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# QUASI-EXCEPTIONAL SETS AND EQUIVARIANT COHERENT SHEAVES ON THE NILPOTENT CONE

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Abstract. A certain $t$-structure on the derived category of equivariant coherent sheaves on the nil-cone of a simple complex algebraic group is introduced by the author in the paper *Perverse coherent sheaves* (the so-called perverse $t$-structure corresponding to the middle perversity). In the present note we show that the same $t$-structure can be obtained from a natural quasi-exceptional set generating this derived category. As a consequence we obtain a bijection between the sets of dominant weights and pairs consisting of a nilpotent orbit, and an irreducible representation of the centralizer of this element, conjectured by Lusztig and Vogan (and obtained by other means by the author in the paper *On tensor categories attached to cells in affine Weyl groups*, to be published).

## 1. Introduction

Let $G$ be a simple complex algebraic group, $\Lambda$ the weight lattice of $G$, $\Lambda^+ \subset \Lambda$ the subset of dominant weights, $\mathfrak{g}$ the Lie algebra of $G$, and $\mathcal{N} \subset \mathfrak{g}$ the subvariety of nilpotent elements.

Let $\mathbf{O}$ be the set of pairs $(O, L)$, where $O \subset \mathcal{N}$ is a $G$-orbit, and $L$ is an irreducible representation of the centralizer $Z_G(n)$, $n \in O$ (up to conjugacy).

Lusztig and Vogan conjectured (independently) that there exists a natural bijection between the sets $\mathbf{O}$ and $\Lambda^+$. (Since the meaning of the word "natural" is not specified, this formulation of the conjecture is not precise.)

Existence of such a bijection follows from the main result of [B1], and also from [B3] (the relation between the main result of [B1] and the bijection $\mathbf{O} \leftrightarrow \Lambda^+$ is explained in [L], 10.8). The arguments of [B1], [B3] use perverse sheaves on the affine flag variety of the Langlands dual group, and some deep results of the geometric theory of Langlands correspondence (in particular, the construction of [G]). In this note we construct a bijection $\mathbf{O} \leftrightarrow \Lambda^+$ by more direct and elementary means. (Coincidence of the two bijections is checked in [B3].)

Let us now describe the content of the paper. We provide a new ("exotic") $t$-structure on the triangulated categories $D^b(Coh^G(\mathcal{N}))$, the derived category of $G$-equivariant coherent sheaves on $\mathcal{N}$. The core $\mathcal{P}$ of this $t$-structure is an abelian category of finite type (i.e. all objects have finite length); moreover, it is a *quasi-hereditary* (or *Kazhdan-Lusztig type*) category. This means, in particular, that $\mathcal{P}$ has a preferred ordered set of objects called standard objects, another one of costandard

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objects, and both these sets are in canonical bijection with the set of (isomorphism classes of) irreducible objects (see section 2.1 below for precise definitions).

The "exotic" $t$-structure admits two different descriptions. On the one hand it is the perverse $t$-structure on equivariant coherent sheaves (in the sense of Deligne) corresponding to the middle perversity; see [B2]. This makes clear that (isomorphism classes of) irreducible objects in $\mathcal{P}$ are numbered by $\mathbf{O}$; for $(O,L)\in\mathbf{O}$ let $IC_{O,L}$ denote the corresponding irreducible object of $\mathcal{P}$.

On the other hand, given an abstract triangulated category $\mathfrak{D}$ with an ordered set of objects $\nabla=\{\nabla^i\}$ satisfying certain conditions (a set satisfying those conditions is called a dualizable quasi-exceptional set generating $\mathfrak{D}$) one can produce a $t$-structure on $\mathfrak{D}$, called the $t$-structure of a quasi-exceptional set. The core of this $t$-structure is quasi-hereditary, and $\nabla$ is the set of its costandard objects. We show that for $\mathfrak{D}=D^b(Coh^G(\mathcal{N}))$, the set $\nabla=\{\nabla^\lambda\}$, $\lambda\in\Lambda^+$ consisting of direct images of positive line bundles under the Springer map $\pi:\tilde{\mathcal{N}}\to\mathcal{N}$ is a quasi-exceptional set generating $D^b(Coh(\mathcal{N}))$; and that the corresponding $t$-structure coincides with the one described in the previous paragraph.

Thus the bijection between the sets of irreducible and costandard objects in a quasi-hereditary category yields a bijection

$$\mathbf{O}=\{IC_{O,L}\}\leftrightarrow\nabla=\Lambda^+.$$

We note that our approach is closely related to that of [O]. We recall briefly the set-up of *loc. cit.* Let $K^G(\mathcal{N})=K^0\left(Coh^G(\mathcal{N})\right)$, $K^{G\times\mathbb{C}^*}(\mathcal{N})=K^0\left(Coh^{G\times\mathbb{C}^*}(\mathcal{N})\right)$ denote respectively the Grothendieck groups of the category of $G$-equivariant coherent sheaves on $\mathcal{N}$, and of the category of $G\times\mathbb{C}^*$-equivariant coherent sheaves on $\mathcal{N}$ (where $\mathbb{C}^*$ acts on $\mathcal{N}$ by $t:n\to t^2n$). Then $K^G(\mathcal{N})$ is freely generated by the classes of $\nabla^\lambda$ ($AJ_\lambda$ in notation of [O]), $\lambda\in\Lambda^+$; and $K^{G\times\mathbb{C}^*}(\mathcal{N})$ is freely generated as a $\mathbb{Z}[v,v^{-1}]$-module by the classes of $\tilde{\nabla}_\lambda$. Here $\tilde{\nabla}_\lambda$ is a natural lift of $\nabla^\lambda$ to $Coh^{G\times\mathbb{C}^*}(\mathcal{N})$ (i.e. a $G\times\mathbb{C}^*$ equivariant coherent sheaf, which gives $\nabla^\lambda$ upon restricting the equivariance to $G$), and the action of the indeterminate $v$ on $K^{G\times\mathbb{C}^*}(\mathcal{N})$ corresponds to the twist by the tautological character of $\mathbb{C}^*$. We call the set of classes $\{[\tilde{\nabla}_\lambda]\}$ the costandard basis of $K^{G\times\mathbb{C}^*}(\mathcal{N})$.

In [O] Ostrik conjectures existence of another $\mathbb{Z}[v,v^{-1}]$-basis of $K^{G\times\mathbb{C}^*}(\mathcal{N})$, which he calls the canonical basis. The latter is characterized by properties similar to those characterizing the Kazhdan-Lusztig basis in the (affine) Hecke algebra. In particular, it is in bijection with the standard basis; and the transition matrix between the two bases is upper triangular in a natural order, and is congruent to the identity matrix modulo $v^{-1}$.

In many known examples a $\mathbb{Z}[v,v^{-1}]$ module with two bases as above arises as a Grothendieck group of a quasi-hereditary graded category, with canonical and costandard basis formed respectively by the classes of irreducible and costandard objects, so one may ask whether this also happens in the case under consideration. Indeed, a straightforward generalization of our construction provides a $t$-structure on $D^b(Coh^{G\times\mathbb{C}^*}(\mathcal{N}))$ such that $\tilde{\nabla}_\lambda$ are the costandard objects of its core $\tilde{\mathcal{P}}$, and $\tilde{IC}_{O,L}$ are its irreducible objects; here $\tilde{IC}_{O,L}$ is a natural lift of $IC_{O,L}$ to $D^b(Coh^{G\times\mathbb{C}^*}(\mathcal{N}))$. The classes of $\tilde{IC}_{O,L}$ form a basis of $K^{G\times\mathbb{C}^*}(\mathcal{N})$; this basis is obtained from the costandard basis by an upper-triangular transformation. In fact, $\{[\tilde{IC}_{O,L}]\}$ is the canonical basis, whose existence is conjectured in [O]; this follows from results of [B3]. One can also deduce this fact from positivity of the grading on the space $Ext^1(IC_{O,L}, IC_{O',L'})$ provided by the $\mathbb{C}^*$ action (i.e. from the fact that the graded components of non-positive degree vanish); however, I was not able to find a direct argument proving this.

We remark that the methods of this note originate from the results of [B3], where the "exotic" $t$-structure on $D^b(Coh^G(\mathcal{N}))$ appears in the context of geometric Langlands duality.

Finally, we mention that results of [AB] yield also "exotic" $t$-structures on the triangulated category $D^b(Coh^G(\tilde{\mathcal{N}}))$. Those $t$-structures have Artinian (finite type) cores, and can be described in terms of (quasi-)exceptional sets; however, I do not know an analogue of the description of the $t$-structures in terms of perverse coherent sheaves, or precise structure of the irreducible objects of their cores.

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## 2. Quasi-exceptional sets and quasi-hereditary categories

Most of this section is a restatement of the result of [BBD] on glueing of $t$-structures. The results are most probably well known to the experts, and appear in some form in the literature (cf., e.g., [PS]; I have learned many of them from L. Positselskii); we sketch the argument for the sake of completeness. We work in a generality slightly greater than usual (allowing possibly infinite exceptional sets), as this does not require any additional efforts (for the application below it would suffice to consider finite quasi-exceptional sets only).

2.1. **Quasi-hereditary categories.** An abelian category $\mathcal{A}$ will be called of *finite type* if any object of $\mathcal{A}$ has finite length. Let $\mathcal{A}$ be an abelian category of finite type with a fixed ordering on the set $I$ of isomorphism classes of irreducible objects. We fix a representative $L_i$ in each isomorphism class $i \in I$.

For $n \in I$ let $\mathcal{A}_{\leq n}$, $\mathcal{A}_{<n}$ be the Serre subcategory in $\mathcal{A}$ generated by $L_i$ with $i \leq n$ or $i < n$ respectively. Thus $\mathcal{A}_{\leq n}$, $\mathcal{A}_{<n}$ are strictly full abelian subcategories of $\mathcal{A}$, and $X \in \mathcal{A}$ lies in $\mathcal{A}_{\leq n}$ (respectively in $\mathcal{A}_{<n}$) iff any irreducible subquotient of $X$ is isomorphic to $L_i$ for some $i \leq n$ (respectively $i < n$).

**Definition 1.** A pair $(M_n, \phi_n)$, where $M_n$ is an object of $\mathcal{A}_{\leq n}$, and $\phi_n : M_n \to L_n$ is a nonzero morphism is called a standard cover of $L_n$ if the following two properties hold:

i) $Ker(\phi_n) \in \mathcal{A}_{<n}$.

ii) We have $\mathrm{Hom}(M_n, L_i) = 0 = \mathrm{Ext}^1(M_n, L_i)$ for $i < n$.

A pair $(N^n, \phi^n)$, where $N^n \in \mathcal{A}_{\leq n}$, and $\phi^n : L_n \to N^n$ is a nonzero morphism is called a costandard hull of $L_n$ if the following two properties hold:

i′) $CoKer(\phi^n) \in \mathcal{A}_{<n}$.

ii′) We have $\mathrm{Hom}(L_i, N^n) = 0 = \mathrm{Ext}^1(L_i, N^n)$ for $i < n$.

We will say that an object $M$ is standard (costandard) if some morphism to (from) an irreducible object from (to) $M$ is a standard cover (respectively, a costandard hull).

**Lemma 1.** *A (co)standard cover (hull) is unique up to a unique isomorphism if it exists.*

*Proof.* If $\phi_n : M_n \to L_n$ and $\phi'_n : M'_n \to L_n$ are two standard covers, then $\mathrm{Hom}(M_n, Ker(\phi'_n)) = 0 = Ext^1(M_n, Ker(\phi'_n))$ implies that

$$\mathrm{Hom}(M_n, M'_n) \widetilde{\longrightarrow} \mathrm{Hom}(M_n, L_n).$$

In particular, there exists a unique morphism $M_n \to M'_n$ compatible with $\phi_n$, $\phi'_n$, which proves the claim about standard covers. The argument for costandard hulls is parallel. $\square$

**Definition 2.** A quasi-hereditary category is a finite type abelian category with an ordering on the set of isomorphism classes of irreducible objects, such that a standard cover, and a costandard hull exist for any irreducible object of $\mathcal{A}$.

2.2. **Quasi-exceptional sets.** We first fix some notation partly borrowed from [BBD]. Let $\mathfrak{D}$ be a triangulated category. We write $\mathrm{Hom}^n(X, Y) = \mathrm{Hom}(X, Y[n])$, and denote the graded abelian group $\bigoplus\limits_n \mathrm{Hom}(X, Y[n])$ by $\mathrm{Hom}^\bullet(X, Y)$; also, $\mathrm{Hom}^{>0}(X, Y) = \bigoplus\limits_{n>0} \mathrm{Hom}(X, Y[n])$, etc.

For an object $X$ of a category we write $[X]$ for its isomorphism class. For a category $\mathcal{C}$ let $[\mathcal{C}]$ be the set of isomorphism classes of $\mathcal{C}$.

Let $\mathfrak{D}$ be a triangulated category.

If $\mathbf{X}, \mathbf{Y}$ are subsets of $[\mathfrak{D}]$, then $\mathbf{X} * \mathbf{Y}$ denotes the subset of $[\mathfrak{D}]$ consisting of classes of all objects $Z$, for which there exists an exact triangle $X \to Z \to Y \to X[1]$ with $[X] \in \mathbf{X}$, $[Y] \in \mathbf{Y}$. The octahedron axiom implies (see [BBD], Lemma 1.3.10) that the $*$-operation is associative, so $\mathbf{X}_1 * \mathbf{X}_2 \cdots * \mathbf{X}_n$ makes sense. For a subset $\mathbf{X} \subset [\mathfrak{D}]$ let $\langle X \rangle \subset \mathfrak{D}$ be the strictly full subcategory defined by

$$[\langle \mathbf{X} \rangle] = \bigcup_n \mathbf{X} * \mathbf{X} * \cdots * \mathbf{X},$$

where $\mathbf{X}$ appears $n$ times in the right-hand side.

For $\mathbf{X} \subset [\mathfrak{D}]$ the triangulated subcategory generated by $\mathbf{X}$ is the smallest strictly full triangulated subcategory $\mathfrak{D}_{\mathbf{X}} \subset \mathfrak{D}$, such that $[\mathfrak{D}_{\mathbf{X}}] \supset \mathbf{X}$. Thus

$$\mathfrak{D}_{\mathbf{X}} = \langle \bigcup_{n \in \mathbb{Z}} \mathbf{X}[n] \rangle,$$

where $\mathbf{X}[n] = \{[X[n]]\ ,\ [X] \in \mathbf{X}\}$. We will say “the triangulated subcategory generated by objects/subcategories” instead of “the triangulated category generated by the corresponding set of isomorphism classes”, and write $\langle X \rangle$ instead of $\langle \{[X]\} \rangle$, etc.

An ordered subset $\nabla = \{\nabla^i,\ i \in I\}$ of $\mathrm{Ob}(\mathfrak{D})$ is called *quasi-exceptional* if we have $\mathrm{Hom}^\bullet(\nabla^i, \nabla^j) = 0$ for $i < j$, $\mathrm{Hom}^{<0}(\nabla^i, \nabla^i) = 0$, and $\mathrm{End}(\nabla^i)$ is a division algebra for all $i$.

For a quasi-exceptional set $\nabla$, and $\nabla^i \in \nabla$ we set $\mathfrak{D}_{\leq i} = \mathfrak{D}_{\{\nabla_j\ |\ j \leq i\}}$, $\mathfrak{D}_{<i} = \mathfrak{D}_{\{\nabla_j\ |\ j<i\}}$.

For a full triangulated subcategory $\mathfrak{D}' \subset \mathfrak{D}$ we will denote by $\mathfrak{D}/\mathfrak{D}'$ the factor category; then $\mathfrak{D}/\mathfrak{D}'$ is again a triangulated category (see [V], 2.2.10). For $X, Y \in \mathfrak{D}$ we will denote by $X \mod \mathfrak{D}'$ the image of $X$ in $\mathfrak{D}/\mathfrak{D}'$, and we write $X \cong Y \mod \mathfrak{D}'$ instead of $(X \mod \mathfrak{D}') \cong (Y \mod \mathfrak{D}')$.

Let $\nabla = \{\nabla^i,\ i \in I\}$ be a quasi-exceptional set, and let $\Delta = \{\Delta_i,\ i \in I\}$ be another subset of $\mathrm{Ob}(\mathfrak{D})$ (in bijection with $\nabla$).

We say that $\Delta$ is *dual* to $\nabla$ if

$$\mathrm{Hom}^\bullet(\Delta_n, \nabla^i) = 0 \ \text{ for } \ n > i, \tag{1}$$

and there exists an isomorphism

$$\Delta_n \cong \nabla^n \mod \mathfrak{D}_{<n}. \tag{2}$$

We set

$$\diamondsuit_n \overset{\text{def}}{=} \Delta_n \mod \mathfrak{D}_{<n} \cong \nabla^n \mod \mathfrak{D}_{<n} \in \mathfrak{D}_{\le n}/\mathfrak{D}_{<n}.$$

**Lemma 2.** *If $\nabla$ is a quasi-exceptional set, and $\Delta$ is a dual set, then:*

*a)* $\mathrm{Hom}^\bullet(\Delta_i, X) = 0 = \mathrm{Hom}^\bullet(X, \nabla^i)$ *for all* $X \in \mathfrak{D}_{<i}$.

*b)* $\mathrm{Hom}^\bullet(\Delta_i, \nabla^j) = 0$ *unless* $i = j$.

*c) For all* $X \in \mathfrak{D}_{\le i}$ *we have*

$$\begin{aligned} &\mathrm{Hom}^\bullet(\Delta_i, X) \xrightarrow{\sim} \mathrm{Hom}^\bullet(\diamondsuit_i, X \mod \mathfrak{D}_{<i}); \\ &\mathrm{Hom}^\bullet(X, \nabla^i) \xrightarrow{\sim} \mathrm{Hom}^\bullet(X \mod \mathfrak{D}_{<i}, \diamondsuit_i). \end{aligned} \tag{3}$$

*d)* $\mathrm{Hom}^\bullet(\Delta_i, \Delta_j) = 0$ *for* $i > j$, *and*

$$\mathrm{Hom}^\bullet(\Delta_i, \Delta_i) \cong \mathrm{Hom}^\bullet(\diamondsuit_i, \diamondsuit_i) \cong \mathrm{Hom}^\bullet(\nabla^i, \nabla^i) \cong \mathrm{Hom}^\bullet(\Delta_i, \nabla^i). \tag{4}$$

*The induced isomorphisms* $\mathrm{End}(\Delta_i) \cong \mathrm{End}(\nabla^i) \cong \mathrm{End}(\diamondsuit_i)$ *are isomorphisms of algebras.*

*e) Let $\nabla$ be a quasi-exceptional set, and let $\Delta$, $\Delta'$ be two dual sets. Then*

$$\Delta_i \cong \Delta'_i \tag{5}$$

*for all $i$; moreover, there exists a unique isomorphism* (5) *compatible with a fixed isomorphism* (2).

*f) Assume $\nabla$ is well-ordered (i.e. every subset of $\Delta$ has a minimal element). Then we have $\mathfrak{D}_{\le n} = \mathfrak{D}_{\{\Delta_i | i \le n\}}$, $\mathfrak{D}_{<n} = \mathfrak{D}_{\{\Delta_i | i<n\}}$. In particular, if $\nabla$ generates $\mathfrak{D}$ (as a triangulated category), then so does $\Delta$.*

*Proof.* (a) is immediate from the definition.

If $i > j$, then (b) follows from the first equality in (a); while if $i < j$, then it follows from the second equality in (a).

By [V], chapitre II, Proposition 2.3.3(a) part (a) of the lemma implies part (c).

(d) and (e) follow from (c).

Finally, (f) follows from the definition by (transfinite) induction. □

*Remark* 1. Let $\nabla = \{\nabla^i |\ i \in I\}$ be a quasi-exceptional set, and let $\overline{I}$ be the set $I$ with the opposite ordering. Statement (d) of the lemma shows that if $\Delta$ is a dual set for $\nabla$, then $\Delta$ is a quasi-exceptional set indexed by $\overline{I}$. We say that a quasi-exceptional set $\nabla$ is *dualizable*, if a dual set exists.

*Remark* 2. A quasi-exceptional set is called *exceptional* (see e.g. [BK]) if $\mathfrak{D}$ is $k$-linear for a field $k$, and $\mathrm{Hom}^\bullet(\nabla^i, \nabla^i) = k$. It is proved in [BK] that if $\mathrm{Hom}^\bullet(X, Y)$ is a finite-dimensional $k$ vector space for all $X, Y \in \mathfrak{D}$, then any (finite) exceptional set in $\mathfrak{D}$ is dualizable.

**Example 1.** The reader can keep in mind the following example. Let $\mathfrak{D}$ be a full subcategory in the bounded derived category of sheaves of $k$-vector spaces on a reasonable topological space (or of etale sheaves on a reasonable scheme), consisting of complexes whose cohomology is smooth along a fixed stratification. Assume for simplicity that the strata $\Sigma_i$ are connected and simply-connected; we write $j < i$ if $\Sigma_j$ lies in the closure of $\Sigma_i$. Let $\mathrm{j}_i$ denote the imbedding of $\Sigma_i$ in the space. Let $p_i$ be arbitrary integers. Then objects $\nabla^i = \mathrm{j}_{i*}(\underline{k}[p_i])$ form a quasi-exceptional set generating $\mathfrak{D}$, and $\Delta_i = \mathrm{j}_{i!}(\underline{k}[p_i])$ is the dual set. $\nabla, \Delta$ are exceptional iff the strata are acyclic.

**Proposition 1.** *Let $\mathfrak{D}$ be a triangulated category. Let $I$ be a well ordered set, and $\nabla^i$, $i \in I$ a dualizable quasi-exceptional set in $\mathfrak{D}$, which generates $\mathfrak{D}$ as a triangulated category; let $\Delta$ be the dual set.*

*There exists a unique $t$-structure $(\mathfrak{D}^{\geq 0}, \mathfrak{D}^{<0})$ on $\mathfrak{D}$, such that $\nabla^i \in \mathfrak{D}^{\geq 0}$; $\Delta^i \in \mathfrak{D}^{\leq 0}$. Moreover, $\mathfrak{D}^{\geq 0}$, $\mathfrak{D}^{<0}$ are given by*

$$\mathfrak{D}^{\geq 0} = \langle \{ \nabla^i[d] \ , \ i \in I, \ d \leq 0 \} \rangle; \tag{6}$$

$$\mathfrak{D}^{<0} = \langle \{ \Delta_i[d] \ , \ i \in I, \ d > 0 \} \rangle. \tag{7}$$

We will need two lemmas to prove the proposition. The first one settles the case when $\nabla = \Delta$ consists of one element ("the base of induction"); the second one allows us to use gluing of $t$-structures (see [BBD]) to make an induction step.

**Lemma 3.** [P] *a) Let $\mathfrak{D}$ be a triangulated category, and $\mathcal{A} \subset \mathfrak{D}$ a full semisimple abelian subcategory, which generates $\mathfrak{D}$ as a triangulated category. Suppose that*

$$\mathrm{Hom}^{<0}(X, Y) = 0 \quad \text{for} \quad X, Y \in \mathcal{A}. \tag{8}$$

*Then there exists a unique $t$-structure on $\mathfrak{D}$ whose core contains $\mathcal{A}$; it is given by*

$$\begin{aligned} \mathfrak{D}^{\leq 0} &= \langle \bigcup_{i \geq 0} \mathcal{A}[i] \rangle; \\ \mathfrak{D}^{\geq 0} &= \langle \bigcup_{i \leq 0} \mathcal{A}[i] \rangle; \\ \mathfrak{D}^{\leq 0} \cap \mathfrak{D}^{\geq 0} &= \langle \mathcal{A} \rangle. \end{aligned} \tag{9}$$

*b) The set of isomorphism classes of simple objects of $\mathcal{A}$ coincides with the set of isomorphism classes of simple objects of $\langle \mathcal{A} \rangle$.*

*Proof.* (a) follows from [BBD], Remarque 1.3.14. More precisely, *loc. cit.* shows that the conclusion of (a) holds for any full subcategory $\mathcal{A} \subset \mathfrak{D}$ which satisfies (8) and such that

$$[\mathcal{A}] * [\mathcal{A}][1] \subset [\mathcal{A}][1] * [\mathcal{A}] \tag{10}$$

(a subcategory satisfying (8), (10) is called *admissible* in [BBD], Definition 1.2.5). A semisimple full abelian subcategory satisfying (8) is readily seen to be admissible in this sense; indeed, for such a category we have

$$[\mathcal{A}] * [\mathcal{A}[1]] = \{ X \oplus Y[1] \mid X, Y \in \mathcal{A} \}.$$

Recall from [BBD], Proposition 1.2.4 that a sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in an admissible abelian subcategory in $\mathfrak{D}$ is exact iff there exists a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1];$$

in particular, this is true for the subcategory $\langle \mathcal{A} \rangle = \mathfrak{D}^{\leq 0} \cap \mathfrak{D}^{\geq 0}$, as the core of any $t$-structure is admissible. Hence every object of $\langle \mathcal{A} \rangle$ has a finite filtration whose subquotients are simple in $\mathcal{A}$. It remains to see that these objects are also simple in $\langle\mathcal{A}\rangle$. But if $L \in \mathcal{A}$ is not simple in $\langle\mathcal{A}\rangle$, then there exists a simple object $L' \in \mathcal{A}$, and a nonzero morphism $L' \to L$ which is not an isomorphism; so $L$ is not simple in $\mathcal{A}$. $\square$

**Corollary 1.** *If $\mathfrak{D} = \mathfrak{D}_{\{X\}}$ for an object $X \in \mathfrak{D}$ such that $\mathrm{Hom}^{<0}(X, X) = 0$, and $\mathrm{End}(X)$ is a division algebra, then there exists a unique $t$-structure on $\mathfrak{D}$ whose core contains $X$. It is given by*

$$\begin{aligned} \mathfrak{D}^{\leq 0} &= \langle\{X[i]\rangle \mid i \geq 0\}; \\ \mathfrak{D}^{\geq 0} &= \langle\{X[i]\rangle \mid i \leq 0\}\rangle; \\ \mathfrak{D}^{\leq 0} \cap \mathfrak{D}^{\geq 0} &= \langle X\rangle. \end{aligned} \tag{11}$$

*$X$ is a simple object of the core of this $t$-structure.*

*Proof.* Apply the previous Lemma to $\mathcal{A} = \{X^{\oplus n} \mid n \in \mathbb{Z}_{\geq 0}\}$. $\square$

For a subcategory $\mathcal{A}$ in an additive category $\mathcal{C}$ let us (following [BK]) write $\mathcal{A}^\perp = (\mathcal{A}^\perp)_\mathcal{C}$ (respectively ${}^\perp\mathcal{A} = ({}^\perp\mathcal{A})_\mathcal{C}$) for the strictly full subcategory in $\mathcal{C}$ consisting of objects $X$ for which $\mathrm{Hom}(A, X) = 0$ (respectively $\mathrm{Hom}(X, A) = 0$) for all $A \in \mathcal{A}$. The subcategories $\mathcal{A}^\perp$, ${}^\perp\mathcal{A}$ are called respectively right and left orthogonal of $\mathcal{A}$.

Set $\mathfrak{D}_n = \mathfrak{D}_{\{\Delta_n\}}$, $\mathfrak{D}^n = \mathfrak{D}_{\{\nabla^n\}}$.

**Lemma 4.** *a) We have*

$$[\mathfrak{D}_{\leq n}] = [\mathfrak{D}_n] * [\mathfrak{D}_{<n}];$$
$$[\mathfrak{D}_{\leq n}] = [\mathfrak{D}_{<n}] * [\mathfrak{D}^n].$$

*b) We have*

$$\mathfrak{D}_n = ({}^\perp\mathfrak{D}_{<n})_{\mathfrak{D}_{\leq n}};$$
$$\mathfrak{D}^n = (\mathfrak{D}_{<n}^\perp)_{\mathfrak{D}_{\leq n}};$$
$$(\mathfrak{D}_n^\perp)_{\mathfrak{D}_{\leq n}} = \mathfrak{D}_{<n} = (\mathfrak{D}_n^\perp)_{\mathfrak{D}_{\leq n}}.$$

*c) $\mathfrak{D}_{<n}$ is a thick (saturated) subcategory in $\mathfrak{D}_{\leq n}$.*

*d) The projection $\Pi = \Pi_n : \mathfrak{D}_{\leq n} \to \mathfrak{D}_{\leq n}/\mathfrak{D}_{<n}$ induces equivalences of triangulated categories*

$$\Pi|_{\mathfrak{D}_n} : \mathfrak{D}_n \xrightarrow{\sim} \mathfrak{D}_{\leq n}/\mathfrak{D}_{<n};$$
$$\Pi|_{\mathfrak{D}^n} : \mathfrak{D}^n \xrightarrow{\sim} \mathfrak{D}_{\leq n}/\mathfrak{D}_{<n}.$$

*$\Pi$ has a left adjoint $\Pi^l$ and a right adjoint $\Pi^r$. Moreover, $\Pi^l$ maps $\mathfrak{D}_{\leq n}/\mathfrak{D}_{<n}$ to $\mathfrak{D}_n$ and induces an equivalence inverse to $\Pi|_{\mathfrak{D}_n}$; while $\Pi^r$ maps $\mathfrak{D}_{\leq n}/\mathfrak{D}_{<n}$ to $\mathfrak{D}^n$ and induces an equivalence inverse to $\Pi|_{\mathfrak{D}^n}$.*

*e) The inclusion functor $\iota : \mathfrak{D}_{<n} \hookrightarrow \mathfrak{D}_{\leq n}$ has a left adjoint $\iota^l$ and a right adjoint $\iota^r$. Functors $\iota^l$, $\iota^r$ are triangulated (i.e., send distinguished triangles into distinguished triangles).*

*Proof.* It is obvious that $\mathfrak{D}_n \subset ({}^\perp\mathfrak{D}_{<n})_{\mathfrak{D}_{\leq n}}$, $\mathfrak{D}^n = (\mathfrak{D}_{<n}^\perp)_{\mathfrak{D}_{\leq n}}$. Hence

$$[\mathfrak{D}_{<n}] * [\mathfrak{D}_n] \subset [\mathfrak{D}_n] * [\mathfrak{D}_{<n}];$$
$$[\mathfrak{D}^n] * [\mathfrak{D}_{<n}] \subset [\mathfrak{D}_{<n}] * [\mathfrak{D}^n].$$

Now (a) follows from the fact that $\mathfrak{D}_{\leq n}$ is generated by $\mathfrak{D}_n$, $\mathfrak{D}_{<n}$, as well as by $\mathfrak{D}_{<n}$, $\mathfrak{D}^n$, by associativity of the star operation.

(b) is immediate from (a); (c) follows from (b) because both left and right orthogonal of a triangulated category is a thick subcategory. The rest of the lemma follows, e.g., from [V], chapitre II, Proposition 2.3.3 (see also [BK] 1.5–1.9).

Let us recall the construction of adjoint functors $\Pi^l$, $\Pi^r$, $\iota^l$, $\iota^r$ for further reference. By part (a) of the lemma for $X \in \mathfrak{D}_{\leq n}$ there exist distinguished triangles

$$X_n \to X \to X_{<n} \to X_n[1];$$
$$X^{<n} \to X \to X^n \to X^{<n}[1]$$

with $X_n \in \mathfrak{D}_n$, $X_{<n}$, $X^{<n} \in \mathfrak{D}_{<n}$, $X^n \in \mathfrak{D}^n$. Then we have the following canonical isomorphisms

$$\begin{aligned} &\iota^l(X) \cong X_{<n}; \quad \iota^r(X) \cong X^{<n}; \\ &\Pi^l \circ \Pi(X) \cong X_n; \quad \Pi^r \circ \Pi(X) \cong X^n. \end{aligned} \tag{12}$$

□

*Proof of Proposition 1.* To prove (a) it suffices to construct a $t$-structure on $\mathfrak{D}$ satisfying (6), (7). Then for another $t$-structure $(\mathfrak{D}_1^{\geq 0}, \mathfrak{D}_1^{<0})$ such that $\Delta^i \in \mathfrak{D}_1^{\leq 0}$, $\nabla_i \in \mathfrak{D}_1^{\geq 0}$ we have $\mathfrak{D}^{<0} \subseteq \mathfrak{D}_1^{<0}$, $\mathfrak{D}^{\geq 0} \subseteq \mathfrak{D}_1^{\geq 0}$, which implies $\mathfrak{D}^{<0} = \mathfrak{D}_1^{<0}$, $\mathfrak{D}^{\geq 0} = \mathfrak{D}_1^{\geq 0}$ (recall that for a triangulated category $\mathfrak{D}$ with a $t$-structure $(\mathfrak{D}^{\geq 0}, \mathfrak{D}^{<0})$ for an object $X \in \mathfrak{D}$ we have $X \in \mathfrak{D}^{\geq 0} \iff \mathrm{Hom}(Y, X) = 0 \ \ \forall Y \in \mathfrak{D}^{<0}$; $X \in \mathfrak{D}^{<0} \iff \mathrm{Hom}(X, Y) = 0 \ \ \forall Y \in \mathfrak{D}^{\geq 0}$).

We construct by induction a $t$-structure on $\mathfrak{D}_{\leq n}$ with

$$\mathfrak{D}^{\geq 0}_{\leq n} = \langle \{\nabla^i[d]\ ,\ i \leq n,\ d \leq 0\} \rangle; \tag{13}$$

$$\mathfrak{D}^{\leq 0}_{\leq n} = \langle \{\Delta_i[d]\ ,\ i \leq n,\ d > 0\} \rangle; \tag{14}$$

since $\mathfrak{D}_{<n} = \bigcup\limits_{i<n} \mathfrak{D}_{\leq i}$ we can assume that a $t$-structure on $\mathfrak{D}_{<n}$ is already defined.

Lemma 4 implies that the functors $\mathfrak{D}_{<n} \xrightarrow{\iota} \mathfrak{D}_{\leq n} \xrightarrow{\Pi} \mathfrak{D}_{\leq n}/\mathfrak{D}_{<n}$ satisfy the requirements of [BBD] 1.4.3 (to pass from our notation to those of [BBD] one should set $\Pi = j^* = j^!$, $\Pi^l = j_!$, $\Pi^r = j_*$; $\iota = i_* = i_!$; $\iota^l = i^*$; $\iota^r = i^!$). Thus the construction of gluing of $t$-structures (*loc. cit.* Theorem 1.4.10) is applicable.

We endow $\mathfrak{D}_{<n}$ with the $t$-structure obtained by the induction assumption; and $\mathfrak{D}_{\leq n}/\mathfrak{D}_{<n} \cong \mathfrak{D}^n$ with the unique $t$-structure which has $\diamondsuit_n$ in its core (see Lemma 3). Then [BBD] Theorem 1.4.10 provides $\mathfrak{D}_{\leq n}$ with a $t$-structure given by

$$\mathfrak{D}^{\leq 0}_{\leq n} = \{X \in \mathfrak{D}_{\leq n} \mid \iota^l(X) \in \mathfrak{D}^{\leq 0}_{<n} \qquad \Pi(X) \in (\mathfrak{D}_{\leq n}/\mathfrak{D}_{<n})^{\leq 0}\};$$
$$\mathfrak{D}^{\geq 0}_{\leq n} = \{X \in \mathfrak{D}_{\leq n} \mid \iota^r(X) \in \mathfrak{D}^{\geq 0}_{<n} \qquad \Pi(X) \in (\mathfrak{D}_{\leq n}/\mathfrak{D}_{<n})^{\geq 0}\}.$$

In view of (12), (11) we have

$$[\mathfrak{D}^{\leq 0}_{\leq n}] = [\langle \Delta_n[i] \ |i \geq 0\rangle] * [\mathfrak{D}^{\leq 0}_{<n}];$$
$$[\mathfrak{D}^{\geq 0}_{\leq n}] = [\mathfrak{D}^{\geq 0}_{<n}] * [\langle \Delta_n[i] \mid i \leq 0\rangle]\,,$$

which implies (14), (13). The proposition is proved. □

We will call the $t$-structure defined by (6), (7) *the $t$-structure of the quasi-exceptional set* $\nabla$.

*Remark* 3. The $t$-structure of the quasi-exceptional set introduced in Example 1 is the "perverse" $t$-structure [BBD] corresponding to perversity $p = (p_i)$.

*Remark* 4. It follows from the axioms of a $t$-structure that the $t$-structure of a quasi-exceptional set $\nabla$ can be alternatively described as follows. For $X \in \mathfrak{D}$ we have

$$X \in \mathfrak{D}^{\geq 0} \iff \mathrm{Hom}^{<0}(\Delta_i, X) = 0\ \forall i \in I; \quad X \in \mathfrak{D}^{<0} \iff \mathrm{Hom}^{\leq 0}(X, \nabla^i) = 0\ \forall i \in I. \tag{15}$$

In the situation of Example 1, (15) turns into the usual definition of a perverse sheaf by a condition on stalks and costalks.

We keep the assumptions of Proposition 1. Let $\mathcal{A}$ be the core of the $t$-structure of the quasi-exceptional set $\nabla$, $\tau$ the corresponding truncation functors, and $H^m = \tau_{\leq m} \circ \tau_{\geq m} : \mathfrak{D} \to \mathcal{A}$ the cohomology functor.

Define $M_i, N^i \in \mathcal{A}$ by $M_i = \tau_{\geq 0}(\Delta_i) = H^0(\Delta_i)$, and $N^i = \tau_{\leq 0}(\Delta^i) = H^0(\Delta^i)$. Isomorphism (4) provides a morphism $\Phi_i : \Delta_i \to \nabla^i$, which goes to $Id_{\Delta_i}$ under (4), and thus also a morphism $H^0(\Phi_i) : M_i \to N^i$. Also, set $\mathcal{A}_{<n} = \mathcal{A} \cap \mathfrak{D}_{<n}$, $\mathcal{A}_{\leq n} = \mathcal{A} \cap \mathfrak{D}_{\leq n}$, and let $\mathcal{A}_n$ be the core of the unique $t$-structure on $\mathfrak{D}_{\leq n}/\mathfrak{D}_{<n}$ such that $\mathcal{A}_n \ni \diamondsuit_n$.

**Proposition 2.** *Let $L_i$ be the image of $H^0(\Phi_i) : M_i \to N^i$. Then $L_i$ is irreducible, and any irreducible object of $\mathcal{A}$ is isomorphic to $L_i$ for some $i$.*

*The order on $I$ induces an order on $\{[L_i]\}$, and $\mathcal{A}$ with this ordering on the set of isomorphism classes of irreducible objects is a quasi-hereditary category. The canonical morphisms $\phi_n : M_n \twoheadrightarrow L_n$ and $\phi^n : L_n \hookrightarrow N^n$ are the standard cover and the costandard hull of $L_n$ respectively.*

*Proof.* For any $t$-category $\mathfrak{D}$ with the core $\mathcal{A}$ and $X \in \mathcal{A}$, $\Delta \in \mathfrak{D}^{\leq 0}$ we have

$$\mathrm{Hom}(H^0(\Delta), X) \widetilde{\longrightarrow} \mathrm{Hom}(\Delta, X); \quad \mathrm{Ext}^1_{\mathcal{A}}(H^0(\Delta), X) \hookrightarrow \mathrm{Hom}(\Delta, X[1]); \tag{16}$$

and dually for $\nabla \in \mathfrak{D}^{\geq 0}$ we have

$$\mathrm{Hom}(X, H^0(\nabla)) \widetilde{\longrightarrow} \mathrm{Hom}(X, \nabla); \quad \mathrm{Ext}^1_{\mathcal{A}}(X, H^0(\nabla)) \hookrightarrow \mathrm{Hom}(X, \nabla[1]). \tag{17}$$

Thus we have

$$\mathrm{Hom}(M_n, X) = 0 = \mathrm{Ext}^1(M_n, X); \tag{18}$$

$$\mathrm{Hom}(X, N^n) = 0 = \mathrm{Ext}^1(X, N^n) \tag{19}$$

for $X \in \mathcal{A}_{<n}$. Let us now show that $L_n$ is simple in $\mathcal{A}$. Assume that $0 \to X \to L_n \to Y \to 0$ is a short exact sequence. Pick the minimal $i$ such that $X, Y \in \mathcal{A}_{\leq i}$. If $i > n$, we get a contradiction because applying the exact functor $\Pi_i$ to the exact sequence we get an exact sequence

$$0 \to X \mod \mathfrak{D}_{<i} \to 0 \to Y \mod \mathfrak{D}_{<i} \to 0$$

in $\mathcal{A}_i$, which shows that $X \mod \mathfrak{D}_{<i} = 0 = Y \mod \mathfrak{D}_{<i}$, so $X, Y \in \mathfrak{D}_{\leq j}$ for some $j < i$. Thus $i = n$; so we have an exact sequence in $\mathcal{A}_n$

$$0 \to X \mod \mathfrak{D}_{<n} \to \diamondsuit_n \to Y \mod \mathfrak{D}_{<n} \to 0.$$

Since $\diamondsuit_n$ is irreducible by Corollary 1, we see that either $X \in \mathcal{A}_{<n}$ or $Y \in \mathcal{A}_{<n}$. However, $X$ is a subobject of $N^n$, while $Y$ is a factor-object of $M_n$; thus we get a contradiction with either (18) or (19).

We claim that

$$\mathcal{A} = \langle L_n \mid n \in I \rangle. \tag{20}$$

Notice that (20) implies the statement of the lemma: since a sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{A}$ is exact iff there exists a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1],$$

(20) shows that any object of $\mathcal{A}$ has a finite filtration with every subquotient isomorphic to $L_n$ for some $n$. To check (20) observe that the isomorphism $L_n \cong \nabla^n \mod \mathfrak{D}_{<n}$ implies by induction that $L_i$, $i \leq n$ generate $\mathfrak{D}_{\leq n}$ as a triangulated category. So (20) follows from Lemma 3. $\square$

*Remark* 5. Fix $i \in I$, and set $(\nabla')^j = \nabla^j[1]$, $(\nabla'')^j = \nabla^j[-1]$ for $j < i$; $(\nabla')^j = \nabla^j = (\nabla'')^j$ for $j \geq i$. Then $\nabla'$, $\nabla''$ are dualizable quasi-exceptional sets; let $\tau'$, $\tau''$ be the truncation functors for the corresponding $t$-structures. One can show that

$$\tau'_{\leq 0}(\nabla^i) \cong L_i \cong \tau''_{\geq 0}(\Delta_i).$$

*Remark* 6. In the situation of Example 1, Proposition 2 provides the standard description of a Goresky-MacPherson IC-sheaf $j_{!*}(\mathcal{L})$ (where $\mathcal{L}$ is a local system) as the image of the canonical morphism $H^{p,0}(j_!(\mathcal{L})) \to H^{p,0}(j_*(\mathcal{L}))$; while Remark 5 describes $j_{!*}(\mathcal{L})$ as a result of successive applications of the direct image and truncation functors; cf. [BBD], Proposition 2.1.11 (cf. also *loc. cit.* 2.1.9).

## 3. Main result

The pull-back and push-forward functors for coherent sheaves are understood to be the corresponding derived functors, unless stated otherwise.

We return to the set-up and notation of the Introduction. In particular, $\pi : \tilde{\mathcal{N}} = T^*(G/B) \to \mathcal{N}$ is the moment map from the cotangent bundle of the flag variety $G/B$ to the nil-cone (the Springer-Grothendieck resolution); also, let $p : \tilde{\mathcal{N}} \to G/B$ be the projection.

From now on we set $\mathfrak{D} = D^b(Coh^G(\mathcal{N}))$.

For a weight $\lambda \in \Lambda$ let $\mathcal{O}_{G/B}(\lambda)$ be the corresponding $G$-equivariant line bundle on $G/B$ (thus $\mathcal{O}_{G/B}(\lambda+\rho)$ is ample for $\lambda \in \Lambda^+$); for a parabolic $P \subset G$ we will write $\mathcal{O}_{G/P}(\lambda)$ for the unique equivariant line bundle on $G/P$ whose pull-back to $G/B$ is $\mathcal{O}_{G/B}(\lambda)$ if such a line bundle exists; for a variety $X$ with a map $f : X \to G/P$ we will denote $f^*(\mathcal{O}_{G/P}(\lambda))$ by $\mathcal{O}_X(\lambda)$; we will write $\mathcal{F}(\lambda)$ instead of $\mathcal{F} \otimes \mathcal{O}_X(\lambda)$ for $\mathcal{F} \in D^b(Coh_X)$, etc. For $\lambda \in \Lambda^+$ we set $V_\lambda = H^0(G/B, \mathcal{O}_{G/B}(\lambda))$.

We define $A_\lambda \in \mathfrak{D}$ by $A_\lambda = R\pi_*(\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda))$.

Let $W$ be the Weyl group. For $\lambda \in \Lambda$ we denote its $W$ orbit by $W(\lambda)$; let $conv(\lambda)$ be the intersection of the convex hull of $W(\lambda)$ with $\Lambda$, and $conv^0(\lambda)$ the complement to $W(\lambda)$ in $conv(\lambda)$.

For a subset $S \subset \Lambda$ let

$$\mathfrak{D}_S = \mathfrak{D}_{\{A_\lambda,\ \lambda \in S\}} = \langle A_\lambda[n],\ \lambda \in S,\ n \in \mathbb{Z} \rangle$$

be the triangulated subcategory of $\mathfrak{D}$ generated by $A_\lambda$, $\lambda \in S$.

**Proposition 3.** *For $w \in W$ there exists a canonical isomorphism*

$$A_\lambda \cong A_{w(\lambda)} \mod \mathfrak{D}_{conv^0(\lambda)}. \tag{21}$$

Our proof of the Proposition is a variation of a classical argument going back at least to [Dem].

*Proof.* Let $\alpha$ be a simple root, $s_\alpha \in W$ the corresponding simple reflection. It suffices to construct (21) for $w = s_\alpha$ and $\lambda \in \Lambda$ such that $s_\alpha(\lambda) = \lambda - n\alpha$, $n > 0$.

Let $\mathrm{pr}_\alpha : G/B \to G/P_\alpha$ be the projection, where $P_\alpha$ is the minimal parabolic corresponding to $\alpha$. Let $G' \to G$ be the universal covering, and let $\Lambda' \supset \Lambda$ be the weight lattice of $G'$. There exists $\lambda' \in \Lambda'$ such that $s_\alpha(\lambda') = \lambda' - (n-1)\alpha$. Set $\mathcal{V}_{\lambda'} = \mathrm{pr}_\alpha^* \mathrm{pr}_{\alpha *} \left(\mathcal{O}_{G/B}(\lambda')\right)$. Thus $\mathcal{V}_{\lambda'}$ is a $G'$-equivariant vector bundle; it has a $G'$-invariant filtration with subquotients $s_\alpha(\lambda')$, $s_\alpha(\lambda') + \alpha, \dots, \lambda'$. We claim that

$$\pi_* \left(p^*(\mathcal{V}_{\lambda'})(\lambda - \lambda')\right) \cong \pi_* \left(p^*(\mathcal{V}_{\lambda'}) \left(\lambda - \lambda' - \alpha\right)\right). \tag{22}$$

Since $p^*(\mathcal{V}_{\lambda'})(\lambda - \lambda')$ is a $G$-equivariant vector bundle on $\tilde{\mathcal{N}}$ equipped with a filtration whose subquotients are $\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda - k\alpha)$, $k = 0, \dots, n-1$, we have

$$[\pi_* \left(p^*(\mathcal{V}_{\lambda'})(\lambda - \lambda')\right)] \in [A_{\lambda-(n-1)\alpha}] * \cdots * [A_\lambda],$$

and

$$\pi_* \left(p^*(\mathcal{V}_{\lambda'})(\lambda - \lambda')\right) \cong A_\lambda \mod \mathfrak{D}_{conv^0(\lambda)}.$$

Similarly,

$$[\pi_* \left(p^*(\mathcal{V}_{\lambda'}) \left(\lambda - \lambda' - \alpha\right)\right)] \in [A_{\lambda-n\alpha}] * \cdots * [A_{\lambda-\alpha}];$$

hence

$$\pi_* \left(p^*(\mathcal{V}_{\lambda'}) \left(s_\alpha(\lambda - \lambda')\right)\right) \cong A_{\lambda-n\alpha} = A_{s_\alpha(\lambda)} \mod \mathfrak{D}_{conv^0(\lambda)}.$$

Thus (22) yields (21).

It remains to check (22). Set $\tilde{\mathcal{N}}_\alpha = T^*(G/P_\alpha) \times_{G/P_\alpha} G/B$; the differential of $\mathrm{pr}_\alpha$ provides a closed imbedding $\tilde{\mathcal{N}}_\alpha \hookrightarrow \tilde{\mathcal{N}}$. We have an exact sequence in $Coh^G(\tilde{\mathcal{N}})$,

$$0 \to \mathcal{O}_{\tilde{\mathcal{N}}}(\alpha) \to \mathcal{O}_{\tilde{\mathcal{N}}} \to \mathcal{O}_{\tilde{\mathcal{N}}_\alpha} \to 0. \tag{23}$$

Tensoring it with $p^*(\mathcal{V}_{\lambda'})(\lambda - \lambda' - \alpha)$ we see that to check (22) it suffices to verify that

$$\pi_* \left(\mathcal{O}_{\tilde{\mathcal{N}}_\alpha} \otimes p^*(\mathcal{V}_{\lambda'})(\lambda - \lambda' - \alpha)\right) = 0.$$

We claim that in fact a stronger equality

$$\pi'_* \left(\mathcal{O}_{\tilde{\mathcal{N}}_\alpha} \otimes p^*(\mathcal{V}_{\lambda'})(\lambda - \lambda' - \alpha)\right) = 0$$

holds, where $\pi'$ is the projection

$$\pi' : \tilde{\mathcal{N}}_\alpha = T^*(G/P_\alpha) \times_{G/P_\alpha} G/B \longrightarrow T^*(G/P_\alpha).$$

Indeed, the fibers of $\pi'$ are projective lines, and $\mathcal{O}_{\tilde{\mathcal{N}}_\alpha} \otimes p^*(\mathcal{V}_{\lambda'})(\lambda - \lambda' - \alpha)$ is readily seen to be isomorphic to a sum of several copies of $\mathcal{O}_{\mathbb{P}^1}(-1)$ when restricted to any fiber of $\pi'$. $\square$

**Proposition 4.** *a) Let* $\lambda \in \Lambda^+$*. Then* $\mathrm{Hom}(A_\lambda, A_\lambda) = \mathbb{C}$*, and* $\mathrm{Hom}^{<0}(A_\lambda, A_\lambda) = 0$*. Also for any* $\mu \in \Lambda$ *we have*

$$\mathrm{Hom}^\bullet(A_\mu, A_\lambda) = 0 \quad \text{if} \quad \lambda \notin conv(\mu). \tag{24}$$

*b) For* $\lambda, \mu \in \Lambda^+$*,* $\lambda \neq \mu$ *we have*

$$\mathrm{Hom}^\bullet(A_{w_0(\mu)}, A_\lambda) = 0, \tag{25}$$

*where* $w_0 \in W$ *is the element of maximal length.*

We will need the following known fact.

**Fact 1** ([Br], [K]). *a) For dominant $\lambda$ we have $H^i(\tilde{\mathcal{N}}, \mathcal{O}(\lambda)) = 0$ for $i \neq 0$, and*

$$\dim \operatorname{Hom}_G(V_\mu, H^0(\tilde{\mathcal{N}}, \mathcal{O}(\lambda)) = n^\mu_\lambda, \tag{26}$$

*where $n^\mu_\lambda$ is the multiplicity of weight $\lambda$ in $V_\mu$.*

*b) $R^\bullet \pi_* \mathcal{O}_{\tilde{\mathcal{N}}} = \mathcal{O}_{\mathcal{N}}$.* □

*Remark* 7. We will only use (26) in the case when $\mu \notin conv(\lambda)$, so both sides vanish.

*Proof of Proposition* 4(a). By Fact 1, $A_\lambda$ is a sheaf (rather than a complex) for $\lambda \in \Lambda^+$; thus, of course, $\operatorname{Hom}^{<0}(A_\lambda, A_\lambda) = 0$. Also, $A_\lambda$ is torsion free and has generic rank 1; hence $\operatorname{Hom}^G(A_\lambda, A_\lambda) = \mathbb{C}$, because $\mathcal{N}$ has an open orbit. It remains to check (24).

From Fact 1 it follows that if $\lambda \in \Lambda^+$, then $R^i\Gamma(A_\lambda) = H^i(\tilde{\mathcal{N}}, \mathcal{O}(\lambda)) = 0$ for $i \neq 0$, and $\operatorname{Hom}_G(V_\mu, \Gamma(A_\lambda)) = \operatorname{Hom}_G(V_\mu, \Gamma(\tilde{\mathcal{N}}, \mathcal{O}(\lambda))) = 0$ unless $\lambda \in conv(\mu)$. Thus we get

$$\operatorname{Hom}^\bullet(V_\mu \otimes \mathcal{O}, A_\lambda) = \operatorname{Hom}_G(V_\mu, R^\bullet\Gamma(A_\lambda)) = 0 \quad \text{if} \quad \lambda \notin conv(\mu). \tag{27}$$

Introduce a (nonstandard) order on $\Lambda$ by $\nu_1 \preceq \nu_2$ if $\nu_1 \in conv(\nu_2)$. We fix $\lambda$, and proceed by induction in $\mu$ in that order. We can assume that (24) holds for all $\mu' \in conv^0(\mu)$. Now notice that $V_\mu \otimes \mathcal{O}_{G/B}$ carries a filtration whose associated graded is

$$\operatorname{gr}(V_\mu \otimes \mathcal{O}_{G/B}) = \bigoplus \mathcal{O}_{G/B}(\nu)^{\oplus n^\mu_\nu}.$$

Hence

$$[V_\mu \otimes \mathcal{O}_{\mathcal{N}}] = [\pi_* p^*(V_\mu \otimes \mathcal{O}_{G/B})] \in \{[A_{\nu_1}^{n^\mu_{\nu_1}}]\} * \cdots * \{[A_{\nu_k}^{n^\mu_{\nu_k}}]\},$$

where $\nu_1, \ldots, \nu_k$ are weights on $V_\mu$. The induction assumption shows that $\operatorname{Hom}^\bullet(A_\nu, A_\lambda) = 0$ for all $\nu \in conv^0(\mu)$. Thus the last equality implies that

$$\begin{aligned}[0] &= [\operatorname{Hom}^\bullet(V_\mu \otimes \mathcal{O}_{\mathcal{N}}, A_\lambda)] \in \{[\operatorname{Hom}^\bullet(A_{\nu_k}^{n^\mu_{\nu_k}}, A_\lambda)]\} * \cdots * \{[\operatorname{Hom}^\bullet(A_{\nu_k}^{n^\mu_{\nu_1}}, A_\lambda)]\} \\ &= \{[\operatorname{Hom}^\bullet(A_\mu, A_\lambda)]\} * \{[\operatorname{Hom}^\bullet(A_{w_1(\mu)}, A_\lambda)]\} * \cdots * \{[\operatorname{Hom}^\bullet(A_{w_s(\mu)}, A_\lambda)]\},\end{aligned} \tag{28}$$

where $\mu$, $w_1(\mu), \ldots, w_s(\mu)$ are extremal weights of $V_\mu$ (here we view $\operatorname{Hom}^\bullet$ as an object of $D^+(Vect_{\mathbb{C}})$). By Proposition 3, $A_{w(\mu)} \cong A_\mu \mod \mathfrak{D}_{conv^0(\mu)}$; thus by the induction assumption, $\operatorname{Hom}^\bullet(A_{w(\mu)}, A_\lambda) \cong \operatorname{Hom}^\bullet(A_\mu, A_\lambda)$ for all $w \in W$. So (28) can be rewritten as

$$[0] \in \{[\operatorname{Hom}^\bullet(A_\mu, A_\lambda)]\} * \{[\operatorname{Hom}^\bullet(A_\mu, A_\lambda)]\} * \cdots * \{[\operatorname{Hom}^\bullet(A_\mu, A_\lambda)]\},$$

where the number of terms in the right-hand side is the number of extremal weights in $V_\mu$. Now (24) follows from the next standard lemma, applied to $\mathcal{A} = Vect$, $V = \operatorname{Hom}^\bullet(A_\mu, A_\lambda)$. □

**Lemma 5.** *Let $V \in D^+(\mathcal{A})$ for an abelian category $\mathcal{A}$. If*

$$[0] \in \{[V]\} * \{[V]\} * \cdots * \{[V]\}$$

*(where $V$ is repeated $n$ times, $n \geq 1$), then $V = 0$.*

*Proof.* Otherwise, if $i$ is minimal, such that $H^i(V) \neq 0$, then $H^i(V) \hookrightarrow H^i(X)$ for $[X] \in \{[V]\} * \{[V]\} * \cdots * \{[V]\}$. □

*Proof of Proposition* 4(b). If $\lambda \notin conv(\mu)$, then (25) follows from (24). Otherwise, $w_0(-\mu) \notin conv(-\lambda)$. Recall that the Grothendieck-Serre duality $\mathcal{S}$ is an anti-autoequivalence of $\mathfrak{D}$, such that $\mathcal{S}(A_\lambda) = A_{-\lambda}[\dim \mathcal{N}]$; see section 3.1 below. Thus we have

$$\mathrm{Hom}^\bullet(A_{w_0(\mu)}, A_\lambda) = \mathrm{Hom}^\bullet(\mathcal{S}(A_\lambda), \mathcal{S}(A_{w_0(\mu)})) = \mathrm{Hom}^\bullet(A_{-\lambda}, A_{w_0(-\mu)}),$$

which again vanishes by (24). □

**Proposition 5.** *$\mathfrak{D}$ is generated by $A_\lambda$, $\lambda \in \Lambda^+$.*

To prove the Proposition we need two auxiliary lemmas.

**Lemma 6.** *Let $X$ be an algebraic variety over $\mathbb{C}$, and $p : Y \to X$ a vector bundle; let $G$ be a linear algebraic group acting on $X$, $Y$, so that $p$ is $G$-equivariant. Then $D^b(Coh^G(Y))$ is generated as a triangulated category by objects of the form $p^*(\mathcal{F})$, $\mathcal{F} \in Coh^G(X)$.*

*Proof.* See, e.g., [CG], p. 266 (last paragraph). □

**Corollary 2.** *$D^b(Coh^G(\tilde{\mathcal{N}}))$ is generated by the objects $\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)$, $\lambda \in \Lambda$.*

*Proof.* The category $Coh^G(G/B)$ is identified with the category of representations of $B$; in particular, any object of $Coh^G(G/B)$ is a vector bundle, which carries a filtration with subquotients being $\mathcal{O}_{G/B}(\lambda)$, $\lambda \in \Lambda$. Now apply Lemma 6 to $X = G/B$, $Y = \tilde{\mathcal{N}}$. □

**Lemma 7.** *The image of the functor $\pi_* : D^b(Coh^G(\tilde{\mathcal{N}})) \to \mathfrak{D}$ generates $\mathfrak{D}$ as a triangulated category.*

*Proof.* (cf., e.g., [O], Lemma 2.2). It is enough to show that for $\mathcal{F} \in \mathfrak{D}$ there exists $\tilde{\mathcal{F}} \in D^b(Coh^G(\tilde{\mathcal{N}}))$ and a morphism $\phi : \mathcal{F} \to \pi_*(\tilde{\mathcal{F}})$ such that the support of its cone $Cone(\phi)$ is strictly smaller than the support of $\mathcal{F}$. We can assume that $\mathcal{F} \in Coh^G(\mathcal{N}) \subset \mathfrak{D}$, and also that the scheme-theoretic support of $\mathcal{F}$ is reduced. Let $O \subset \mathcal{N}$ be a $G$-orbit, which is open in the support of $\mathcal{F}$, and $\overline{O}$ its closure. It is well known that there exists a parabolic subgroup $P_O$, and a $G$-equivariant sub-bundle $\tilde{\mathcal{N}}_O \subset T^*(G/P_O)$, such that $\pi_O : \tilde{\mathcal{N}}_O \to \overline{O}$ is birational (and thus is a resolution of singularities of $\overline{O}$); here $\pi_O$ is the restriction to $\tilde{\mathcal{N}}_O$ of the moment map $T^*(G/P_O) \to \mathcal{N}$ (more precisely, for $x \in O$, one can define $P_O$, $\tilde{\mathcal{N}}_O$ by $\mathfrak{p}_O = \mathfrak{g}_{\geq 0}$, $\tilde{\mathcal{N}}_O = \mathfrak{g}_{\geq 2} \times_{P_O} G$; here $\mathfrak{g}_{\geq i}$ are the terms of the Jacobson-Morozov-Deligne filtration on the Lie algebra $\mathfrak{g}$ associated to the nilpotent operator $\mathrm{ad}(x)$, see [De], 1.6, and $\mathfrak{p}_O$ is the Lie algebra of $P_O$). Let $\tilde{i}$ be the imbedding $\tilde{\mathcal{N}}_O \times_{G/P_O} G/B \hookrightarrow \tilde{\mathcal{N}}$, and set $\tilde{\mathcal{F}} = (\pi \circ \tilde{i})^*(\mathcal{F})$ (the non-derived pull-back). We have a canonical adjunction morphism $\mathcal{F} \to (\pi \circ \tilde{i})_*(\tilde{\mathcal{F}})$ (where we again consider the non-derived direct image). The composition

$$\mathcal{F} \to (\pi \circ \tilde{i})_*(\tilde{\mathcal{F}}) = R^0(\pi \circ \tilde{i})_*(\tilde{\mathcal{F}}) \to R\pi_*(\tilde{i}_*\tilde{\mathcal{F}})$$

is an isomorphism on $O$, because the fiber of $\pi \circ \tilde{i}$ over a point of $O$ is $P_O/B$, (the flag variety for the Levi subgroup), and the structure sheaf of $P_O/B$ is acyclic; hence the cone of this composition has smaller support. □

*Proof of Proposition 5.* It follows directly from Lemma 7 and Corollary 2 that $\mathfrak{D}$ is generated by $A_\lambda$, $\lambda \in \Lambda$. So it is enough to show that for $\lambda \in \Lambda^+$ the category

$\mathfrak{D}_{conv(\lambda)}$ is generated by $A_\mu$, $\mu \in \Lambda^+ \cap conv(\lambda)$. This follows by induction in $\lambda$ (with respect to the standard partial order on $\Lambda^+$) from Proposition 3. □

Propositions 4, 5 and 3 yield the following

**Theorem 1.** *Let us equip $\Lambda^+$ with any total ordering $\leq$ compatible with the standard partial order (i.e., $\lambda \in conv(\mu) \Rightarrow \lambda \leq \mu$). Then the set $\{A_\lambda \mid \lambda \in \Lambda^+\}$ is a quasi-exceptional set generating $\mathfrak{D}$. The set*

$$\{A_{w_0(\lambda)} \mid \lambda \in \Lambda^+\}$$

*is a dual quasi-exceptional set.*

*Remark* 8. The set $\{A_\lambda\}$ is not exceptional. For example, one can show that if $G = SL(2)$, and $\lambda = 1$, then $\mathrm{Ext}^i(A_\lambda, A_\lambda) \cong \mathbb{C}$ for all $i \geq 0$. It is also easy to see, that $\mathfrak{D} = D^b(Coh^G(\mathcal{N}))$ is not generated by any exceptional set (for otherwise $\mathrm{Hom}^\bullet(X, Y)$ would be finite-dimensional for all $X, Y \in \mathfrak{D}$, while this is not so in the above example $X = Y = A_1$, $G = SL(2)$). Notice, however, that the "larger" category $D^b(Coh^G(\tilde{\mathcal{N}}))$ is generated by the set $\mathcal{O}(\lambda)$, $\lambda \in \Lambda$, which can be shown to be exceptional for any ordering on $\Lambda$, which is compatible with the standard partial order.

*Remark* 9. Let $\Pi_\lambda : \mathfrak{D}_{\leq\lambda} \to \mathfrak{D}_{\leq\lambda}/\mathfrak{D}_{<\lambda}$ be the projection; by Theorem 1, Lemma 4(d) we have left and right adjoint functors $\Pi^l_\lambda : \mathfrak{D}_{\leq\lambda}/\mathfrak{D}_{<\lambda} \to \mathfrak{D}_{A_{w_0(\lambda)}}$, and $\Pi^r_\lambda : \mathfrak{D}_{\leq\lambda}/\mathfrak{D}_{<\lambda} \to \mathfrak{D}_{A_\lambda}$. For $\lambda \in \Lambda^+$ set

$$\mathbf{A}_\lambda = \Pi^r_\lambda(V_\lambda \otimes \mathcal{O}_{\mathcal{N}});$$
$$\mathbf{A}_{w_0(\lambda)} = \Pi^l_\lambda(V_\lambda \otimes \mathcal{O}_{\mathcal{N}}).$$

Then we have

$$\mathrm{Hom}^\bullet(\mathbf{A}_{w_0(\lambda)}, A_\mu) = 0 = \mathrm{Hom}^\bullet(A_{w_0(\lambda)}, \mathbf{A}_\mu)$$

for $\lambda \neq \mu \in \Lambda^+$, and

$$\mathrm{Hom}^\bullet(\mathbf{A}_{w_0(\lambda)}, A_\lambda) = \mathbb{C} = \mathrm{Hom}^\bullet(A_{w_0(\lambda)}, \mathbf{A}_\lambda),$$

where the latter equality follows from (26).

We claim that $\mathbf{A}_\lambda$, $\mathbf{A}_{w_0(\lambda)}$ admit the following geometric description.

For $\lambda \in \Lambda$ let $P_\lambda$ be the largest parabolic such that $\mathcal{O}_{G/B}(\lambda)$ is isomorphic to the pull-back of a line bundle on $G/P_\lambda$; let $\mathfrak{p}_\lambda$ be its Lie algebra. Set $\tilde{\mathfrak{g}}_\lambda = \mathfrak{p}_\lambda \times_{P_\lambda} G$, and let $\pi_\lambda : \tilde{\mathfrak{g}}_\lambda \to \mathfrak{g}$ be the projection. Then we have

$$\mathbf{A}_\lambda \cong i^*\pi_{\lambda *}(\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda)); \tag{29}$$

$$\mathbf{A}_{w_0(\lambda)} \cong i^*\pi_{\lambda *}\left(\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(w_0(\lambda))\right), \tag{30}$$

where $i : \mathcal{N} \to \mathfrak{g}$ is the imbedding, and $\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda)$ is defined by means of the obvious projection $\tilde{\mathfrak{g}}_\lambda \to G/P_\lambda$.

Indeed, the familiar morphism $V_\lambda \otimes \mathcal{O}_{G/P_\lambda} \to \mathcal{O}_{G/P_\lambda}(\lambda)$ yields a morphism $V_\lambda \otimes \mathcal{O}_{\tilde{\mathfrak{g}}_\lambda} \to \mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda)$, and hence also morphisms

$$V_\lambda \otimes \mathcal{O}_{\mathfrak{g}} \to \pi_{\lambda *}(\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda)),$$
$$V_\lambda \otimes \mathcal{O}_{\mathcal{N}} \to i^*\pi_{\lambda *}(\mathcal{O}_{\tilde{\mathfrak{g}}_\lambda}(\lambda));$$

and thus a morphism $\phi$ from the left-hand side to the right-hand side of (29). Since both objects in question lie in $\langle A_\lambda \rangle$ and have length $\dim(H^\bullet(G/P_\lambda))$, it suffices to check that this morphism is injective. This would follow if we show that the composition $A_{w_0(\lambda)} \to V_\lambda \otimes \mathcal{O}_{\mathcal{N}} \xrightarrow{\phi} \mathbf{A}_\lambda$ is nonzero, where the first arrow is the only (up to a constant) nonzero morphism $A_{w_0(\lambda)} \to V_\lambda$. Thus it suffices to see that $\phi|_{\mathcal{N}_0}$ is surjective, where $\mathcal{N}_0 \subset \mathcal{N}$ is the open orbit. Surjectivity of $\phi|_{\mathcal{N}_0}$ follows from the next lemma. Finally, (30) follows from (29) by Grothendieck-Serre duality.

**Lemma 8.** *Let $e \in \mathfrak{g}$ be a regular nilpotent, let $\tilde{\mathfrak{g}}^e_\lambda$ be the preimage of $e$ under $\pi_\lambda$, and $(G/P_\lambda)^e$ the image of $\tilde{\mathfrak{g}}^e_\lambda$ in $G/P_\lambda$ (thus $(G/P_\lambda)^e$ is a nilpotent scheme of length $\dim H^*(G/P_\lambda)$).*

*Then the restriction map*

$$V_\lambda = \Gamma(G/P_\lambda, \mathcal{O}(\lambda)) \to \Gamma((G/P_\lambda)^e, \mathcal{O}(\lambda)) \tag{31}$$

*is surjective.*

*Proof.* The structure sheaf $\mathcal{O}_{(G/P_\lambda)^e}$ has the Koszul resolution

$$0 \to \Omega^{\mathrm{top}}_{G/P} \to \cdots \to \Omega^1_{G/P} \to \mathcal{O}_{G/P}.$$

Twisting this resolution by $\mathcal{O}(\lambda)$ and considering the standard spectral sequence we see that the statement of the lemma follows from the equality

$$H^i(\Omega^j_{G/P} \otimes \mathcal{O}(\lambda)) = 0 \;\text{ for }\; i \geq j > 0 \;\text{ and }\; \text{for }\; i > j = 0.$$

The latter vanishing was proved by Broer: for $i > j$ it follows [Br1], Theorem 1(i); and for $i = j$ from [Br], Proposition 3.7(4) and Lemma 3.9. □

*Remark* 10. Victor Ginzburg pointed out to us that the surjection (31) probably admits the following alternative description. One can realize $V_\lambda$ as the total cohomology of an irreducible perverse sheaf $IC_\lambda$ on the affine Grassmanian $\mathcal{G}\mathfrak{r}$ of the Langlands dual group ${}^LG$, equivariant under the maximal bounded subgroup ${}^LG(O)$ in the loop group ${}^LG(K)$; see [Gi], [MV]. Ginzburg conjectures that one can identify $\Gamma((G/P_\lambda)^e, \mathcal{O}(\lambda))$ with cohomology (with constant coefficients) of the open ${}^LG(O)$ orbit $\mathcal{G}\mathfrak{r}_\lambda$ in the support of $IC_\lambda$, so that (31) is identified with the restriction map

$$H^\bullet(IC_\lambda) \to H^\bullet(IC_\lambda|_{\mathcal{G}\mathfrak{r}_\lambda}) = H^\bullet(\mathcal{G}\mathfrak{r}_\lambda). \tag{32}$$

Notice that it is easy to see that

$$H^\bullet(\mathcal{G}\mathfrak{r}_\lambda) \cong H^\bullet({}^LG/{}^LP_{\check{\lambda}}) \cong H^\bullet(G/P_\lambda) \cong \Gamma((G/P_\lambda)^e, \mathcal{O}),$$

where $\check{\lambda}$ is a weight of ${}^LG$ obtained from $\lambda$ by means of an invariant quadratic form on $\mathfrak{g}$. Thus at least the dimensions of the target spaces in (31) and (32) coincide.

3.1. **Comparison with perverse coherent $t$-structure.** Recall the coherent perverse $t$-structure on $\mathfrak{D}$, corresponding to the middle perversity, $p(O) = -\frac{\dim(O)}{2}$ for a $G$-orbit $O \subset \mathcal{N}$; see [B2]. We let $\mathfrak{D}^{p,>0}$, $\mathfrak{D}^{p,\leq 0}$ denote the corresponding positive and negative subcategories, and we let $\mathcal{P} = \mathfrak{D}^{p,\geq 0} \cap \mathfrak{D}^{p,\leq 0}$ be its core.

Let $\mathcal{S} : \mathfrak{D} \to \mathfrak{D}^{op}$ be the Grothendieck-Serre duality; $\mathcal{S} : X \mapsto \underline{RHom}(X, \mathbb{D}C)$, where $\mathbb{D}C$ is the *equivariant dualizing complex*; cf. [B2], Definition 1 (we assume that the dualizing complex is normalized in the standard way, i.e., $\mathbb{D}C = pr^!(\underline{\mathbb{C}})$, where $pr$ is the projection to $Spec(\mathbb{C})$).

Set $d = \dim(\mathcal{N})/2$.

**Lemma 9.** *We have $A_\lambda[d] \in \mathcal{P}$ for all $\lambda$.*

*Proof.* We have $\mathcal{S}(A_\lambda)[d] = A_{-\lambda}[d]$, because duality commutes with proper direct images, and the (equivariant) dualizing sheaf on $\tilde{\mathcal{N}}$ is isomorphic to $\mathcal{O}_{\tilde{\mathcal{N}}}[2d]$. Thus it is enough to check conditions on stalks for $A_\lambda[d]$; i.e., for an orbit $O \subset \mathcal{N}$ to see that

$$i^*_{O_{\rm gen}}(A_{-\lambda}) \in \mathfrak{D}^{\leq p(O)+d}(\mathcal{O}_{O_{\rm gen}} - mod),$$

where $i_{O_{\rm gen}} : O_{\rm gen} \hookrightarrow O$ is the imbedding of the generic point of $O$ into $\mathcal{N}$ (see [B2], Definition 2, Lemma 5(a)). This follows from the two well-known facts: that $\pi$ is a semi-small morphism, (i.e., $\dim(\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}) = \dim \tilde{\mathcal{N}}$, so that $\operatorname{co}\dim(\pi^{-1}(x)) \leq \frac{1}{2}\operatorname{co}\dim(O) = p(O) + d$ for an orbit $O \subset \mathcal{N}$, $x \in O$); and that the homological dimension of the direct image functor $\pi_*$ for coherent sheaves under a proper morphism $\pi$ of algebraic varieties over a field equals the dimension of $\pi$ (maximal dimension of a fiber of $\pi$); see, e.g., [CG], 3.3.20, 8.9.19; and [Ha] Corollary 11.2 respectively. □

For $(O, L) \in \mathbf{O}$ (notation of the Introduction) let $\mathcal{L}$ be the $G$-equivariant vector bundle on $O$ corresponding to $L$, and $j : O \hookrightarrow \mathcal{N}$ the imbedding. We set

$$IC_{O,L} = j_{!*}(\mathcal{L}[p(O)]) \in D^b(Coh);$$

see [B2], 3.2 for the definition of the minimal (Goresky-MacPherson) extension functor $j_{!*}$ for coherent sheaves.

*Remark* 11. We do not know an explicit description of the object $IC_{O,L}$, in general; however, in the particular case when $L = \mathbb{C}$ is the trivial representation they are easy to describe. Namely, we claim that for any orbit $O \overset{j}{\hookrightarrow} \mathcal{N}$ we have

$$IC_{O,L} \cong j_*\mathcal{O}_O[p(O)] = N_*(\mathcal{O})[p(O)], \tag{33}$$

where $j_*$ stands for the *non-derived* direct image, and $N$ is the normalization morphism for $\overline{O}$ (cf. Conjecture 4 in [O]). Indeed, the result of [Hi], [Pa] implies that the normalization of $\overline{O}$ is Cohen-Macaulay; hence

$$\mathcal{S}(N_*(\mathcal{O})[p(O)]) \in Coh^G(\mathcal{N})[p(O)],$$

which yields (33).

Ostrik pointed out to us that a similar statement is probably true for $(O, L) \in \mathbf{O}$ if $L$ has finite image, due to the result of [Br2].

**Corollary 3.** *a) The perverse $t$-structure on $D^b(Coh^G(\mathcal{N}))$ corresponding to the middle perversity (*[B2]*, Theorem 1; Example 1) coincides with the $t$-structure of the dualizable quasi-exceptional set $\nabla_\lambda = A_\lambda[d]$.*

*b) The core $\mathcal{P}$ of this $t$-structure is a quasi-hereditary category.*

*The set of isomorphism classes of irreducible objects in $\mathcal{P}$ equals $\{[IC_{O,L}] \mid (O, L) \in \mathbf{O}\}$.*

*The set of isomorphism classes of costandard objects equals $\{[\nabla^\lambda] = [A_\lambda[d]] \mid \lambda \in \Lambda^+\}$; and that of standard objects equals $\{[\Delta_\lambda] = [A_{w_0(\lambda)}[d]] \mid \lambda \in \Lambda^+\}$.*

*Proof.* (a) follows directly from Lemma 9 and Proposition 1. The second statement in part (b) is a particular case of Corollary 4 in [B1]. The rest follows from Proposition 2. □

**Corollary 4.** *a) The Grothendieck group $K^0(\mathfrak{D})$ is a free abelian group; each of the sets $\{[A_\lambda] \mid \lambda \in \Lambda^+\}$, $\{[IC_{O,L}] \mid (O, L) \in \mathbf{O}\}$ forms a basis of this group.*

*b) There exists a unique bijection between $\Lambda^+$ and $\mathbf{O}$, $\lambda \mapsto (O_\lambda, L_\lambda)$ satisfying any of the following equivalent properties:*

*(i)* $\mathrm{Hom}(IC_{O_\lambda,L_\lambda}, A_\lambda[d]) \neq 0$.

*(i′)* $\mathrm{Hom}(A_{w_0(\lambda)}[d], IC_{O_\lambda,L_\lambda}) \neq 0$.

*(ii) There exists a morphism $IC_{O_\lambda,L_\lambda} \to A_\lambda[d]$ whose cone lies in $\mathfrak{D}_{conv^0(\lambda)}$.*

*(ii′) There exists a morphism $A_{w_0(\lambda)}[d] \to IC_{O_\lambda,L_\lambda}$ whose cone lies in $\mathfrak{D}_{conv^0(\lambda)}$.*

*(iii) $IC_{O_\lambda,L_\lambda} \in \mathfrak{D}_{conv(\lambda)}$ for all $\lambda$.*

*(iv) $[IC_{O_\lambda,L_\lambda}]$ lies in the span of $[A_\mu]$, $\mu \in \Lambda^+ \cap conv(\lambda)$, i.e., the transformation matrix between the two bases is upper triangular.*

*(v) $[IC_{O_\lambda,L_\lambda}] - (-1)^d [A_\lambda]$ lies in the span of $[A_\mu]$, $\mu \in conv^0(\lambda) \cap \Lambda^+$.* □

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