorphism identification of the center with the space of Weyl invariant regular functions on the symmetric algebra of a Cartan subalgebra, and (iii) as invariant distributions on the group supported at the identity.

1.2. By following J. Bernstein, one can also consider the center $\mathfrak{Z}(\mathfrak{U}(\mathfrak{g}))$ from a ring and categorical point of view. If A is an algebra with identity, then the center of A is naturally isomorphic to the algebra of endomorphisms of the identity functor of the category of all A-modules. In particular, $\mathfrak{Z}(\mathfrak{U}(\mathfrak{g}))$ is isomorphic to the algebra of endomorphisms of the identity functor of the category of modules of the Lie algebra. This formulation of the center of the enveloping algebra is the starting point for Bernstein's construction of an analogue of the center of the enveloping algebra for reductive p-adic groups. If G is a reductive p-adic group G, let $\mathcal{S}(G)$ denote the category of smooth representations of G. The Bernstein center $\mathfrak{Z}(G)$ of G is defined to be the algebra of endomorphisms of the identity functor of $\mathcal{S}(G)$ There are two realizations of Bernstein center. Both are much less concrete than in the real case.

(i) The most explicit description of the Bernstein center of G is in terms of algebraic geometry. Let $\tilde{G}$ denote the non-unitary dual of G, the set of all equivalence classes of smooth irreducible representations. It carries a natural topology (see [T]). Let $\Omega(G)$ denote the Hausdorffization of $\tilde{G}$. The natural algebraic group structure which exists on characters of Levi subgroups of G, defines an algebraic variety structure on $\Omega(G)$. The Bernstein

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center is the algebra of regular functions on $\Omega(G)$. As a direct consequence of this description, we define a family of ideals $\mathfrak{Z}(\Omega)$ in $\mathfrak{Z}(G)$, indexed by connected components Ω of $\Omega(G)$, as follows: Define

$$\text{(1.2.1)} \qquad \mathfrak{Z}(\Omega) := \text{the regular functions on } \Omega(G) \\ \text{supported on the connected component } \Omega.$$

It is obvious from (1.1) that $\mathfrak{Z}(\Omega)$ is a direct summand in $\mathfrak{Z}(G)$. Furthermore, multiplication by the function on $\Omega(G)$ which is the characteristic function of Ω defines a projection map $z \mapsto z_\Omega$ from $\mathfrak{Z}(G)$ to $\mathfrak{Z}(\Omega)$.

(ii) A second description of the Bernstein center of G is in terms of distributions. An element of the Bernstein center is a G-invariant distribution D on G which is essentially compact; i.e., convolution of D against a compactly supported locally constant function f on G results in a compactly supported locally constant function on G. The relationship between viewing an element of the Bernstein center as a G-invariant essentially compact distribution D and as a regular function on $\Omega(G)$ is Fourier Transform. More precisely, the Fourier Transform of D is a scalar operator at each point of $\Omega(G)$ (see (2.3.2)). The delta distribution supported at the identity is an example of a G-invariant essentially compact distribution from which we see that distributions in the Bernstein center are, in general, not represented by locally integrable functions on the group. It is a nontrivial task to find other distributions in the Bernstein center and it is an interesting question of how to find sufficient conditions for a distribution in the Bernstein center to be representable by a locally integrable function on the group.

1.3. One of our main goals (Theorem 2.5) is to show the family of ideals $\mathfrak{Z}(\Omega)$ in $\mathfrak{Z}(G)$, indexed by connected components of Ω of $\Omega(G)$, satisfy the properties:

(i) Each distribution in $\mathfrak{Z}(\Omega)$ is represented by a locally integrable function on G.

(ii) Suppose f is a locally constant compactly supported function on G. Then $z_\Omega(f) \neq 0$ for only finitely many connected components Ω of $\Omega(G)$. In particular, for an arbitrary family $z_\Omega \in \mathfrak{Z}(\Omega)$, the sum $\sum_\Omega z_\Omega$ is a well-defined distribution lying in $\mathfrak{Z}(G)$.

(iii) Each distribution in $z \in \mathfrak{Z}(\Omega)$ can be represented as in (iii) by a unique family z_Ω, $\Omega \subseteq \Omega(G)$.

We obtain in this fashion a description of the entire Bernstein center completely in terms of distributions represented by locally integrable functions.

1.4. As a consequence of our description of the Bernstein center in terms of locally integrable functions, for the special linear group $SL(2)$, we explicitly compute locally integrable functions which describe elements in $\mathfrak{Z}(\Omega)$.

1.5. This paper is organized as follows: In section two we introduce notation, and show in terms of invariant distributions, the ideals $\mathfrak{Z}(\Omega)$ satisfy (i), (ii), and (iii). For a fixed connect component $\Omega \subset \Omega(G)$, properties (i), and (ii) imply the existence of a projector function $e_\Omega \in \mathfrak{Z}(\Omega)$ such that for any $z \in \mathfrak{Z}(G)$ we have $z_\Omega = z \star e_\Omega$. Equivalently, e_Ω is the G-invariant essentially compact distribution whose Fourier Transform is the characteristic function of Ω. In section three we explicitly compute the functions e_Ω when $G = SL(2, F)$.

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2. On local integrability of elements from the Bernstein center

In this section, we obtain a description of the Bernstein center in terms of locally integrable functions.

2.1. Notation and Plancherel formula.

2.1.1. Let F denote a non-archimedean local field. Let $|\,|_F$ denote the modulus character of F. Let G denote the group of F-rational points of a connected reductive group defined over F. In the sequel, all algebraic subgroups of G will be the F-rational points of an algebraic subgroup defined over F.

2.1.2. Suppose π is an admissible representation of G. Let Θ_π denote its character. We assume char $(F) = 0$ in the sequel. Under this assumption, Harish-Chandra showed in [HC2] that the distribution Θ_π is given by a locally integrable function, which is locally constant on regular semisimple elements. We follow the usual convention and denote the locally integrable function also as Θ_π. If f is a function on G, then we define $\check{f}$ on G by $\check{f}(g) = f(g^{-1})$.

2.1.3. We establish notation for the unramified characters of G. Set

$$G^0 := \{g \in G \mid |\chi(g)|_F = 1 \text{ for all rational characters of } G\},$$
$$\Lambda(G) := G/G^0.$$

The group $\Lambda(G)$ is a free abelian of rank r equal to the rank of a maximal split torus in the center of G. A character of G is unramified if it is trivial on G^0. Let $\Psi(G)$ denote the commutative group of unramified characters. Clearly, $\Psi(G) \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda(G), \mathbb{C}^\times)$. In particular, $\Psi(G)$ has the structure of a complex algebraic group (isomorphic to $(\mathbb{C}^\times)^r$). Let $\Psi^u(G)$ denote the subgroup of unitary characters in $\Psi(G)$.

2.1.4. Let $\tilde{G} \supset \hat{G}$ denote respectively the smooth and unitary dual of G. For $\pi \in \tilde{G}$ set

$$\mathrm{Stab}(\pi) \ := \ \{\ \chi \in \Psi(G) \mid \pi \cong \chi\pi\ \}.$$

Then, $\mathrm{Stab}(\pi)$ is a finite group, hence contained in $\Psi^u(G)$, and $\chi\mathrm{Stab}(\pi) \mapsto \chi\pi$ is a one-to-one mapping of $\Psi(G)/\mathrm{Stab}(\pi)$ onto $\Psi(G)\pi \subseteq \tilde{G}$.

2.1.5. Next, we recall notation for parabolic subgroups and roots. We fix a minimal parabolic subgroup P_0 (with radical U) of G and a Levi factor M_0 of P_0 so $P_0 = M_0U$. A parabolic subgroup $P \supset P_0$ is called standard (with respect to P_0). For such a parabolic subgroup P, there is a unique Levi factor $M \supset M_0$. Let N denote the unipotent radical of P. The standard Levi decomposition of P is $P = MN$. We shall assume all Levi decompositions of standard parabolic subgroups are of this type. Let A_M denote the maximal split torus in the center of M and set $W_M := N_G(A_M)/M = N_G(M)/Z_G(A_M)$, where $N_G(A_M)$ (resp. $Z_G(A_M)$) denotes the normalizer (resp. the centralizer) of A_M in G. Let $|W_M|$ be the cardinal number of W_M. For convenience, we shorten A_{M_0} to A_0 and W_{M_0} to W. Denote as Σ the set of roots of A_0 in G. Let $\Sigma^+ \subset \Sigma$ (resp. $\Delta \subset \Sigma^+$) denote the set of positive (resp. simple) roots determined by the selection of the minimal parabolic P_0. There is a bijection between subsets of Δ and standard parabolic subgroups of G. For $\Phi \subseteq \Delta$, let $P_\Phi = M_\Phi N_\Phi$ denote the corresponding standard parabolic subgroup (note that $P_\emptyset = P_0$ and $M_\emptyset = M_0$). Define $W(\Phi) := \{\, w \in W \mid w(\Phi) = \Phi \,\}$.

2.1.6. Fix a maximal special parahoric subgroup K_0 of G, which is in a good relative position with M_0 (see [W] for a precise description of K_0). A simple description of K_0 can also be formulated using affine buildings as follows. Let $\mathcal{B}(G) \supset \mathcal{B}(M_0)$ denote the extended affine buildings of G and M_0 respectively. The group K_0 is then a parahoric subgroup associated to any special point x of $\mathcal{B}(G)$ which lies in $\mathcal{B}(M_0)$.

If S is a closed subgroup of G and ds a Haar measure on S (left or right), then we shall always assume that it is normalized so the measure of $S \cap K_0$ is one. We now recall the definition [HC1] of Harish-Chandra's γ and c-factors. For $P = MN$ a parabolic subgroup, let $\bar{P}$ (resp. $\bar{N}$) denote the opposite parabolic subgroup (with respect to M) (resp. the unipotent radical of $\bar{P}$). Let $d\bar{n}$ denote the Haar measure on $\bar{N}$. Extend the modular function δ_P of P to a function δ'_P on all G by the decomposition $G = PK_0$ (i.e., $\delta'_P(pk) = \delta_P(p)$ for $p \in P$ and $k \in K_0$). Then Harish-Chandra's γ-factor $\gamma(G|M)$ is defined as

$$\gamma(G|M) = \int_{\bar{N}} \delta'_P(\bar{n})\, d\bar{n}\,. \tag{2.1.7}$$

To define Harish-Chandra's c-factor $c(G|M)$, let α be a reduced root of A_M in G. Define A_α to be the connected component of the kernel of α (in A_M), and set M_α equal to the centralizer of A_α in G. Then $M_\alpha \supset M$ is a Levi subgroup in G and $M_\alpha \cap K_0$ is a maximal compact subgroup of M_α. We define $\gamma(M_\alpha|M)$ as above. Then

$$c(G|M) \;:=\; \gamma(G|M)^{-1} \prod \gamma(M_\alpha|M), \tag{2.1.8}$$

where the product runs over all reduced roots of A_0 in P.

2.1.9. For a smooth representation σ of the Levi factor M of a standard parabolic subgroup $P = MN$ of G, let $\mathrm{Ind}_P^G(\sigma)$ denote the normalized induced representation under right translations. We recall the formula [vD] for $\Theta_{\mathrm{Ind}_P^G(\pi)}$, the induced character. Let H be a Cartan subgroup of G; i.e, H is the F-rational points of a maximal F-rational torus of the algebraic group associated to G.

Let $D_G : G \longrightarrow F$ denote the Weyl discriminant. In particular, the set of regular elements G' of G are those $\gamma \in G$ satisfying $D_G(\gamma) \neq 0$. Suppose $\gamma \in H \cap G'$. If H is not G-conjugate to a subgroup of M, then $\Theta_{\mathrm{Ind}_P^G(\pi)}(\gamma) = 0$. Otherwise we can assume γ and H belong to M. In this case, let A_H denote the maximal F-split part of H, and let $W(A_M, A_H)$ be the set of embeddings $s : A_M \longrightarrow A_H$ with the property that each embedding s can be realized as $s(a) = hah^{-1}$ for some $h \in G$. For $s \in W(A_M, A_H)$ let $s\,\pi$ denote the representation of $M^s := Ad(h)M = hMh^{-1}$ defined by $(s,\pi)(m) = \pi(h^{-1}mh)$. Then [vD, Theorem 3]

$$\Theta_{\mathrm{Ind}_P^G(\pi)}(\gamma) = \sum_{s \in W(A_M, A_H)} \Theta_{s\pi}(\gamma) \frac{|D_{M^s}(\gamma)|^{1/2}}{|D_G(\gamma)|^{1/2}}. \tag{2.1.10}$$

2.1.11. We now review Plancherel measures. Let $M = M_\Phi$ be a standard Levi and σ an irreducible square integrable modulo center representation of M_Φ and $w \in W(\Phi)$. For $w \in W$ denote $N_w = N_\emptyset \cap w\bar{N}_\Phi w^{-1}$, where $\bar{N}_\Phi$ is the opposite unipotent radical of N_Φ. Then there exists an open nonempty subset of unramified characters χ of M such that the integral $A_w(\chi\sigma) = \int_{N_w} f(w^{-1}ng)dn$ converges for all $f \in \mathrm{Ind}_{P_\Phi}^G(\chi\sigma)$. The integral has a meromorphic continuation to the set of all unramified characters. The operator $A_w(\chi\sigma)$ intertwines $\mathrm{Ind}_{P_\Phi}^G(\chi\sigma)$ and $\mathrm{Ind}_{P_\Phi}^G(w(\chi\sigma))$. Let w_Φ be the longest element in $W(\Phi)$ and let μ be the meromorphic function on $\Psi(M)\,\sigma$ satisfying

$$A_{w_\Phi}(\chi\sigma)A_{w_\Phi^{-1}}(w_\Phi(\chi\sigma)) = \mu(\chi\sigma)^{-1}\gamma(G|M_\Phi)^2\ I\,. \tag{2.1.12}$$

On the right side, I is the identity operator. Then,

$$\text{Plancherel measure } \mu\ :=\ \text{the restriction of } \mu \text{ to } \Psi^u(M)\sigma\,. \tag{2.1.13a}$$

Let $\mathcal{E}_2(M)$ denote the square integrable modulo center classes in $\hat{M}$. For $\omega \in \mathcal{E}_2(M)$, set

$$d(\omega)\ :=\ \text{the formal degree of } \omega\ . \tag{2.1.13b}$$

There is an obvious bijection of $\Psi^u(M)/\mathrm{Stab}(\omega)$ with $\Psi^u(M)\,\omega \subseteq \mathcal{E}_2(M)$. Set

$$\begin{aligned} d\,\omega\ :=\ &\text{measure on } \Psi^u(M)\,\omega \text{ transferred from the} \\ &\text{normalized Haar measure on } \Psi^u(M)/\mathrm{Stab}(\omega)\ . \end{aligned} \tag{2.1.13c}$$

2.1.14. Let f be a function in Harish-Chandra's Schwartz space of G. For our purposes it is sufficient to take $f \in C_c^\infty(G)$. For a standard parabolic subgroup $P = MN$ set

$$a(G|M) := c(G|M)^{-2}\gamma(G|M)^{-1}|W_M|^{-1} \tag{2.1.15}$$

and

$$\begin{aligned} f_M(1) &= a(G|M)\int_{\mathcal{E}_2(M)} \mathrm{trace}(\mathrm{Ind}_P^G(\omega)(\check{f}))\ d(\omega)\ \mu(\omega)\ d\,\omega \\ &= a(G|M)\int_{\mathcal{E}_2(M)} \Theta_{\mathrm{Ind}_P^G(\omega)}(\check{f})\ d(\omega)\ \mu(\omega)\ d\,\omega \\ &= a(G|M)\int_{\mathcal{E}_2(M)} \left(\int_G \Theta_{\mathrm{Ind}_P^G(\omega)}(g)f(g^{-1})dg\right)\ d(\omega)\ \mu(\omega)\ d\,\omega \\ &= a(G|M)\int_{\mathcal{E}_2(M)} \left(\int_G \check{\Theta}_{\mathrm{Ind}_P^G(\omega)}(g)f(g)dg\right)\ d(\omega)\ \mu(\omega)\ d\,\omega \end{aligned} \tag{2.1.16}$$

(recall $\check{f}(g) := f(g^{-1})$). Harish-Chandra's Plancherel Formula states that

$$f(1) = \sum_M f_M(1), \tag{2.1.17}$$

where the sum runs over all Levi subgroups up to conjugacy.

2.2. Bernstein center - distributions.

2.2.1. Define

$$\mathcal{Z}(G) := \{\ G\text{-invariant distributions } z \mid z \star f \in C_c^\infty(G) \text{ for all } f \in C_c^\infty(G)\ \}.$$

We remark, that given any $f \in C_c^\infty(G)$, there exists a small enough open compact subgroup J so that $f = \mathrm{ch}_J \star f = f \star \mathrm{ch}_J$ where ch_J denotes the characteristic function of J. If z is a distribution, then $z \star f = z \star (\mathrm{ch}_J \star f) = (z \star \mathrm{ch}_J) \star f$. In particular, in the definition of the space $\mathcal{Z}(G)$, it is sufficient to only require that $z \star \mathrm{ch}_J \in C_c^\infty(G)$ for some system of neighborhoods of identity of compact open subgroups J. Furthermore, if (π, V_π) is a smooth representation of G, for any $v \in V_\pi$ there is an open compact subgroup J which fixes v and $\pi(z)v := \pi(z \star \mathrm{ch}_J)\, v$ is well-defined; hence $\mathcal{Z}(G)$ acts in a natural way in each smooth representation. In this way each $z \in \mathcal{Z}$ defines an endomorphism of the identity functor of the category of smooth representations of G, i.e., an element of the Bernstein center. Every element of the Bernstein center is obtained in this way.

2.2.2. Let $z \in \mathcal{Z}(G)$ and let (π, V_π) be an irreducible smooth representation of G. Schur's lemma implies that $\pi(z)$ is a scalar operator. More generally, when the Bernstein center acts by scalars in a smooth representation π, we say that π has an infinitesimal character. The corresponding homomorphism of the Bernstein center is called the infinitesimal character of π and denoted as χ_π.

2.3. Bernstein center - regular functions.

2.3.1. Consider pairs (M, ρ), where M is a Levi subgroup of G and ρ is an irreducible cuspidal representation of M. Let $[M, \rho]_G$ denote the equivalence class of (M, ρ) under the natural adjoint action of G and let $\Omega(G)$ denote the set of the equivalence class of $[M, \rho]_G$. For $[M, \rho]_G \in \Omega(G)$, the set $\{[M, \chi\rho]_G | \chi \in \Psi(M)\}$ is called a connected component in $\Omega(G)$ (of $[M, \rho]_G$ or sometimes by abuse of notation of (M, ρ)). The map $\psi\mathrm{Stab}(\rho) \mapsto [M, \psi\rho]_G$ from $\Psi(M)/\mathrm{Stab}(\rho)$ to $\Omega(G)$ has finite fibers, and allows one to define a complex algebraic variety structure on the component Ω of $\Omega(G)$ containing $[M, \rho]_G$. The set $\Omega(G)$ is a disjoint union of connected components and so we view it as a complex algebraic variety also. If L is a Levi subgroup of G, the mapping $[H, \rho]_L \mapsto [H, \rho]_G$ (H a Levi subgroup of L) defines a morphism of algebraic varieties $\Omega(M) \to \Omega(G)$, with finite fibers. Denote this mapping by i_{GM}. Denote the algebra of all regular functions on $\Omega(G)$ by $\mathfrak{Z}(G)$ and let $\mathfrak{Z}(G)_0$ be the subalgebra of all regular functions supported on only finitely many components. It is elementary that an infinitesimal character χ of $\mathfrak{Z}(G)$ is completely determined by its restriction to $\mathfrak{Z}(G)_0$.

2.3.2. For $\pi \in \tilde{G}$ one can choose $[M, \rho]_G \in \Omega(G)$ such that π is a subquotient of $\mathrm{Ind}_P^G(\rho)$, where P is a parabolic subgroup (of G) containing M as a Levi factor. The equivalence class of $[M, \rho]_G \in \Omega(G)$ is uniquely determined by π. Let $\Pi_{G}(\pi) := [M, \rho]_G$ denote this class. The map $\Pi_G : \tilde{G} \mapsto \Omega(G)$ has finite fibers. If Ω is a connected component of $\Omega(G)$, consider $\Pi_G^{-1}(\Omega)$. Under a natural topology on $\tilde{G}$ (see [T]), this inverse image is a connected component of $\tilde{G}$. Each $z \in \mathcal{Z}(G)$ defines a map $\hat{z} : \Omega(G) \longrightarrow \mathbb{C}$,

$$\hat{z}([M, \rho]_G) := \chi_{\mathrm{Ind}_P^G(\rho)}(z)\ . \tag{2.3.3}$$

Clearly, $\hat{z}$ is the operator Fourier Transform, with the operator being a scalar. This mapping is an isomorphism of $\mathcal{Z}(G)$ with $\mathfrak{Z}(G)$ (see [BD]). Let $\mathcal{Z}(G)_0$ denote the space of essentially compact distributions corresponding to elements in $\mathfrak{Z}(G)_0$.

2.4. Bernstein center - inversion formula.

2.4.1. Let $z \in \mathcal{Z}(G)$ and $\varphi \in C_0^\infty(G)$. For $x \in G$, let $\lambda_x\varphi$ (resp. $\rho_x\varphi$) denote left (resp. right) translation of the function φ by x; i.e., $(\lambda_x\varphi)(y) := \varphi(x^{-1}y)$ and $(\rho_x\varphi)(y) := \varphi(y)$. We recall that

$$\text{(2.4.2)} \qquad \begin{aligned} &\text{(i)} \quad z \star \varphi(x) \;:=\; z(\lambda_x\check{\varphi}) \qquad (\check{\varphi}(g) := \varphi(g^{-1})),\\ &\text{(ii)} \quad \varphi \star z(x) \;:=\; z(\rho_{x^{-1}}\check{\varphi}),\\ &\text{(iii)} \quad z(\varphi) \;=\; (z \star \check{\varphi})(1) \;=\; (\check{\varphi} \star z)(1)\ . \end{aligned}$$

For ease of notation, we shall abbreviate the induced representation $\mathrm{Ind}_P^G(\omega)$ to π_ω, trace to tr and $\hat{z}(i_{GM}(\Pi_{_M}(\omega)))$ to $\hat{z}(\omega)$. An application of the Plancherel formula to $z \star \check{\varphi} \in C_0^\infty(G)$ gives

$$\text{(2.4.3)} \qquad \begin{aligned} z(\varphi) \;&=\; (z \star \check{\varphi})(1) = \sum_M (z \star \check{\varphi})_M(1)\\ &= \sum_M a(G|M) \int_{\mathcal{E}_2(M)} \mathrm{tr}(\pi_\omega(z \star \check{\varphi}))\ d(\omega)\ \mu(\omega)\ d\omega\\ &= \sum_M a(G|M) \int_{\mathcal{E}_2(M)} \mathrm{tr}(\ \pi_\omega(z)\ \pi_\omega(\check{\varphi}))\ d(\omega)\ \mu(\omega)\ d\omega\\ &= \sum_M a(G|M) \int_{\mathcal{E}_2(M)} \hat{z}(\omega)\ \mathrm{tr}(\pi_\omega(\check{\varphi}))d(\omega)\mu(\omega)\ d\omega\\ &= \sum_M a(G|M) \int_{\mathcal{E}_2(M)} \hat{z}(\omega)\ \left(\int_G \varphi(g)\check{\Theta}_{\pi_\omega}(g)dg \right)\ d(\omega)\ \mu(\omega)\ d\,\omega. \end{aligned}$$

This is just the formula from Remark 2.17 in [BD].

2.5. Distributions in $\mathcal{Z}(G)_0$.

2.5.1. For each component $\Omega \subseteq \Omega(G)$, let $\mathcal{Z}(\Omega)$ denote the vector space of distributions in $\mathcal{Z}(G)$ corresponding to regular functions (supported) on Ω. Then

Theorem 2.5. (i) $\mathcal{Z}(G)_0 = \bigoplus_{\Omega\subseteq\Omega(G)} \ \mathcal{Z}(\Omega)$.

(ii) *Each distribution in $\mathcal{Z}(G)_0$ is given by an invariant locally integrable function. In particular, each distribution in $\mathcal{Z}(\Omega)$ is given by an invariant locally integrable function.*

(iii) *Suppose that we have a family $z_\Omega \in \mathcal{Z}(\Omega)$, $\Omega \in \Omega(G)$. Then for any $\varphi \in C_c^\infty(G)$, $z_\Omega(\varphi) \neq 0$ for only finitely many connected components $\Omega \subseteq \Omega(G)$ and*

$$\varphi \mapsto \sum_{\Omega\subseteq\Omega(G)} z_\Omega(\varphi)$$

defines an invariant distribution. This distribution lies in $\mathcal{Z}(G)$.

(iv) *Any distribution in $\mathcal{Z}(G)$ can be presented in the same way as above, and the presentation above is unique.*

Thus, to understand $\mathcal{Z}(G)$, it is enough to understand $\mathcal{Z}(G)_0$. By the above theorem, elements of the algebra $\mathcal{Z}(G)_0$ are given by invariant locally integrable functions.

Proof. Assertion (i) is obvious from the definitions of $\mathcal{Z}(G)_0$, $\mathcal{Z}(\Omega)$ and, the corresponding fact for $\mathfrak{Z}(G)_0$.

To prove statement (ii), we make the obvious reduction to the situation $z \in \mathcal{Z}(\Omega)$, where Ω is a connected component of $\Omega(G)$. We recall again the mapping $i_{GM} : \Omega(M) \to \Omega(G)$ has finite fibers. We also recall that each irreducible essentially square integrable representation of M is elliptic; this follows, for example, from the orthogonality relations among characters (see [K]). It follows that its infinitesimal character is discrete (with respect to M) in the sense of 3.1 of [BDK]. Recall that there is only finitely many $\Psi(M)$-orbits of discrete characters by Proposition 3.1 of [BDK]. From this it easily follows that one can find only finitely many irreducible square integrable representations ω_i of M_i, $i = 1, \ldots, n$, such that each irreducible essentially square integrable representation of any Levi subgroup M' of G, whose infinitesimal character after induction lies in Ω, is contained in $\bigcup_{i=1}^n \Psi(M_i)\,\omega_i$. Considering central characters, we see that all irreducible square integrable representations of Levi subgroups M' of G, whose infinitesimal characters after induction lie in Ω, are contained in $\bigcup_{i=1}^n \Psi^u(M_i)\,\omega_i$. For each $i \in \{1, \ldots, n\}$ fix a parabolic subgroup P_i such that M_i is a Levi factor of P_i. Now the inversion formula (2.4.3) for the Bernstein center for $z \in \mathcal{Z}(\Omega)$ gives

$$z(\varphi) = \sum_{i=1}^{n} a(G|M_i) \int_{\Psi(M_i)^u\,\omega_i} \hat{z}(\omega) \left(\int_G \varphi(g)\ \check{\Theta}_{\pi_\omega}(g)\,dg \right) \mu(\omega)\ d(\omega)\ d\,\omega. \tag{2.5.2}$$

(Recall $\hat{z}(\omega) = \hat{z}(i_{GM_i}(\Pi_{M_i}(\omega)))$ and $\pi_\omega = \mathrm{Ind}_{P_i}^G(\omega)$.) Fix $i \in \{1, \ldots, n\}$. We know that μ is a continuous nonnegative function on $\Psi(M_i)^u\,\omega_i$ [W, Lemma V.2.1]. Further, $\omega \mapsto d(\omega)$ is constant on $\Psi(M_i)^u\,\omega_i$, and the function $\omega \mapsto \hat{z}(i_{GM_i}(\Pi_{M_i}(\omega)))$ is continuous on $\Psi(M_i)^u\omega_i$. From the formula for induced representation, it follows that

$$(\omega, g) \mapsto \check{\Theta}_{\mathrm{Ind}_{P_i}^G(\omega)}(g)$$

is measurable on $(\Psi(M_i)^u\,\omega_i) \times G$. Now, obviously, the function

$$(\omega, g) \mapsto \mu(\omega)\ d(\omega)\ \hat{z}(i_{GM_i}(\Pi_{M_i}(\omega)))\,\check{\Theta}_{\mathrm{Ind}_{P_i}^G(\omega)}(g)\ \varphi(g)$$

is measurable on $(\Psi(M_i)^u\,\omega_i) \times G$. We prove that

$$(\omega, g) \mapsto |\mu(\omega)\ d(\omega)\ \hat{z}(i_{GM_i}(\Pi_{M_i}(\omega)))\,\check{\Theta}_{\mathrm{Ind}_{P_i}^G(\omega)}(g)\ \varphi(g)| \tag{2.5.3}$$

is integrable on $(\Psi(M_i)^u\,\omega_i) \times G$. To do this, it is enough to show that

$$(\omega, g) \mapsto |\check{\Theta}_{\mathrm{Ind}_{P_i}^G(\omega)}(g)\ \varphi(g)| \tag{2.5.4}$$

is integrable on $(\Psi(M_i)^u\,\omega_i) \times G$ because the factor $\mu(\omega)\,d(\omega)\,\hat{z}(i_{GM_i}(\Pi_{M_i}(\omega)))$ is bounded on $\Psi(M_i)^u\,\omega_i$. More precisely, $\mu(\omega)\,d(\omega)\,\hat{z}(i_{GM_i}(\Pi_{M_i}(\omega)))$ is continuous on $\Psi(M_i)^u\,\omega_i$, and $\Psi(M_i)^u\,\omega_i$ is compact. The choice of ω_i as a base point in $(\Psi(M_i)^u\,\omega_i)$ results in the question of integrability being equivalent to the integrability of $(\chi, g) \mapsto |\check{\Theta}_{\mathrm{Ind}_{P_i}^G(\chi\omega_i)}(g)\ \varphi(g)|$ on $\Psi(M_i)^u \times G$. Heuristically, $(\chi, g) \mapsto |\check{\Theta}_{\mathrm{Ind}_{P_i}^G(\chi\omega_i)}(g)|$ is a locally integrable function on $\Psi(M_i)^u \times G$ and φ is a compactly supported function on G, so their product is integrable. More precisely, and for the sake of completeness, we follow the calculation in [vD, §5, page 237]. We use

Weyl's integration formula. Let Γ be a maximal torus in G defined over F. For $\gamma \in \Gamma \cap G'$ define the orbital integral

$$F_{\varphi}^{G/\Gamma}(\gamma) \;=\; |D_G(\gamma)|^{1/2} \int_{G/\Gamma} f(x\gamma x^{-1})\, dx \;. \tag{2.5.5}$$

Set $W_G(\Gamma) := N_G(\Gamma)/\Gamma$. Let $\mathcal{C}_G$ be a set of representatives for the conjugacy classes of maximal (F)-tori in G. We have

$$\begin{aligned}
&\int_{\Psi^u(M_i)} \left(\int_G |\check{\Theta}_{\mathrm{Ind}_{P_i}^G(\chi\omega_i)}(g)\ \varphi(g)| dg \right) d\chi \qquad (2.5.6)\\
&\quad \leq \int_{\Psi^u(M_i)} \Bigg(\sum_{\Gamma\in\mathcal{C}_G} |W_G(\Gamma)|^{-1} \int_\Gamma |D_G(\gamma)|^{1/2} \\
&\qquad\qquad |\Theta_{\mathrm{Ind}_{P_i}^G(\chi\omega_i)}(\gamma)||F_{\check{\varphi}}^{G/\Gamma}(\gamma)| d\gamma \Bigg) d\chi \\
&\quad \leq \int_{\Psi^u(M_i)} \Bigg(\sum_{\Gamma\in\mathcal{C}_G} |W_G(\Gamma)|^{-1} \int_\Gamma \sum_{s\in W(A_{M_i},A_\Gamma)} \qquad (2.5.7)\\
&\qquad\qquad |\Theta_{s(\chi\omega_i)}(\gamma)||D_{M_i^s}(\gamma)|^{1/2}|F_{\check{\varphi}}^{G/\Gamma}(\gamma)| d\gamma \Bigg) d\chi \\
&\quad = \left(\int_{\Psi^u(M_i)} d\chi \right) \Bigg(\sum_{\Gamma\in\mathcal{C}_G} |W_G(\Gamma)|^{-1} \int_\Gamma \sum_{s\in W(A_{M_i},A_\Gamma)} \\
&\qquad\qquad |\Theta_{s(\omega_i)}(\gamma)||D_{M_i^s}(\gamma)|^{1/2}|F_{\check{\varphi}}^{G/\Gamma}(\gamma)| d\gamma \Bigg).
\end{aligned}$$

We have used above that $|\Theta_{s(\chi\omega_i)}(\gamma)| = |\Theta_{s(\chi)s(\omega_i)}(\gamma)| = |s(\chi)(\gamma)\Theta_{s(\omega_i)}(\gamma)| = |\Theta_{s(\omega_i)}(\gamma)|$. The last integral converges by [vD, §5, page 237]. Now, Fubini's theorem implies that

$$(\omega, g) \mapsto |\mu(\omega)\ d(\omega)\ \hat{z}(i_{GM_i}(\Pi_{M_i}(\omega)))\ \ \check{\Theta}_{\mathrm{Ind}_{P_i}^G(\omega)}(g)\ \varphi(g)| \tag{2.5.8}$$

is integrable on $(\Psi(M_i)^u\, \omega_i) \times G$ and the function

$$g \mapsto \varphi(g)\ a(G|M_i) \int_{\Psi(M_i)^u\, \omega_i} \hat{z}(i_{GM_i}(\Pi_{M_i})(\omega))\ \check{\Theta}_{\mathrm{Ind}_{P_i}^G(\omega)}(g)\mu(\omega)d(\omega)d\omega \tag{2.5.9}$$

is well-defined almost everywhere and integrable on G. We can therefore define a function f_z by the formula
(2.5.10)

$$f_z : g \mapsto \sum_{i=1}^n a(G|M_i) \int_{\Psi(M_i)^u\, \omega_i} \hat{z}(i_{GM_i}(\Pi_{M_i})(\omega))\ \check{\Theta}_{\mathrm{Ind}_{P_i}^G(\omega)}(g)\mu(\omega)d(\omega)d\omega.$$

This function is well-defined almost everywhere and locally integrable on G. Furthermore,

$$z(\varphi) = \int_G \varphi(g) f_z(g) dg \tag{2.5.11}$$

for $\varphi \in C_c^\infty(G)$. Since the character is a class function, so is f_z. This completes the proof of (ii)

To prove assertion (iii), let $z_\Omega \in \mathcal{Z}(\Omega)$ be a family indexed by the connected components $\Omega \subseteq \Omega(G)$. Note (since $z_\Omega \in \mathcal{Z}(\Omega)$) that by (2.5.2) we have

$$z_\Omega(\varphi) = \sum_{i=1}^{n} a(G|M_i) \int_{\Psi(M_i)^u\,\omega_i} \hat{z}_\Omega(i(\omega))\ \mathrm{tr}\big(\pi_\omega(\check{\varphi})\big)\ \mu(\omega)\ d(\omega)\ d\omega, \tag{2.5.12}$$

where, for ease of notation, we have used $\hat{z}_\Omega(\omega)$ (resp. $\mathrm{tr}(\pi_\omega(\check{\varphi}))$) as obvious abbreviations for $\hat{z}_\Omega(i_{GM_i}(\Pi_{M_i}))$ (resp. $\mathrm{trace}\big(\mathrm{Ind}_{P_i}^G(\omega)(\check{\varphi})\big)$. For a given open compact subgroup J, only finitely many connected components of the smooth dual $\tilde{G}$ possess representations with nontrivial J-invariant vectors. This well-known fact and the above formula imply that $\pi_\omega(\check{\varphi})$ is nonzero; hence $z_\Omega(\varphi)$ is nonzero for only finitely many components Ω. We conclude that the distribution $z := \sum_\Omega z_\Omega$ is well-defined.

To complete the proof of (iii), it remains to show that z is essentially compact; i.e., $z \star \varphi$ is compactly supported for any $\varphi \in C_c^\infty(G)$. We do this by again using (2.5.12). By the remark in section 2.2, it suffices to show for an arbitary open compact subgroup J of G that $z \star \mathrm{ch}_J$ is a compactly supported function. To do this, it suffices to show the function $z_\Omega \star \mathrm{ch}_J = \mathrm{ch}_J \star z_\Omega$ is nonzero for only finitely many $\Omega \subseteq \Omega(G)$. We have

$$\begin{aligned} z_\Omega \star \mathrm{ch}_J(g) &= z_\Omega(\lambda_g((\mathrm{ch}_J)\check{\ })) = z_\Omega(\lambda_g(\mathrm{ch}_J)) \\ &= \sum_{i=1}^{n} a(G|M_i) \int_{\Psi(M_i)^u\,\omega_i} \hat{z}_\Omega(i(\omega))\ \mathrm{tr}\big(\pi_\omega((\lambda_g(\mathrm{ch}_J))\check{\ })\big)\ \mu(\omega)\ d(\omega)\ d\omega. \end{aligned} \tag{2.5.13}$$

The function $(\lambda_g(\mathrm{ch}_J))\check{\ }$ is left J-invariant and so $\pi_\omega((\lambda_g(\mathrm{ch}_J))\check{\ })$ lies in $\big(\mathrm{Ind}_{P_i}^G(\omega)\big)^J$. But this later space is nonzero for only finitely many components Ω, and by consequence $\mathrm{tr}\big(\pi_\omega((\lambda_g(\mathrm{ch}_J))\check{\ })\big)$ and hence the integral in (2.5.13) is nonzero for only finitely many components Ω. This proves (iii).

We turn finally to the proof of statement (iv). Suppose $z \in \mathcal{Z}(G)$. Consider $\hat{z} : \Omega(G) \to \mathbb{C}$. For $\Omega \subseteq \Omega(G)$ let $\hat{z}_\Omega$ denote the function on $\Omega(G)$ which coincides with $\hat{z}$ on Ω and is zero elsewhere. Let $z_\Omega \in \mathcal{Z}(G)$ denote the distribution corresponding to $\hat{z}_\Omega$. By (iii), we know that $z - \sum_\Omega z_\Omega \in \mathcal{Z}(G)$. Moreover, by construction $(z - \sum_\Omega z_\Omega)\hat{\ } = 0$. Thus, $z = \sum_\Omega z_\Omega$. In the same way, one sees that if $\sum_\Omega z_\Omega = 0$ for $z_\Omega \in \mathcal{Z}(\Omega)$, then $z_\Omega = 0$ for all Ω. This implies (iv). □

3. $SL(2, F)$

In this section we shall determine explicitly the locally integrable functions that determine the Bernstein center in the case of $G = SL(2, F)$. More precisely, we shall consider a natural basis of regular functions on $\Omega(G)$, and determine explicitly distributions in the Bernstein center as locally integrable functions on G which, after Fourier Transform, give corresponding elements of the natural basis.

3.1. Cuspidal components. We consider connected components of $\Omega(G)$. Fix an irreducible cuspidal representation ρ of G. Then $\{[G, \rho]_G\}$ is a (cuspidal) connected component of $\Omega(G)$. Let $e_0 = e_0^{(\rho)}$ denote the characteristic function of $[G, \rho]_G \in \Omega(G)$. Let $d(\rho)$ denote the formal degree of ρ. In the inversion formula, we have

$\gamma(G|G) = c(G|G) = 1$, $\mu(\rho) = 1$ and $|W_G| = 1$; so the inversion formula gives

$$(d(\rho)\check{\Theta}_\pi)\hat{} = e_0 \ .$$

3.2. Characters of principal series representations. Let A_0 (resp. P_0) denote the diagonal (resp. upper triangular) matrices in G. Let λ be a character of $F^\times$. For convenience, we also denote as λ the character of A_0 and P_0 given by

$$\text{diag}(a, a^{-1}) \mapsto \lambda(a) \ .$$

Let $\pi = \pi(\lambda) = \text{Ind}_{P_0}^G(\lambda)$ denote the principal series representation. It is well-known that the character Θ_π of π is supported on the adjoint orbit of A_0. If $g \in G$ is conjugate to $\text{diag}(a, a^{-1})$ for some $a \in F^\times$, then

$$\Theta_\pi(g) = \frac{\lambda(a) + \lambda(a^{-1})}{|a - a^{-1}|_F}.$$

We rewrite this formula as follows: Fix a uniformizing element ϖ, i.e. a generator of the maximal ideal in the ring of integers $\mathcal{O}$ of F. Denote $q = |\varpi|_F^{-1}$ ($|\,|_F$ is a modulus character of F). Let λ_0 denote the restriction of λ to $\mathcal{O}^\times$ and set $s = \lambda(\varpi) \in \mathbb{C}^\times$. Since λ_0 and s completely determine λ, we shall also denote λ by (λ_0, s). For $o \in \mathcal{O}^\times$ and $k \in \mathbb{Z}$, the above formula gives

$$\Theta_\pi(\text{diag}(\varpi^k o, \varpi^{-k} o^{-1})) = \frac{\lambda_0(o)s^k + \lambda_0(o^{-1})s^{-k}}{|\varpi^k o^2 - \varpi^{-k}|_F}$$

$$= \begin{cases} \frac{\lambda_0(o)s^k + \lambda_0(o^{-1})s^{-k}}{q^{|k|}} & \text{when } k \neq 0, \\ \frac{\lambda_0(o) + \lambda_0(o^{-1})}{|o^2 - 1|_F}. & \text{when } k = 0. \end{cases}$$

We note that

$$\gamma(G|A_0) = \frac{q+1}{q}, \qquad c(G|A_0) = 1, \qquad |W_{A_0}| = 2, \qquad d(\lambda) = 1.$$

3.3. Non-cuspidal components. We can parameterize the non-cuspidal connected components in $\Omega(G)$ as the set of character pairs $\big\{\{\lambda_0, \lambda_0^{-1}\} \,\big|\, \lambda_0 \in (\mathcal{O}^\times)\hat{}\big\}$. To $\{\lambda_0, \lambda_0^{-1}\}$, we associate the component

$$\{\ \big[A_0, (\lambda_0, s)\big]_G \mid s \in \mathbb{C}^\times\ \} \ = \ \{\ \big[A_0, (\lambda_0^{-1}, s)\big]_G \mid s \in \mathbb{C}^\times\ \} \ .$$

Fix $\{\lambda_0, \lambda_0^{-1}\}$ and let $\Omega = \Omega(\{\lambda_0, \lambda_0^{-1}\})$ denote the associated component. We consider connected components according to three cases:

(i) $\lambda_0^2 \neq 1$,
(ii) λ_0 has order two,
(iii) $\lambda_0 = 1$.

3.4. Regular non-cuspidal components. If $\lambda_0^2 \neq 1$, then there are two natural ways to identify Ω with $\mathbb{C}^\times$:

$$s \mapsto (\lambda_0, s) \qquad \text{and} \qquad s \mapsto (\lambda_0^{-1}, s) \ .$$

We fix such an identification. Then, regular functions on Ω have as a natural basis the functions (recall, the uniformizing element is fixed)

$$e_n = e_n^{(\lambda_0)} : s \mapsto s^n, \quad n \in \mathbb{Z}.$$

Let $z_n = z_n^{(\lambda_0)} \in \mathcal{Z}(G)$ be the distribution corresponding to e_n. Note that $e_n^{(\lambda_0)}$ and $z_n^{(\lambda_0)}$ depend on the choice of uniformizing element ϖ.

For $\varphi \in C_c^\infty(G)$, and $z \in \mathcal{Z}(\Omega)$, the Plancherel inversion formula yields

$$z(\varphi) = \frac{q}{2(q+1)} \sum_{\epsilon \in \{\pm 1\}} \int_{\{t \in \mathbb{C} \mid |t|=1\}} \hat{z}(i_{GM_\emptyset}(\Pi_{M_\emptyset}((\lambda_0^\epsilon, t^\epsilon)))) \\ \times \left(\int_G \varphi(g) \check{\Theta}_{\mathrm{Ind}_{P_\emptyset}^G((\lambda_0^\epsilon, t^\epsilon))}(g) dg \right) \mu((\lambda_0^\epsilon, t^\epsilon))\, dt,$$

for $\varphi \in C_c^\infty(G)$. In the above formula dt denotes the Haar measure on $\{t \in \mathbb{C}; |t| = 1\}$. It follows directly from the definition that $\mu((\lambda_0^{-1}, t^{-1})) = \mu((\lambda_0, t))$. Further, $\mu((\lambda_0, t))$ is constant on Ω, and this constant is equal to $\mu_\Omega := (\frac{q+1}{q})^2 q^{f(\lambda_0)}$, where $f(\lambda_0)$ is the conductor of λ_0. Observe also that $\Pi_{M_\emptyset}((\lambda_0^{-1}, t^{-1})) = \Pi_{M_\emptyset}((\lambda_0, t))$; so $\check{\Theta}_{\mathrm{Ind}_{P_\emptyset}^G((\lambda_0,t))}(g) = \Theta_{\mathrm{Ind}_{P_\emptyset}^G((\lambda_0,t))}(g)$ and

$$z(\varphi) = \frac{q}{(q+1)} \int_{\{t \in \mathbb{C} \mid |t|=1\}} \hat{z}(i_{GM_\emptyset}(\Pi_{M_\emptyset}((\lambda_0, t)))) \\ \times \left(\int_G \varphi(g) \Theta_{\mathrm{Ind}_{P_\emptyset}^G((\lambda_0,t))}(g) dg \right) \mu_\Omega\, dt.$$

Identifying Ω with $\mathbb{C}^\times$ we get

$$z(\varphi) = \mu_\Omega \frac{q}{q+1} \frac{1}{2\pi i} \int_{\{t \in \mathbb{C} \mid |t|=1\}} \hat{z}(s) \left(\int_G \varphi(g) \Theta_{\mathrm{Ind}_{P_\emptyset}^G((\lambda_0,s))}(g) dg \right) \frac{ds}{s}.$$

Here, ds denotes the complex integral along the unit circle $\{t \in \mathbb{C} \mid |t| = 1\}$ and $i = \sqrt{-1}$. By Fubini's theorem we have

$$z(\varphi) = \int_G \varphi(g) \left(\left(\frac{q+1}{q}\right) q^{f(\lambda_0)} \ \frac{1}{2\pi i} \int_{\{t \in \mathbb{C} \mid |t|=1\}} \hat{z}(s) \Theta_{\mathrm{Ind}_{P_\emptyset}^G((\lambda_0,s))}(g) \frac{ds}{s} \right) dg,$$

and therefore the locally integrable function

$$f_z(g) = \left(\frac{q+1}{q}\right) q^{f(\lambda_0)} \ \frac{1}{2\pi i} \int_{\{t \in \mathbb{C} \mid |t|=1\}} \hat{z}(s) \Theta_{\mathrm{Ind}_{P_\emptyset}^G((\lambda_0,s))}(g) \frac{ds}{s}$$

represents the distribution z.

We now compute $f_{z_n} = f_{z_n^{(\lambda_0)}}$. Clearly, f_{z_n} is conjugation invariant. Note that $f_{z_n}(g)$ is zero if g is not conjugate to an element from $A_\emptyset$. We have

$$f_{z_n}(\mathrm{diag}(\varpi^k o, \varpi^{-k} o^{-1})) = \left(\frac{q+1}{q}\right) q^{f(\lambda_0)} \ \frac{1}{2\pi i} \int_{|s|=1} s^n \frac{\lambda_0(o) s^k + \lambda_0(o^{-1}) s^{-k}}{|\varpi^k o^2 - \varpi^{-k}|_F} \frac{ds}{s}.$$

For $k = 0$ we get

$$f_{z_n}(\mathrm{diag}(o, o^{-1})) = \left(\frac{q+1}{q}\right) q^{f(\lambda_0)} \ \frac{1}{2\pi i} \int_{|s|=1} s^n \frac{\lambda_0(o) + \lambda_0(o^{-1})}{|o^2 - 1|_F} \frac{ds}{s}$$

(3.4.1)

$$= \begin{cases} \left(\frac{q+1}{q}\right) q^{f(\lambda_0)} \ \frac{\lambda_0(o) + \lambda_0(o^{-1})}{|o^2-1|_F} & \text{when } n = 0, \\ 0 & \text{when } n \neq 0\ . \end{cases}$$

For $k \neq 0$ we get

$$f_{z_n}(\mathrm{diag}(\varpi^k o, \varpi^{-k} o^{-1})) = \big(\frac{q+1}{q}\big)\, q^{f(\lambda_0)}\ \frac{1}{2\pi i}\int_{|s|=1} s^n \frac{\lambda_0(o)s^k + \lambda_0(o^{-1})s^{-k}}{|\varpi^k o^2 - \varpi^{-k}|_F}\frac{ds}{s}$$

$$(3.4.2)\qquad = \begin{cases} 0 & \text{when } k \neq \pm n, \\ \big(\frac{q+1}{q}\big)\, q^{f(\lambda_0)}\ \frac{\lambda_0(o^{-1})}{|\varpi^k o^2 - \varpi^{-k}|_F} & \text{when } k = n, \\ \big(\frac{q+1}{q}\big)\, q^{f(\lambda_0)}\ \frac{\lambda_0(o)}{|\varpi^k o^2 - \varpi^{-k}|_F} & \text{when } k = -n \end{cases}$$

$$= \begin{cases} 0 & \text{when } k \neq \pm n, \\ \big(\frac{q+1}{q}\big)\, q^{f(\lambda_0)}\ q^{-|n|}\lambda_0(o^{-1}) & \text{when } k = n, \\ \big(\frac{q+1}{q}\big)\, q^{f(\lambda_0)}\ q^{-|n|}\lambda_0(o) & \text{when } k = -n. \end{cases}$$

In particular, the minimal projector $e_\Omega = f_{z_0}$ is

$$(3.4.3)\qquad f_{z_0}(\mathrm{diag}(\varpi^k o, \varpi^{-k} o^{-1})) = \begin{cases} \big(\frac{q+1}{q}\big)\, q^{f(\lambda_0)}\ \frac{\lambda_0(o)+\lambda_0(o^{-1})}{|o^2-1|_F} & \text{when } k = 0, \\ 0 & \text{when } k \neq 0. \end{cases}$$

Observe that if $n \neq 0$, then the formula (3.4.1) holds for all $k \in \mathbb{Z}$ (including $k = 0$).

3.5. Irregular ramified principal series components. Let λ_0 be a nontrivial character of $\mathcal{O}^\times$ of order two. Note that $s_1, s_2 \in \mathbb{C}^\times$ give conjugate pairs $(A_0, (\lambda_0, s_1))$ and $(A_0, (\lambda_0, s_2))$ if and only if $s_1 = s_2^{\pm 1}$. In this way we shall parameterize Ω (parameters are in $\mathbb{C}^\times$). For the basis of polynomials one can take here

$$e_n = e_n^{(\lambda_0)} : s \mapsto s^n + s^{-n},\ n \in \mathbb{Z}_+.$$

The function μ is the constant $\mu_\Omega := \big(\frac{q+1}{q}\big)^2 q$. The Plancherel inversion formula for $z \in \mathcal{Z}(\Omega)$ gives (after identification of Ω with $\mathbb{C}^\times$)

$$\begin{aligned} z(\varphi) &= \mu_\Omega \frac{q}{2(q+1)}\ \frac{1}{2\pi i}\int_{\{s\in\mathbb{C}\ |\ |s|=1\}} \hat{z}(s)\bigg(\int_G \varphi(g)\, \check{\Theta}_{\mathrm{Ind}_{P_\emptyset}^G((\lambda_0,s))}(g)\, dg\bigg)\frac{ds}{s} \\ &= \int_G \varphi(g)\ \bigg(\frac{q+1}{2}\int_{\{s\in\mathbb{C}\ |\ |s|=1\}} \hat{z}(s)\, \check{\Theta}_{\mathrm{Ind}_{P_\emptyset}^G((\lambda_0,s))}(g)\frac{ds}{s}\bigg)\, dg, \end{aligned}$$

for $\varphi \in C_c^\infty$. Thus

$$f_{e_n} \ = \ f_{e_n^{(\lambda_0)}} \ = \ \frac{q+1}{2}\int_{\{s\in\mathbb{C};|s|=1\}} (s^n + s^{-n})\, \check{\Theta}_{\mathrm{Ind}_{P_\emptyset}^G((\lambda_0,s))}(g)\frac{ds}{s}.$$

The function f_{z_n} is invariant, and equals zero on elements which are not conjugate to elements of A_0. We have further

$$(3.5.1)\qquad \begin{aligned} f_{z_n}(\mathrm{diag}(\varpi^k o, \varpi^{-k} o^{-1})) &= \frac{q+1}{2}\frac{1}{2\pi i}\int_{|s|=1} (s^n + s^{-n})\frac{\lambda_0(o)s^k + \lambda_0(o^{-1})s^{-k}}{|\varpi^k o^2 - \varpi^{-k}|_F}\frac{ds}{s} \\ &= (q+1)\ \frac{1}{2\pi i}\int_{|s|=1} s^n \frac{\lambda_0(o)(s^k + s^{-k})}{|\varpi^k o^2 - \varpi^{-k}|_F}\frac{ds}{s}. \end{aligned}$$

For $k=0$ we get

$$
\begin{aligned}
f_{z_n}(\mathrm{diag}(o,o^{-1})) &= (q+1)\frac{1}{2\pi i}\int_{|s|=1} s^n \frac{2\lambda_0(o)}{|o^2-1|_F}\frac{ds}{s} \\
&= \begin{cases} (q+1)\frac{2\lambda_0(o)}{|o^2-1|_F} & \text{when } n=0, \\ 0 & \text{when } n\neq 0. \end{cases}
\end{aligned}
\tag{3.5.2}
$$

For $k\neq 0$ we get from (3.5.1)

$$
\begin{aligned}
f_{z_n}(\mathrm{diag}(\varpi^k o,\varpi^{-k}o^{-1})) &= \begin{cases} 0 & \text{when } k\neq \pm n, \\ (q+1)\ \frac{\lambda_0(o)}{|\varpi^k o^2-\varpi^{-k}|_F} & \text{when } k=\pm n \end{cases} \\
&= \begin{cases} 0 & \text{when } k\neq \pm n, \\ (q+1)\ q^{-|n|}\lambda_0(o) & \text{when } k=n. \end{cases}
\end{aligned}
\tag{3.5.3}
$$

In particular, observe that the minimal projector $e_\Omega = f_{z_0}$ is

$$
f_{z_0}(\mathrm{diag}(\varpi^k o,\varpi^{-k}o^{-1})) = \begin{cases} (q+1)\frac{\lambda_0(o)}{|o^2-1|_F} & \text{when } k=0, \\ 0 & \text{when } k\neq 0. \end{cases}
\tag{3.5.4}
$$

Again if $n\neq 0$, the formula (3.5.2) holds also for $k=0$.

3.6. Unramified principal series component. Let Ω be the unramified component of $\Omega(G)$. We attach to $s\in\mathbb{C}^\times$ the character $(1_{\mathcal{O}^\times},s):\mathrm{diag}(\varpi^k o,\varpi^{-k}o^{-1})\mapsto s^k, o\in\mathcal{O}^\times, k\in\mathbb{Z}$. Then s and s^{-1} give the same element of Ω, and this is the only case when this happens. The regular functions on this component have for a natural basis

$$
e_n = e_n^{(1_{\mathcal{O}^\times})} : s\mapsto s^n+s^{-n},\ n\in\mathbb{Z}_+.
$$

For $\varphi\in C_c^\infty(G)$ and $z\in\mathcal{Z}(\Omega)$, the Plancherel inversion formula gives

$$
\begin{aligned}
z(\varphi) = &\frac{q+1}{2q}\frac{1}{2\pi i}\int_{|s|=1}\hat{z}(s) \\
&\times\left(\int_G \varphi(g)\,\check{\Theta}_{\mathrm{Ind}_{P_\emptyset}^{SL(2,F)}((1_{\mathcal{O}^\times},s))}(g)dg\right)\frac{(1-s)(1-s^{-1})}{(q-s)(q-s^{-1})}\frac{ds}{s} \\
&+ d(St_{SL(2)})\hat{z}(q)\int_G \varphi(g)\check{\Theta}_{St_{SL(2,F)}}(g)dg \\
= &\int_G \varphi(g)\left(\frac{q+1}{2q}\frac{1}{2\pi i}\int_{|s|=1}\hat{z}(s)\check{\Theta}_{\mathrm{Ind}_{P_\emptyset}^{SL(2,F)}((1_{\mathcal{O}^\times},s))}(g)\frac{(1-s)(1-s^{-1})}{(q-s)(q-s^{-1})}\frac{ds}{s}\right)dg \\
&\quad + \int_G \varphi(g)\left((q-1)\hat{z}(q)\check{\Theta}_{St_{SL(2,F)}}(g)\right)dg\ .
\end{aligned}
$$

So,

$$
f_{e_n} \;=\; f'_{e_n} + f''_{e_n}
\tag{3.6.1}
$$

where

$$f'_{e_n}(g) = \frac{q+1}{2q}\frac{1}{2\pi i}\int_{|s|=1}(s^n+s^{-n})\,\check{\Theta}_{\mathrm{Ind}_{P_\emptyset}^{SL(2,F)}((1_{\mathcal{O}^\times},s))}(g)\frac{(1-s)(1-s^{-1})}{(q-s)(q-s^{-1})}\frac{ds}{s}$$
$$= \frac{q+1}{q}\frac{1}{2\pi i}\int_{|s|=1} s^n\,\check{\Theta}_{\mathrm{Ind}_{P_\emptyset}^{SL(2,F)}((1_{\mathcal{O}^\times},s))}(g)\frac{(1-s)(1-s^{-1})}{(q-s)(q-s^{-1})}\frac{ds}{s}$$

and

$$f''_{e_n}(g) \ = \ (q-1)(q^n+q^{-n})\,\check{\Theta}_{St_{SL(2,F)}}(g).$$

Both these functions are invariant. In particular,

$$(3.6.2)\qquad f''_{e_n}(g) = \begin{cases} (q-1)(q^n+q^{-n})\big(\frac{q^k+q^{-k}}{|\varpi^k o^2-\varpi^{-k}|_F}-1\big) & \text{if } g \text{ is conjugate to } \\ & \mathrm{diag}(\varpi^k o, \varpi^{-k}o^{-1}) \\ -(q-1)(q^n+q^{-n}) & \text{otherwise .} \end{cases}$$

The function f'_{e_n} has support in $Ad(G)A_0$. If g is conjugate to $\mathrm{diag}(\varpi^k o, \varpi^{-k}o^{-1})$, we have

$$f'_{e_n}(g) \ = \ \frac{q+1}{q} \ \ \frac{2}{|\varpi^k o^2-\varpi^{-k}|_F} \ \ \frac{1}{2\pi i} \ \ \int_{|s|=1} s^n(s^k+s^{-k})\frac{(1-s)(1-s^{-1})}{(q-s)(q-s^{-1})}\frac{ds}{s}\ .$$

Set

$$(3.6.3)\qquad I_l = \frac{1}{2\pi i}\int_{|s|=1} s^l\frac{(1-s)(1-s^{-1})}{(q-s)(q-s^{-1})}\frac{ds}{s}.$$

Thus

$$(3.6.4)\qquad f'_{e_n}(\mathrm{diag}(\varpi^k o, \varpi^{-k}o^{-1})) = \frac{q+1}{q}\frac{2}{|\varpi^k o^2-\varpi^{-k}|_F}(I_{n+k}+I_{n-k}).$$

We now compute I_l. From

$$s^{l-1}\frac{(1-s)(1-s^{-1})}{(q-s)(q-s^{-1})} \ = \ s^{l-1}\frac{(1-s)(s-1)}{(q-s)(sq-1)} \ = \ \frac{1}{q}s^{l-1}\frac{(1-s)^2}{(q-s)(1/q-s)}$$
$$= \ \frac{1}{q}\ s^{l-1}\frac{(1-s)^2}{(1-s/q)(1-sq)}\ ,$$

we see that the above function has exactly one pole in the circle $|s|=1$ if $l \geq 1$. It is at $s=q^{-1}$. If $l \leq 0$, there is an additional pole at $s=0$. We now compute the residues. The residue at $1/q$ is

$$\lim_{s\to 1/q}\ (s-1/q)\frac{1}{q}\ s^{l-1}\frac{(1-s)^2}{(q-s)(1/q-s)} = \lim_{s\to 1/q} -\frac{1}{q}s^{l-1}\frac{(1-s)^2}{(q-s)}$$
$$= -\frac{1}{q}\ q^{1-l}\frac{(1-1/q)^2}{(q-1/q)} = -q^{-l-1}\frac{(q-1)^2}{(q^2-1)}$$
$$= -q^{-l-1}\frac{(q-1)}{(q+1)}.$$

Now suppose that $l \leq 0$. Then

$$\begin{aligned}
\frac{1}{q}\ s^{l-1}\frac{(1-s)^2}{(1-s/q)(1-sq)} &= \frac{1}{q}\ (s^{l-1}-2s^l+s^{l+1})\left(\sum_{a=0}^{\infty}\frac{s^a}{q^a}\right)\left(\sum_{b=0}^{\infty}q^b s^b\right) \\
&= \frac{1}{q}\ (s^{l-1}-2s^l+s^{l+1})\left(\sum_{m=0}^{\infty}\left(\sum_{b=0}^{m}q^{2b-m}\right)s^m\right) \\
&= \frac{1}{q}\ (s^{l-1}-2s^l+s^{l+1})\left(\sum_{m=0}^{\infty}\left(q^{-m}\frac{q^{2m+2}-1}{q^2-1}\right)s^m\right) \\
&= \frac{1}{q^2-1}\ (s^{l-1}-2s^l+s^{l+1})\left(\sum_{m=0}^{\infty}(q^{m+1}-q^{-m-1})s^m\right)
\end{aligned}$$

and we see the residue at 0 is

$$\begin{cases} \frac{1}{q^2-1}(q-q^{-1}) & \text{when } l=0, \\ \frac{1}{q^2-1}\left((q^2-q^{-2})+(q-q^{-1})\right) & \text{when } l=-1, \\ \frac{1}{q^2-1}\left((q^{-l+1}-q^{l-1})+(q^{-l}-q^{l})+(q^{-l-1}-q^{l+1})\right) & \text{when } l<-1; \end{cases}$$

i.e.,

$$\begin{cases} \frac{1}{q^2-1}(q-q^{-1}) & \text{when } l=0, \\ \frac{1}{q^2-1}\sum_{i=-l-1}^{-l+1}(q^{-i}-q^{i}) & \text{when } l\leq -1. \end{cases}$$

Thus

$$I_l = \begin{cases} -\ q^{-l-1}\frac{(q-1)}{(q+1)} & \text{when } l\geq 1, \\ -\ q^{-l-1}\frac{(q-1)}{(q+1)} + \frac{1}{q^2-1}(q-q^{-1}) & \text{when } l=0, \\ -\ q^{-l-1}\frac{(q-1)}{(q+1)} + \frac{1}{q^2-1}\sum_{i=-l-1}^{-l+1}(q^{-i}-q^{i}) & \text{when } l\leq -1. \end{cases} \tag{3.6.5}$$

Now (3.6.1), (3.6.2), (3.6.5) and (3.6.4) yield f_{e_n} ((3.6.5) and (3.6.4) yield f'_{e_n}). The minimal projector e_Ω equals $\frac{1}{2}f_{e_0} = \frac{1}{2}f'_{e_0} + \frac{1}{2}f''_{e_0}$ and

$$\frac{1}{2}f'_{e_0}\left(\operatorname{diag}(\varpi^k o, \varpi^{-k}o^{-1})\right) = \frac{q+1}{q}\frac{1}{|\varpi^k o^2 - \varpi^{-k}|_F}(I_k + I_{-k})$$

(3.6.6)

$$= \frac{q+1}{q}\frac{1}{|\varpi^k o^2 - \varpi^{-k}|_F}\begin{cases} \frac{2}{q(q+1)} & \text{when } k=0, \\ \frac{2q^{k+1}-q^k+2q^{k-1}-q^{-k+1}-q^{-k}-q^{-k-1}}{q^2-1} & \text{when } k>0\,. \end{cases}$$

Remark. With the exception of the unramified component, the functions representing the minimal projectors of all the other components have support in the compact elements of G.

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