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CHARACTER VALUES, SCHUR INDICES AND CHARACTER SHEAVES

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Abstract. In this paper we are concerned with the problem of determining the character values and Schur indices of a finite group of Lie type over $\mathbb{F}_q$. We show that (under some conditions on q) these values lie in the ring of algebraic integers generated by $(1+\sqrt{\pm q})/2$ and roots of unity of order prime to q. Furthermore, we determine the Schur indices for some of the (nonrational) unipotent characters in exceptional groups. Our results, combined with previous results due to Gow, Ohmori and Lusztig, imply that there are only 6 cases left where the Schur index of a cuspidal unipotent character remains unknown. Our methods rely, in an essential way, on Lusztig's theory of character sheaves.

1. Introduction

Let G be a connected reductive group defined over the finite field $\mathbb{F}_q$ with q elements; let $F\colon G \to G$ be the corresponding Frobenius map. We denote by $\mathrm{Irr}(G^F)$ the set of complex irreducible characters of the finite group G^F. This paper is concerned with the following two problems (which are general problems in the character theory of finite groups). Let $\chi \in \mathrm{Irr}(G^F)$.

(a) Determine the character field $\mathbb{Q}(\chi) := \mathbb{Q}(\chi(g) \mid g \in G^F) \subseteq \mathbb{C}$.
(b) Determine the Schur index $m_{\mathbb{Q}}(\chi)$, that is, the smallest possible degree of a field extension of $\mathbb{Q}(\chi)$ over which χ can be realized.

It is clear that all character values of G^F lie in the field K_0 generated by the eth roots of unity over $\mathbb{Q}$ where e is the exponent of G^F; furthermore, by a famous theorem of Brauer (see [15, 10.3]), every irreducible character can be realized over K_0. Of course, a minimal field containing all character values actually may be much smaller than K_0. To get an idea of what we might expect to be true for our group G^F, in general (as far as character values are concerned), we consider the following three examples.

Example 1.1. Let $G^F = \mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q)$, where q is any prime power. Then it is known that all irreducible characters of G^F are rational linear combinations of the Deligne–Lusztig generalized characters $R_{T,\theta}$ (see, for example, the description by Fong–Srinivasan [9]). Thus, we are reduced to determine the character values of $R_{T,\theta}$ where $T \subseteq G$ is an F-stable maximal torus and $\theta \in \mathrm{Irr}(T^F)$. Let $g = us = su \in G^F$, where $s \in G^F$ is semisimple and $u \in G^F$ is unipotent. Then the character formula in [3, 7.2.8] and the rationality of Green functions (see [3, 7.6]) show that $R_{T,\theta}(g)$ can be expressed as a rational linear combination of terms $\theta(t)$

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where $t \in T^F$ is conjugate to s. In particular, we see that all character values lie in $\mathbb{Q}(\varepsilon)$ where $\varepsilon \in \mathbb{C}$ is a root of unity of order prime to q.

Example 1.2. While the general linear groups behave quite smoothly as far as character values are concerned, new arithmetical problems arise when we consider the special linear groups. Consider the simplest possible case where $G^F = \mathrm{SL}_2(q)$ and q is odd. Then the character table of G^F is explicitly known and can be found, for example, in [8, §15.9]. There are irreducible characters χ of degree $(q+1)/2$ such that $\chi(u) = (1+\sqrt{\pm q})/2$ for a regular unipotent element $u \in G^F$ (where the sign is such that $\pm q \equiv 1 \bmod 4$). This phenomenon is related to the fact that the center of G is not connected.

Example 1.3. Let $G^F = \mathrm{E}_8(q)$ where q is an odd power of a prime p. Let $B \subseteq G$ be an F-stable Borel subgroup and consider the corresponding Hecke algebra $\mathcal{H}_\mathbb{C} = \mathrm{End}_{G^F}(\mathbb{C}[G^F/B^F])$. Let $\mathrm{Irr}(\mathcal{H}_\mathbb{C})$ be the set of irreducible characters of $\mathcal{H}_\mathbb{C}$. Then, for each $\rho \in \mathrm{Irr}(\mathcal{H}_\mathbb{C})$ we have a corresponding $\chi_\rho \in \mathrm{Irr}(G^F)$ whose restriction to B^F contains the trivial character as a constituent. Now, if $\rho \in \mathrm{Irr}(\mathcal{H}_\mathbb{C})$ is of degree 4096, then ρ can be realized over $\mathbb{Q}(\sqrt{q})$ but not over $\mathbb{Q}$ (see [13, 9.2.3] and the references there). Consequently, for such ρ, we have $\mathbb{Q}(\chi_\rho) = \mathbb{Q}(\sqrt{q})$. (See Proposition 5.6 for more details.)

One of our aims is to show that these examples do indeed give the correct general idea. We introduce the following notation. Let $\omega_q \in \mathbb{C}$ be the algebraic number defined by

$$\omega_q = \begin{cases} 1 & \text{if } q \text{ is a square or a power of 2,} \\ \frac{1}{2}(1+\sqrt{\delta(q)q}) & \text{otherwise, where } \delta(q) = (-1)^{(q-1)/2}. \end{cases}$$

Note that ω_q is an algebraic integer in the field $\mathbb{Q}(\omega_q)$ but, in general, ω_q will not generate the ring of algebraic integers in that field.

Theorem 1.4. *There exists a finite set $\flat \subseteq \mathbb{Z}$ of prime numbers, depending only on the Dynkin diagram of G, such that the following holds. Assume that q is a power of a prime which is not in $\flat$. Let $g \in G^F$. Then we have*

$$\chi(g) \in \mathbb{Z}[\omega_q, \varepsilon] \qquad \text{for all } \chi \in \mathrm{Irr}(G^F),$$

where $\varepsilon \in \mathbb{C}$ is a root of unity of order dividing the order of the semisimple part of g. In particular, we have $\chi(g) \in \mathbb{Z}[\omega_q]$ if g is unipotent.

The proof will be given in Section 3; it uses, in an essential way, Lusztig's theory of character sheaves [23] together with its applications to generalized Gelfand–Graev representations in [26]. Note that, in general, the problem of determining the Schur index of χ is very subtle; see, for example, the papers by Gow [14], Ohmori [32], Lusztig [28] and Turull [41].

It is likely that the set $\flat$ in Theorem 1.4 only contains the bad primes for G. Recall that a prime p is "good" for G if p is good for each simple factor of the derived subgroup of G, where the conditions for the various simple types are as follows:

$$\begin{aligned} A_n: &\quad \text{no condition,} \\ B_n, C_n, D_n: &\quad p \neq 2, \\ G_2, F_4, E_6, E_7: &\quad p \neq 2, 3, \\ E_8: &\quad p \neq 2, 3, 5. \end{aligned}$$

TABLE 1. Character fields and bounds for the Schur indices of cuspidal unipotent characters (notation of [3, 13.7])

TYPE OF (G,F)	CUSPIDAL UNIPOTENT χ	$\mathbb{Q}(\chi)$	$m_{\mathbb{Q}}(\chi)$
$B_n, C_n, D_n, {}^2D_n$	(at most one χ)	$\mathbb{Q}$	1
${}^2A_n$	(at most one χ)	$\mathbb{Q}$	1 or $2^{(*)}$
G_2	$G_2[1], G_2[-1]$	$\mathbb{Q}$	1
	$G_2[\theta], G_2[\theta^2]$	$\mathbb{Q}(\theta)$	1
${}^3D_4$	${}^3D_4[1], {}^3D_4[-1]$	$\mathbb{Q}$	1
F_4	$F_4^I[1], F_4^{II}[1], F_4[-1]$	$\mathbb{Q}$	1
	$F_4[i], F_4[-i]$	$\mathbb{Q}(i)$	$1, 2$ or $4^{(*)}$
	$F_4[\theta], F_4[\theta^2]$	$\mathbb{Q}(\theta)$	1
E_6	$E_6[\theta], E_6[\theta^2]$	$\mathbb{Q}(\theta)$	1
${}^2E_6$	${}^2E_6[1]$	$\mathbb{Q}$	1
	${}^2E_6[\theta], {}^2E_6[\theta^2]$	$\mathbb{Q}(\theta)$	1
E_7	$E_7[\xi], E_7[-\xi]$	$\mathbb{Q}(\xi)$	$1, 2$ or $4^{(*)}$
E_8	$E_8^I[1], E_8^{II}[1], E_8[-1]$	$\mathbb{Q}$	1
	$E_8[-\theta], E_8[-\theta^2]$,	$\mathbb{Q}(\theta)$	1 or 2
	$E_8[\theta], E_8[\theta^2]$	$\mathbb{Q}(\theta)$	$1, 2, 3$ or $6^{(*)}$
	$E_8[i], E_8[-i]$	$\mathbb{Q}(i)$	$1, 2$ or $4^{(*)}$
	$E_8[\zeta], E_8[\zeta^2], E_8[\zeta^3], E_8[\zeta^4]$	$\mathbb{Q}(\zeta)$	1

$i := \sqrt{-1}$, $\theta := \exp(2\pi i/3)$, $\xi := \sqrt{-q}$, $\zeta := \exp(2\pi i/5)$.

We remark that the problem of determining character and realization fields for finite groups of Lie type has also been studied independently by Tiep and Zalesski [40], using completely different methods. See also the Ph.D. Thesis of J. Kelly (University of Warwick, U.K., 1975), under the direction of G. Lusztig.

A special role in the character theory of finite groups of Lie type is played by the *cuspidal unipotent characters*. For example, a classical group has at most one such character (see [19, 8.11]) which then must be rational-valued. The Schur indices of rational-valued unipotent characters are completely known by Ohmori [32] (in type ${}^2A_n$) and Lusztig [28] (in general). Here, we determine explicitly the character fields and some upper bounds for the Schur indices of (nonrational) cuspidal unipotent characters of exceptional groups,[1] without any assumption on q; the results are summarized in Table 1. In Proposition 5.6, we extend this to noncuspidal unipotent characters.

The entries marked by $(*)$ in Table 1 are trivial consequences of the Benard–Schacher Theorem [5, 74.26]: the field $\mathbb{Q}(\chi)$ has to contain all roots of unity of order $m_{\mathbb{Q}}(\chi)$. The assertions on $\mathbb{Q}(\chi)$ will be proved in Section 5, using the results on the eigenvalues of Frobenius obtained by Digne–Michel [7] and Lusztig [18]. The knowledge of the character fields has strong implications on the Schur index, when

[1] The results on $\mathbb{Q}(\chi)$ are probably known to the specialists. But, as far as I know, they have not yet been published.

certain congruence conditions are satisfied. This can be used to obtain some more precise information concerning the entries marked by $(*)$. As a summary, we state the following result.

Corollary 1.5. *Assume that q is a power of a prime p such that $p \not\equiv 1 \bmod 3$ and $p \not\equiv 1 \bmod 4$ and that G has a simple derived subgroup. (For example, q may be any power of 2 or 3.) Then every unipotent character has Schur index 1 except in type ${}^2A_n$ or ${}^2E_6$ where there exist unipotent characters with Schur index 2. (In all cases, the Schur indices are at most 2.)*

For the proof, see (6.7). The problem of determining the Schur indices $m_{\mathbb{Q}}(\chi)$ will be considered in detail in Section 6. Besides the theory of Hasse invariants and the Benard–Schacher Theorem, our strategy is to look at the multiplicities with which an irreducible character of G^F occurs in certain generalized Gelfand–Graev characters of G^F or, more generally, in the induction of certain generalized Gelfand–Graev characters from centralizers of isolated semisimple elements in G^F. To compute these multiplicities, we work in the framework provided by Lusztig's theory of character sheaves, using the results in Section 2 (especially Theorem 2.10). As a further application of the results in Section 2, we show that the "algebraic numbers of absolute value 1" occurring in Shoji's proof [34], [35] of Lusztig's conjecture on character sheaves and almost characters are actually roots of unity whose order is divisible by bad primes only; see Proposition 4.2 (for G of classical type) and Proposition 4.4 (for G of exceptional type). The proof is an adaptation of the arguments in the proof of [27, Theorem 0.8].

As far as the Schur indices for the Suzuki and Ree groups are concerned, the situation is as follows. By [14, p. 119], every irreducible character of a Suzuki or Ree group of type ${}^2B_2$ or ${}^2G_2$ has Schur index 1. Now assume that q is an odd power of $\sqrt{2}$ and consider the Ree group ${}^2F_4(q^2)$. By [21, Appendix], there is a unique cuspidal unipotent character which occurs with even multiplicity in every Deligne–Lusztig generalized character $R_{T,1}$. In Malle's table [29], this character is denoted χ_{21}. We have $\mathbb{Q}(\chi_{21}) = \mathbb{Q}$ and so $m_{\mathbb{Q}}(\chi) \leqslant 2$ by the Brauer–Speiser Theorem [5, 74.27]. Using the methods in [28], it could not be decided if χ_{21} can be realized over $\mathbb{Q}$ (see the remarks in [28, 2.23]). We shall prove the following result.

Theorem 1.6. *The cuspidal unipotent character $\chi_{21} \in \operatorname{Irr}({}^2F_4(q^2))$ cannot be realized over $\mathbb{R}$ and so we have $m_{\mathbb{Q}}(\chi_{21}) = 2$.*

The proof will be given in Section 7; see Proposition 7.5. Ohmori (unpublished) has shown that all the remaining unipotent characters of ${}^2F_4(q^2)$ have Schur index 1. We will give a slightly different proof in Section 7.

2. Generalized Gelfand–Graev representations

In this section we present some basic results concerning Kawanaka's generalized Gelfand–Graev characters [17]. These results rely on the deep results of Lusztig [26]. In particular, Theorem 2.5 (which is proved in [26]) provides a link between generalized Gelfand–Graev characters and certain "cuspidal" functions on G^F which are unipotently supported. In Theorem 2.10, we present a variant of that result for "cuspidal" functions which are not necessarily unipotently supported. That such a variant exists is mentioned briefly by Lusztig at the end of the proof of Theorem 0.8 in [27, p. 985].

2.1. Unipotently supported class functions. Let C be any conjugacy class in G. For $g \in C$, we write $A_G(g) = Z_G(g)/Z_G^\circ(g)$; this is a finite group. Now assume that C is F-stable and $g \in C^F$. Then F induces a group automorphism of $A_G(g)$ which we denote by the same symbol. For each $y \in A_G(g)$, we denote by g_y an element in C^F obtained by twisting the given representative g with y. Then it is well known that the correspondence $y \mapsto g_y$ induces a bijection between the F-conjugacy classes of $A_G(g)$ and the G^F-conjugacy classes contained in C^F; see [38, I.2].

Now let $\psi \in \mathrm{Irr}(A_G(g))$ be F-invariant. Then we can extend ψ to a character $\tilde{\psi}$ of the semidirect product $A_G(g) \rtimes \langle F \rangle$ (see [15, 11.22]). We now define a class function $Y^G_{(C,\psi)}$ on G^F by the requirement that

(a) $$Y^G_{(C,\psi)}(h) = \begin{cases} \tilde{\psi}(yF) & \text{if } h \text{ is conjugate to } g_y \text{ for some } y \in A_G(g), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{N}_G$ be the set of pairs (C, ψ) where C is a unipotent class and ψ is an irreducible character of $A_G(u)$ (for a chosen representative $u \in C$). If $\iota \in \mathcal{N}_G$, we also write $\iota = (C_\iota, \psi_\iota)$. We denote by $\mathcal{N}_G^F$ the set of all pairs $(C, \psi) \in \mathcal{N}_G$ such that C is F-stable and $\psi \in \mathrm{Irr}(A_G(u))$ is invariant under F, where $u \in C^F$. Then the set of functions $\{Y_\iota^G \mid \iota \in \mathcal{N}_G^F\}$ is a basis of the space of unipotently supported class functions of G^F; see [23, 24.2.7]. (Note that it depends on the choice of extensions $\tilde{\psi}$; thus, it is only well defined up to nonzero scalar multiples.)

What can we say about the values of a class function $\tilde{\psi}$ as above? A first approach is given by the following simple remark. (We will come back to this point in Lemma 3.5 below.)

Remark 2.2. Let A be a finite group and $\sigma\colon A \to A$ be an automorphism. Let $\psi \in \mathrm{Irr}(A)$ be a linear character, that is, the map $\psi\colon A \to \mathbb{C}^\times$ is a group homomorphism. Assume now that ψ is σ-invariant; this means that $\psi(\sigma y \sigma^{-1}) = \psi(y)$ for any $y \in A$. Then we have a canonical extension of ψ to $A \rtimes \langle \sigma \rangle$. Indeed, it is readily checked that the map

$$\tilde{\psi}\colon A \rtimes \langle \sigma \rangle \to \mathbb{C}^\times, \qquad y\sigma \mapsto \psi(y),$$

defines a group homomorphism extending ψ. In particular, the character values of that extension are given by the character values of ψ.

2.3. Generalized Gelfand–Graev representations. Assume that q is a power of a good prime for G. Let us fix a pair $\iota = (C, \psi) \in \mathcal{N}_G^F$. Let $y_1, \ldots, y_d$ be representatives for the F-conjugacy classes of $A_G(u)$, where $y_1 = 1$. For each $r \in \{1, \ldots, d\}$, let $u_r \in C^F$ be an element obtained by twisting u with y_r. Then $u_1, \ldots, u_d$ are representatives for G^F-conjugacy classes contained in C^F. Let $\Gamma^G_{u_1}, \ldots, \Gamma^G_{u_d}$ be the characters of the corresponding generalized Gelfand–Graev representations. We don't need to recall the exact definition from [17]; there exists an F-stable closed unipotent subgroup $U \subseteq G$ (which depends on C but not on u_r) such that

(a) $$\Gamma^G_{u_r} = \mathrm{Ind}_{U^F}^{G^F}(\eta_r) \quad \text{where } \eta_r \in \mathrm{Irr}(U^F) \text{ for } 1 \leqslant r \leqslant d.$$

In particular, each $\Gamma^G_{u_r}$ is an actual representation of G^F. As in [26, (7.5)], we define the following twisted version of generalized Gelfand–Graev characters:

(b) $$\Gamma^G_\iota = \sum_{r=1}^{d} |A_G(u_r) : A_G(u_r)^F|\, Y^G_\iota(u_r)\, \Gamma^G_{u_r} \quad \text{where } \iota = (C, \psi) \in \mathcal{N}_G^F.$$

Using an inversion formula for the functions Y_ι^G, we can write

(c) $$\Gamma_{u_r}^G = \frac{1}{|A_G(u)|}\sum_\iota Y_\iota^G(u_r^{-1})\,\Gamma_\iota^G \quad \text{for all } 1 \leqslant r \leqslant d,$$

where the sum is over all $\iota \in \mathcal{N}_G^F$ with $C_\iota = C$; see [26, 7.5].

2.4. Blocks. As in [23, 24.1], let $\mathcal{J}_G$ be the set of all pairs (L, ι_0) (up to G-conjugacy), where $L \subseteq G$ is a Levi complement in some parabolic subgroup of G and $\iota_0 \in \mathcal{N}_L$ is "cuspidal" in the sense of [22, 6.2]. By the generalized Springer correspondence, we have a natural surjective map $\tau\colon \mathcal{N}_G \to \mathcal{J}_G$. If $\iota \in \mathcal{N}_G$ and $\tau(\iota) = (L, \iota_0)$ where $\iota_0 \in \mathcal{N}_L$, we set

$$b_\iota = \dim G - \dim C - \dim Z_L^\circ \qquad \text{where } Z_L = \text{center of } L.$$

We shall now state some results concerning generalized Gelfand–Graev characters which are formal consequences of Lusztig's results in [26]. Assume that q is large enough, so that the results of [26] can be applied; in particular, q is a power of a good prime for G. Denote by $\langle\ ,\ \rangle_{G^F}$ the usual hermitian product for class functions on G^F. Furthermore, let D_G be the Alvis–Curtis–Kawanaka duality operation on the character ring of G^F; see [3, 8.2]. Then the following hold.

(a) For any $\iota = (C, \psi) \in \mathcal{N}_G^F$, we have $D_G(\Gamma_\iota^G)(g) = 0$ unless $g \in G^F$ is unipotent and C is contained in the closure of the class of g.

(b) Let $\iota, \iota' \in \mathcal{N}_G^F$ be such that $C_\iota = C_{\iota'}$. Then we have

$$\langle D_G(\Gamma_\iota^G), Y_{\iota'}^G\rangle_{G^F} = |A_G(u)|\,\zeta_\iota'\,q^{-b_\iota/2}\,\delta_{\iota,\iota'} \quad (u \in C_\iota),$$

where $\zeta_\iota' \in \mathbb{C}$ is a 4th root of unity. The above relations follow from the results in [26]; see [11, 2.3] for some remarks concerning the proofs. Note that (a) and (b) hold for any normalisation of the function Y_ι^G. Finally, let $\{u_\alpha \mid \alpha \in A\}$ be a set of representatives of the unipotent classes of G^F. Then we have that

(c) the characters $\{\Gamma_{u_\alpha}^G \mid \alpha \in A\}$ form a basis of the space of unipotently supported class functions on G^F.

(This is an easy consequence of (a) and (b); see the argument in [10, 3.6].)

In the case of a cuspidal pair, we have the following basic relation with generalized Gelfand–Graev characters.

Theorem 2.5 (Lusztig [26, 7.6])**.** *Assume that q is large enough, so that the results of* [26] *can be applied; in particular, q is a power of a good prime for G. Let $\iota_0 = (C, \psi) \in \mathcal{N}_G^F$ be a cuspidal pair and $u \in C$. Then we have*

$$|A_G(u)|\,|(Z_G^\circ)^F|\,q^{b_\iota/2}\,Y_{\iota_0}^G = \zeta\,\Gamma_{\iota_0}^G \quad \textit{where } \zeta \in \mathbb{C},\ \zeta^4 = 1.$$

In particular, the class function on the left-hand side is a linear combination of $\mathrm{Irr}(G^F)$, where the coefficients lie in the ring $\mathbb{Z}[\zeta, Y_\iota^G(u)\ (u \in C^F)]$. Note that this holds for any normalisation of the function Y_ι^G.

Proof. The above statement is proved in [26, 7.6] under the assumption that G is semisimple. The proof is based on the formula [26, 7.5(b)]. The same argument also works in the case where G is not semisimple and it yields the formula stated above. $\square$

Remark 2.6. Assume that q is a power of a good prime for G. Let $u \in G^F$ be a regular unipotent element. Then Γ_u^G is an ordinary Gelfand–Graev character. In this case, the relations (a) and (b) in (2.4) are known to hold without any further condition on q. Indeed, as far as (a) is concerned, see [8, 14.33]; the fact that (b) holds in general is shown by Digne, Lehrer and Michel [6, 2.7]. (More generally, Digne, Lehrer and Michel have studied the problem of computing character values at regular unipotent elements; see [6] and the references there.)

Now assume, furthermore, that all irreducible components of the root system of G are of type A. Then any prime is good for G. Furthermore, if $\iota_0 = (C, \psi) \in \mathcal{N}_G$ is cuspidal, then C must be the class of regular unipotent elements (see [22, 10.3]). Hence $\Gamma_{\iota_0}^G$ is a linear combination of the ordinary Gelfand–Graev characters of G^F. We have $D_G(Y_{\iota_0}^G) = \pm Y_{\iota_0}^G$, by the definition of D_G and the results of Bonnafé [2]. Using [23, 24.4(d)], we conclude that Theorem 2.5 remains valid in this case, without any assumption on q.

In order to extend Theorem 2.5 to "cuspidal" pairs which are not necessarily unipotently supported, we begin by recalling some results from Lusztig's papers on character sheaves [23].

2.7. Characteristic functions. In the following discussion, we assume that q is a power of a prime p such that the main result in [23, 23.1] holds; for example, it will be enough to assume that p is a good prime.

Let Σ be a subset of G and $\mathcal{E}$ a $\overline{\mathbb{Q}}_\ell$-local system on Σ such that $(\Sigma, \mathcal{E})$ is a cuspidal pair in the sense of [23, 7.1]. (Here, ℓ is a prime not dividing q; however, we will identify $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$ using a choice of an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$.) If Σ is F-stable and $\mathcal{E}$ is isomorphic to its inverse image under F, then the choice of an isomorphism $\varphi\colon F^*\mathcal{E} \to \mathcal{E}$ gives rise to a "characteristic function" $\chi_{(\Sigma,\mathcal{E})}$ which is a class function on G^F; see [23, 8.4]. Since $(\Sigma, \mathcal{E})$ is clean (see [23, 7.7 and 23.1]), we actually have that

(a) $$\chi_{(\Sigma,\mathcal{E})}(g) = \begin{cases} (-1)^{\dim \Sigma}\,\mathrm{Tr}(\varphi_g, \mathcal{E}_g) & \text{if } g \in \Sigma^F, \\ 0 & \text{otherwise;} \end{cases}$$

here, φ_g denotes the induced isomorphism on the stalk $\mathcal{E}_g$. We assume that φ is chosen such that the eigenvalues of φ_g ($g \in \Sigma^F$) are of the form $q^{(\dim G - \dim \Sigma)/2}$ times a root of unity; this is possible by [23, 25.1]. Note that this determines $\chi_{(\Sigma,\mathcal{E})}$ up to multiplication by a root of unity and we have $\langle \chi_{(\Sigma,\mathcal{E})}, \chi_{(\Sigma,\mathcal{E})}\rangle_{G^F} = 1$; see [23, 25.7].

The above formula for $\chi_{(\Sigma,\mathcal{E})}$ can be made more explicit; see [23, 25.5]. Let $s_0 \in G^F$ be the semisimple part of an element in Σ and set

(b) $$H := Z_G(s_0) \quad \text{and} \quad C_1 := \{u \in H \mid u \text{ unipotent}, s_0u \in \Sigma\}.$$

We shall assume that s_0 lies in the center of G or that the derived subgroup of G is simply-connected. Then H is connected (see [3, 3.5.6]), C_1 is a single conjugacy class in H (see [23, 7.11]) and s_0 is "isolated", that is, H is not contained in a Levi complement of a proper parabolic subgroup of G (see [23, 3.12]). In particular, it follows that C_1 is F-stable. Let $u_1 \in C_1^F$ and consider the induced action of F on $A_H(u_1)$. We set $g_0 := s_0u_1 \in \Sigma^F$. Then we have $Z_G(g_0) = Z_H(u_1)$ and so $A_G(g_0) \cong A_H(u_1)$ canonically. Now, via this isomorphism, $\mathcal{E}$ corresponds to an F-invariant $\psi_1 \in \mathrm{Irr}(A_H(u_1))$; see the discussion in [22, 2.10].

Finally, there is a tame local system $\mathcal{G}$ of rank 1 on Z_G° and a cuspidal local system $\mathcal{E}_1$ on C_1 such that the inverse image of $\mathcal{E}$ under the natural map $Z_G^\circ \times C_1 \to \Sigma$, $(z,u) \mapsto zs_0u$, is $\mathcal{G} \boxtimes \mathcal{E}_1$; see [23, 7.11] and [22, 2.5]. Here, $\mathcal{E}_1$ corresponds to $\psi_1 \in \mathrm{Irr}(A_H(u_1))$ and $\mathcal{G}$ is F-stable and corresponds to some $\theta \in \mathrm{Irr}((Z_G^\circ)^F)$. We set $\iota_1 = (C_1, \psi_1) \in \mathcal{N}_H$ and define $Y_{\iota_1}^H$ as in (2.1)(a). Then $\varphi\colon F^*\mathcal{E} \to \mathcal{E}$ can be chosen such that

(c) $$\chi_{(\Sigma,\mathcal{E})}(zs_0u) = q^{(\dim G - \dim \Sigma)/2}\hat{\theta}(zs_0)\, Y_{\iota_1}^H(u)$$

for all $z \in (Z_G^\circ)^F$ and all unipotent $u \in H^F$. Here, $\hat{\theta}$ is an extension of θ to an irreducible character of Z_H^F; note that we have $Z_G^\circ = Z_H^\circ$; see [8, 14.11]. The above formula is a special case of the general character formula in [23, 8.5]; see also the discussion in [23, 25.5].

The above discussion shows that the computation of $\chi_{(\Sigma,\mathcal{E})}$ can be reduced to the computation of the function $Y_{\iota_1}^H$ and, hence, via Theorem 2.5, to generalized Gelfand–Graev characters. In order to make this more explicit, we introduce the following notation.

Definition 2.8. Let $s_0 \in G^F$ be isolated semisimple and set $H = Z_G(s_0)$. Assume that H is connected. Then we have $Z_H^\circ = Z_G^\circ$ (see [8, 14.11]). Let $\theta \in \mathrm{Irr}((Z_G^\circ)^F)$ and $f\colon H^F \to \mathbb{C}$ be a unipotently supported class function. Then we define a class function $\tilde{f}_{s_0}^\theta$ on G^F by the requirement that

$$\tilde{f}_{s_0}^\theta(su) = \begin{cases} \theta(z)\, f(u) & \text{if } s = zs_0 \text{ with } z \in Z_G^\circ, \\ 0 & \text{if } s \text{ is not } G^F\text{-conjugate to } s_0z \text{ with } z \in Z_G^\circ. \end{cases}$$

Here, $g = su$ (where s is semisimple and u is unipotent) is the Jordan decomposition of an element $g \in G^F$. Thus, in the setting of (2.7), we have

$$\chi_{(\Sigma,\mathcal{E})} = q^{(\dim G - \dim \Sigma)/2}\, \hat{\theta}(s_0)\, (\widetilde{Y_{\iota_1}^H})_{s_0}^\theta \qquad \text{where } \iota_1 = (C_1, \psi_1) \in \mathcal{N}_H^F$$

and $\hat{\theta}$ is an extension of θ to an irreducible character of Z_H^F. We shall need the following elementary result.

Lemma 2.9. *In the setting of Definition 2.8, assume that there exists an F-stable closed unipotent subgroup $U \subseteq H$ such that*

$$f = \mathrm{Ind}_{U^F}^{H^F}(\eta) \qquad \textit{where } \eta\colon U^F \to \mathbb{C} \textit{ is a class function.}$$

Let $\mathrm{Irr}(Z_H^F \mid \theta)$ be the set of all $\lambda \in \mathrm{Irr}(Z_H^F)$ whose restriction to $(Z_H^\circ)^F$ equals θ. Then we have

$$\frac{1}{|(Z_H^\circ)^F|}\, \tilde{f}_{s_0}^\theta = \sum_{\lambda \in \mathrm{Irr}(Z_H^F|\theta)} \lambda(s_0^{-1})\, \mathrm{Ind}_{Z_H^F \times U^F}^{G^F}(\lambda \boxtimes \eta).$$

Proof. Using the transitivity of induction, we can write the right-hand side of the above identity in the form

(1) $$\mathrm{Ind}_{H^F}^{G^F}(f') \quad \text{where} \quad f' = \sum_{\lambda \in \mathrm{Irr}(Z_H^F|\theta)} \lambda(s_0^{-1})\, \mathrm{Ind}_{Z_H^F \times U^F}^{H^F}(\lambda \boxtimes \eta).$$

Let $h = su = us \in H^F$ where $s \in H^F$ is semisimple and $u \in H^F$ is unipotent. If su is not H^F-conjugate to an element in $Z_H^F \times U^F$, then we certainly have $f'(h) = 0$. So it remains to consider the case where $s \in Z_H^F$ and $u \in U^F$. Using the formula for the induction of class functions (see [15, p. 64]), it is straightforward to check that

$$\mathrm{Ind}_{Z_H^F \times U^F}^{H^F}(\lambda \boxtimes \eta)(su) = \frac{1}{|Z_H^F|}\,\lambda(s)\,\mathrm{Ind}_{U^F}^{H^F}(\eta)(u). \tag{2}$$

Now let us fix some element $\hat{\theta} \in \mathrm{Irr}(Z_H^F \mid \theta)$. Then we can write any $\lambda \in \mathrm{Irr}(Z_H^F \mid \theta)$ in the form $\hat{\theta} \otimes \bar{\lambda}$ where $\bar{\lambda} \in \mathrm{Irr}(Z_H^F)$ is trivial on $(Z_H^\circ)^F$ and, hence, can be regarded as an irreducible character of $\bar{Z}_H^F := Z_H^F/(Z_H^\circ)^F$. Using the second orthogonality relations for the irreducible characters of $\bar{Z}_H^F$, we obtain that

$$f'(su) = \begin{cases} \dfrac{1}{|(Z_H^\circ)^F|}\theta(z)\, f(u) & \text{if } s = zs_0 \text{ with } z \in Z_H^\circ, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Using once more the formula for the induction of class functions, we find that $\tilde{f}_{s_0} = |(Z_H^\circ)^F|\,\mathrm{Ind}_{H^F}^{G^F}(f')$, as desired. □

Now we can formulate the promised variant of Theorem 2.5 (see also the discussion in [26, §9]).

Theorem 2.10. *Let $(\Sigma, \mathcal{E})$ be an F-stable cuspidal pair as in (2.7). Assume that $s_0 \in Z_G$ or that G has a simply-connected derived subgroup. Furthermore, assume that q is such that Theorem 2.5 holds for the associated cuspidal pair $\iota_1 \in \mathcal{N}_H^F$ where $H = Z_G(s_0)$. Then the class function*

$$|A_G(g_0)|\,\chi_{(\Sigma,\mathcal{E})} \colon G^F \to \mathbb{C}, \qquad \textit{as in } (2.7)(c),$$

is an R-linear combination of $\mathrm{Irr}(G^F)$ where

$$R := \mathbb{Z}[\zeta, \bar{\lambda}(\bar{s}_0)\ (\bar{\lambda} \in \mathrm{Irr}(\bar{Z}_H^F)), Y_{\iota_1}^H(u)\ (u \in C_1^F)] \subseteq \mathbb{C},$$

where $\bar{s}_0$ denotes the image of s_0 in $\bar{Z}_H^F := Z_H^F/(Z_H^\circ)^F$. In particular, R is a subring of $\mathbb{C}$ consisting of cyclotomic integers.

Proof. Let the notation be as in (2.7) so that we can express $\chi_{(\Sigma,\mathcal{E})}$ as in Definition 2.8. Thus, noting that $\dim H - \dim Z_H - \dim C_1 = \dim G - \dim \Sigma$, we have that

$$\chi_{(\Sigma,\mathcal{E})} = q^{(\dim H - \dim Z_H - \dim C_1)/2}\,\hat{\theta}(s_0)\,(\widetilde{Y_{\iota_1}^H})_{s_0}^{\theta} \tag{1}$$

where $\iota_1 = (C_1, \psi_1) \in \mathcal{N}_H^F$ and $u_1 \in C_1^F$. Next, applying Theorem 2.5 to the cuspidal pair ι_1 and noting that $A_G(g_0) \cong A_H(u_1)$, we have

$$|A_G(u_1)|\,q^{(\dim H - \dim Z_H - \dim C_1)/2}\,Y_{\iota_1}^H = \frac{\zeta}{|(Z_G^\circ)^F|}\,\Gamma_{\iota_1}^H$$

where $\zeta \in \mathbb{C}$, $\zeta^4 = 1$. Now we place ourselves in the setting of (2.3) and let $u_1, \ldots, u_d$ be representatives for the H^F-classes contained in C_1^F. Let

$$\eta = \sum_{r=1}^{d} a_r\, Y_{\iota_1}^H(u_r)\,\eta_r, \qquad a_r := |A_H(u_r) : A_H(u_r)^F| \in \mathbb{Z},$$

where η_r is as in (2.3)(a). Then we can write $\Gamma_{\iota_1}^H = \mathrm{Ind}_{U^F}^{H^F}(\eta)$. In particular, we are in a situation where Lemma 2.9 can be applied. This yields

$$|A_G(g_0)|\,\chi_{(\Sigma,\mathcal{E})} = \zeta\,\hat{\theta}(s_0) \sum_{\lambda \in \mathrm{Irr}(Z_H^F \mid \theta)} \lambda(s_0^{-1})\,\mathrm{Ind}_{Z_H^F \times U^F}^{G^F}(\lambda \boxtimes \eta).$$

Now any λ in the above sum is of the form $\hat{\theta}\otimes\bar{\lambda}$ where $\bar{\lambda}\in\operatorname{Irr}(\bar{Z}_H^F)$ is regarded as an irreducible character of Z_H^F which is trivial on $(Z_H^\circ)^F$. Then $\hat{\theta}(s_0)\lambda(s_0^{-1})=\bar{\lambda}(s_0^{-1})$ and we can rewrite the above identity as follows:

$$|A_G(g_0)|\,\chi_{(\Sigma,\mathcal{E})}=\sum_{r=1}^{d}\sum_{\bar{\lambda}\in\operatorname{Irr}(\bar{Z}_H^F)}\xi_{r,\bar{\lambda}}\operatorname{Ind}_{Z_H^F\times U^F}^{G^F}\Big((\hat{\theta}\otimes\bar{\lambda})\boxtimes\eta_r\Big), \tag{2}$$

where the coefficients $\xi_{r,\bar{\lambda}}:=a_r\,\zeta\,\bar{\lambda}(s_0^{-1})\,Y_{\iota_1}^H(u_r)$ lie in the ring R. Expanding the induced characters as linear combinations of $\operatorname{Irr}(G^F)$, we obtain an expression of $|A_G(g_0)|\,\chi_{(\Sigma,\mathcal{E})}$ as required. □

3. The proof of Theorem 1.4

The aim of this section is to present a proof of Theorem 1.4. This will proceed in several steps. First we show in (3.1) that we can reduce to the case of unipotent elements. So it remains to show that the values of any $\chi\in\operatorname{Irr}(G^F)$ at unipotent elements lie in the ring $\mathbb{Z}[\omega_q]$, where ω_q is defined as in Section 1. The case where q is even can be settled rather easily, using generalized Gelfand–Graev characters; see (3.3). In order to deal with the remaining cases, we shall need a basis of the space of unipotently supported class functions on G^F which satisfies additional integrality conditions. This is provided by the generalized Green functions; the required integrality property will be established in Lemmas 3.5 and 3.7. The proof of Theorem 1.4 will then be completed in (3.9).

3.1. Reduction to unipotent elements. Assume that the assertion in Theorem 1.4 holds whenever $g\in G^F$ is unipotent. We show that it then holds in general. Indeed, let $\chi\in\operatorname{Irr}(G^F)$ and $g\in G^F$. We write $g=us=su$ where $s\in G^F$ is semisimple and $u\in G^F$ is unipotent. Let $\varepsilon\in\mathbb{C}$ be a root of unity of order equal to the order of s. We set $H:=Z_G(s)^\circ$. Then H is a closed subgroup invariant under F. We consider the restriction of χ to H^F and write

$$\operatorname{Res}_{H^F}^{G^F}(\chi)=\sum_{\psi\in\operatorname{Irr}(H^F)}m_\psi\,\psi\qquad\text{where } m_\psi\in\mathbb{Z}.$$

Now, we have $s\in H^F$ (see [3, 3.5.1]), $u\in H^F$ (see [8, 13.15(a)]) and, hence, $g\in H^F$. This implies that $\chi(g)=\sum_\psi m_\psi\,\psi(su)$ and so it is enough to show that $\psi(su)\in\mathbb{Z}[\omega_q,\varepsilon]$ for all $\psi\in\operatorname{Irr}(H^F)$.

Now s lies in the center of H. So there exists some $\lambda\in\operatorname{Irr}(Z_H^F)$ such that $\psi(su)=\lambda(s)\psi(u)$. By our choice of ε, we have $\lambda(s)\in\mathbb{Z}[\varepsilon]$. Hence it is enough to show that $\psi(u)\in\mathbb{Z}[\omega_q]$ for all $\psi\in\operatorname{Irr}(H^F)$. Now note that H is a connected reductive group defined over $\mathbb{F}_q$ (see [3, 3.5]). Thus, assuming that Theorem 1.4 holds for unipotent elements, we see that it holds in general.

Let us now turn to character values at unipotent elements. The following result allows us to pass from generalized Gelfand–Graev characters to irreducible characters and vice versa.

Lemma 3.2. *Assume that q satisfies the conditions in* (2.4). *Let*

$$K_1:=\mathbb{Q}(\Gamma_u^G(g)\mid u,g\in G^F\ \textit{unipotent}),$$
$$K_2:=\mathbb{Q}(\chi(g)\mid \chi\in\operatorname{Irr}(G^F),g\in G^F\ \textit{unipotent}).$$

Then we have $K_1=K_2$.

Proof. Since each generalized Gelfand–Graev character is the character of an actual representation of G^F, it is clear that $K_1 \subseteq K_2$. Now let $\chi \in \mathrm{Irr}(G^F)$. We must show that $\chi(g) \in K_1$ for all unipotent $g \in G^F$. Using the notation in (2.4)(c), we write

$$\sum_{\alpha \in A} \frac{\Gamma^G_{u_\beta}(u_\alpha)\,\chi(u_\alpha^{-1})}{|Z_G(u_\alpha)^F|} = \langle \Gamma^G_{u_\beta}, \chi\rangle_{G^F} \in \mathbb{Z} \qquad \text{for } \beta \in A.$$

Since the matrix $(\Gamma^G_{u_\beta}(u_\alpha))_{\alpha,\beta\in A}$ is invertible over K_1, we conclude that $\chi(u_\alpha^{-1}) \in K_1$ for all α, as required. $\square$

3.3. Proof of Theorem 1.4: The case where q is even. Assume that the results in [26] happen to hold in the case where q is a power of 2. (This is likely to be the case if the derived subgroup of G is a product of groups of type A.) Let $\chi \in \mathrm{Irr}(G^F)$. By (3.1), it is enough to show that $\chi(g) \in \mathbb{Z}$ for all unipotent elements $g \in G^F$. Then, by Lemma 3.2, it is sufficient to show that $\Gamma^G_u(g) \in \mathbb{Q}$ for all unipotent $u, g \in G^F$. This is seen as follows. The explicit construction of Γ^G_u and the formula in [26, p. 142] show that $\Gamma^G_u(g)$ can be expressed as a rational linear combination of the values of an irreducible character of the additive group of $\mathbb{F}_q$; see also [17, 3.1.12]. However, since q is a power of 2, these character values are integers and so we have $\Gamma^G_u(g) \in \mathbb{Q}$.

We shall now use the classification of unipotent classes in simple algebraic groups to obtain some more precise information on the groups $A_G(u)$ and the values of a class function Y^G_ι for $\iota \in \mathcal{N}^F_G$.

Lemma 3.4. *Assume that q is a power of a good prime for G. Let C be a unipotent class in G. Then the order of $A_G(u)$ $(u \in C)$ is divisible only by primes which are bad for G or divide Z_G/Z_G° (where Z_G =center of G).*

Proof. First assume that G has a connected center. Let $\pi\colon G \to \bar{G}$, $g \mapsto \bar{g}$ be the adjoint quotient. Then, for any unipotent $u \in G$, we have an induced isomorphism $A_G(u) \cong A_{\bar{G}}(\bar{u})$. Now, we can write $\bar{G} = G_1 \times \cdots \times G_t$ where G_i are simple algebraic groups of adjoint type. Writing $\bar{u} = u_1 \cdots u_t$ with $u_i \in G_i$, we have $A_{\bar{G}}(\bar{u}) \cong A_{G_1}(u_1) \times \cdots \times A_{G_t}(u_t)$. Now the tables in [3, 13.1] show that the order of the group $A_{G_i}(u_i)$ is divisible only by primes which are bad for G_i. It remains to note that the notion of bad primes is compatible with direct products. Thus, the assertion is proved in the case where the center of G is connected.

In the general case, we can embed G as a closed subgroup into a connected reductive group G' with a connected center, such that G and G' have the same derived subgroup (see [21, 8.8]). Now let $u \in G$ be unipotent. Then, as in [11, 3.6], we have a natural surjective map $A_G(u) \to A_{G'}(u)$ whose kernel is given by the image of Z_G/Z_G° in $A_G(u)$. Thus, the order of $A_G(u)$ divides $|Z_G/Z_G^\circ|$ times the order of $A_{G'}(u)$. $\square$

We note that the above result is also proved in [38, III.3.19].

Lemma 3.5. *Assume that q is a power of a good prime for G. Let C be an F-stable unipotent class in G. Then there exists an element $u \in C^F$ such that the following holds. For any F-invariant $\psi \in \mathrm{Irr}(A_G(u))$, we can find an extension $\tilde{\psi}$ such that the class function $Y^G_{(C,\psi)}$ defined in* (2.1)(a) *has values in $\mathbb{Z}[\mu]$, where $\mu \in \mathbb{C}$ is a root of unity of order prime to q. More precisely, for G simple, the order of μ can be specified as follows:*

Type of G	Order of μ
A_{n-1}	a divisor of n prime to q
C_n	1
B_n, D_n	a divisor of 4
E_6, E_7	a divisor of 6
G_2, F_4, E_8	1

Proof. First we show that the proof can be reduced to the case where G is simple of simply-connected type. So let us assume that the above assertions hold in this case. Now assume that G is semisimple of simply-connected type. Then we can write $G = G_1 \times \cdots \times G_t$ where G_i are simple simply-connected groups and F permutes the factors G_i. For each i, let μ_i be a root of unity such that the above assertion holds for any Frobenius map on G_i. Then a standard reduction argument (see, for example, the proof of [21, 3.2]) shows that the values of $Y_{(C,\psi)}^G$ can be chosen to lie in $\mathbb{Z}[\mu_1, \ldots, \mu_t]$. Thus, the above assertions also hold in this case. Next assume that G has a simply-connected derived subgroup. Denote that derived subgroup by G'. We have $G = G'S$ where $S \subseteq G$ is a central torus. Hence the embedding $G' \subseteq G$ induces a surjective homomorphism $A_{G'}(u) \to A_G(u)$ for any unipotent element $u \in G'$. Consequently, the assertions also hold in this case. Finally, consider the general case. Then we can find a connected reductive group $\tilde{G}$ with a simply-connected derived subgroup, and a surjective homomorphism $\pi\colon \tilde{G} \to G$ such that the kernel is a central torus and $\tilde{G}$ and π are defined over $\mathbb{F}_q$; see [21, 8.8] or [27, 0.1]. For any unipotent element $\tilde{u} \in \tilde{G}$, the map π induces an isomorphism $A_{\tilde{G}}(\tilde{u}) \to A_G(\pi(\tilde{u}))$. Hence, the above assertions hold in general.

So it remains to consider the case where G is simple of simply-connected type. We use the classification of unipotent classes for simple algebraic groups and the knowledge of the groups $A_G(u)$ in all cases. The relevant information for groups of classical type can be found in [22]; for exceptional types see the tables in [37]. We have already seen in Lemma 3.4 that the order of $A_G(u)$ is divisible only by primes which are bad for G or which divide Z_G/Z_G°.

We can now settle many cases using Remark 2.2. Assume first that $G = \mathrm{SL}_n$. Then $A_G(u)$ is a cyclic group of order dividing the part n' of n prime to q (see [22, 10.3]). Thus, all irreducible characters of $A_G(u)$ are of degree 1 and we can apply Remark 2.2. The same argument applies to G of type C_n, where the groups $A_G(u)$ are elementary abelian 2-groups (see [22, 10.4]).

Now assume that G is of type B_n or D_n. If G is of type D_4 and F is the triality automorphism, then we have $|A_G(u)| \leqslant 2$ for every unipotent element $u \in G$. So the assertion is easily seen to hold in this case. If F has order 1 or 2 we can identify G with a spin group associated to a nonsingular (split or nonsplit) quadratic form over $\mathbb{F}_q$; see [19, 6.6 and 8.1]. We have a surjective homomorphism $\beta\colon G \to \bar{G}$ where $\bar{G}$ is a special orthogonal group such that $\bar{G}$ and β are defined over $\mathbb{F}_q$ and $\ker(\beta)$ has order 2. Denote the Frobenius map on $\bar{G}$ again by F. In [36, 3.2, 3.3], it is shown that $u \in G^F$ can be chosen such that F acts trivially on $A_{\bar{G}}(\bar{u})$ where $\bar{u} = \beta(u)$. Now, by [22, 14.3], we have a surjective homomorphism $A_G(u) \to A_{\bar{G}}(\bar{u})$ whose kernel is of order 1 or 2 and is given by the image of Z_G. Since F induces the identity on $A_{\bar{G}}(\bar{u})$, an easy computation shows that $F^2 = 1$ on $A_G(u)$. Now, by [22, 14.3], the group $A_G(u)$ either is an elementary abelian 2-group or a central extension of such a group with a kernel of order 2. In any case, the exponent of $A_G(u)$ divides 4. Since F induces the identity on $A_{\bar{G}}(\bar{u})$, a simple computation shows that the exponent of $A_G(u) \rtimes \langle F\rangle$ also divides 4. Hence every irreducible character of that semidirect product can be realized over $\mathbb{Q}(\sqrt{-1})$ (by Brauer's theorem). This yields the entries for type B_n and D_n.

Finally, assume that G is of exceptional type. Let $\pi\colon G \to \bar{G}$ be the adjoint quotient such that $\bar{G}$ and π are defined over $\mathbb{F}_q$. We denote the Frobenius map on $\bar{G}$ again by F. By [1], it is known that $u \in G^F$ can be chosen such that F acts trivially on $A_{\bar{G}}(\bar{u})$ where $\bar{u} = \pi(u)$ and all character values of $A_{\bar{G}}(\bar{u})$ are rational. Furthermore, it is remarked in [37, 5.4] that $A_G(u) \cong \ker(\pi) \times A_{\bar{G}}(\bar{u})$ canonically. Thus, if $\ker(\pi) = \{1\}$, the entry in the above table must be 1. This applies to G of type G_2, F_4 and E_8. It remains to consider G of type E_6 or E_7, where $\ker(\pi)$ has order 3 or 2, respectively. Since F induces the identity on $A_{\bar{G}}(\bar{u})$, we conclude that $F^2 = 1$. Now the tables in [37] show that $A_G(u)$ is a direct product with at most one factor isomorphic to $\mathfrak{S}_3$ and possibly some factors which are cyclic of order 2 or 3. We can apply Remark 2.2 to all linear characters. Furthermore, an F-invariant irreducible character of degree 2 of $A_G(u)$ can be trivially extended to $A_G(u) \rtimes \langle F\rangle$ (since F acts trivially on $A_{\bar{G}}(\bar{u})$ and $A_G(u) \cong \ker(\pi) \times A_{\bar{G}}(\bar{u})$). This yields the entries for E_6 and E_7. $\square$

3.6. Generalized Green functions. Let us now assume that q is such that the main result of [25] holds. Let $L \subseteq G$ be an F-stable closed subgroup which is a Levi complement in some (not necessarily F-stable) parabolic subgroup of G. We consider a class function $Y_{\iota_0}^L\colon L^F \to \mathbb{C}$ as in (2.1)(a), where the pair $\iota_0 \in \mathcal{N}_L^F$ is cuspidal. By [25, 1.14], the corresponding generalized Green function is given by

$$Q_{L,\iota_0}^G(g) := R_L^G(Y_{\iota_0}^L)(g) \quad \text{for all } g \in G^F,$$

where R_L^G denotes the operation of twisted induction defined, for example, in [8, Chap. 11]. The character formula in [25, 1.7] shows that $Q_{L,\iota_0}^G(g) = 0$ unless g is unipotent. Furthermore, by [23, 25.4], the functions Q_{L,ι_0}^G (for various pairs L, ι_0 as above) span the space of all unipotently supported class functions on G^F. In order to select a basis, we introduce the following equivalence relation on the set of pairs (L, ι_0) as above. We write $(L, \iota_0) \sim (L', \iota_0')$ if there exists some $g \in G^F$ such that $L' = gLg^{-1}$ and conjugation by g transforms ι_0 to ι_0'. Then we have the following orthogonality relations:

$$|(Z_L^\circ)^F|\,\langle Q_{L,\iota_0}^G, Q_{L',\iota_0'}^G\rangle_{G^F} = \begin{cases} |N_G(L)^F/L^F|\,q^{-b_{\iota_0}} & \text{if } (L,\iota_0) \sim (L',\iota_0'), \\ 0 & \text{otherwise.}\end{cases}$$

This follows from [23, 9.11 and 25.6.2]. Thus, if we take a set of representatives for the equivalence classes of pairs (L, ι_0), then the functions Q_{L,ι_0}^G form a basis of the space of unipotently supported class functions.

Lemma 3.7. *In the above setting, assume that $\mu \in \mathbb{C}$ is a root of unity such that the values of $Y_{\iota_0}^L$ lie in $\mathbb{Z}[\mu]$. Then we also have*

$$Q_{L,\iota_0}^G(g) \in \mathbb{Z}[\mu] \quad \textit{for all } g \in G^F.$$

Proof. The character formula in [25, 1.7] and the fact that the two-variable Green function involved in that formula is integer valued (see [8, 10.6]) imply that the values of Q_{L,ι_0}^G lie in $\mathbb{Q}(\mu)$. Hence it will be enough to show that the values of Q_{L,ι_0}^G are algebraic integers.

To see this, we argue as follows. Write $\iota_0 = (C_1, \psi_1)$. Then ψ_1 corresponds to a G-equivariant irreducible cuspidal local system $\mathcal{E}_1$ on C_1 which is isomorphic to its inverse image under the Frobenius map. Then $Y_{\iota_0}^L$ is the characteristic function (as in [23, 24.2.3]) corresponding to an isomorphism $\varphi\colon F^*\mathcal{E}_1 \to \mathcal{E}_1$ which induces maps of finite order on the stalks of $\mathcal{E}_1$ at any point in C_1^F. Thus, we are in a setting as in [23, 24.2]. We claim that there exists a root of unity $\varepsilon \in \mathbb{C}$ and integers c_ι such that

$$\text{(a)} \qquad \varepsilon\, R_L^G(Y_{\iota_0}^L)(g) = \sum_{\iota \in \mathcal{N}_G^F} c_\iota\, \chi_{A_\iota,\phi_{A_\iota}}(g) \quad \text{for } g \in G^F \text{ unipotent},$$

where A_ι are the character sheaves defined in [23, 24.1.1] and the isomorphisms $\phi_{A_\iota}\colon F^*A_\iota \to A_\iota$ are chosen as in [23, 24.2.1]. This follows by combining [23, 10.4.5 and 10.6.1] with [25, 1.7]. We have $c_\iota = \mathrm{Tr}(\theta_w \circ \sigma_{A_\iota}, V_{A_\iota})$ in the notation of [23, 10.4.5], and these are integers by the normalisation of ϕ_{A_ι}. The root of unity ε takes into account that the function $Y_{\iota_0}^L$ is only well defined up to multiplication by roots of unity. Thus, (a) follows.

Hence it will be enough to show that the values of $\chi_{A_\iota,\phi_{A_\iota}}$ at unipotent elements are algebraic integers. Now, we have

$$\text{(b)} \qquad \chi_{A_\iota,\phi_\iota}(g) = q^{(a_0+r)/2}\, X_\iota^G(g) \quad \text{for } g \in G^F \text{ unipotent},$$

where X_ι^G is the unipotently supported class function defined in [23, 24.2.8] and the integers a_0 and r are defined in [23, 24.2.2]. We have $a_0 = b_\iota - \dim G$ (see (2.4)) and $r = \dim \mathrm{supp}\ A_\iota = \dim G - b_{\iota_0}$ (using [23, 4.3.1] and the dimension formula in [22, 3.1]). Thus, we have $a_0 + r = b_\iota - b_{\iota_0}$. It is remarked in the proof of [23, 24.8] that $a_0 + r$ is even. In fact, we have that

$$\text{(c)} \qquad a_0 + r = b_\iota - b_{\iota_0} \text{ is even and nonnegative.}$$

Indeed, let $\iota = (C, \psi)$. Since the restriction of A_ι to C is nonzero, we have $C \subseteq \mathrm{supp}\ A_\iota$. But we also have $Z_L^\circ \subseteq \mathrm{supp}\ A_\iota$ (by construction) and so $\dim C + \dim Z_L^\circ \leqslant \dim \mathrm{supp}\ A_\iota$, as claimed.

Now, by (24.2.4), (24.2.9) and (24.5.2) in [23], we know that the values of X_ι^G are algebraic integers. So, using (b) and (c), we conclude that the values of $\chi_{A_\iota,\phi_{A_\iota}}$ at unipotent elements are also algebraic integers, as required. $\square$

Finally, we collect the following classical number-theoretic results.

Remark 3.8. Let $\zeta_p \in \mathbb{C}$ be a primitive pth root of unity for some odd prime p. Then we have

$$\text{(a)} \qquad \sqrt{\delta(p)p} \in \mathbb{Z}[\zeta_p] \qquad \text{where } \delta(p) = (-1)^{(p-1)/2};$$

see [16, I.11.1]. Now let q be a power of p. Then we claim that we also have

$$\text{(b)} \qquad \sqrt{\delta(q)q} \in \mathbb{Z}[\zeta_p] \qquad \text{where } \delta(q) = (-1)^{(q-1)/2}.$$

Indeed, assume first that q is an even power of p. This means that $q \equiv 1 \bmod 4$ and so $\sqrt{\delta(q)q}$ is an integer. Next assume that q is an odd power of p and write $q = p^{2n+1}$. Since then $q \equiv p \bmod 4$, we also have $\sqrt{\delta(q)q} = p^n\sqrt{\delta(p)p} \in \mathbb{Z}[\zeta_p]$, as desired. Finally, let $\mu \in \mathbb{C}$ be a root of unity of order prime to p. Then we have

$$\text{(c)} \qquad \left\{ \begin{array}{rcl} \mathbb{Q}(\sqrt{\delta(q)q}, \mu) \cap \mathbb{Q}(\zeta_p) & = & \mathbb{Q}(\sqrt{\delta(q)q}), \\ \mathbb{Q}(\sqrt{\delta(q)q}, \mu^m) \cap \mathbb{Q}(\mu) & = & \mathbb{Q}(\mu^m) \quad (m \in \mathbb{Z}). \end{array} \right.$$

This follows from (b) and the fact that $\mathbb{Q}(\mu)$ and $\mathbb{Q}(\zeta_p)$ are linearly disjoint field extensions of $\mathbb{Q}$; see [16, I.10.3].

3.9. Completion of the proof of Theorem 1.4. Let $\chi \in \mathrm{Irr}(G^F)$. By (3.1), it is enough to show that $\chi(g) \in \mathbb{Z}[\omega_q]$ for all unipotent $g \in G^F$. By (3.3), we may further assume that q is odd. Let $\flat \subseteq \mathbb{Z}$ be the finite set of prime numbers p such that (a) any p not in $\flat$ is a good prime for G, (b) any p not in $\flat$ is coprime to the order of the Weyl group of G, and (c) the results in [25] and [26] hold for G whenever q is a power of a prime which is not in $\flat$. (Note that, indeed, this set is finite and it only depends on the Dynkin diagram of G.)

Now assume that q is a power of a prime p which is not in $\flat$. Let $\mathbb{Z}_{(p)} \subseteq \mathbb{Q}$ be the local ring with maximal ideal generated by p. We claim that there exists a root of unity $\mu \in \mathbb{C}$ of order prime to p such that

$$(*) \qquad \chi(g) \in \mathbb{Z}_{(p)}[\sqrt{q}, \mu] \quad \text{for all unipotent } g \in G^F.$$

This is seen as follows. Consider the set of pairs (L, ι_0) as in (3.6). By Lemma 3.5, we can normalize the functions $Y_{\iota_0}^L$ such that their values lie in $\mathbb{Z}[\mu]$, where $\mu \in \mathbb{C}$ is a root of unity whose order is divisible by 4 and prime to p. Since the corresponding generalized Green functions form a basis of the space of unipotently supported class functions on G^F, there exist $\xi_{(L,\iota_0)} \in \mathbb{C}$ such that

$$\chi(g) = \sum_{(L,\iota_0)} \xi_{(L,\iota_0)}\, q^{b_{\iota_0}/2}\, Q_{L,\iota_0}^G(g) \quad \text{for all unipotent } g \in G^F,$$

where (L, ι_0) runs over a set of representatives of the equivalence classes of pairs as in (3.6). Now fix such a pair (L, ι_0) and consider the inner product of $q^{b_{\iota_0}/2}\, Q_{L,\iota_0}^G$ with χ. The formula in (3.6) yields that

$$|(Z_L^\circ)^F|\, \langle q^{b_{\iota_0}/2}\, Q_{L,\iota_0}^G, \chi\rangle_{G^F} = \bar{\xi}_{(L,\iota_0)}\, |N_G(L)^F/L^F|,$$

where the bar denotes complex conjugation. The group $N_G(L)/L$ is a subquotient of the Weyl group of G and, hence, its order is prime to p. The same also holds for the order of $(Z_L^\circ)^F$. Finally, by Theorem 2.5, Lemma 3.4 and (2.3)(b), the class function $q^{b_{\iota_0}/2}\, Y_{\iota_0}^L$ is a $\mathbb{Z}_{(p)}[\mu]$-linear combination of $\mathrm{Irr}(L^F)$. Since R_L^G takes generalized characters of L^F to generalized characters of G^F, we conclude that the class function $q^{b_{\iota_0}/2}\, Q_{L,\iota_0}^G$ is a $\mathbb{Z}_{(p)}[\mu]$-linear combination of $\mathrm{Irr}(G^F)$. Hence the inner product of that class function with χ lies in $\mathbb{Z}_{(p)}[\mu]$. Combining these facts, we obtain $\xi_{(L,\iota_0)} \in \mathbb{Z}_{(p)}[\mu]$; so $(*)$ follows from Lemma 3.7.

Now we can argue as follows. If q is a square, then $(*)$ implies $\chi(g) \in \mathbb{Z}[\mu]$. Now, since g is unipotent, we certainly have that $\chi(g) \in \mathbb{Z}[\zeta']$ where $\zeta' \in \mathbb{C}$ is a root of unity whose order is a power of p. Using Remark 3.8(c) and the fact that character values are algebraic integers, we conclude that $\chi(g) \in \mathbb{Z}$, as required.

Now assume that q is not a square. Since the order of μ is divisible by 4 and q is odd, we have $\mathbb{Z}_{(p)}[\sqrt{q}, \mu] = \mathbb{Z}_{(p)}[\sqrt{\delta(q)q}, \mu]$ where $\delta(q) = (-1)^{(q-1)/2}$. As before, since g is unipotent, we certainly have that $\chi(g) \in \mathbb{Z}[\zeta']$ where $\zeta' \in \mathbb{C}$ is a root of unity whose order is a power of p. By Remark 3.8(b), we have $\sqrt{\delta(q)q} \in \mathbb{Z}[\zeta']$. Combining this with $(*)$ and the fact that $\mathbb{Q}(\zeta')$ and $\mathbb{Q}(\mu)$ are linearly disjoint, we conclude that

$$\chi(g) \in \mathbb{Z}_{(p)}[\sqrt{\delta(q)q}] \quad \text{for all unipotent } g \in G^F.$$

Now let us write $q = p^{2n+1}$ where n is a nonnegative integer. Then $\sqrt{\delta(q)q} = p^n\sqrt{\delta(p)p}$ generates an extension of degree 2 over $\mathbb{Q}$ and the above statement means that we can write uniquely $\chi(g) = x + yp^n\sqrt{\delta(p)p}$ where $x, y \in \mathbb{Z}_{(p)}$. On the other hand, $\chi(g)$ is an algebraic integer in $\mathbb{Q}(\sqrt{\delta(p)p})$ and, hence, can be written uniquely as

$$(\dagger) \qquad \chi(g) = u + \frac{1}{2}v(1 + \sqrt{\delta(p)p}) \quad \text{where } u, v \in \mathbb{Z};$$

see [16, I.9.2]. Comparing this with the expression $\chi(g) = x + yp^n\sqrt{\delta(p)p}$, we deduce that $2u + v = 2x$ and $v = 2yp^n$. Since $p \neq 2$, this forces that p^n divides v. Inserting this into (†), we obtain $\chi(g) \in \mathbb{Z}[\omega_q]$, as required. This completes the proof of Theorem 1.4.

We close with the following remark. Assume that q is such that the statement in Theorem 1.4 holds. Then Fong's Theorem [15, 10.13] immediately implies that every $\chi \in \mathrm{Irr}(G^F)$ can be realized over $\mathbb{Q}(\omega_q, \varepsilon_0)$ (if q is odd) or $\mathbb{Q}(\sqrt{-1}, \varepsilon_0)$ (if q is even), where $\varepsilon_0 \in \mathbb{C}$ is a root of unity whose order is the least common multiple of the orders of all semisimple elements in G^F.

4. Character sheaves and almost characters

We shall now apply the results of Section 2 to the problem (formulated by Lusztig [23]) of relating characteristic functions of cuspidal character sheaves and almost characters of G^F. This will also be used in Section 6 where we consider Schur indices of unipotent characters of groups of exceptional type.

4.1. Almost characters. Assume that the center of G is connected. Then the *almost characters* defined in [21, 4.24.1] form an orthonormal basis of the space of class functions on G^F; denote that basis by $\{R_x \mid x \in X_0\}$. By definition, the transition matrix from this new basis to the basis consisting of $\mathrm{Irr}(G^F)$ is explicitly known; it involves the entries of certain nonabelian Fourier transforms. We don't need to recall the precise definition. However, we note that

$$\text{(a)} \qquad \text{each } R_x \text{ is a } \mathbb{Q}(\alpha)\text{-linear combination of } \mathrm{Irr}(G^F),$$

where $\alpha \in \mathbb{C}$ is a root of unity whose order is divisible by bad primes only. Now assume that q is a power of a good prime for G. Then Shoji [34], [35] has established Lusztig's conjecture [23] concerning characteristic functions of character sheaves and almost characters. In particular, this means that for any F-stable cuspidal pair $(\Sigma, \mathcal{E})$ in G with corresponding characteristic function as in (2.7)(a), there exists some $x \in X_0$ such that

$$\text{(b)} \qquad \chi_{(\Sigma,\mathcal{E})} = \xi\, R_x \text{ where } \xi \in \mathbb{C} \text{ has absolute value } 1.$$

See also [27], where cuspidal character sheaves are considered without any restriction on the center of G (but with q "large").

Proposition 4.2. *Assume that q is as in* (2.5)*. Furthermore, assume that G has a connected center and a simple derived subgroup, of classical type. Let $(\Sigma, \mathcal{E})$ be an F-stable cuspidal pair such that $\chi_{(\Sigma,\mathcal{E})} = \xi R_x$ as in* (4.1)(b)*. Then $\chi_{(\Sigma,\mathcal{E})}$ can be normalized such that $\xi^4 = 1$ and $\chi_{(\Sigma,\mathcal{E})}(g) = \pm 1$ for all $g \in \Sigma^F$.*

Proof. The whole line of reasoning is inspired from a similar argument in the proof of [27, Theorem 0.8]. First we show that we may assume, without loss of generality, that the derived subgroup of G is simply-connected. Indeed, if this is not the case, we can find a connected reductive group $\tilde{G}$ with a simply-connected derived subgroup, and a surjective homomorphism $\pi \colon \tilde{G} \to G$ such that the kernel is a central torus and $\tilde{G}$ and π are defined over $\mathbb{F}_q$; see [21, 8.8] or [27, 0.1]. Let $\tilde{\Sigma} \subseteq \tilde{G}$ be the inverse image of Σ and $\tilde{\mathcal{E}}$ the pull-back of $\mathcal{E}$ under π. Then, by [22, 2.5], $(\tilde{\Sigma}, \tilde{\mathcal{E}})$ is a cuspidal pair for $\tilde{G}$. The pull-back of $\chi_{(\Sigma,\mathcal{E})}$ under the restriction of π to $\tilde{G}^F$ is a characteristic function for $(\tilde{\Sigma}, \tilde{\mathcal{E}})$. A similar relation also holds for almost characters; see [21, 8.8]. Hence it is enough to prove the assertion in the case where G has a simply-connected derived subgroup. We now claim that, for a suitable normalisation of $\chi_{(\Sigma,\mathcal{E})}$, we have

(1) $\qquad R \subseteq \mathbb{Z}[\sqrt{-1}] \qquad$ where R is the ring in Theorem 2.10.

This is seen as follows. We use the notation and set-up in (2.7). Let $H = Z_G(s_0)$. By [23, 23.2], we know that H is a product of at most two groups of classical type and Z_H/Z_H° has exponent at most 2. First of all, this implies that the function $Y_{\iota_1}^H$ in (2.7)(c) can be normalized so that the values are ± 1; see Remark 2.2. Next, we certainly have $\lambda(s_0) = \pm 1$ for all $\lambda \in \mathrm{Irr}(Z_H^F)$ which are trivial on $(Z_H^\circ)^F$. Thus, (1) is proved.

Furthermore, since H is a product of groups of classical type, we know that $A_G(g_0) \cong A_H(u_1)$ is a 2-group. Now Theorem 2.10 shows that there exists some $a \geqslant 0$ such that

(2) $\qquad 2^a\, \chi_{(\Sigma,\mathcal{E})}$ is a $\mathbb{Z}[\sqrt{-1}]$-linear combination of $\mathrm{Irr}(G^F)$.

On the other hand, by the definition of almost characters and the explicit form of the Fourier transform for classical groups (see [21, Chap. 4]), we know that R_x is a rational linear combination of $\mathrm{Irr}(G^F)$, where the coefficients are of the form $\pm 1/2^b$ for some $b \geqslant 0$. Combining this with (2) yields that

(3) $\qquad \xi = 2^{b-a}(r + s\sqrt{-1}) \qquad$ for some $r, s \in \mathbb{Z}$.

Now we can argue as follows. We know that the absolute value of ξ is 1. Hence we must have $r^2 + s^2 = 4^{(a-b)}$. It is easily checked that this implies that $r = 0$ or $s = 0$. Consequently, we must have $\xi \in \{\pm 1, \pm\sqrt{-1}\}$. $\square$

Our next aim is to prove an analogue of Proposition 4.2 for groups of exceptional type. This will require some detailed information from the classification of cuspidal character sheaves in these groups. A major part of this information is contained in Table 2.

4.3. Remarks concerning Table 2. Assume that G has a connected center and that the derived subgroup is simple, simply-connected of exceptional type. Assume also that q is a power of a good prime. Let $(\Sigma, \mathcal{E})$ be a cuspidal pair in G. Then it is known that $\mathcal{E}$ is a local system of rank 1 in all cases; see [34, 6.2.5] for type G_2; [34, 6.2.4] for type F_4; [23, Proof of (20.3)] for types E_6, E_7; [35, 4.7] for type E_8 (plus the reduction arguments in [23, 17.9, 17.10]). Now we place ourselves in the setting of (2.7). Let $g_0 \in \Sigma$ and write $g_0 = s_0 u_1$ where $s_0 \in G$ is semisimple, $u_1 \in G$ is unipotent. Let $H = Z_G(s_0)$ and let C_1 be the conjugacy class of u_1 in H. Then $\mathcal{E}$ corresponds to a cuspidal local system $\mathcal{E}_1$ on C_1 and, hence, to some $\psi_1 \in \mathrm{Irr}(A_H(u_1))$. Since $\mathcal{E}$ has rank 1, we conclude that ψ_1 is a linear character. Then the above references also show that

(a) $\qquad$ the order of ψ_1 and the exponent of Z_H/Z_H° divide n_0,

where $n_0 \geqslant 1$ denotes the order of the image of s_0 in G/Z_G. The possibilities for $(\Sigma, \mathcal{E})$ are listed in Table 2. The second column describes the type of $Z_G(s_0)$, the

TABLE 2. Cuspidal pairs in exceptional algebraic groups; cf. (4.3)

TYPE OF G	$Z_G(s_0)$	$A_G(g_0)$	$\operatorname{codim}\Sigma$	n_0	$(x,\sigma)\in\mathcal{M}(\mathcal{G})$
G_2 ($\mathcal{G}=\mathfrak{S}_3$)	G_2	$\mathfrak{S}_3$	4	1	$(1,\varepsilon)$
	$A_1{\times}A_1$	$\mathbb{Z}/2$	2	2	(g_2,ε)
	A_2	$\mathbb{Z}/3$	2	3	$(g_3,\theta),(g_3,\theta^2)$
F_4 ($\mathcal{G}=\mathfrak{S}_4$)	F_4	$\mathfrak{S}_4$	12	1	$(1,\lambda^3)$
	B_4	$\mathrm{Dih}(8)$	8	2	(g_2',ε)
	$A_1{\times}C_3$	$\mathbb{Z}/2$	6	2	(g_2,ε)
	$A_1{\times}A_3$	$\mathbb{Z}/4$	4	4	$(g_4,i),(g_4,-i)$
	$A_2{\times}A_2$	$\mathbb{Z}/3$	4	3	$(g_3,\theta),(g_3,\theta^2)$
E_6 ($\mathcal{G}=\mathfrak{S}_3$)	$A_2{\times}A_2{\times}A_2$	$\mathbb{Z}/3$	6	3	$(g_3,\theta),(g_3,\theta^2)$
E_7 ($\mathcal{G}=\mathfrak{S}_2$)	$A_1{\times}A_3{\times}A_3$	$\mathbb{Z}/4$	7	4	$(g_2,1),(g_2,\varepsilon)$
E_8 ($\mathcal{G}=\mathfrak{S}_5$)	E_8	$\mathfrak{S}_5$	40	1	$(1,\lambda^4)$
	$A_1{\times}E_7$	$\mathbb{Z}/2$	22	2	$(g_2,-\varepsilon)$
	D_8	$\mathrm{Dih}(8)$	20	2	(g_2',ε)
	$A_2{\times}E_6$	$\mathbb{Z}/3$	14	3	$(g_3,\varepsilon\theta),(g_3,\varepsilon\theta^2)$
	$A_3{\times}D_5$	$\mathbb{Z}/4$	10	4	$(g_4,i),(g_4,-i)$
	$A_1{\times}A_2{\times}A_5$	$\mathbb{Z}/6$	8	6	$(g_6,-\theta),(g_6,-\theta^2)$
	$A_4{\times}A_4$	$\mathbb{Z}/5$	8	5	(g_5,ζ^a) $(1\leqslant a\leqslant 4)$

third column gives the isomorphism type of $A_G(g_0)$ $(g_0\in\Sigma)$, the fourth column gives $\operatorname{codim}\Sigma=\dim G-\dim\Sigma$, and the fifth column specifies the number n_0. This information is extracted from the results of Lusztig [23, §20, §21], [22, 15.6], and Shoji [34, §6], [35, §4]. (Note that the order of $A_G(g_0)$ in type E_6 and E_7 is as shown since we assume that G has a connected center *and* a simply-connected derived subgroup. I also thank Frank Lübeck for independently checking these assertions.)

We now explain the information in the remaining columns. Let G^* be the Langlands dual of G. Then, by [27, 1.2], one can associate with $(\Sigma,\mathcal{E})$ a well-defined semisimple element $s\in G^*$ (up to conjugacy) and a family $\mathcal{F}$ of irreducible characters of W_s, where W_s denotes the Weyl group of $Z_{G^*}(s)$. We have $s\in Z_{G^*}$ in all cases and so $W_s\cong W$ (the Weyl group of G); furthermore, all cuspidal pairs belong to a fixed $\mathcal{F}$. Let $\mathcal{G}_{\mathcal{F}}$ be the finite group associated with $\mathcal{F}$ as in [21, 4.14]. Then $(\Sigma,\mathcal{E})$ corresponds to a pair (x,σ) in the set $\mathcal{M}(\mathcal{G}_{\mathcal{F}})$ under a bijection as in [23, Theorem 23.1], where $\mathcal{M}(\mathcal{G}_{\mathcal{F}})$ is defined in [21, 4.3]. The group $\mathcal{G}=\mathcal{G}_{\mathcal{F}}$ is specified in the first column of Table 2 and the pairs $(x,\sigma)\in\mathcal{M}(\mathcal{G})$ in the last column.

Now assume that Σ is F-stable and $g_0\in\Sigma^F$. Then we can evaluate $\chi_{(\Sigma,\mathcal{E})}$ as in (2.7)(c). The fact that ψ_1 is linear implies that we can evaluate the class function $Y^H_{(C_1,\psi_1)}$ in a very simple way, using an extension of ψ_1 as in Remark 2.2. Thus, in the special case where the central character θ is trivial, $\chi_{(\Sigma,\mathcal{E})}$ can be normalized such that

(b) $$\chi_{(\Sigma,\mathcal{E})}(zs_0u_y)=q^{(\dim G-\dim\Sigma)/2}\,\psi_1(y)\quad\text{for } z\in Z_G^F,\ y\in A_H(u_1);$$

here $u_y \in C_1^F$ is obtained by twisting $u_1 \in H^F$ with $y \in A_H(u_1)$; see (2.1)(a) and (2.7)(c).

We have remarked in (4.3) that every cuspidal pair $(\Sigma, \mathcal{E})$ corresponds to a semisimple element $s \in G^*$. A similar result also holds for the irreducible characters of G^F which are partitioned into "Lusztig series"; see [21, 8.4.4]. The transformation from $\mathrm{Irr}(G^F)$ to the basis $\{R_x \mid x \in X_0\}$ consisting of almost characters takes place inside the various "Lusztig series"; see [21, Chap. 4]. Now the statement in (4.1)(b) implicitly contains the assertion that, if we have $\chi_{(\Sigma,\mathcal{E})} = \xi R_x$ for some $\xi \in \mathbb{C}$, then $(\Sigma, \mathcal{E})$ and R_x are associated with the same $s \in G^*$ (up to conjugacy).

Now, in the case of exceptional groups, any cuspidal pair $(\Sigma, \mathcal{E})$ lies in a series where $s \in G^*$ is central. The reduction arguments in [23, 17.10] then show that it is actually enough to deal with the case where $s = 1$.

Proposition 4.4. *Assume that q is as in* (2.5)*; see also Remark 4.5 below. Furthermore, assume that G has a connected center and a simple derived subgroup of exceptional type. Let $(\Sigma, \mathcal{E})$ be an F-stable cuspidal pair such that $\chi_{(\Sigma,\mathcal{E})} = \xi R_x$ as in* (4.1)(b)*, where R_x is a combination of unipotent characters. Then $\chi_{(\Sigma,\mathcal{E})}$ can be normalized such that*

$$\xi^{n_0} = \pm 1 \quad \textit{and} \quad \chi_{(\Sigma,\mathcal{E})}(g) \in \mathbb{Z}[\zeta_{n_0}, q^{(\dim G - \dim \Sigma)/2}] \textit{ for all } g \in G^F,$$

where $n_0 \geqslant 1$ denotes the order of the image in G/Z_G of the semisimple part of an element in Σ and $\zeta_{n_0} \in \mathbb{C}$ is a primitive root of unity of order n_0.

Proof. As in the proof of Proposition 4.2, we may assume that the derived subgroup of G is simply-connected. (In type G_2, F_4 or E_8, we may even assume that $Z_G = \{1\}$.) First we claim that

(1) $\qquad \xi \in \mathbb{Q}(\varepsilon) \quad \text{and} \quad |A_G(g_0)|\, \chi_{(\Sigma,\mathcal{E})}$ is a $\mathbb{Z}[\varepsilon]$-combination of $\mathrm{Irr}(G^F)$,

where $\varepsilon \in \mathbb{C}$ is a root of unity of order prime to q. Indeed, consider the ring R in Theorem 2.10. Using Lemma 3.5, we see that R is generated over $\mathbb{Z}$ by roots of unity of order prime to q (note that 2 is a bad prime). By the definition in [21, Chap. 4], R_x can also be expressed as a linear combination of $\mathrm{Irr}(G^F)$, where the coefficients are rational combinations of roots of unity of order prime to q (note that q is a power of a good prime). Thus, the above claim follows from the identity $\chi_{(\Sigma,\mathcal{E})} = \xi R_x$.

Now let $g_0 \in \Sigma^F$ and write $g_0 = s_0 u_1$ where $s_0 \in G^F$ is semisimple and $u_1 \in G^F$ is unipotent. Furthermore, let p be the prime such that $q = p^f$. We claim that

(2) $\qquad R_x(g_0) \in \mathbb{Q}(\zeta', \zeta_{n_0}), \quad \text{where } (\zeta')^{p^a} = 1 \text{ for some } a \geqslant 1.$

This is seen as follows. Let $\pi\colon G \to \bar{G}$ be the adjoint quotient of G, such that $\bar{G}$ and π are defined over $\mathbb{F}_q$. We have $G/Z_G \cong \bar{G}$ as abstract groups. We denote the Frobenius map on $\bar{G}$ again by F. Now, by [20, Prop. 3.15], the restriction of an (irreducible) unipotent character of $\bar{G}^F$ to $\pi(G^F)$ is irreducible and the lift of this character to G^F is a unipotent; furthermore, all unipotent characters of G^F are obtained in this way. Thus, we have $\chi(g_0) = \bar{\chi}(\pi(g_0))$ for a unipotent character $\bar{\chi} \in \mathrm{Irr}(\bar{G}^F)$. Since $g_0 = s_0 u_1$, we clearly have $\bar{\chi}(\pi(g_0)) \in \mathbb{Q}(\zeta', \zeta_{n_0})$ where $\zeta' \in \mathbb{C}$ is a root of unity whose order is a power of p. Thus, we see that $\chi(g_0) \in \mathbb{Q}(\zeta', \zeta_{n_0})$. Now, as in (4.3), the pair $(\Sigma, \mathcal{E})$ is associated with a family $\mathcal{F} \subseteq \mathrm{Irr}(W)$ and a finite group $\mathcal{G}_{\mathcal{F}}$. If $\mathcal{G}_{\mathcal{F}}$ is isomorphic to $\mathfrak{S}_2$, $\mathfrak{S}_3$ or $\mathfrak{S}_4$, then the Fourier matrix associated with $\mathcal{G}_{\mathcal{F}}$ has rational entries; see [3, 13.6]. Hence we also have $R_x(g_0) \in \mathbb{Q}(\zeta', \zeta_{n_0})$.

It remains to consider the case where $\mathcal{G}_\mathcal{F} \cong \mathfrak{S}_5$. This only occurs in type E_8 and the entries of the corresponding Fourier matrix lie in $\mathbb{Q}(\zeta_5)$, where ζ_5 is a primitive 5th root of unity. However, all coefficients are actually rational except those which correspond to the pairs $(x,\sigma) \in \mathcal{M}(\mathcal{F}_\mathcal{G})$ where x has order 5. But in these latter cases, we also have $n_0 = 5$ (see Table 2). Thus, (2) is proved.

Now we continue as follows. Our hypotheses imply that R_x is a linear combination of unipotent characters. Hence, since all these have Z_G^F in their kernel (see [20, Prop. 3.15]), the central character θ is trivial. So, by (4.3)(b), we have $\chi_{(\Sigma,\mathcal{E})}(g_0) \in \mathbb{Z}[q^{(\dim G-\dim\Sigma)/2}, \zeta_{n_0}]$. Now $\dim G - \dim\Sigma$ is even except in type E_7. But in type E_7, we have $n_0 = 4$ (see Table 2). Hence we have $\chi_{(\Sigma,\mathcal{E})}(g_0) \in \mathbb{Q}(\sqrt{\delta(q)q}, \zeta_{n_0}) \subseteq \mathbb{Q}(\zeta', \zeta_{n_0})$ in all cases (see also Remark 3.8). The identity $\chi_{(\Sigma,\mathcal{E})}(g_0) = \xi R_x(g_0)$ and (2) now yield $\xi \in \mathbb{Q}(\zeta', \zeta_{n_0})$. Combining this with (1) and using once more Remark 3.8(c), we obtain

$$\xi \in \mathbb{Q}(\zeta_{n_0}). \tag{3}$$

Now we go through the various possibilities for n_0. If $n_0 \in \{1,2\}$, then $\xi \in \mathbb{Q}$. Since ξ has absolute value 1, this implies $\xi = \pm 1$, as required. Now assume that $n_0 = 4$. Then, using the information in Table 2, we see that $A_G(g_0)$ is a 2-group of order 2^a say, and the ring R in Theorem 2.10 is contained in $\mathbb{Z}[\sqrt{-1}]$. So $2^a\chi_{(\Sigma,\mathcal{E})}$ is a $\mathbb{Z}[\sqrt{-1}]$-linear combination of $\mathrm{Irr}(G^F)$. On the other hand, using the Fourier matrices in [3, 13.6] (where we only need to consider the coefficients in rows or columns corresponding to pairs (x,σ) as in Table 2), we see that 2^bR_x (for some $b \geqslant 0$) is a $\mathbb{Z}$-linear combination of unipotent characters of G^F, where some coefficient is ± 1. Thus, we have $\xi = 2^{b-a}(r + s\sqrt{-1})$ for some $r, s \in \mathbb{Z}$. Arguing as in the proof of Proposition 4.2, we see that $\xi^4 = 1$, as required.

Next, assume that $n_0 \in \{3,6\}$. Then, using once more the information in Table 2, we see that the group $A_G(g_0)$ has order 3 or 6 in all these cases. On the other hand, using the explicit form of the Fourier matrices, we see that $6R_x$ is a $\mathbb{Z}$-linear combination of unipotent characters of G^F, where some coefficients will be ± 1 or ± 2. Thus, using (1), we conclude that $2\xi \in \mathbb{Z}[\varepsilon]$, that is, 2ξ is an algebraic integer. But the ring of algebraic integers in $\mathbb{Q}(\zeta_{n_0})$ is $\mathbb{Z}[\zeta_{n_0}]$ (see [16, I.10.4]). So (3) yields that $2\xi \in \mathbb{Z}[\zeta_{n_0}]$. Now, since $n_0 \in \{3,6\}$, we can check that $\pm\xi$ must be a power of ζ_{n_0}. (Indeed, $\mathbb{Z}[\zeta_{n_0}]$ has $\mathbb{Z}$-basis $1, \zeta_{n_0}$ and we have $\zeta_{n_0}^2 \pm \zeta_{n_0} + 1 = 0$. So, writing $2\xi = r + \zeta_{n_0}s$ with $r, s \in \mathbb{Z}$, the absolute value of 2ξ is 4 on the one hand and $r^2 + s^2 \pm rs$ on the other hand. A simple verification shows $r = 0$ or $s = 0$, as required.)

Finally, assume that $n_0 = 5$. Then, by the explicit form of the coefficients in the Fourier matrix corresponding to the pairs occurring in Table 2, we see that $5R_x$ is a $\mathbb{Z}[\zeta_5]$-linear combination of unipotent characters, where some coefficient is ± 1. On the other hand, we have $|A_G(g_0)| = 5$. It follows that $\xi \in \mathbb{Z}[\varepsilon]$ and so ξ is an algebraic integer in $\mathbb{Q}(\zeta_5)$. But an algebraic integer in $\mathbb{Z}[\zeta_5]$ of absolute value 1 is easily seen to be a power of ζ_5 (up to sign). $\square$

Remark 4.5. The assumption on q in Proposition 4.4 comes (via Theorem 2.10) from the condition that the formula in Theorem 2.5 should hold for a cuspidal pair $(C_1, \psi_1) \in \mathcal{N}_H$ where $H = Z_G(s_0)$ and s_0 is the semisimple part of an element in Σ. Now assume that H is a product of groups of type A. Then, by [22, 10.3], C_1 is the class of regular unipotent elements in H. So Theorem 2.5 holds in this case without any assumption on q, as pointed out in Remark 2.6. This applies to all cuspidal pairs in type E_6 or E_7 and to some cuspidal pairs in type G_2, F_4 or E_8. Furthermore, in type F_4, the generalized Gelfand–Graev characters for the relevant subgroups H (in good characteristic) have been computed explicitly by Wings [42], and these computations show that Theorem 2.5 holds.

Example 4.6. Let G be of type E_6 where F is of split or nonsplit type. There are two nonrational cuspidal unipotent characters denoted by ${}^{\delta}E_6[\theta]$ and ${}^{\delta}E_6[\theta^2]$; see Table 1. Now consider the two unipotent almost characters corresponding to the F-stable cuspidal pairs in G; by Table 2, they are associated with the pairs (g_3, θ) and (g_3, θ^2) in $\mathcal{M}(\mathfrak{S}_3)$. Using the entries of the corresponding Fourier matrix, we find that

$$R_{(g_3,\theta)} - R_{(g_3,\theta^2)} = \pm({}^{\delta}E_6[\theta] - {}^{\delta}E_6[\theta^2]).$$

Now, by Proposition 4.4, the values of $R_{(g_3,\theta)}$ and $R_{(g_3,\theta^2)}$ lie in the field $\mathbb{Q}(\theta)$, which also is the character field of ${}^{\delta}E_6[\theta]$ and of ${}^{\delta}E_6[\theta^2]$ by Table 1. Furthermore, ${}^{\delta}E_6[\theta] - {}^{\delta}E_6[\theta^2]$ cannot be rational-valued since ${}^{\delta}E_6[\theta] + {}^{\delta}E_6[\theta^2]$ is rational-valued. We have thus shown that some character values of ${}^{\delta}E_6[\theta]$ and ${}^{\delta}E_6[\theta^2]$ on elements in Σ^F involve θ. Similar arguments also work for all the nonrational unipotent characters in Table 1.

5. Character fields of unipotent characters

The aim of this section is to determine the character fields for unipotent characters of G^F, without any restriction on q. The proof uses the results on the eigenvalues of Frobenius obtained by Digne–Michel [7] and Lusztig [18]. We will recall the necessary background in (5.2) and (5.3). The proof for the entries concerning $\mathbb{Q}(\chi)$ in Table 1 will then be given in (5.4). Finally, we consider noncuspidal unipotent characters in Proposition 5.6.

Recall that $\chi \in \mathrm{Irr}(G^F)$ is called unipotent if χ occurs with nonzero multiplicity in some Deligne–Lusztig generalized character $R_{T,1}$; see [8, 13.19], for example. First note that unipotent characters are "insensitive" to the center of G (see [20, Prop. 3.15]). So it is enough to deal with the case where G has a simple derived subgroup. We begin with the case of cuspidal unipotent characters. As in [21, Appendix] or [3, 13.7], they will be denoted by ${}^{\delta}X[\alpha]$, where X stands for the type of G, δ denotes the order of the induced action of F on the Weyl group of G and α is a certain complex number. (If $\delta = 1$, we also write $X[\alpha]$.)

5.1. Character fields: First steps. Let $\chi \in \mathrm{Irr}(G^F)$ be cuspidal unipotent and $E \supseteq \mathbb{Q}(\chi)$ be a finite Galois extension such that χ can be realized over E. Then, for any $\tau \in \mathrm{Gal}(E/\mathbb{Q}(\chi))$, we also have $\chi^\tau \in \mathrm{Irr}(G^F)$ where χ^τ is the class function defined by $\chi^\tau(g) := \tau(\chi(g))$ for $g \in G^F$; see [15, 9.16]. The characterisation of cuspidal characters in [3, 9.1.2] shows that χ^τ also is a cuspidal character. Furthermore, the character formula in [3, 7.2.8] and the rationality of Green functions (see [3, 7.6]) imply that the Deligne–Lusztig generalized characters $R_{T,1}$ are rational-valued. Thus, χ and χ^τ have the same multiplicity in any $R_{T,1}$. In particular, χ^τ is also unipotent. This discussion implies the following criterion. Assume that there are $e \geqslant 1$ cuspidal unipotent characters of G^F of the same degree as χ. Then $[\mathbb{Q}(\chi) : \mathbb{Q}] \leqslant e$. In particular, if χ is the unique cuspidal unipotent character of G^F, then $\mathbb{Q}(\chi) = \mathbb{Q}$.

First of all, these remarks show that $\mathbb{Q}(\chi) = \mathbb{Q}$ if G is of classical type. (Indeed, by [19, 8.11], there is at most one cuspidal unipotent character.) Now consider a group G of exceptional type. Going through the tables in [3, 13.7], a simple verification yields that $e=1$ and, hence, $\mathbb{Q}(\chi)=\mathbb{Q}$ for

$$^3D_4[\pm 1], G_2[\pm 1], F_4^I[1], F_4^{II}[1], F_4[-1], {}^2E_6[1], E_8^I[1], E_8^{II}[1], E_8[-1].$$

In the remaining cases, we have $[\mathbb{Q}(\chi):\mathbb{Q}] \leqslant 2$ for

$$F_4[\pm i], E_8[\pm i], E_7[\pm\xi], G_2[\theta^a], F_4[\theta^a], E_6[\theta^a], {}^2E_6[\theta^a], E_8[\pm\theta^a] \; (a=1,2),$$

and $[\mathbb{Q}(\chi):\mathbb{Q}] \leqslant 4$ for $E_8[\zeta^a]$ $(1 \leqslant a \leqslant 4)$.

In order to find out in which extension of $\mathbb{Q}$ the characters values of χ lie, we use the results on the eigenvalues of Frobenius obtained by Digne–Michel [7] and Lusztig [18]. Let W be the Weyl group of G (with respect to some F-stable maximal torus which is contained in some F-stable Borel subgroup of G). For any $w \in W$, we have a corresponding Deligne–Lusztig variety X_w. Then the ℓ-adic cohomology spaces $H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$ (where ℓ is a prime not dividing q) are equipped with commuting actions of G^F and F^δ. Choosing an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, we may regard these cohomology spaces as $\mathbb{C}G^F$-modules.

5.2. Coxeter orbits. Let $w \in W$ be a Coxeter element as in [18, §1]. Then, by [18, 6.1], F^δ acts as a semisimple automorphism of $\bigoplus_i H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$ and the corresponding eigenspaces are mutually nonisomorphic, simple G^F-modules. To fix notation, let $\mu_1, \ldots, \mu_h$ be the distinct eigenvalues of F^δ and denote by $V_1, \ldots, V_h$ the corresponding eigenspaces; assume that V_j is a subspace of $H_c^{n(j)}(X_w, \overline{\mathbb{Q}}_\ell)$. Now we define a class function N_w on G^F by

$$N_w(g) := \sum_{i \geqslant 0} (-1)^i \operatorname{Trace}(gF^\delta, H_c^i(X_w, \overline{\mathbb{Q}}_\ell)) \quad \text{for } g \in G^F.$$

As in [18, 8.6], we can evaluate N_w as follows. Let $g \in G^F$. Then we have

$$N_w(g) = \sum_{j=1}^h (-1)^{n(j)} \operatorname{Trace}(gF^\delta, V_j) = \sum_{j=1}^h (-1)^{n(j)} \mu_j \operatorname{Trace}(g, V_j).$$

Thus, we obtain the decomposition $N_w = \sum_{j=1}^h (-1)^{n(j)} \mu_j \chi_j$, where χ_j denotes the character of V_j. Using the orthogonality relations for the characters of G^F, this yields

$$\frac{1}{|G^F|} \sum_{g \in G^F} N_w(g)\, \chi_j(g^{-1}) = (-1)^{n(j)} \mu_j.$$

Now note that N_w is a rational-valued class function on G^F. (This follows from the fact that gF^δ also is Frobenius map on G and by using Grothendieck's trace formula; see [8, Chap. 10].) Thus, we conclude that

$$(*) \qquad \mu_j \in \mathbb{Q}(\chi_j) \qquad \text{for } 1 \leqslant j \leqslant h.$$

In order to deal with unipotent characters which are not covered by the above results, we use the following constructions.

5.3. Shintani descent. Assume that G is of split type. For any $\varphi \in \operatorname{Irr}(W)$ we define the corresponding almost character by

$$R_\varphi := \frac{1}{|W|} \sum_{w \in W} \varphi(w)\, R_w,$$

where R_w denotes the generalized character afforded by $\sum_i (-1)^i H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$. If $w = 1$, then R_1 is nothing but the permutation character of G^F on the cosets of a Borel subgroup; see [8, Chap. 11]. Having fixed a square root $\sqrt{q} \in \overline{\mathbb{Q}}_\ell$, we have a well-defined decomposition

$$R_1 = \sum_{\varphi \in \mathrm{Irr}(W)} \varphi(1)\, \chi_\varphi \qquad \text{where } \chi_\varphi \in \mathrm{Irr}(G^F).$$

(Indeed, by the theory of Hecke algebras, we have a decomposition of R_1 where the constituents are parametrized by the irreducible characters of the endomorphism algebra of R_1, and these are in bijection with $\mathrm{Irr}(W)$ by Tits' Deformation theorem; see [13, 8.1.7, 8.4.7]. Furthermore, these correspondences only depend on a choice of a square root of q by [13, 9.3.5].)

Finally, denote by $n_{F/F}$ the bijection on the set of conjugacy classes of G^F defined in [7, Chap. I, 7.2]. This map gives rise to the *Shintani descent* map on the space of class functions on G^F. By [7, Chap. III, 2.3], we have

$$\chi_\varphi \circ n_{F/F} = \sum_{\substack{\chi \in \mathrm{Irr}(G^F) \\ \text{unipotent}}} \langle R_\varphi, \chi \rangle_{G^F}\, \lambda_\chi\, \chi.$$

Here, $\lambda_\chi \in \overline{\mathbb{Q}}_\ell$ is the root of unity associated with χ as in [21, Chap. 11] (using the eigenvalues of F on the cohomology spaces $H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$). Now note that the multiplicities $\langle R_\varphi, \chi \rangle_{G^F}$ are rational numbers by the rationality of the characters of W. Hence, using the orthogonality relations for the characters of G^F and arguing as in (5.2), we find that

$$(\dagger) \qquad \lambda_\chi \in \mathbb{Q}(\chi, \chi_\varphi) \qquad \text{if } \langle R_\varphi, \chi \rangle_{G^F} \neq 0.$$

5.4. Determination of $\mathbb{Q}(\chi)$. We go case by case through the list of cuspidal unipotent characters in Table 1. As before, let us write $\chi = {}^\delta X[\alpha]$. The information contained in (5.1) shows that $\mathbb{Q}(\chi) = \mathbb{Q}$ if α is rational.

Now consider, for example, the cuspidal unipotent character $G_2[\theta]$ for G of type G_2. This character occurs in $\bigoplus_i H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$ with eigenvalue θq and where w is a Coxeter element; see Table I in [20, p. 32]. Hence (5.2)(∗) shows that $\theta \in \mathbb{Q}(G_2[\theta])$. On the other hand, we have already seen in (5.1) that $[\mathbb{Q}(G_2[\theta]) : \mathbb{Q}] \leqslant 2$. So we can now conclude that $\mathbb{Q}(G_2[\theta]) = \mathbb{Q}(\theta)$, as claimed in Table 1. The same kind of argument applies to all cuspidal unipotent characters occurring in [20, p. 32] and it yields the corresponding entries in Table 1.

The only cuspidal unipotent characters which are not covered by the above arguments are the characters

$$E_8[i], E_8[-i], E_8[\theta], E_8[\theta^2] \quad \text{for } G \text{ of type } E_8.$$

Assume that χ is one of these characters. Then, by [21, 11.2], the corresponding eigenvalue λ_χ is i, $-i$, θ or θ^2, respectively. We place ourselves in the setting of (5.3) and let $\phi = \phi_{4480,16} \in \mathrm{Irr}(W)$ (notation of [3, pp. 484]). Then we find $\langle R_\varphi, \chi \rangle_{G^F} \neq 0$ using the multiplicity formula in [21, 4.23]. Furthermore, by [28, 1.8], the principal series character $\chi_\varphi \in \mathrm{Irr}(G^F)$ is rational-valued. Hence, (5.3)(†) shows that $\lambda_\chi \in \mathbb{Q}(\chi)$. Using the bounds on $[\mathbb{Q}(\chi) : \mathbb{Q}]$ in (5.1), we conclude that $\mathbb{Q}(\chi) = \mathbb{Q}(\lambda_\chi)$, as required.

Thus, we have proved that $\mathbb{Q}(\chi)$ is as asserted in Table 1.

Our next aim is to obtain a result on character fields and Schur indices which is valid for all unipotent characters of G^F. For this purpose, we use the fact that every $\chi \in \mathrm{Irr}(G^F)$ belongs to a Harish-Chandra series; see [3, Chap. 9]. This means that there exists an F-stable parabolic subgroup $P \subseteq G$ with a Levi decomposition $P = U_P \rtimes L$ such that L is F-stable and there exists a cuspidal $\psi \in \mathrm{Irr}(L^F)$ such that χ occurs with nonzero multiplicity in the Harish-Chandra induction $R_L^G(\psi)$. If χ is unipotent, then ψ must be unipotent also (see [21, 8.5.1]).

We begin with the following general result which provides a reduction to the case of cuspidal characters (at least if the center of G is connected).

Proposition 5.5. *Assume that G has a connected center. Let $\chi \in \mathrm{Irr}(G^F)$ be a constituent of the Harish–Chandra induction $R_L^G(\psi)$, where $\psi \in \mathrm{Irr}(L^F)$ is cuspidal. Let $\mathcal{W}$ be the stabilizer of ψ in $N_G(L)^F/L^F$. Then $\mathcal{W}$ is a finite Weyl group by* [21, 8.6]*. Assume that ψ can be realized over a subfield $k \subseteq \mathbb{C}$. Then we have $k(\chi) \subseteq k(\sqrt{q})$ and χ can be realized over $k(\chi)$. Furthermore, we have $k(\chi) = k$ unless $\sqrt{q} \notin k$ and $\mathcal{W}$ has a component of type E_7 or E_8.*

Proof. This will require some detailed results on Hecke algebras from [3, Chap. 10], [21, Chap. 8] and [13, Chap. 9]. Let $\rho\colon L^F \to \mathrm{End}_k(V)$ be a representation affording ψ and consider the Hecke algebra

$$H_\rho = \mathrm{End}_{kG^F}(R_L^G(V)).$$

Now we extend scalars from k to $\mathbb{C}$. To simplify notation, we indicate tensoring with $\mathbb{C}$ by a superscript $\mathbb{C}$ and regard a k-vectorspace E as a subspace of $E^{\mathbb{C}}$. In particular, we have $H_\rho^{\mathbb{C}} = \mathrm{End}_{\mathbb{C}G^F}(R_L^G(V^{\mathbb{C}}))$. Now, the assumption that χ occurs in $R_L^G(\psi)$ means that there exists a primitive idempotent $e_\chi \in H_\rho^{\mathbb{C}}$ such that $e_\chi R_L^G(V)$ affords χ. Then the left ideal $E_\chi := H_\rho^{\mathbb{C}} e_\chi \subseteq H_\rho^{\mathbb{C}}$ is a simple $H_\rho^{\mathbb{C}}$-module. Assume that E_χ can be realized over a subfield $k_1 \subseteq \mathbb{C}$ such that $k \subseteq k_1$. Then the fact that $H_\rho^{\mathbb{C}}$ is a symmetric algebra (see [3, §10.9]) and the formulas for idempotents in [13, 7.2.7] imply that e_χ can be chosen to be defined over k_1 and so χ can be realized over k_1. Thus, we are reduced to a question concerning rationality properties of representations of Hecke algebras.

Now $H_\rho^{\mathbb{C}}$ is an Iwahori–Hecke algebra associated with the finite Weyl group $\mathcal{W}$. More precisely, by [21, 8.6], we have a canonically defined set of simple reflections in $\mathcal{W}$. Let $l\colon \mathcal{W} \to \mathbb{N}_0$ be the corresponding length function. Then there exists a basis $\{T_w \mid w \in \mathcal{W}\}$ of $H_\rho^{\mathbb{C}}$ and integers $c_w > 0$ for $w \in \mathcal{W}$ with $l(w) = 1$ such that the following relations hold. We have

$$T_w^2 = q^{c_w} T_1 + (q^{c_w} - 1) T_w \quad \text{if } l(w) = 1$$

and $T_w = T_{w_1} T_{w_2}$ whenever $w = w_1 w_2$ and $l(w) = l(w_1) + l(w_2)$. Now recall that we regard H_ρ as a subspace of $H_\rho^{\mathbb{C}}$. We claim that

$$T_w \in H_\rho \qquad \text{for all } w \in \mathcal{W}.$$

To see this, we recall the construction of T_w from [3, Chap. 10]. Let $w \in \mathcal{W}$. Then, by [3, 10.8.3], there exists some $0 \neq \xi_w \in \mathbb{C}$ such that $B_w = \xi_w T_w$, where B_w is constructed in [3, 10.1.4]; that construction actually shows that $B_w \in H_\rho$. The scaling factor ξ_w is obtained by first choosing a suitable scalar multiple of B_w so that the cocycle conditions in [3, 10.3.4] are satisfied and then using [3, 10.8.3]. Hence, it will be enough to show that $\xi_w \in k$ for all $w \in \mathcal{W}$. The above multiplication rules show that it is even enough to prove this in the case where $l(w) = 1$. In this case, we have

$$B_w^2 = (\xi_w T_w)^2 = \xi_w^2 q^{c_w} \xi_1^{-1} B_1 + \xi_w (q^{c_w} - 1) B_w.$$

Since B_w is defined over k and $c_w > 0$, we conclude that $\xi_w \in k$ and so T_w is defined over k, as desired. Thus, the above claim is proved.

The above discussion implies that $H_\rho^{\mathbb{C}}$ is a specialisation of a suitable generic Iwahori–Hecke $\mathcal{H}_\rho$ (see [13, 8.1.4]). Indeed, let $A = \mathbb{C}[v, v^{-1}]$ be the ring of Laurent polynomials over $\mathbb{C}$ in an indeterminate v. Let $\mathcal{H}_\rho$ be a free A-module with a basis $\{T_w \mid w \in \mathcal{W}\}$. Then $\mathcal{H}_\rho$ is an associative algebra where the product is given as follows. We have

$$T_w^2 = v^{2c_w} T_1 + (v^{2c_w} - 1) T_w \quad \text{if } l(w) = 1$$

and $T_w = T_{w_1} T_{w_2}$ whenever $w = w_1 w_2$ and $l(w) = l(w_1) + l(w_2)$. Let $\theta\colon A \to \mathbb{C}$ be the ring homomorphism sending v to a square root of q. Then we can regard $\mathbb{C}$ as an A-module via θ and we have

$$H_\rho^{\mathbb{C}} = \mathbb{C} \otimes_A \mathcal{H}_\rho.$$

Let $K = \mathbb{C}(v)$ be the field of fractions of A. Then we know by [13, 9.3.5] that $\mathcal{H}_\rho^K := K \otimes_A \mathcal{H}_\rho$ is a split semisimple algebra. So Tits' Deformation Theorem shows that we have a canonical bijection between the isomorphism classes of simple modules of $\mathcal{H}_\rho^K$ and $H_\rho^{\mathbb{C}}$ (see [13, 8.1.7]). The argument in the proof of [13, 7.4.6] shows that this bijection is given as follows. Let us fix a simple $\mathcal{H}_\rho^K$-module. Since A is a principal ideal domain with field of fractions K, we can realize that module over A (see [13, 7.3.3]). Extending scalars from A to $\mathbb{C}$ via θ, we obtain a simple $H_\rho^{\mathbb{C}}$-module. By construction, if the original $\mathcal{H}_\rho^K$-module can be realized over a subring $A_0 \subseteq A$, then the corresponding $H_\rho^{\mathbb{C}}$-module is realized over the subring $\theta(A_0) \subseteq C$.

Now, since $\mathcal{W}$ is a Weyl group, we actually have that every simple $\mathcal{H}_\rho^K$-module can be realized over $K_0 = k(v)$; see once more [13, 9.3.5]. Since $A_0 = k[v, v^{-1}]$ is still a principal ideal domain, we can realize every simple $\mathcal{H}_\rho^K$-module over A_0. Hence the above argument shows that every simple $H_\rho^{\mathbb{C}}$-module can be realized over $\theta(A_0) = k(\sqrt{q})$. Consequently, our irreducible character $\chi \in \mathrm{Irr}(G^F)$ can be realized over $k(\sqrt{q})$ and we have $k(\chi) \subseteq k(\sqrt{q})$. Thus, the proof is complete if $\sqrt{q} \in k$.

Finally, assume that $\sqrt{q} \notin k$ and χ cannot be realized over k. We claim that $k(\chi) = k(\sqrt{q})$. Indeed, since we already know that $k(\chi) \subseteq k(\sqrt{q})$, it is enough to show that $k(\chi) \neq k$. This is seen as follows. The assumption that χ cannot be realized over k implies that the simple $H_\rho^{\mathbb{C}}$-module E_χ cannot be realized over k either. Let τ be the field automorphism of $k(\sqrt{q})$ which sends $\sqrt{q}$ to $-\sqrt{q}$. Let E_χ^τ be the simple $H_\rho^{\mathbb{C}}$-module obtained from E_χ via algebraic conjugation with τ. Similarly, let $\chi^\tau \in \mathrm{Irr}(G^F)$ be obtained from χ via algebraic conjugation. Then χ^τ corresponds to E_χ^τ and we have $E_\chi \cong E_\chi^\tau$ if and only if $\chi = \chi^\tau$. Hence it is enough to show that $E_\chi \not\cong E_\chi^\tau$. But, if we had $E_\chi \cong E_\chi^\tau$, this would mean that E_χ has Schur index 2 over k, contradicting the fact that E_χ occurs with multiplicity 1 in the representation induced from the index representation of some proper parabolic subalgebra of $H_\rho^{\mathbb{C}}$; see [13, 9.1.9 and §6.3]. Thus, we have $k(\chi) = k(\sqrt{q}) \neq k$.

It remains to show that G must have a component of type E_7 or E_8 in this case. This is seen as follows. We write $\mathcal{W} = \mathcal{W}_1 \times \cdots \times \mathcal{W}_t$ where $\mathcal{W}_i$ are irreducible Weyl groups. Then we also have $H_\rho = H_1 \otimes_k \cdots \otimes_k H_t$ where H_i is the parabolic subalgebra corresponding to $\mathcal{W}_i$. We can express E_χ in the form $E_\chi = E_1 \boxtimes \cdots \boxtimes E_t$ where each E_i is a simple $H_i^{\mathbb{C}}$-module. As already pointed out above, the assumption that χ cannot be realized over k implies that the simple $H_\rho^{\mathbb{C}}$-module E_χ cannot be realized over k either. Hence there must be some i such that E_i cannot be realized over k. By [13, 9.3.4], this shows that $\mathcal{W}_i$ has type G_2, E_7 or E_8. However, if $\mathcal{W}_i$ is of type G_2, then the sum of the two integers c_s corresponding to the simple reflections in $\mathcal{W}_i$ is even by [21, 8.2.3]. But then [13, 8.3.1] shows that all simple modules of $H_i^{\mathbb{C}}$ can be realized over k. Thus, $\mathcal{W}$ must have a component of type E_7 or E_8 and E_χ must be one of the simple modules of dimension 512 or 4096, respectively. $\square$

Proposition 5.6. *Assume that G has a simple derived subgroup. Let $\chi \in \operatorname{Irr}(G^F)$ be unipotent and assume that χ occurs in the Harish–Chandra induction $R_L^G(\psi)$, where $\psi \in \operatorname{Irr}(L^F)$ is cuspidal unipotent. Then we have $\mathbb{Q}(\chi) = \mathbb{Q}(\psi)$ except in the following cases (notation of* [21, Appendix]*):*

Type of G	χ unipotent	$\mathbb{Q}(\chi)$
E_7	$[512_a], [512'_a]$	$\mathbb{Q}(\sqrt{q})$
E_8	$[4096_x], [4096'_x], [4096_z], [4096'_z]$	$\mathbb{Q}(\sqrt{q})$

Furthermore, assuming that $m_{\mathbb{Q}}(\psi)$ is as specified in Table 1 (note that L also has simple derived subgroup), we have $m_{\mathbb{Q}}(\chi) = m_{\mathbb{Q}}(\psi)$.

Proof. By [20, Prop. 3.15], we may assume, without loss of generality, that G has a connected center and a simple derived subgroup. If G is of classical type, then every unipotent character can be realized over $\mathbb{Q}$, except possibly in type ${}^2A_n$; see [28, 1.12 and 1.13]. If G^F is a finite unitary group, then every unipotent character χ is rational-valued and the equality $m_{\mathbb{Q}}(\chi) = m_{\mathbb{Q}}(\psi)$ is proved by Ohmori [32]. So it remains to consider a group G of exceptional type. If χ itself is cuspidal, there is nothing to prove. So let us now assume that χ is not cuspidal. We go through the various possibilities, using the tables in [3, 13.7].

First we consider the case where G is of split type and the cuspidal unipotent character $\psi \in \operatorname{Irr}(L^F)$ is rational-valued. We claim that

$$(*) \qquad \mathbb{Q}(\chi) = \mathbb{Q} \text{ if } \chi \notin \{[512_a], [512'_a], [4096_x], [4096'_x], [4096_z], [4096'_z]\}.$$

Indeed, since L also is of split type, we can apply [28, 1.10] and obtain $\lambda_\psi = \pm 1$, where λ_ψ is the eigenvalue of Frobenius attached to ψ as in [21, Chap. 11]. Then, by [21, 11.3], we also have $\lambda_\chi = \pm 1$. Now, if χ is not one of the exceptional characters mentioned above, then [28, 1.10] shows that χ can also be realized over $\mathbb{Q}$. In particular, $(*)$ is proved.

Next consider the case where χ lies in the principal series (i.e., ψ is the trivial character of a maximally split torus). If (G, F) is of type ${}^3D_4$ or ${}^2E_6$, then χ can be realized over $\mathbb{Q}$ by Proposition 5.5. (In these cases, the Hecke algebra is of type G_2 or F_4, respectively.) If G is of type E_7 or E_8, then the argument in the proof of $(*)$ shows that χ can be realized over $\mathbb{Q}$ except possibly if χ is one of the exceptional characters mentioned above. If χ is one of these exceptional characters, we argue as follows. By Proposition 5.5, χ can be realized over $\mathbb{Q}(\sqrt{q})$. On the other hand, by [28, 1.9], we have $\mathbb{Q}(\chi) \neq \mathbb{Q}$ if q is not a square. Thus, χ can be realized over $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{q})$ and so $m_{\mathbb{Q}}(\chi) = 1$, as required.

The only cases which are not covered by the above arguments are those where ψ cannot be realized over $\mathbb{Q}$.

Let us first consider the case where (G, F) is of type ${}^2E_6$ and L is of type ${}^2A_5$. (See also [31].) The cuspidal unipotent character $\psi \in \mathrm{Irr}(L^F)$ cannot be realized over $\mathbb{Q}$ by [32]. On the other hand, we have $R_L^G(\psi) = \chi_1 + \chi_2$ where $\chi_1 \neq \chi_2$ are irreducible characters of G^F. Now, since ψ is rational-valued, the same also holds for $R_L^G(\psi)$. Since χ_1 and χ_2 have different degrees (see the table in [3, p. 481]), a simple argument using algebraic conjugate characters shows that they must be rational-valued. By the Brauer–Speiser Theorem [5, 74.27], each χ_i can be realized over a field $k \subseteq \mathbb{C}$ such that $[k : \mathbb{Q}] \leqslant 2$. Assume, if possible, that one of these characters, χ_1 say, can be realized over $\mathbb{Q}$. Then the Harish–Chandra restriction ${}^*R_L^G(\chi_1)$ can also be realized over $\mathbb{Q}$; see [8, 4.6]. Furthermore, since Harish–Chandra induction and restriction are adjoint functors, we see that ψ occurs with multiplicity 1 in ${}^*R_L^G(\chi_1)$. This would imply that ψ can be realized over $\mathbb{Q}$ (see [15, 10.2]), a contradiction. So we have $m_{\mathbb{Q}}(\chi_i) = m_{\mathbb{Q}}(\psi) = 2$ for $i = 1, 2$, as required.

Next assume that L is of type E_6 and $\psi \in \{E_6[\theta], E_6[\theta^2]\}$. In these cases, the Hecke algebra is of type A_1 or A_2. By Table 1, ψ can be realized over $\mathbb{Q}(\psi) = \mathbb{Q}(\theta)$. Hence Proposition 5.5 implies that any constituent χ of $R_L^G(\psi)$ can also be realized over $\mathbb{Q}(\theta)$. So the only remaining question is whether $\mathbb{Q}(\chi) = \mathbb{Q}$ or $\mathbb{Q}(\theta)$. Assume, if possible, that $\mathbb{Q}(\chi) = \mathbb{Q}$. Let τ be a field automorphism of $\mathbb{Q}(\psi)$ which does not fix ψ. Then ψ^τ also is a cuspidal unipotent character and χ occurs in both $R_L^G(\psi)$ and $R_L^G(\psi^\tau)$. By [3, 9.1.5], this would imply that ψ and ψ^τ are conjugate in $N_G(L)^F$, which is impossible by [21, 8.6].

Finally, if L is of type E_7 and $\psi \in \{E_7[\xi], E_7[-\xi]\}$, then $R_L^G(\psi) = \chi_1 + \chi_2$ where $\chi_1 \neq \chi_2$ are irreducible characters of G^F. A simple argument about algebraically conjugate characters shows that $\mathbb{Q}(\chi_i) \subseteq \mathbb{Q}(\psi)$ for $i = 1, 2$. Since $\mathbb{Q}(\psi)$ has degree 2 over $\mathbb{Q}$, we have $\mathbb{Q}(\chi_i) = \mathbb{Q}$ or $\mathbb{Q}(\chi_i) = \mathbb{Q}(\psi)$. The first possibility is excluded by using the same argument as above. So we have $\mathbb{Q}(\chi_i) = \mathbb{Q}(\psi) = 1$. Since χ_i occurs with multiplicity 1 in $R_L^G(\psi)$, we have $m_{\mathbb{Q}}(\chi_i) \leqslant m_{\mathbb{Q}}(\psi)$ by [15, 10.2]. On the other hand, ψ also occurs with multiplicity 1 in the Harish–Chandra restriction ${}^*R_L^G(\chi_i)$ (for $i = 1, 2$) and so we have $m_{\mathbb{Q}}(\psi) \leqslant m_{\mathbb{Q}}(\chi_i)$. This yields $m_{\mathbb{Q}}(\chi_i) = m_{\mathbb{Q}}(\psi)$, as desired. □

6. On the Schur indices of cuspidal unipotent characters

This section is concerned with the problem of determining the Schur indices of unipotent characters of G^F. By Proposition 5.6, it is enough to consider a cuspidal unipotent character $\chi \in \mathrm{Irr}(G^F)$. If $\mathbb{Q}(\chi) = \mathbb{Q}$, the answer is known and will be recalled in (6.2). So our task will be to consider the nonrational cuspidal unipotent characters occurring in Table 1. The information on character fields in Table 1 in combination with the Benard–Schacher Theorem [5, 74.26] already provides strong restrictions on the Schur index $m_{\mathbb{Q}}(\chi)$: the field $\mathbb{Q}(\chi)$ has to contain all roots of unity of order $m_{\mathbb{Q}}(\chi)$. Using some deeper results on the Schur index, one can show that $m_{\mathbb{Q}}(\chi) = 1$ if certain congruence conditions on q are satisfied; see Proposition 6.3. The proofs for the remaining cases in Table 1 will then be completed in (6.5).

We begin by recalling some classical results concerning local Schur indices. Let H be any finite group and $\chi \in \mathrm{Irr}(H)$. Let $K = \mathbb{Q}(\chi)$; this is a finite algebraic extension of $\mathbb{Q}$. Let ℓ be a rational prime number or $\ell = \infty$ and consider a corresponding completion $\mathbb{K}_\ell$ of K. Thus, if $\ell = \infty$, then $\mathbb{K}_\ell$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ (depending on whether K can be embedded in $\mathbb{R}$ or not). If $\ell > 0$, then $\mathbb{K}_\ell$ is a finite extension of $\mathbb{Q}_\ell$ which depends on the prime ideal decomposition of the ideal generated by ℓ in the ring of algebraic integers of K. (For all these facts, see [16, Chap. 2].) We define $m_\ell(\chi)$ to be the Schur index of χ over $\mathbb{K}_\ell$. By the Benard–Schacher Theorem [5, 74.26], $m_\ell(\chi)$ only depends on ℓ and not on the chosen completion. Furthermore, by [5, 74.13], we have

$$m_{\mathbb{Q}}(\chi) = \text{least common multiple of all } m_\ell(\chi).$$

The following result is an important tool for the computation of Schur indices. Its proof relies on the theory of Hasse invariants; see [5, §74B] and the references there.

Lemma 6.1. *Let H be any finite group and $\chi \in \mathrm{Irr}(H)$. Assume that there exists a prime number $p > 0$ such that the following two conditions hold:*

(a) *We have $m_\ell(\chi) = 1$ whenever $\ell \neq p$ (including $\ell = \infty$).*
(b) *The ideal generated by p in the ring of algebraic integers of $\mathbb{Q}(\chi)$ is a power of a prime ideal.*

Then we also have $m_p(\chi) = 1$.

Note that (b) is automatically satisfied, for example, if $\mathbb{Q}(\chi) = \mathbb{Q}$.

Proof. This immediately follows from the sum formula [5, 74.14] for the Hasse invariants. By (a), all terms in that sum are 0 except possibly those which correspond to the completions of $\mathbb{Q}(\chi)$ with respect to the prime ideals above p. However, by (b), there is only one such completion and so the sum formula reduces to the statement that $[r_p/m_p(\chi)]$ is 0 in $\mathbb{Q}/\mathbb{Z}$, where r_p is an integer coprime to $m_p(\chi)$. Hence we must have $m_p(\chi) = 1$, as desired. □

Now we return to the case where G is a connected reductive group and $F\colon G \to G$ is the Frobenius map corresponding to some $\mathbb{F}_q$-rational structure on G. Throughout, we assume that the derived subgroup of G is simple.

6.2. The case where $\mathbb{Q}(\chi) = \mathbb{Q}$. Assume that $\chi \in \mathrm{Irr}(G^F)$ is cuspidal unipotent and rational-valued. Then the Brauer–Speiser Theorem [5, 74.27] shows that the Schur index $m_{\mathbb{Q}}(\chi)$ is at most 2. Lusztig has shown that we actually have $m_{\mathbb{Q}}(\chi) = 1$ whenever G is of split type or of type ${}^2D_n$; see [28, 0.2 and 1.13]. The proof uses Lemma 6.1 in an essential way, where p is the prime dividing q. The case of finite unitary groups is settled by Ohmori [32]. (In these groups, the Schur index of χ is 1 if and only if the semisimple $\mathbb{F}_q$-rank of G is even.)

If G is of type ${}^3D_4$ or ${}^2E_6$, the semisimple $\mathbb{F}_q$-rank of G is even and χ has inner product ± 1 with some Deligne–Lusztig generalized character $R_{T,1}$, by [28, 2.19]. Hence the argument in the proof of [28, Theorem 2.22] (see also [32]) still works and it yields that the characters ${}^3D_4[\pm 1]$ and ${}^2E_6[1]$ can also be realized over $\mathbb{Q}$. (As far as ${}^3D_4[-1]$ is concerned, this is also mentioned in [18, 7.6].)

As a further application of Lemma 6.1, we prove the following result.

Proposition 6.3. *Assume that $\chi \in \mathrm{Irr}(G^F)$ is cuspidal unipotent such that $\mathbb{Q}(\chi)$ is generated over $\mathbb{Q}$ by an mth root of unity, where $m \geqslant 2$. Let p be the prime dividing q and assume that one of the following conditions hold:*

(a) *m is a power of p;*
(b) *m is not divisible by p and the smallest $r \geqslant 1$ satisfying $p^r \equiv 1 \bmod m$ is the degree of $\mathbb{Q}(\chi)$ over $\mathbb{Q}$.*

If, furthermore, we have $\mathbb{Q}(\chi) \neq \mathbb{Q}$ (that is, $m \geqslant 3$), then $m_{\mathbb{Q}}(\chi) = 1$.

Proof. We check that the hypotheses of Lemma 6.1 are satisfied for the prime p which divides q. Let $\ell > 0$ be a prime number not equal to p. Then, by [28, 2.18, 2.19], χ occurs with multiplicity ± 1 in some Deligne–Lusztig generalized character $R_{T,1}$. These characters are integral linear combinations of characters which are realized over $\mathbb{Q}_\ell$ by construction. Hence we have $m_\ell(\chi) = 1$ in all these cases. Now let $\ell = \infty$. By inspection of Table 1, the field $\mathbb{Q}(\chi)$ cannot be embedded into $\mathbb{R}$. Hence, clearly, we have $m_\infty(\chi) = 1$.

Hence, it remains to show that the ideal generated by p in the ring of algebraic integers of $\mathbb{Q}(\chi)$ is a power of a prime ideal. But this holds due to our assumptions on m; see [16, I.10.1] (if m is as in (a)) and [16, Chap. 1, §10, Ex. 5] (if m is as in (b)). ☐

Example 6.4. Assume that G is of type E_7 and that $\chi = E_7[\pm\xi]$. Then we have $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{-q})$; see Table 1. This is the only case where $\mathbb{Q}(\chi)$ might not be generated over $\mathbb{Q}$ by a root of unity. Let p be the prime dividing q. We claim that

$$\begin{array}{ll} m_{\mathbb{Q}}(\chi) = 1 & \text{if } p \not\equiv 1 \bmod 4 \text{ or if } q \text{ is not a square,} \\ m_{\mathbb{Q}}(\chi) = 1, 2 \text{ or } 4 & \text{otherwise.} \end{array}$$

This is seen as follows. Assume first that q is a square. Then we have $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{-1})$. Hence the Benard–Schacher Theorem [5, 74.26] implies that the Schur index is 1, 2 or 4. Furthermore, Proposition 6.3 yields that $m_{\mathbb{Q}}(\chi) = 1$ if $p = 2$ or if $p \equiv -1 \bmod 4$. It remains to consider the case where q is not a square. Then we have $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{-p})$. The ideal generated by p in the ring of algebraic integers of $\mathbb{Q}(\sqrt{-p})$ is a power of a prime ideal (see [16, §I.7]). We can now argue as in the proof of Proposition 6.3 to conclude that $m_{\mathbb{Q}}(\chi) = 1$.

6.5. On the bounds for the Schur indices in Table 1. Assume that G has a connected center and a simple simply-connected derived subgroup. Let $\chi \in \mathrm{Irr}(G^F)$ be cuspidal unipotent. By (6.2), we can assume that $\mathbb{Q}(\chi) \neq \mathbb{Q}$. As already remarked in Section 1, the entries marked by $(*)$ in Table 1 are simple consequences of the Benard–Schacher Theorem.

For G of type E_7, see Example 6.4. Next we consider the following cases. Let χ be one of the characters $G_2[\theta]$, $G_2[\theta^2]$ (in type G_2), $F_4[\theta]$, $F_4[\theta^2]$ (in type F_4) or ${}^\delta E_6[\theta]$, ${}^\delta E_6[\theta^2]$ (in type E_6 or ${}^2E_6$). In all these cases, we have $\mathbb{Q}(\chi) = \mathbb{Q}(\theta)$; see Table 1. We show that $m_{\mathbb{Q}}(\chi) = 1$. If q is a power of 2 or 3, this holds by Proposition 6.3. So we can now assume that q is a power of a good prime. By general results concerning the Schur index (see [15, 10.2]), it will be enough to show that there exist characters $\gamma_1, \ldots, \gamma_m$ of G^F (for some $m \geqslant 1$) such that each γ_i is afforded by a representation which can be realized over $\mathbb{Q}(\theta)$ and the greatest common divisor of all multiplicities $\langle \gamma_i, \chi \rangle_{G^F}$ is 1. We will find such characters using Proposition 4.4 and Theorem 2.10.

Indeed, using Proposition 4.4 (see also Remark 4.5) and the Fourier matrix in type G_2, F_4 or E_6, we see that there exists a cuspidal pair $(\Sigma, \mathcal{E})$ as in Table 2 (with trivial central character θ) such that

$$\langle \chi_{(\Sigma,\mathcal{E})}, \chi \rangle_{G^F} = \pm \frac{1}{3}\xi \quad \text{where } \xi \in \mathbb{C},\ \xi^3 = 1.$$

Now Theorem 2.10 shows that $|A_G(g_0)|\chi_{(\Sigma,\mathcal{E})}$ is an R-linear combination of $\mathrm{Irr}(G^F)$, where $R \subseteq \mathbb{C}$ is generated by algebraic integers. We consider equation (2) in the proof of Theorem 2.10:

$$|A_G(g_0)|\, \chi_{(\Sigma,\mathcal{E})} = \sum_{r=1}^{d} \sum_{\bar{\lambda} \in \mathrm{Irr}(\bar{Z}_H^F)} \xi_{r,\bar{\lambda}}\, \mathrm{Ind}_{Z_H^F \times U^F}^{G^F} \Big((\hat{\theta} \otimes \bar{\lambda}) \boxtimes \eta_r \Big)$$

where $\xi_{r,\bar{\lambda}} \in R$. We now continue the proof assuming that, for any $1 \leqslant r \leqslant d$ and any $\bar{\lambda} \in \mathrm{Irr}(\bar{Z}_H^F)$, we have that

$$(*) \qquad \mathrm{Ind}_{Z_H^F \times U^F}^{H^F} \Big((\hat{\theta} \otimes \bar{\lambda}) \boxtimes \eta_r \Big) \text{ can be realized over } \mathbb{Q}(\theta).$$

This condition implies that the Schur index of χ divides the multiplicity of χ of any induced character as in $(*)$; see [15, 10.2]. Consequently, the Schur index divides $\langle |A_G(g_0)|\, \chi_{(\Sigma,\mathcal{E})}, \chi \rangle_G^F = \pm |A_G(g_0)|\xi/3$ in the ring $R \subseteq \mathbb{C}$. Since $|A_G(g_0)| = 3$ (see Table 2) and ξ is a root of unity, we conclude that the Schur index of χ is 1, as required.

It remains to prove $(*)$. Let $s_0 \in G^F$ be the semisimple part of an element in Σ and set $H := Z_G(s_0)$. Then all irreducible components of the root system of H have type A_2 and so, as pointed out in Remark 2.6, C_1 must be the class of regular unipotent elements in H, where $\iota_1 = (C_1, \psi_1) \in \mathcal{N}_H$ is the associated cuspidal pair. This means that $\gamma_r := \mathrm{Ind}_{U^F}^{H^F}(\eta_r)$ is an ordinary Gelfand–Graev character of H^F. Now, any ordinary Gelfand–Graev character of a simple simply-connected group of type A_2 or ${}^2A_2$ can be realized over $\mathbb{Q}$ by [32, Prop. 1(iii)]. Hence the reduction arguments in the proof of [32, Lemma 2] show that γ_r can also be realized over $\mathbb{Q}$. Using the formula (2) in the proof of Lemma 2.9, we see that each induced character as in $(*)$ has its values in $\mathbb{Q}(\theta)$. Note that we can take $\hat{\theta}$ to be the trivial character of Z_H^F and that $\bar{Z}_H \cong A_H(u) \cong A_G(g_0) \cong \mathbb{Z}/3$ (see the proof of [8, 14.24] for the first isomorphism). Hence, it will be enough to show that each irreducible constituent, $\rho \in \mathrm{Irr}(H^F)$ say, of an induced character as in $(*)$ has Schur index 1. (This follows from a general property of the Schur index; see [15, 10.2(b)].) Now, such an induced character is a direct summand of γ_r and so ρ will also occur in γ_r. But γ_r is known to be multiplicity–free (see [8, 14.30]) and so ρ has Schur index 1, as desired.

Finally, it remains to consider a group of type E_8. First assume that χ is one of the characters $E_8[-\theta]$ or $E_8[-\theta^2]$. Then we have $\mathbb{Q}(\chi) = \mathbb{Q}(\theta)$; see Table 1. If q is a power of 2, 3 or 5, we have $m_{\mathbb{Q}}(\chi) = 1$ by Proposition 6.3. So we can assume that q is a power of a good prime. We show that χ has Schur index 1 over $K = \mathbb{Q}(\sqrt{\delta(p)p}, \theta)$. We use again the Hasse principle. If ℓ is a prime not dividing q, then $m_\ell(\chi) = 1$ by [28, 2.18]. Since $\mathbb{Q}(\chi)$ cannot be embedded into $\mathbb{R}$, we also have $m_\infty(\chi) = 1$. So it remains to show that $m_p(\chi) = 1$, where p is the prime dividing q. Now we proceed by an argument similar to that above. The relevant centralizer H is of type $A_1 \times A_2 \times A_5$ and so any induced character as in $(*)$ is a summand of an ordinary Gelfand–Graev character of H^F. By [30, Prop. 1], we know that all these Gelfand–Graev characters can be realized over any p-adic completion of K. Consequently, any induced character as in $(*)$ can be realized over any p-adic completion of K. Now we can argue as before to conclude that the Schur index of χ over K is 1.

On the other hand, if χ is one of the characters $E_8[\zeta^a]$ ($1 \leqslant a \leqslant 4$), we can argue as follows. We have $\mathbb{Q}(\chi) = \mathbb{Q}(\zeta)$; see Table 1. The cases where q is a power of 2, 3 or 5 are covered by Proposition 6.3. So we can assume that q is a power of a good prime from now on. Then we can apply a similar argument as above. The Fourier matrix shows that there exists a cuspidal pair $(\Sigma, \mathcal{E})$ as in Table 2 (with trivial central character θ) such that

$$\langle \chi_{(\Sigma,\mathcal{E})}, \chi \rangle_{G^F} = \pm \frac{1}{10}(3+\sqrt{5})\,\xi \quad \text{where } \xi \in \mathbb{C},\ \xi^5 = 1.$$

Now, since H has type $A_4 \times A_4$, we can use again [32, Prop. 1(iii)] to conclude that every induced character as in $(*)$ can be realized over $\mathbb{Q}(\zeta)$. Since $|A_G(g_0)| = 5$, the Schur index of χ will divide $(3+\sqrt{5})/2$ in the ring of algebraic integers. Hence it also divides $(3-\sqrt{5})/2$ and so equals 1, as claimed.

Remark 6.6. The methods employed in (6.5) show that it would be very useful to know over which field a generalized Gelfand–Graev character Γ_u^G of G^F can be realized. Assume that q is such that the methods in [26] can be applied and Theorem 1.4 holds. Then we certainly have

$$\Gamma_u^G(g) \in \mathbb{Q}(\sqrt{\delta(q)q}) \qquad \text{for all unipotent elements } u, g \in G^F.$$

Any knowledge on a minimal field over which Γ_u^G can be realized would also be useful in connection with the multiplicity formula in [26, 11.2]. To be more precise, let $\chi \in \mathrm{Irr}(G^F)$ and C_χ be the "unipotent support" of $\pm D_G(\chi)$ where D_G denotes the duality operation on the character ring of G^F. Then there exists some $u \in C^F$ such that the multiplicity of χ in Γ_u^G is "small". For groups of exceptional type, these multiplicities can be computed explicitly.

For example, if G is of type F_4, then the cuspidal unipotent characters $\chi = F_4[\pm i]$ occur with multiplicity 1 in some generalized Gelfand–Graev character Γ_u^G; see [42, p. 133]. Hence χ can be realized over a field $K \supseteq \mathbb{Q}(\chi)$ such that Γ_u^G can be realized over K. Similarly, if we are in type E_7, then it can be checked that the cuspidal unipotent characters $E_7[\pm\xi]$ occur with multiplicity 1 in some generalized Gelfand–Graev character.

6.7. Proof of Corollary 1.5. Using the reduction argument provided by Proposition 5.6, it is enough to prove the assertion for a cuspidal unipotent character $\chi \in \mathrm{Irr}(G^F)$. Note that a group ${}^2E_6$ has unipotent characters with Schur index 2. (They lie in the Harish–Chandra series determined by a Levi subgroup of type ${}^2A_5$; see [31].)

If we are in type E_7 and q is not a square, we have $m_{\mathbb{Q}}(\chi) = 1$ by Example 6.4. Hence, by inspection of Table 1, we see that the only cases which remain to be considered are those where $\mathbb{Q}(\chi) = \mathbb{Q}(\theta)$ or $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{-1})$. These cases can be settled using Proposition 6.3. Indeed, assume first that $\mathbb{Q}(\chi) = \mathbb{Q}(\theta)$. If $p = 3$, we are in case (a) of Proposition 6.3. If $p \neq 3$, we only need to make sure that $p \not\equiv 1 \bmod 3$. A similar argument also works for $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{-1})$. If $p \neq 2$, we only have to make sure that $p \not\equiv 1 \bmod 4$. Thus, Corollary 1.5 is proved.

7. On the Ree groups ${}^2F_4(q^2)$

In this section, let G be a simple algebraic group of type F_4 over an algebraic closure of the finite field with 2 elements. Then, for any odd power q of $\sqrt{2}$, there exists an isogeny $F'\colon G \to G$ such that the fixed point set $G^{F'}$ is the Ree group of type ${}^2F_4(q^2)$. By [21, Appendix], we have 21 unipotent characters $\chi_1, \ldots, \chi_{21}$. The character values have been computed explicitly by Malle [29]. By inspection we find the character fields

$$\mathbb{Q}(\chi_i) = \begin{cases} \mathbb{Q}(\sqrt{-1}) & \text{for } i = 5,6,7,8,15,16,17,18, \\ \mathbb{Q}(\sqrt{-3}) & \text{for } i = 19,20, \\ \mathbb{Q} & \text{otherwise.} \end{cases}$$

The character χ_{21} is uniquely determined by the condition that it has even multiplicity in all Deligne–Lusztig generalized characters $R_{T,1}$. Our aim is to show that $m_{\mathbb{Q}}(\chi_{21}) = 2$ and $m_{\mathbb{Q}}(\chi_i) = 1$ for $1 \leqslant i \leqslant 20$. (In an unpublished note, Ohmori has also proved that $m_{\mathbb{Q}}(\chi_i) = 1$ for $i \neq 21$; the proof below is slightly different as far as χ_{12}, χ_{13} and χ_{14} are concerned.)

7.1. Noncuspidal unipotent characters. The 7 characters χ_1, χ_2, χ_3, χ_4, χ_9, χ_{10}, χ_{11} lie in the principal series. They all have Schur index 1. To see this, one can use an argument involving the Hecke algebra (as in the proof of Proposition 5.5, but note that here we only have to deal with the Hecke algebra of a permutation module where we can use [13, §8.4] to construct a suitable basis of that algebra).

The 4 characters χ_5, χ_6, χ_7, χ_8 lie in the Harish–Chandra series determined by the two cuspidal unipotent characters ψ_j $(j = 1,2)$ of a Levi subgroup $L \subset G$ of type ${}^2B_2$. Now, the Harish–Chandra induction $R_L^G(\psi_j)$ is multiplicity–free (in fact, there are just two irreducible constituents). Furthermore, ψ_1 and ψ_2 have character field $\mathbb{Q}(\sqrt{-1})$ (see Suzuki [39]) and Schur index 1 (see Gow [14]). Hence we conclude that χ_5, χ_6, χ_7 and χ_8 all have Schur index 1.

7.2. Nonrational cuspidal unipotent characters. Assume that χ_i is cuspidal and that $\mathbb{Q}(\chi_i) = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$. Then we have $m_{\mathbb{Q}}(\chi_i) = 1$. Indeed, we know that $m_{\mathbb{Q}}(\chi_i)$ is the least common multiple of all local Schur indices $m_\ell(\chi_i)$ (where $\ell = \infty$ or ℓ is a prime). Since $\mathbb{Q}(\chi_i)$ cannot be embedded in $\mathbb{R}$, we have $m_\infty(\chi_i) = 1$. Furthermore, the table in [21, p. 374] shows that each χ_i occurs with multiplicity ± 1 in some Deligne–Lusztig generalized character $R_{T,1}$. Since $R_{T,1}$ can be realized over $\mathbb{Q}_\ell$ by construction (where ℓ is an odd prime), we see that $m_\ell(\chi_i) = 1$ for all $\ell \neq 2$. Now we can argue in exactly the same way as in the proof of Proposition 6.3 to conclude that $m_{\mathbb{Q}}(\chi_i) = 1$.

7.3. Rational cuspidal unipotent characters. The only cases which are left to be considered are the rational-valued cuspidal unipotent characters χ_{12}, χ_{13}, χ_{14} and χ_{21}. We claim that

$$\begin{array}{rclcrcll} m_\infty(\chi_i) & = & 1 & \Rightarrow & m_{\mathbb{Q}}(\chi_i) & = & 1 & \quad \text{for } i = 12,13,14, \\ m_\infty(\chi_{21}) & = & 2 & \Rightarrow & m_{\mathbb{Q}}(\chi_{21}) & = & 2. & \end{array}$$

Indeed, by the Brauer–Speiser Theorem, we already know that $m_{\mathbb{Q}}(\chi_{21}) \leqslant 2$. Hence, in order to show that $m_{\mathbb{Q}}(\chi_{21}) = 2$, it is enough to show that $m_\infty(\chi_{21}) = 2$. Furthermore, the table in [21, p. 374] shows that $m_\ell(\chi_i) = 1$ for all odd primes ℓ and $i = 12, 13, 14$. Hence, in order to show that these three characters have Schur index 1, it is enough to show that $m_\infty(\chi_i) = 1$ (see Lemma 6.1).

Thus, in order to complete the computation of the Schur indices, it remains to determine $m_\infty(\chi_i)$. By a well-known criterion due to Frobenius and Schur (see [15, Chap. 4]), we can do this by computing $\nu_2(\chi_i)$ where

$$\nu_2(\gamma) := \frac{1}{|G^{F'}|} \sum_{g \in G^{F'}} \gamma(g^2) \qquad \text{for any class function } \gamma\colon G^{F'} \to \mathbb{C}.$$

In principle, $\nu_2(\gamma)$ can be computed once the values of γ are known. However, this involves the determination of the 2-power map $C \mapsto C^{[2]}$ on the set of conjugacy classes of $G^{F'}$, where we set $C^{[2]} := \{g^2 \mid g \in C\}$ for any conjugacy class $C \subset G^{F'}$. In general, this is quite a computational task. Still, we will compute $\nu_2(\chi_i)$ explicitly, by reducing the problem to the case of unipotent classes.

7.4. Unipotent classes of $G^{F'}$. By Shinoda [33, Theorem 2.1], there are 19 unipotent classes in $G^{F'}$ and these fall into 10 unipotent classes in the algebraic group G which are invariant under F'. As in [33], we denote representatives of the 19 classes by $u_0, \ldots, u_{18}$ where $u_0 = 1$. They are grouped together into classes of G as follows:

$$\begin{gathered} \{u_0\} = C_0, \quad \{u_1\} \subset C_1, \quad \{u_2\} \subset C_2, \quad \{u_3, u_4\} \subset C_3, \quad \{u_5\} \subset C_5, \\ \{u_6\} \subset C_6, \quad \{u_7, u_8, u_9\} \subset C_7, \quad \{u_{10}, u_{11}, u_{12}\} \subset C_{10}, \\ \{u_{13}, u_{14}\} \subset C_{13}, \quad \{u_{15}, u_{16}, u_{17}, u_{18}\} \subset C_{15}, \end{gathered}$$

where we denote by $C_0, C_1, \ldots$ the ten F'-invariant unipotent classes in G. With this notation, we claim that

$$\begin{gathered} C_1^{[2]} = C_2^{[2]} = C_0 = \{1\}, \\ C_3^{[2]} \subseteq C_1, \quad C_5^{[2]} \subseteq C_1, \quad C_6^{[2]} \subseteq C_1, \\ C_7^{[2]} \subseteq C_2, \quad C_{10}^{[2]} \subseteq C_5, \quad C_{13}^{[2]} \subseteq C_7, \quad C_{15}^{[2]} \subseteq C_{10}. \end{gathered}$$

This can be proved as follows. By [33], the ten F'-invariant unipotent classes in G do not depend on the choice of q. Hence we can see the power map already in the smallest possible case where we take $q = \sqrt{2}$. But in this case, the character table and the power maps are explicitly known and printed in the Cambridge Atlas [4, p. 75]. So the only problem is to determine the correspondence between the 19 unipotent classes in that table with the representatives u_i. However, this is easily done by looking at centralizer orders and some character values (setting $q = \sqrt{2}$ in Malle's tables). For example, consider u_5. We have $|C_G(u_5)^{F'}| = q^{16}$ by [33]. Setting $q = \sqrt{2}$, we obtain centralizer order 2^8. Comparing with the Atlas table, we see that u_5 must lie in the class denoted `4B` in that table. The information concerning the 2-power map in that table shows that u_5^2 lies in the class denoted `2A`, with centralizer of order $2^{12} \cdot 5$. Setting $q = \sqrt{2}$ in Shinoda's tables [33], we see that `2A` must correspond to the class C_1. Thus, we see that $C_5^{[2]} \subseteq C_1$. The remaining cases are treated by similar methods. We omit further details.

We can even obtain some more precise information on the 2-power map of the unipotent classes in the finite group $G^{F'}$. Indeed, by [33, Theorem 2.1], the representatives u_i already lie in ${}^2F_4(2) \subseteq G^{F'}$. Thus, if we know that u_i^2 is conjugate to u_j in ${}^2F_4(2)$ (for some i, j), then this will also hold in $G^{F'}$. For example, we have that

$$(*) \qquad u_{15}^2 \text{ is conjugate to } u_{11} \text{ or } u_{12} \text{ in } G^{F'}.$$

Indeed, using the Atlas table we see that, for all $g \in C_{15}^{F'}$ (the elements of order 16), the centralizer of g^2 has order 16. On the other hand, we have already seen above that g^2 must be conjugate to u_{10}, u_{11} or u_{12}. By [33], the orders of the centralizers of these elements are given by $2q^8$, $4q^8$ and $4q^8$, respectively. Setting $q = \sqrt{2}$ we see that u_{15}^2 cannot be conjugate to u_{10}.

Proposition 7.5. *We have $\nu_2(\chi_i) = 1$ for $i = 12, 13, 14$ and $\nu_2(\chi_{21}) = -1$.*

Proof. Using Malle's notation [29], we define the following linear combinations of unipotent characters of $G^{F'}$.

$$\begin{aligned}
\Phi_1 &:= \chi_9 + \chi_{10} - 2\chi_{11} - \chi_{12} - \chi_{13} - 2\chi_{14} + \chi_{15} + \chi_{16} + \chi_{17} + \chi_{18},\\
\Phi_2 &:= 3\chi_9 + 3\chi_{10} - 6\chi_{11} + \chi_{12} + \chi_{13} + 2\chi_{14} - 3\chi_{15}\\
&\qquad\qquad - 3\chi_{16} - 3\chi_{17} - 3\chi_{18} + 4\chi_{19} + 4\chi_{20} + 4\chi_{21}.
\end{aligned}$$

Using the table of character values in [29], it is a straightforward matter to check the following statements:

(1) Every $\mathbb{C}$-linear combination of unipotent characters which is zero on all elements of $G^{F'}$ except possibly those which are conjugate to u_{10}, u_{11}, u_{12} is a scalar multiple of Φ_1. The values of Φ_1 on u_{10}, u_{11}, u_{12} are $4q^4$, $-4q^4$, $-4q^4$, respectively (and 0 otherwise).

(2) Every $\mathbb{C}$-linear combination of unipotent characters which is zero on all elements of $G^{F'}$ except possibly those which are conjugate to u_7, u_8, u_9 is a scalar multiple of Φ_2. The values of Φ_4 on u_7, u_8, u_9 are $-12q^6$, $12q^6$, $-12q^6$, respectively (and 0 otherwise).

(How did we get the idea to look for these linear combinations? As a heuristic principle, we just assumed that the theory of cuspidal character sheaves and its connection with almost characters would also hold for $G^{F'}$. Then, by using the information on the support of cuspidal character sheaves in bad characteristic in [34, 7.2], we are lead to consider linear combinations of unipotent characters whose values satisfy conditions (1) and (2). There are further such possible conditions, but only the above will be of interest to us. Thus, up to some scaling factor, the class functions Φ_i may be regarded as "characteristic functions of F'-stable cuspidal character sheaves of G". The actual computations were performed using the CHEVIE system [12].)

Now, each of the above functions is supported on very few conjugacy classes only. So we can try to compute the 2-power map related to these classes and evaluate $\nu_2(\Phi_i)$. First of all, we can make the following simplifications. We have $\mathbb{Q}(\chi_i) \not\subseteq \mathbb{R}$ for $i \in \{5, 6, 7, 8, 15, 16, 17, 18, 19, 20\}$. Hence we have $\nu_2(\chi_i) = 0$ in theses cases. Furthermore, by (7.1), χ_9, χ_{10} and χ_{11} lie in the principal series of $G^{F'}$ and they have Schur index 1. Hence they can be realized over $\mathbb{R}$ and so $\nu_2(\chi_i) = 1$ for $i = 9, 10, 12$. So the above relations yield the following equations:

$$\begin{aligned}
\nu_2(\Phi_1) &= -\nu_2(\chi_{12}) - \nu_2(\chi_{13}) - 2\chi_{14},\\
\nu_2(\Phi_2) &= \nu_2(\chi_{12}) + \nu_2(\chi_{13}) + 2\chi_{14} + 4\nu_2(\chi_{21}).
\end{aligned}$$

We first try to compute $\nu_2(\Phi_1)$. For this purpose, we have to look at all elements $g \in G^{F'}$ such that g^2 is conjugate to u_{10}, u_{11} or u_{12}, that is, $g^2 \in C_{10}$. But, any such g must be unipotent also, and the information in (7.4) shows that $g \in C_{15}$. Furthermore, by [33], we know that $|C_G(g)^{F'}| = 4q^4$ for $g \in C_{15}^{F'}$. Since the values of Φ_1 are $\pm 4q^4$ we conclude that $\nu_2(\Phi_1) = \pm 1 \pm 1 \pm 1 \pm 1$. Assume, if possible, that $\nu_2(\Phi_1) = 4$. Then (1) shows that g^2 must be conjugate to u_{10} in $G^{F'}$ for any $g \in C_{15}^{F'}$. But this contradicts (7.4)($*$). Hence we have

$$(\dagger) \qquad \nu_2(\Phi_1) < 4.$$

Now let us turn to $\nu_2(\Phi_2)$. Here, we have to look at all elements $g \in G^{F'}$ such that g^2 is conjugate to u_7, u_8 or u_9, that is, $g^2 \in C_7$. Again, any such g must be unipotent also, and the information in (7.4) shows that $g \in C_{13}$. By [33], we know that $|C_G(g)^{F'}| = 2q^6$ for $g \in C_{13}^{F'}$. Since the values of Φ_2 are $\pm 12q^6$, we conclude that $\nu_2(\Phi_2) = \pm 6 \pm 6$. Now, since $\nu_2(\chi) \in \{0, 1, -1\}$ for any irreducible character χ (of any finite group), we certainly have $|\nu_2(\Phi_2)| \leqslant 8$. Hence the only possibility is that $\nu_2(\Phi_2) = 0$.

Now the above relations also show that $\nu_2(\Phi_1) = 4\nu_2(\chi_{21}) - \nu_2(\Phi_2) = 4\nu_2(\chi_{21}) = \pm 4$. But, by (†), we have $\nu_2(\Phi_1) < 4$. Hence we conclude that $\nu_2(\Phi_1) = -4$ and so $\nu_2(\chi_i) = 1$ for $i = 12, 13, 14$ and $\nu_2(\chi_{21}) = -1$, as claimed. □

By combining the above results, we see that $m_{\mathbb{Q}}(\chi_{21}) = 2$ and $m_{\mathbb{Q}}(\chi_i) = 1$ for $i \neq 21$. Thus, Theorem 1.6 is proved.

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