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QUANTUM LOOP MODULES

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Dedicated to Anthony Joseph on the occasion of his 60th birthday

ABSTRACT. We classify the simple infinite-dimensional integrable modules with finite-dimensional weight spaces over the quantized enveloping algebra of an untwisted affine algebra. We prove that these are either highest (lowest) weight integrable modules or simple submodules of a loop module of a finite-dimensional simple integrable module and describe the latter class. Their characters and crystal basis theory are discussed in a special case.

0. INTRODUCTION

The aim of this paper is to study irreducible integrable modules for quantum affine algebras, which have finite-dimensional weight spaces. The best known examples of such representations are the highest weight representations $\widehat{V(\lambda)}$ (cf. [16, 24]) on which the center of the quantum affine algebra acts via a positive integer power of q. These representations have many pleasant properties; for instance, it is known that they admit a canonical global or crystal basis. Another family of integrable modules for the quantum affine algebra are the finite-dimensional modules which have been studied, amongst others, in [1, 8, 11, 12, 13, 18, 25, 26, 27]. However, unlike the highest weight representations, these finite-dimensional representations do not respect the natural $\mathbf{Z}$-grading on the quantum affine algebra which arises from the adjoint action of the element of the torus corresponding to the Euler operator. Thus it is natural to look for graded analogues of finite-dimensional modules. Besides, in certain cases one has to consider these graded infinite-dimensional modules instead of finite-dimensional ones. For example, a finite-dimensional module cannot appear as a submodule of the ring of linear endomorphisms of $\widehat{V(\lambda)}$ whilst any simple graded integrable module with nontrivial zero weight space can be embedded in such a ring provided that λ is sufficiently large (cf., for example, [20]). Such embeddings arise in the theory of quantum affine analogues of the Parthasarathy-Ranga Rao-Varadarajan determinants ([22]).

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank n and let $\mathbf{U}_q$ be the associated quantum affine algebra. Examples of infinite-dimensional integrable modules which are not highest weight modules are easy to construct. Namely, let V be a finite-dimensional module over $\mathbf{U}_q$. Then the space $L(V) = V \otimes \mathbf{C}(q)[t, t^{-1}]$ admits an obvious structure of a $\mathbf{Z}$-graded $\mathbf{U}_q$-module, which respects the grading on $\mathbf{U}_q$ induced by the Euler operator. However, even though V is irreducible, the resulting representation $L(V)$ need not remain so.

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The irreducible finite-dimensional representations of $\mathbf{U}_q$ are known (cf. [8]) to be parametrized by n-tuples of polynomials in an indeterminate u, with constant term 1. For instance, for the quantum affine algebra associated to $\mathfrak{sl}_2$, we have just one polynomial $\pi(u)$. If we take $\pi(u) = (1-u)(1+u)$, and V to be the corresponding irreducible finite-dimensional representation of $\mathbf{U}_q$, then it is not hard to see that the graded infinite-dimensional module $L(V)$ is a direct sum of two simple components. More generally, the module $L(V)$ is reducible (moreover, completely reducible) if the roots of the polynomials associated to V satisfy a certain condition which involves roots of unity.

The paper is organized as follows. In Section 2 we show that if V is an irreducible finite-dimensional representation of the quantum affine algebra, then the corresponding graded representation on $L(V)$ is completely reducible and we describe its irreducible components (cf. Theorem 1 and Lemma 2.8). Our analysis is based on the result on irreducibility of tensor products of finite-dimensional simple $\mathbf{U}_q$-modules obtained in [5], which allows one to construct an action of the cyclic group $\mathbf{Z}/m\mathbf{Z}$ on $L(V)$ commuting with that of the quantum affine algebra, the integer m being determined by the family of polynomials corresponding to V. Namely, these turn out to be polynomials in u^m. The irreducible components of $L(V)$, as in the classical case (cf. [4, 6, 14]), correspond to different irreducible characters of the finite abelian group $\mathbf{Z}/m\mathbf{Z}$. However, unlike in the classical case when it is induced by the natural action of the symmetric group S_m on V, the action of $\mathbf{Z}/m\mathbf{Z}$ is rather sophisticated and difficult to describe explicitly.

Section 3 is devoted to a classification of irreducible, integrable modules with finite-dimensional weight spaces of the quantum affine algebra. Thus our Theorem 5 establishes that such a module must be isomorphic to either a highest weight module or its (graded) dual, or to an irreducible component of the module $L(V)$, where V is some irreducible finite-dimensional module over $\mathbf{U}_q$. The corresponding result in the classical case was established in [4, 6]. Besides, we show (cf. Proposition 3.5) that, as in the classical case (cf. [21]), all weight spaces of a simple integrable module are finite dimensional if and only if its set of weights satisfies a boundedness condition of [20].

The final section of the paper is concerned with the problem of describing the characters of irreducible components of the module $L(V)$ in terms of the characters of V. More precisely, we are interested in relating the dimension of the weight spaces of an irreducible component of $L(V)$ to the dimension of the weight spaces of V. In the classical case this was done in [14], and the formulae involve a modification of the Euler φ-function. In the quantum case we conjecture an analogous formula and establish it in Proposition 4.7 for certain special modules for the quantum affine algebra associated to $\mathfrak{sl}_{n+1}$. In order to do that, we consider a crystal $q=0$ limit of these modules. Then the argument becomes purely combinatorial and involves the major index of MacMahon. In particular, we also show that the characters of these modules in the quantum case coincide with the characters of their classical analogues. At the end of the section we discuss a crystal basis theory for these modules.

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1. Preliminaries

Throughout this paper, $\mathbf{N}$ (respectively, $\mathbf{N}^+$) denotes the set of nonnegative (respectively, positive) integers. Let q be an indeterminate and let $\mathbf{C}(q)$ be the field of rational functions in q with complex coefficients. For $r, m \in \mathbf{N}$, $m \geq r$, define

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \qquad [m]_q! = [m]_q[m-1]_q \dots [2]_q[1]_q, \qquad \begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{[m]_q!}{[r]_q![m-r]_q!}.$$

1.1. Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra of rank n with a Cartan subalgebra $\mathfrak{h}$ and let W be the Weyl group of $\mathfrak{g}$. Set $I = \{1, 2, \dots, n\}$ and let $A = (d_i a_{ij})_{i,j \in I}$, where the d_i are positive integers, be the $n \times n$ symmetrized Cartan matrix of $\mathfrak{g}$. Let $\{\alpha_i : i \in I\} \subset \mathfrak{h}^*$ (respectively $\{\varpi_i : i \in I\} \subset \mathfrak{h}^*$) be the set of simple roots (respectively of fundamental weights) of $\mathfrak{g}$ with respect to $\mathfrak{h}$. As usual, Q (respectively, P) denotes the root (respectively, weight) lattice of $\mathfrak{g}$. Let $P^+ = \sum_{i \in I} \mathbf{N}\varpi_i$ be the set of dominant weights and set $Q^+ = \sum_{i \in I} \mathbf{N}\alpha_i$. It is well known that $\mathfrak{h}^*$ admits a nondegenerate symmetric W-invariant bilinear form which will be denoted by $(\cdot \,|\, \cdot)$. We assume that $(\alpha_i \,|\, \alpha_i) = d_i a_{ij}$ for all $i, j \in I$.

Let

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d$$

be the untwisted extended affine algebra associated with $\mathfrak{g}$ and let $\hat{A} = (d_i a_{ij})_{i,j \in \hat{I}}$, where $\hat{I} = I \cup \{0\}$, be the extended symmetrized Cartan matrix and $\widehat{W}$ the affine Weyl group. Set $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbf{C}c \oplus \mathbf{C}d$. From now on we identify $\mathfrak{h}^*$ with the subspace of $\widehat{\mathfrak{h}}^*$ consisting of elements which are zero on c and d. Define $\delta \in \widehat{\mathfrak{h}}^*$ by

$$\delta(\mathfrak{h} \oplus \mathbf{C}c) = 0, \qquad \delta(d) = 1.$$

Denote by θ the highest root of $\mathfrak{g}$ and set $\alpha_0 = \delta - \theta$. Then $\{\alpha_i : i \in \hat{I}\}$ is a set of simple roots for $\widehat{\mathfrak{g}}$ with respect to $\widehat{\mathfrak{h}}$ and δ generates its imaginary roots. The bilinear form on $\mathfrak{h}^*$ extends to a $\widehat{W}$-invariant bilinear form on $\widehat{\mathfrak{h}}^*$ which we continue to denote by $(\cdot \,|\, \cdot)$. One has $(\delta \,|\, \alpha_i) = 0$ and $(\alpha_i \,|\, \alpha_j) = d_i a_{ij}$, for all $i, j \in \hat{I}$. Define a set of fundamental weights $\{\omega_i : i \in \hat{I}\} \subset \widehat{\mathfrak{h}}^*$ of $\widehat{\mathfrak{g}}$ by the conditions $(\omega_i | \alpha_j) = d_i \delta_{i,j}$ and $\omega_i(d) = 0$ for all $i, j \in \hat{I}$. Let $\widehat{P} = \sum_{i \in \hat{I}} \mathbf{Z}\omega_i \oplus \mathbf{Z}\delta$ (respectively, $\widehat{P}^+ = \sum_{i \in \hat{I}} \mathbf{N}\omega_i \oplus \mathbf{Z}\delta$) be the corresponding set of integral (respectively, dominant) weights. Extend the ϖ_i, $i \in I$ to elements of $\widehat{P}$ by setting $\varpi_i(c) = \varpi_i(d) = 0$. Set $P^e = P \oplus \mathbf{Z}\delta \subset \widehat{P}$. Denote by $\widehat{Q}$ the root lattice of $\widehat{\mathfrak{g}}$ and set $\widehat{Q}^+ = \sum_{i \in \hat{I}} \mathbf{N}\alpha_i$. Given $\lambda, \mu \in \widehat{P}^+$ (respectively, $\lambda, \mu \in P^+$) we say that $\lambda \leq \mu$ if $\mu - \lambda \in \widehat{Q}^+$ (respectively, $\mu - \lambda \in Q^+$).

1.2. For $i \in \hat{I}$, set $q_i = q^{d_i}$ and $[m]_i = [m]_{q^i}$. The quantum affine algebra $\widehat{\mathbf{U}}_q(\mathfrak{g})$ (cf. [2, 3, 10, 19]) associated to $\mathfrak{g}$, which will be further denoted as $\widehat{\mathbf{U}}_q$, is an associative algebra over $\mathbf{C}(q)$ with generators $x_{i,r}^{\pm}$, $i \in I$, $r \in \mathbf{Z}$, $K_i^{\pm 1}$, $i \in I$, $C^{\pm 1/2}$,

$D^{\pm 1}$, $h_{i,r}$, $i \in I$, $r \in \mathbf{Z}\backslash\{0\}$, and the following defining relations

$$C^{\pm 1/2} \text{ are central,}$$
$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad C^{1/2}C^{-1/2} = C^{-1/2}C^{1/2} = 1,$$
$$K_iK_j = K_jK_i, \quad K_ih_{j,r} = h_{j,r}K_i,$$
$$K_ix_{j,r}^{\pm}K_i^{-1} = q_i^{\pm a_{ij}}x_{j,r}^{\pm},$$
$$[h_{i,r}, h_{j,s}] = \delta_{r,-s}\,\frac{1}{r}[ra_{ij}]_i\,\frac{C^r - C^{-r}}{q_j - q_j^{-1}},$$
$$[h_{i,r}, x_{j,s}^{\pm}] = \pm\frac{1}{r}[ra_{ij}]_i\,C^{\mp|r|/2}x_{j,r+s}^{\pm},$$
$$x_{i,r+1}^{\pm}x_{j,s}^{\pm} - q_i^{\pm a_{ij}}x_{j,s}^{\pm}x_{i,r+1}^{\pm} = q_i^{\pm a_{ij}}x_{i,r}^{\pm}x_{j,s+1}^{\pm} - x_{j,s+1}^{\pm}x_{i,r}^{\pm},$$
$$[x_{i,r}^{+}, x_{j,s}^{-}] = \delta_{i,j}\,\frac{C^{(r-s)/2}\,\psi_{i,r+s}^{+} - C^{-(r-s)/2}\,\psi_{i,r+s}^{-}}{q_i - q_i^{-1}},$$
$$\sum_{\pi\in\Sigma_m}\sum_{k=0}^{m}(-1)^k\begin{bmatrix} m \\ k \end{bmatrix}_i x_{i,r_{\pi(1)}}^{\pm}\ldots x_{i,r_{\pi(k)}}^{\pm}x_{j,s}^{\pm}x_{i,r_{\pi(k+1)}}^{\pm}\ldots x_{i,r_{\pi(m)}}^{\pm} = 0, \qquad \text{if } i \neq j,$$

for all sequences of integers $r_1, \ldots, r_m$, where $m = 1 - a_{ij}$, Σ_m is the symmetric group on m letters, and the $\psi_{i,r}^{\pm}$ are determined by equating powers of u in the formal power series

$$\sum_{r=0}^{\infty}\psi_{i,\pm r}^{\pm}u^{\pm r} = K_i^{\pm 1}\exp\left(\pm(q_i - q_i^{-1})\sum_{s=1}^{\infty}h_{i,\pm s}u^{\pm s}\right).$$

The algebra $\widehat{\mathbf{U}}_q(\mathfrak{g})$ is $\mathbf{Z}$-graded, the lth-graded piece being

$$(\widehat{\mathbf{U}}_q(\mathfrak{g}))_l = \{x \in \widehat{\mathbf{U}}_q(\mathfrak{g}) \,:\, DxD^{-1} = q^lx\}.$$

The subalgebra of $\widehat{\mathbf{U}}_q(\mathfrak{g})$ generated by the elements $x_{i,0}^{\pm}$, $i \in I$ is isomorphic to the quantized enveloping algebra $\mathbf{U}_q(\mathfrak{g})$ of $\mathfrak{g}$. Let $\widehat{\mathbf{U}}_q^{\pm}(\gg)$ (respectively $\widehat{\mathbf{U}}_q^{\pm}(\ll)$, $\widehat{\mathbf{U}}_q^{\pm}(0)$) be the subalgebra of $\widehat{\mathbf{U}}_q$ generated by the elements $x_{i,r}^{+}$, $i \in I$, $\pm r \in \mathbf{N}$ (respectively, $x_{i,r}^{-}$, $i \in I$, $\pm r \in \mathbf{N}$, $h_{i,r}$, $i \in I$, $\pm r \in \mathbf{N}^+$). Let $\widehat{\mathbf{U}}_q^{\circ}$ be the subalgebra generated by $K_i^{\pm 1}$, $i \in I$, $D^{\pm 1}$ and $C^{\pm 1/2}$. We will need the following result which was established in [3].

Proposition. *The subspaces* $\widehat{\mathbf{U}}_q^{\pm} = \widehat{\mathbf{U}}_q^{\pm}(\ll)\widehat{\mathbf{U}}_q^{\pm}(0)\widehat{\mathbf{U}}_q^{\pm}(\gg)$ *are subalgebras of* $\widehat{\mathbf{U}}_q$ *and*

$$\widehat{\mathbf{U}}_q = \widehat{\mathbf{U}}_q^{-}\widehat{\mathbf{U}}_q^{\circ}\widehat{\mathbf{U}}_q^{+}.$$

1.3. We will also need another presentation of $\widehat{\mathbf{U}}_q$. Namely, after [2, 19], the algebra $\widehat{\mathbf{U}}_q$ is isomorphic to an associative $\mathbf{C}(q)$-algebra generated by $E_i, F_i, K_i^{\pm 1}$: $i \in \hat{I}$, $D^{\pm 1}$ and central elements $C^{\pm 1/2}$ satisfying the following relations:

$$C = K_0\prod_{i\in I}K_i^{r_i}, \text{ where } \theta = \sum_{i\in I}r_i\alpha_i,\ r_i \in \mathbb{N}^+,$$
$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \qquad K_iK_j = K_jK_i,$$
$$DD^{-1} = D^{-1}D = 1, \qquad K_iD = DK_i,$$

$$K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j, \qquad K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j,$$
$$DE_jD^{-1} = q^{\delta_{j0}}E_j, \qquad DF_jD^{-1} = q^{-\delta_{j0}}F_j,$$
$$[E_i, F_j] = \delta_{ij}\,\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i (E_i)^r E_j (E_i)^{1-a_{ij}-r} = 0 \qquad \text{if } i \neq j,$$
$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i (F_i)^r F_j (F_i)^{1-a_{ij}-r} = 0 \qquad \text{if } i \neq j.$$

The element E_i (respectively, F_i), $i \in I$ corresponds to $x_{i,0}^+$ (respectively, $x_{i,0}^-$). In particular, the $E_i, F_i, K_i^{\pm 1} : i \in I$ generate $\mathbf{U}_q(\mathfrak{g})$.

It is well known that $\widehat{\mathbf{U}}_q$ is a Hopf algebra over $\mathbf{C}(q)$ with the co-multiplication being given in terms of generators $E_i, F_i, K_i^{\pm 1} : i \in \hat{I}$ by the following formulae:

$$\Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \qquad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i,$$

the $K_i^{\pm 1}$ being group-like. Although explicit formulae for the co-multiplication on generators $x_{i,r}^\pm$, $h_{i,r}$ are not known, we will not need these in the present paper and so we say no more about it.

Lemma. *Take $x \in (\widehat{\mathbf{U}}_q)_k$ and write $\Delta(x) = x_1 \otimes x_2$ in the summation notation. Then we may assume that $x_i \in (\widehat{\mathbf{U}}_q)_{k_i}$ where $k_1 + k_2 = k$.*

Proof. The assertion is obvious for the generators $E_i, F_i, K_i^{\pm 1}$. Since Δ is an algebra homomorphism, it holds for any polynomial in these generators, which is homogeneous with respect to D. □

1.4. Let $\mathbf{U}_q^e$ be the extended quantum loop algebra, namely the graded quotient of $\widehat{\mathbf{U}}_q$ by the graded ideal generated by $C^{\pm 1/2} - 1$. The Hopf algebra structure on $\widehat{\mathbf{U}}_q$ descends to a Hopf algebra structure on $\mathbf{U}_q^e$. Let $\mathbf{U}_q$ be the $\mathbf{C}(q)$-subalgebra of $\mathbf{U}_q^e$ generated by the elements $x_{i,k}^\pm$, $h_{i,r}$, K_i, K_i^{-1}, $i \in I$, $k, r \in \mathbf{Z}$, $r \neq 0$. It is easy to see that $\mathbf{U}_q$ is a Hopf subalgebra of $\mathbf{U}_q^e$. Let $\mathbf{U}_q^e(0)$ be the subalgebra of $\mathbf{U}_q^e$ generated by elements $h_{i,r}$, $K_i^{\pm 1}$, $i \in I$, $r \in \mathbf{Z}$ and $D^{\pm 1}$. Clearly, $\mathbf{U}_q^e(0)$ is a $\mathbf{Z}$-graded subalgebra of $\mathbf{U}_q^e$. Let $\mathbf{U}_q(>)$ (respectively, $\mathbf{U}_q(<)$ and $\mathbf{U}_q(0)$) be the subalgebra of $\mathbf{U}_q^e(0)$ generated by the elements $x_{i,r}^+$, $i \in I$, $r \in \mathbf{Z}$ (respectively, $x_{i,r}^-$, $i \in I$, $r \in \mathbf{Z}$ and $h_{i,r}$, $i \in I$, $r \in \mathbf{Z} \setminus \{0\}$). Then (cf., for example, [8])

$$\mathbf{U}_q^e = \mathbf{U}_q(<)\mathbf{U}_q^e(0)\mathbf{U}_q(>).$$

For $i \in I$, set

$$h_i^\pm(u) = \sum_{k=1}^{\infty} \frac{q^{\pm k} h_{i,\pm k}}{[k]_i} u^k,$$

and

$$P_i^\pm(u) = \exp(-h_i^\pm(u)) = \exp\Big(-\sum_{k=1}^{\infty} \frac{q^{\pm k} h_{i,\pm k}}{[k]_i} u^k\Big).$$

Let $P_{i,\pm r}$ be the coefficient of u^r in $P_i^\pm(u)$. One can show (cf. [3]) that the $P_{i,r}$: $i \in I$, $r \in \mathbf{Z}$ generate $\mathbf{U}_q(0)$. Moreover, by [3] monomials in the $P_{i,r}$, $i \in I$, $r \in \mathbf{Z}$ (or equivalently, monomials in the $h_{i,r}$, $i \in I$, $r \in \mathbf{Z}$) form a basis of $\mathbf{U}_q(0)$.

Lemma. *Let $\chi : \mathbf{U}_q(0) \to \mathbf{C}(q)[t, t^{-1}]$ be a nontrivial homomorphism of $\mathbf{Z}$-graded algebras.*

(i) *There exists a unique $m > 0$ such that the image of χ equals $\mathbf{C}(q)[t^m, t^{-m}]$.*
(ii) *Suppose that the image of χ equals $\mathbf{C}(q)[t^m, t^{-m}]$. Then there exist $i_0 \in I$ such that $\chi(P_{i_0,\pm m}) \neq 0$ and the kernel of χ is generated by the $P_{i,r}$, $i \in I$, $r \neq 0 \pmod m$ and by the elements of the form*

$$P_{i,\pm sm} - (\chi(P_{i_0,\pm m})^{-s}\chi(P_{i,\pm sm}))P^s_{i_0,\pm m}, \qquad i \in I, s \in \mathbf{N}^+.$$

Proof. Let $m, n \in \mathbf{N}^+$ be minimal such that both t^m, t^{-n} lie in the image of χ. Then $t^{m-n} \in \operatorname{Im}\chi$. Since either $0 \le m - n < m$ or $0 \le n - m < n$, it follows that $m = n$ which proves the first part. For the second part, it is enough to observe that the elements listed in the assertion are homogeneous and lie in the kernel of $\operatorname{Im}\chi$. □

1.5. In the rest of this section we summarize some general results from the representation theory of $\widehat{\mathbf{U}}_q$ and $\mathbf{U}_q(\mathfrak{g})$, which will be used later.

For any $\widehat{\mathbf{U}}_q$-module $\widehat{V}$ and any $\mu = \sum_{i\in\hat I} \mu_i\omega_i + l\delta \in \widehat{P}$, set

$$\widehat{V}_\mu = \{v \in \widehat{V} \,:\, D.v = q^l v,\, K_i.v = q_i^{\mu_i} v,\, \forall\, i \in \hat I\}.$$

If $\widehat{V}_\mu \neq 0$, we say that μ is a weight of $\widehat{V}$. The set of weights of $\widehat{V}$ will be denoted by $\Omega(\widehat{V})$. The module $\widehat{V}$ is said to be an *admissible module of type* 1 if

$$\widehat{V} = \bigoplus_{\mu\in\widehat{P}} \widehat{V}_\mu,$$

and $\dim \widehat{V}_\mu < \infty$ for all $\mu \in \widehat{P}$. One has analogous definitions of admissible modules of type 1 for the algebras $\mathbf{U}_q^e$ (with $\widehat{P}$ replaced by P^e), $\mathbf{U}_q$ and $\mathbf{U}_q(\mathfrak{g})$ (with $\widehat{P}$ replaced by P). From now on, all modules will be assumed to be of type 1. A $\widehat{\mathbf{U}}_q$-module $\widehat{V}$ is integrable if for all $i \in \hat I$ the elements E_i and F_i act locally nilpotently on $\widehat{V}$. Similarly, one can define integrable $\mathbf{U}_q^e$, $\mathbf{U}_q$ and $\mathbf{U}_q(\mathfrak{g})$-modules.

1.6. We now recall the construction of highest weight integrable modules over $\widehat{\mathbf{U}}_q$ and $\mathbf{U}_q(\mathfrak{g})$. We work with the presentation of $\widehat{\mathbf{U}}_q$ described in 1.4.

The following result can be found in [24, 3.5].

Proposition. *For every $\lambda = \sum_{i\in\hat I} \lambda_i\omega_i + k\delta \in \widehat{P}^+$ (respectively $\lambda = \sum_{i\in I}\lambda_i\varpi_i \in P^+$) there exists a unique, up to an isomorphism, simple integrable $\widehat{\mathbf{U}}_q$-module $\widehat{V}(\lambda)$ (respectively $\mathbf{U}_q(\mathfrak{g})$-module $V(\lambda)$) of type 1 which is generated by an element v_λ satisfying*

$$E_i.v_\lambda = 0, \qquad K_i.v_\lambda = q_i^{\lambda_i} v_\lambda, \qquad D.v_\lambda = q^k v_\lambda, \qquad F_i^{\lambda_i+1}.v_\lambda = 0, \qquad \forall\, i \in \hat I$$

(respectively, $E_i.v_\lambda = 0$, $K_i.v_\lambda = q_i^{\lambda_i} v_\lambda$, $F_i^{\lambda_i+1}.v_\lambda = 0$, $\forall\, i \in I$).

1.7. The modules $\widehat{V}(\lambda)$ and $V(\lambda)$ are the quantum analogues of the corresponding modules $\widehat{V}_{c\ell}(\lambda)$ and $V_{c\ell}(\lambda)$ over, respectively, $\widehat{\mathfrak{g}}$ and $\mathfrak{g}$, whose characters are given by the Weyl-Kac formula. Namely (cf. [23, Theorem 4.12]), for all $\nu \in \widehat{P}$, (respectively $\nu \in P$), we have $\dim \widehat{V}(\lambda)_\nu = \dim \widehat{V}_{c\ell}(\lambda)_\nu$ (respectively, $\dim V(\lambda)_\nu = \dim V_{c\ell}(\lambda)_\nu$). In particular, both $\widehat{V}(\lambda)$ and $V(\lambda)$ have finite-dimensional weight spaces. Furthermore, a standard argument from the representation theory of $\mathfrak{g}$ yields the following.

Lemma. *Let $\mu \in P^+$ be such that $\lambda - \mu \in Q^+$. Then* $\dim V(\lambda)_\mu \neq 0$.

Finally, we will need the following.

Proposition (cf. [24, 3.5 and 6.3]). (i) *Assume that $V = \bigoplus_{\nu\in\widehat{P}} V_\nu$ is an integrable admissible $\widehat{\mathbf{U}}_q$-module such that $\Omega(V)$ is contained in the set $\bigcup_{i=1}^k \{\mu_i - \nu : \nu \in \widehat{Q}^+\}$, for some $k \in \mathbb{N}^+$ and for some $\mu_1, \dots, \mu_k \in \widehat{P}^+$. Then*

$$V \cong \bigoplus_{\lambda\in\widehat{P}^+} m(\lambda)\widehat{V}(\lambda),$$

for some nonnegative integers $m(\lambda)$. Furthermore, as a $\mathbf{U}_q(\mathfrak{g})$-module, $\widehat{V}(\lambda)$ is a direct sum of simple finite-dimensional highest weight modules $V(\mu)$ for various $\mu \in P^+$.

(ii) *Let M be an integrable $\mathbf{U}_q(\mathfrak{g})$-module. Then M is a sum of simple $\mathbf{U}_q(\mathfrak{g})$-modules of form $V(\lambda)$ for various $\lambda \in P^+$.*

Corollary. *Let M be an integrable $\mathbf{U}_q(\mathfrak{g})$-module with finite dimensional weight spaces. Then M is finite-dimensional.*

Proof. Let $r \in \mathbf{N}^+$ be minimal such that $r\varpi_i \in Q^+$ for all $i \in I$. Then any $\lambda \in P^+$ can be written as $\lambda' + \eta$ where $\eta \in Q^+$ and $\lambda' \in P_r^+ := \{\nu = \sum_{i\in I} k_i\varpi_i \in P^+ : 0 \le k_i < r\}$. Evidently, P_r^+ is a finite set. On the other hand, by Lemma 1.7, for all $\lambda \in P^+$, there exists $\lambda' \in \Omega(V(\lambda)) \cap P_r^+$. Since $\dim M_{\lambda'} < \infty$ and the sum of $V(\lambda)$ is direct, it follows that the multiplicity of each $V(\lambda)$ in M is finite and that the set of $\lambda \in P^+$ such that $V(\lambda)$ occurs in M is also finite. It remains to apply (ii) of the above proposition. □

1.8. Since $\widehat{\mathbf{U}}_q$ is a Hopf algebra, given a $\widehat{\mathbf{U}}_q$-module $V = \bigoplus_{\nu\in\widehat{P}} V_\nu$, we can endow V^* with a structure of a $\widehat{\mathbf{U}}_q$ module via the antipode. If $\dim V_\nu < \infty$ for all $\nu \in \widehat{P}$, then $V^\# = \bigoplus_{\nu\in\widehat{P}} V_\nu^*$, is a $\widehat{\mathbf{U}}_q$-submodule of V^*. The module $V^\#$ is called the graded dual of V. One can prove that the graded dual of $\widehat{V}(\lambda)$, $\lambda \in \widehat{P}^+$ is the unique simple integrable module generated by an element v_λ^* such that

$$F_i.v_\lambda^* = 0, \qquad K_i.v_\lambda^* = q_i^{-\lambda_i} v_\lambda^*, \qquad D.v_\lambda^* = q^{-k} v_\lambda^*, \qquad E_i^{\lambda_i+1} v_\lambda^* = 0.$$

Clearly, $\widehat{V}(\lambda)^\#$ is an integrable module. Results analogous to the ones above hold for the modules $\widehat{V}(\lambda)^\#$. Finally, note that the element C acts on $\widehat{V}(\lambda)$ as q^r id and on $\widehat{V}(\lambda)^\#$ as q^{-r} id where $r = (\lambda \,|\, \delta)$. Notice that $r > 0$ unless $\lambda \in \widehat{P}^+ \cap P^e = \mathbf{Z}\delta$. In the latter case both $\widehat{V}(\lambda)$ and $\widehat{V}(\lambda)^\#$ are one-dimensional.

1.9. Let M be an integrable $\mathbf{U}_q(\mathfrak{g})$ or $\widehat{\mathbf{U}}_q$-module. Following [24, 5.2], one can define $\mathbf{C}(q)$-linear endomorphisms $T_i : i \in I$ (respectively, $i \in \hat{I}$) of M satisfying $T_i M_\lambda = M_{s_i\lambda}$. Moreover, by [24, Chapter 39], the T_i satisfy the relations of the braid group associated with W (respectively, $\widehat{W}$). In particular, the set of weights of M is W (respectively $\widehat{W}$) invariant. Moreover, if M is admissible, then we have $\dim M_\lambda = \dim M_{w\lambda}$ for all $w \in W$ (respectively, $w \in \widehat{W}$).

2. Quantum loop modules

In this section we study a family of irreducible integrable modules of $\widehat{\mathbf{U}}_q$ on which C acts as the identity and hence these are actually modules for the extended quantum loop algebra $\mathbf{U}_q^e$.

2.1. Given a $\mathbf{U}_q$ module V of type 1, one can easily verify that the following formulae define a structure of a $\mathbf{U}_q^e$-module of type 1, which we denote by $L(V;d)$, on the vector space $L(V) = V \otimes \mathbf{C}(q)[t,t^{-1}]$. Namely, fix $d \in \mathbf{Z}$ and define for all $k, r \in \mathbf{Z}$, $x \in (\mathbf{U}_q^e)_k$ and $v \in V$,

$$x(v \otimes t^r) = (xv) \otimes t^{r+k}, \qquad D(v \otimes t^r) = q^{d+r}\, v \otimes t^r.$$

We set $L(V) = L(V;0)$ and call it the quantum loop module associated to V.

Lemma. *Let V be a cyclic $\mathbf{U}_q$ module generated by an element $v \in V$. Then $L(V)$ is generated as a $\mathbf{U}_q^e$-module by the elements $v \otimes t^r$, $r \in \mathbf{Z}$.*

Proof. Immediate. □

If V is a finite-dimensional $\mathbf{U}_q$-module, then the corresponding loop module $L(V)$ is an integrable $\mathbf{U}_q^e$-module.

The main result of this section is the following

Theorem 1. *Let V be an irreducible finite-dimensional $\mathbf{U}_q$-module and let $d \in \mathbb{Z}$. Then there exist $v \in V$ and a unique $m \in \mathbf{N}^+$ such that as $\mathbf{U}_q^e$-modules we have,*

$$L(V;d) = \bigoplus_{s=0}^{m-1} \mathbf{U}_q^e.(v \otimes t^s),$$

where $\mathbf{U}_q^e.(v \otimes t^s)$ is an irreducible $\mathbf{U}_q^e$-module for all $0 \le s \le m-1$.

We prove this theorem in the remainder of this section. For simplicity of notation, we assume that $d = 0$, the general case being identical.

2.2. We need several results about irreducible finite-dimensional representations of $\mathbf{U}_q$ which we now recall (cf. [8]). Let

$$\mathcal{A} = \{\pi = \sum_{m \ge 0} \pi_m u^m \in \mathbf{C}(q)[[u]] \,:\, \pi(0) = 1\}.$$

Definition. We say that a $\mathbf{U}_q$-module V is ℓ-highest weight, with highest weight $(\lambda, \boldsymbol{\pi}^\pm)$, where $\lambda = \sum_{i \in I} \lambda_i \varpi_i$, $\boldsymbol{\pi}^\pm = (\pi_1^\pm(u), \dots, \pi_n^\pm(u)) \in \mathcal{A}^n$, if there exists $0 \ne v \in V_\lambda$ such that $V = \mathbf{U}_q.v$ and

$$x_{i,k}^+.v = 0, \qquad K_i.v = q_i^{\lambda_i} v, \qquad P_i^\pm(u).v = \pi_i^\pm(u) v,$$

for all $i \in I$, $k \in \mathbf{Z}$. Such an element v is called a highest weight vector.

If V is an ℓ-highest weight module, then in fact $V = \mathbf{U}_q(<).v$ and so

$$V_\mu \ne 0 \implies \mu = \lambda - \eta \quad (\eta \in Q^+).$$

For any $\lambda \in P^+$, $\boldsymbol{\pi}^\pm \in \mathcal{A}^n$, there exists a unique (up to an isomorphism) irreducible highest weight $\mathbf{U}_q$-module with highest weight $(\lambda, \boldsymbol{\pi}^\pm)$. Write $\pi_i^\pm(u) = \sum_{r \ge 0} \pi_{i,r}^\pm u^r$ and let $I(\lambda, \boldsymbol{\pi}^\pm)$ be the left ideal in $\mathbf{U}_q$ generated by $\mathbf{U}_q(>)_+$ and the elements $P_{i,\pm r} - \pi_{i,r}^\pm$, $K_i - q_i^{\lambda_i} \,:\, i \in I$, $r \in \mathbf{N}^+$ and let $M(\lambda, \boldsymbol{\pi}^\pm)$ be the quotient of $\mathbf{U}_q$ by $I(\lambda, \boldsymbol{\pi}^\pm)$. Let $\bar v_{\boldsymbol{\pi}}$ be the canonical image of $1 \in \mathbf{U}_q$ in $M(\lambda, \boldsymbol{\pi}^\pm)$. This module is ℓ-highest weight and has a unique irreducible quotient which we denote as $V(\lambda, \boldsymbol{\pi}^\pm)$. Let $v_{\boldsymbol{\pi}}$ be the canonical image of $\bar v_{\boldsymbol{\pi}}$ in $V(\lambda, \boldsymbol{\pi}^\pm)$.

Lemma. *Let $\boldsymbol{\pi}^\pm \in \mathcal{A}$. Take $a \in \mathbf{C}(q)^\times$ and set $\boldsymbol{\pi}_a^\pm = (\pi_1^m(au), \dots, \pi_n^\pm(au))$. Then there exists a canonical isomorphism $\phi_{\boldsymbol{\pi}^\pm,a} : V(\lambda, \boldsymbol{\pi}^\pm) \to V(\lambda, \boldsymbol{\pi}_a^\pm)$ of $\mathbf{U}_q(\mathfrak{g})$-modules, such that*

(i) *For all $x \in (\mathbf{U}_q)_k$, $v \in V(\lambda, \boldsymbol{\pi}^{\pm})$ we have,*

$$\phi_{\boldsymbol{\pi}^{\pm},a}(xv) = a^k x \phi_{\boldsymbol{\pi}^{\pm},a}(v).$$

(ii) *For all $a, b \in \mathbf{C}(q)^{\times}$,*

$$\phi_{\boldsymbol{\pi}_a^{\pm},b} \circ \phi_{\boldsymbol{\pi}^{\pm},a} = \phi_{\boldsymbol{\pi}^{\pm},ab}.$$

Proof. Given $a \in \mathbf{C}(q)^{\times}$, let ϕ_a be the graded algebra automorphism of $\mathbf{U}_q^e$ defined on generators by

$$\phi_a(x_{i,k}^{\pm}) = a^k x_{i,k}^{\pm}, \qquad \phi_a(h_{i,k}) = a^k h_{i,k}, \qquad \phi_a(K_i) = K_i, \qquad \phi_a(D) = D,$$

for all $i \in I$ and $k \in \mathbf{Z}$. Evidently, $\phi_a^{-1} = \phi_{a^{-1}}$. It follows immediately from the definitions that ϕ_a maps $I(\lambda, \boldsymbol{\pi}^{\pm})$ to $I(\lambda, \boldsymbol{\pi}_a^{\pm})$ and hence induces an isomorphism of vector spaces $\phi_a : M(\lambda, \boldsymbol{\pi}^{\pm}) \to M(\lambda, \boldsymbol{\pi}_a^{\pm})$. Since $\phi_a(x) = x$ if $x \in (\mathbf{U}_q)_0$, it follows that this map is an isomorphism of $(\mathbf{U}_q)_0$-modules, in particular, a map of $\mathbf{U}_q(\mathfrak{g})$-modules. Let $L(\lambda, \boldsymbol{\pi}^{\pm})$ be the maximal $\mathbf{U}_q$-submodule of $M(\lambda, \boldsymbol{\pi}^{\pm})$. We claim that $\phi_a(L(\lambda, \boldsymbol{\pi}^{\pm}))$ is a $\mathbf{U}_q$-submodule of $L(\lambda, \boldsymbol{\pi}_a^{\pm})$, even though ϕ_a is not a $\mathbf{U}_q$-module map. Indeed, take $y \in \mathbf{U}_q$ such that $y + I(\lambda, \boldsymbol{\pi}^{\pm}) \in L(\lambda, \boldsymbol{\pi}^{\pm})$. Then for all $x \in \mathbf{U}_q$,

$$\begin{aligned} x\phi_a(y + I(\lambda, \boldsymbol{\pi}^{\pm})) &= x\phi_a(y) + I(\lambda, \boldsymbol{\pi}_a^{\pm}) = \phi_a(\phi_a^{-1}(x)y) + I(\lambda, \boldsymbol{\pi}_a^{\pm}) \\ &= \phi_a(\phi_a^{-1}(x)y + I(\lambda, \boldsymbol{\pi}^{\pm})) \in \phi_a(L(\lambda, \boldsymbol{\pi}^{\pm})). \end{aligned}$$

Since $\phi_a(\bar{v}_{\boldsymbol{\pi}}) = \bar{v}_{\boldsymbol{\pi}_a}$, we conclude that ϕ_a factors through to a map $V(\lambda, \boldsymbol{\pi}^{\pm}) \to V(\lambda, \boldsymbol{\pi}_a^{\pm})$, which is the desired isomorphism $\phi_{\boldsymbol{\pi}^{\pm},a}$. The properties (i) and (ii) of $\phi_{\boldsymbol{\pi}^{\pm},a}$ are immediate from the fact the algebra automorphism ϕ_a satisfies these conditions. □

2.3. The following was proved in [8] (see also [9]).

Theorem 2. *Assume that $\lambda \in P^+$ and $\boldsymbol{\pi}^{\pm} \in \mathcal{A}^n$ satisfy the following:*

(i) *$\boldsymbol{\pi}^+ = (\pi_1, \ldots, \pi_n)$ is an n-tuple of polynomials,*
(ii) *$\lambda_i = \deg \pi_i$,*
(iii) *$\pi_i^-(u) = u^{\deg \pi_i} \pi_i(u^{-1}) / \left(u^{\deg \pi_i} \pi_i(u^{-1}) \right)\big|_{u=0}$.*

Then $V(\lambda, \boldsymbol{\pi}^{\pm})$ is an irreducible finite-dimensional ℓ-highest weight $\mathbf{U}_q$-module with highest weight $(\lambda, \boldsymbol{\pi}^{\pm})$. Moreover, these exhaust the irreducible finite-dimensional ℓ-highest weight modules.

From now, we shall only consider n-tuples of polynomials $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$, which have constant term 1 and split over $\mathbf{C}(q)$ and we denote by $V(\boldsymbol{\pi})$ the irreducible ℓ-highest weight module corresponding to $\boldsymbol{\pi}$.

We also need the following (cf. [9])

Proposition. *Assume that $V(\boldsymbol{\pi})$ and $V(\boldsymbol{\pi}')$ are irreducible highest weight $\mathbf{U}_q$-modules and also that $V(\boldsymbol{\pi}) \otimes V(\boldsymbol{\pi}')$ is irreducible. Then*

$$V(\boldsymbol{\pi}) \otimes V(\boldsymbol{\pi}') \cong V(\boldsymbol{\pi}\boldsymbol{\pi}') \cong V(\boldsymbol{\pi}') \otimes V(\boldsymbol{\pi}),$$

where $\boldsymbol{\pi}\boldsymbol{\pi}' = (\pi_1\pi_1', \ldots, \pi_n\pi_n')$.

2.4. Given such an n-tuple $\boldsymbol{\pi}$, and $d \in \mathbf{Z}$, define a homomorphism $\chi_{\boldsymbol{\pi},d} : \mathbf{U}_q^e(0) \to \mathbf{C}(q)[t, t^{-1}]$ by extending the assignment,

$$\chi_{\boldsymbol{\pi},d}(P_{i,\pm r}) = \pi^{\pm}_{i,r} t^{\pm r}, \qquad \chi_{\boldsymbol{\pi},d}(K_i) = q_i^{\deg \pi_i}, \qquad \chi_{\boldsymbol{\pi},d}(D) = q^d, \tag{2.1}$$

for all $i \in I$, $r \in \mathbf{Z}$ to a homomorphism of $\mathbf{Z}$-graded algebras. We set $\chi_{\boldsymbol{\pi}} = \chi_{\boldsymbol{\pi},0}$.

Lemma. *Let $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ be an n-tuple of polynomials satisfying $\pi_i(0) = 1$. Suppose that $\chi_{\boldsymbol{\pi}} : \mathbf{U}_q^e \to \mathbf{C}(q)[t^m, t^{-m}] \to 0$, for some $m > 1$. Then for all $i \in I$, π_i is actually a polynomial in u^m. In particular, there exists an n-tuple of polynomials $\boldsymbol{\pi}^0 = (\pi_1^0, \ldots, \pi_n^0)$ such that $\pi_i^0(0) = 1$ and*

$$\pi_i = \prod_{s=0}^{m-1} \pi_i^0(\zeta^s u),$$

where $\zeta \in \mathbf{C}$ is an mth primitive root of unity.

Proof. By the definition of $\chi_{\boldsymbol{\pi}}$ and Lemma 1.4, $\pi^{\pm}_{i,r} = 0$ unless m divides r. It follows that the π are polynomials in u^m. In particular, m divides $\deg \pi_i$, hence we can write $\deg \pi_i = m_i m$ for some $m_i \in \mathbf{N}$. Let a_i be a root of π_i in the algebraic closure of $\mathbf{C}(q)$. Then, evidently, $\zeta^s a_i$ is a root of π_i for all $s = 0, \ldots, m-1$. Since $\pi_i(0) = 1$, $a_i \neq 0$, and so we can write

$$\pi_i(u) = \prod_{r=1}^{m_i} \prod_{s=0}^{m-1} (1 - b_{i,r} \zeta^s u),$$

for some $b_{i,r} \in \overline{\mathbf{C}(q)}$. Set

$$\pi_i^0(u) = \prod_{r=1}^{m_i} (1 - b_{i,r} u).$$

It remains to observe that $\pi_i^0 \in \mathbf{C}(q)[u]$. □

Given $\boldsymbol{\pi}$ such that $\chi_{\boldsymbol{\pi}} : \mathbf{U}_q^e \to \mathbf{C}(q)[t^m, t^{-m}] \to 0$ for some $m > 1$, let $\boldsymbol{\pi}^0$ be the n-tuple of polynomials defined in the previous Lemma.

2.5. Our further analysis is based on the following result which was established in [5, Theorem 3].

Theorem 3. *Let $\{\boldsymbol{\pi}_j = (\pi_{j,1}, \ldots, \pi_{j,n}) : 1 \le j \le k\}$ be a set of n-tuples of polynomials with constant term one which are split over $\mathbf{C}(q)$. Assume further, that for all $1 \le j, j' \le k$ and $l \in I$ we have*

$$a \neq b q^{\mathbf{Z}},$$

where a and b are arbitrary roots of $\pi_{j,l}$ and $\pi_{j',l}$. Then, the tensor product $V(\boldsymbol{\pi}_1) \otimes \cdots \otimes V(\boldsymbol{\pi}_k)$ is an irreducible $\mathbf{U}_q$-module.

Proposition. *Let $V(\boldsymbol{\pi})$ be an irreducible finite-dimensional representation of $\mathbf{U}_q$, and assume that $\chi_{\boldsymbol{\pi}} : \mathbf{U}_q^e \to \mathbf{C}(q)[t^m, t^{-m}] \to 0$, for some $m > 1$. Then there exists an n-tuple of polynomials $\boldsymbol{\pi}^0$ such that*

$$V(\boldsymbol{\pi}) \cong V(\boldsymbol{\pi}^0(u)) \otimes V(\boldsymbol{\pi}^0(\zeta u)) \otimes \cdots \otimes V(\boldsymbol{\pi}^0(\zeta^{m-1} u)).$$

Proof. This follows immediately from Theorem 3 and Lemma 2.4. □

2.6. It follows from Theorem 3 and Proposition 2.5 that there exists a nontrivial isomorphism of $\mathbf{U}_q$-modules,

$$\tau_{\boldsymbol{\pi}^0} : V(\boldsymbol{\pi}^0(u)) \otimes V(\boldsymbol{\pi}^0(\zeta u)) \otimes \cdots \otimes V(\boldsymbol{\pi}^0(\zeta^{m-1}u)) \\ \to V(\boldsymbol{\pi}^0(\zeta u)) \otimes V(\boldsymbol{\pi}^0(\zeta^2 u)) \otimes \cdots \otimes V(\boldsymbol{\pi}^0(u)),$$

which maps the tensor product of highest weight vectors on the left to the corresponding element on the right. Set

$$\eta_{\boldsymbol{\pi}^0} = (\phi_{\boldsymbol{\pi}^0_\zeta,\zeta^{m-1}} \otimes \phi_{\boldsymbol{\pi}^0_{\zeta^2},\zeta^{m-1}} \otimes \cdots \otimes \phi_{\boldsymbol{\pi}^0,\zeta^{m-1}}) \circ \tau_{\boldsymbol{\pi}^0}.$$

Then $\eta_{\boldsymbol{\pi}^0}$ is a $\mathbf{U}_q(\mathfrak{g})$-module endomorphism of $V(\boldsymbol{\pi}) \cong V(\boldsymbol{\pi}^0(u)) \otimes V(\boldsymbol{\pi}^0(\zeta u)) \otimes \cdots \otimes V(\boldsymbol{\pi}^0(\zeta^{m-1}u))$ and maps a highest weight vector to its multiple. We may assume, without loss of generality, that $\eta_{\boldsymbol{\pi}^0}$ maps a highest weight vector to itself.

Lemma. *Suppose that $x \in (\mathbf{U}_q)_k$ for some $k \in \mathbf{Z}$. Then*

$$\eta_{\boldsymbol{\pi}^0}(x.v) = \zeta^{-k} x.\eta_{\boldsymbol{\pi}^0}(v).$$

In particular, $\eta^m_{\boldsymbol{\pi}^0} = \mathrm{id}$.

Proof. Let Δ denote the standard co-multiplication on $\mathbf{U}_q$ (cf. 1.4) and set $\Delta^r = (\Delta \otimes \mathrm{id}^{\otimes r-1}) \circ \cdots \circ (\Delta \otimes \mathrm{id})\Delta : \mathbf{U}_q \to \mathbf{U}_q^{\otimes r+1}$. Suppose that $x \in (\mathbf{U}_q)_k$. Then we can write, using the summation notation of Sweedler and Lemma 1.3, $\Delta^{m-1}(x) = x_1 \otimes \cdots \otimes x_m$ where $x_j \in (\mathbf{U}_q)_{k_j}$, $k_j \in \mathbf{Z}$ and $k_1 + \cdots + k_m = k$. Therefore,

$$\begin{aligned} \eta_{\boldsymbol{\pi}^0}(xv) &= (\phi_{\boldsymbol{\pi}^0_\zeta,\zeta^{m-1}} \otimes \phi_{\boldsymbol{\pi}^0_{\zeta^2},\zeta^{m-1}} \otimes \cdots \otimes \phi_{\boldsymbol{\pi}^0,\zeta^{m-1}})(x_1 \otimes \cdots \otimes x_m)\tau_{\boldsymbol{\pi}^0}(v) \\ &= \zeta^{(m-1)k}(x_1 \otimes \cdots \otimes x_m)(\phi_{\boldsymbol{\pi}^0_\zeta,\zeta^{m-1}} \otimes \phi_{\boldsymbol{\pi}^0_{\zeta^2},\zeta^{m-1}} \otimes \cdots \otimes \phi_{\boldsymbol{\pi}^0,\zeta^{m-1}}) \circ \tau_{\boldsymbol{\pi}^0}(v) \\ &= \zeta^{-k} x.\eta_{\boldsymbol{\pi}^0}(v). \end{aligned}$$

Since $V(\boldsymbol{\pi})$ is simple, it follows that $V(\boldsymbol{\pi}) = \mathbf{U}_q v_{\boldsymbol{\pi}}$. On the other hand, $\eta_{\boldsymbol{\pi}^0} v_{\boldsymbol{\pi}} = v_{\boldsymbol{\pi}}$, hence by the above $V(\boldsymbol{\pi})$ is a direct sum of eigenspaces of $\eta_{\boldsymbol{\pi}^0}$ and all the eigenvalues of $\eta_{\boldsymbol{\pi}^0}$ are mth roots of unity. It follows immediately that $\eta^m_{\boldsymbol{\pi}^0} = \mathrm{id}$. □

2.7. Extend $\eta_{\boldsymbol{\pi}^0}$ to $L(V(\boldsymbol{\pi}))$ by setting $\eta_{\boldsymbol{\pi}^0}(v \otimes t^r) = \zeta^r \eta_{\boldsymbol{\pi}^0}(v) \otimes t^r$. From now on we will denote this map by η.

Lemma. *The map $\eta : L(V(\boldsymbol{\pi})) \to L(V(\boldsymbol{\pi}))$ is a $\mathbf{U}_q^e$ module endomorphism.*

Proof. Take $x \in (\mathbf{U}_q^e)_k$ for some $k \in \mathbf{Z}$. Then, for all $v \in V(\boldsymbol{\pi})$ and $r \in \mathbf{Z}$,

$$\begin{aligned} \eta(x(v \otimes t^r)) &= \eta(xv \otimes t^{r+k}) = \zeta^{r+k}\eta_{\boldsymbol{\pi}^0}(xv) \otimes t^{r+k} \\ &= \zeta^r x.\eta_{\boldsymbol{\pi}^0}(v) \otimes t^{r+k} \qquad \text{by Lemma 2.6} \\ &= x.\eta(v \otimes t^r). \end{aligned}$$

Since every $x \in \mathbf{U}_q^e$ can be written, uniquely, as a finite sum of homogeneous elements, we conclude that η commutes with the action of $\mathbf{U}_q^e$. □

2.8. Our Theorem 1 is an immediate consequence of the following

Lemma. *Let $L^s(V(\boldsymbol{\pi})) \subset L(V(\boldsymbol{\pi}))$ be the eigenspace of η corresponding to the eigenvalue ζ^s, $s = 0, \ldots, m-1$. Then*

$$L(V(\boldsymbol{\pi})) = \bigoplus_{s=0}^{m-1} L^s(V(\boldsymbol{\pi})) = \bigoplus_{s=0}^{m-1} \mathbf{U}_q^e.(v_{\boldsymbol{\pi}} \otimes t^s),$$

where the summands are simple $\mathbf{U}_q^e$-modules.

Proof. Since $\eta_{\boldsymbol{\pi}^0}(v_{\boldsymbol{\pi}}) = v_{\boldsymbol{\pi}}$, it follows that $\eta(v_{\boldsymbol{\pi}} \otimes t^s) = \zeta^s v_{\boldsymbol{\pi}} \otimes t^s$, hence $v_{\boldsymbol{\pi}} \otimes t^s \in L^s(V(\boldsymbol{\pi}))$. Furthermore, $\eta \in \operatorname{End}_{\mathbf{U}_q^e} L(V(\boldsymbol{\pi}))$ whence $L^s(V(\boldsymbol{\pi}))$ is a $\mathbf{U}_q^e$-submodule of $L(V(\boldsymbol{\pi}))$. Finally, observe that, by Lemma 2.1, every proper submodule of $L(V(\boldsymbol{\pi}))$ meets $v_{\boldsymbol{\pi}} \otimes \mathbf{C}(q)[t, t^{-1}]$. Therefore, $\mathbf{U}_q^e(v_{\boldsymbol{\pi}} \otimes t^s)$ is simple. □

Remark. Although we have worked over $\mathbf{C}(q)$, the results of the section go through if we specialize q to be a complex number which is *not* a root of unity.

3. Classification of irreducible integrable modules for $\widehat{\mathbf{U}}_q$.

3.1. We begin with the following definition which is analogous to Definition 2.2.

Definition. A $\mathbf{U}_q^e$-module V of type 1 is called ℓ-highest weight if there exists a vector $v \in V$, $\lambda = \sum_{i \in I} \lambda_i \varpi_i + d\delta \in P^e$ and a maximal graded ideal $\mathfrak{M}$ in $\mathbf{U}_q^e(0)$ such that

$$x_{i,k}^+.v = 0, \qquad \mathfrak{M}.v = 0, \qquad K_i.v = q_i^{\lambda_i} v, \qquad Dv = q^d v$$

for all $i \in I$ and $k \in \mathbf{Z}$.

It is not hard to see that the set of maximal graded ideals in $\mathbf{U}_q(0)$ is in bijective correspondence with the set of graded ring homomorphisms $\chi : \mathbf{U}_q(0) \to \mathbf{C}(q)[t, t^{-1}]$. Given such a χ and $\lambda \in P^e$, one can define in the obvious way a universal highest weight module $M(\lambda, \chi)$. Namely $M(\lambda, \chi)$ is the left $\mathbf{U}_q^e$-module obtained by taking the quotient of $\mathbf{U}_q^e$ by the left ideal generated by the elements $x_{i,r}^+$, $i \in I, r \in \mathbf{Z}$, $\ker \chi$, $K_i - q_i^{\lambda_i}$, $i \in I$ and $D - q^d$. Let $\bar{v}_{\lambda,\chi}$ be the image of $1 \in \mathbf{U}_q^e$ in $M(\lambda, \chi)$. Standard methods show that $M(\lambda, \chi)$ has a unique irreducible quotient $V(\lambda, \chi)$. Denote the canonical image of $\bar{v}_{\lambda,\chi}$ in $V(\lambda, \chi)$ by $v_{\lambda,\chi}$.

We need the following result which was proved in [7, Proposition 3.5].

Proposition. *For all $s \geq 0$ we have*

$$(x_{i,\pm 1}^+)^s (x_{i,0}^-)^s = P_{i,\pm s} + \text{terms in } \mathbf{U}_q^e \mathbf{U}_q(>)_+.$$

Theorem 4. *The $\mathbf{U}_q^e$-module $V(\lambda, \chi)$ is integrable if and only if there exists an n-tuple of polynomials $\boldsymbol{\pi} = (\pi_1, \dots, \pi_n)$ in an indeterminate u with constant term one, and $d \in \mathbb{Z}$, such that*

$$\lambda_i = \deg \pi_i, \qquad \chi = \chi_{\boldsymbol{\pi},d},$$

where $\chi_{\boldsymbol{\pi},d}$ was defined in (2.1).

Proof. Suppose that $V(\lambda, \chi)$ is integrable. Then we have

$$(x_{i,0}^-)^s.v_{\lambda,\chi} = 0, \qquad i \in I, \quad s \geq \lambda_i + 1,$$

whence

$$(x_{i,\pm 1}^+)^s (x_{i,0}^-)^s.v_{\lambda,\chi} = 0, \qquad i \in I, \quad s \geq \lambda_i + 1.$$

Using Proposition 3.1, we conclude that

$$P_{i,s}.v_{\lambda,\chi} = 0, \qquad i \in I, \qquad |s| \geq \lambda_i + 1,$$

that is, $P_{i,s} \in \ker \chi$ for all $i \in I$ and $|s| \geq \lambda_i + 1$. Furthermore, since $(x_{i,0}^-)^{\lambda_i}.v_{\lambda,\chi} \neq 0$, and $x_{i,-1}^-(x_{i,0}^-)^{\lambda_i}.v_{\lambda,\chi} = 0$, it follows from the representation theory of $\mathbf{U}_q(\mathfrak{sl}_2)$ that

$$(x_{i,1}^+)^{\lambda_i} (x_{i,0}^-)^{\lambda_i}.v_{\lambda,\chi} \neq 0,$$

hence $P_{i,\lambda_i} \neq 0$. Thus, we can define an n-tuple of polynomials $\boldsymbol{\pi}$ by

$$\pi_i(u) = \left(\frac{1}{\chi(P_{i,\lambda_i})t^{-\lambda_i}}\right) \sum_{r\geq 0} (\chi(P_{i,r})t^{-r})u^r.$$

One can prove then, as in [7], that the eigenvalues of $P_{i,-r}$ are given by the n-tuple $\boldsymbol{\pi}^-$ defined in the previous section. It is now easy to check that $\chi = \chi_{\boldsymbol{\pi},d}$ where $\chi(D) = q^d$ and the result follows.

For the converse statement it suffices to prove that given an n-tuple of polynomials with constant term one and $d \in \mathbf{Z}$, there exists an irreducible integrable module with highest weight $\chi_{\boldsymbol{\pi},d}$. Consider the module $L(V(\boldsymbol{\pi}); d)$ defined in 2.1. It is clear that the element $v_{\boldsymbol{\pi}} \otimes 1$ generates a highest weight $\mathbf{U}_q^e$-module which is integrable. Further, by Theorem 1, it follows that this module is irreducible and hence is isomorphic to $V(\deg \boldsymbol{\pi}, \chi_{\boldsymbol{\pi},d})$. □

We note the following consequence.

Corollary. *Any irreducible integrable ℓ-highest weight $\mathbf{U}_q^e$-module is isomorphic to a simple submodule of a quantum loop module, and in particular has finite-dimensional weight spaces.*

3.2. *We are aiming to prove a classification result which is similar to the one obtained in* [4, 6]. *This can be done only if we work over an algebraically closed field containing $\mathbf{C}(q)$. The classification result also holds if we specialize q to be a nonzero complex number which is not a root of unity. We will assume without further comment that we are in one of these situations until the end of this section.*

We begin with the following

Proposition. *Let $V = \bigoplus_{\nu\in\widehat{P}} V_\nu$ be an integrable $\widehat{\mathbf{U}}_q$-module of type 1, such that $\dim V_\nu < \infty$ for all $\nu \in \widehat{P}$. Then there exists $\lambda \in \widehat{P}$ such that $V_\lambda \neq 0$ and $V_{\lambda+\eta} = 0$ for all $\eta \in Q^+ \setminus \{0\}$. In particular, $(\lambda \,|\, \alpha_i) \geq 0$ for all $i \in I$.*

Proof. Suppose that for each $\mu \in \Omega(V)$ there exists $\nu \in Q^+$ such that $\mu+\nu \in \Omega(V)$. Fix some $\mu \in \Omega(V)$. Then there exists an infinite sequence $\{\eta_r\}_{r\geq 1}$ such that $\eta_r \leq \eta_{r+1}$ and $\mu + \eta_r \in \Omega(V)$ for all $r \geq 1$. Set $W_r := \mathbf{U}_q(\mathfrak{g})V_{\mu+\eta_r}$. Then W_r is an integrable $\mathbf{U}_q(\mathfrak{g})$-module with finite-dimensional weight spaces and hence by Proposition 1.7(ii) and Corollary 1.7,

$$W_r \cong \bigoplus_{\mu_{r,s}\in P^+} m(\mu_{r,s})V(\mu_{r,s}),$$

where $m(\mu_{r,s}) \in \mathbf{N}$ and are nonzero for finitely many $\mu_{r,s}$. Choose s_1 such that $\nu_1 := \mu_{1,s_1} > \mu$ and $m(\mu_{1,s_1}) \neq 0$. Such s_1 exists since $\mu + \eta_1$ is a weight of W_1. Furthermore, let r_2 be the smallest positive integer so that there exists s_2 with $\nu_2 := \mu_{r_2,s_2} > \mu$, $\mu_{1,s_1} \neq \mu_{r_2,s_2}$ and $m(\mu_{r_2,s_2}) \neq 0$. Notice that r_2 always exists since the module W_1 is finite-dimensional and the maximal weights which occur in W_r keep increasing. Repeating this process, we obtain an infinite collection of elements $\nu_k > \mu$, $k \geq 1$ such that $V(\nu_k)$ is isomorphic to an irreducible $\mathbf{U}_q(\mathfrak{g})$-submodule $W(\nu_k)$ of V. By Lemma 1.7 it follows that $V_\mu \cap W(\nu_k) \neq 0$ for all $k \geq 1$. Since all the ν_k are distinct, the sum of $W(\nu_k)$ is direct, which contradicts the finite-dimensionality of V_μ. In particular, $\lambda + \alpha_i$ is not a weight for all $i \in I$. It follows that V_λ generates a highest weight $\mathbf{U}_q(\mathfrak{g})$-module, whence λ is dominant with respect to the α_i, $i \in I$. □

3.3. Let $\mathbf{U}^+_{i,s}$ (respectively, $\mathbf{U}^-_{i,s}$), $i \in I$, $s \in \mathbf{Z}$ be the subalgebra of $\widehat{\mathbf{U}}_q$ generated by $E = x^+_{i,s}$, $F = x^-_{i,-s}$ and $K = C^s K_i$ (respectively, by $E = x^-_{i,s}$, $F = x^+_{i,-s}$ and $K = C^s K_i^{-1}$). One can easily see from the defining relations (cf. 1.2) that $\mathbf{U}^\pm_{i,s}$ is isomorphic to $\mathbf{U}_q(\mathfrak{sl}_2)$, E, F and K being the standard generators of the latter algebra.

The following simple lemma will be used repeatedly in the proof of our main theorem.

Lemma. *Let V be an integrable admissible $\widehat{\mathbf{U}}_q$-module. Assume that $\mu \in \Omega(V)$ satisfies $(\mu \,|\, s\delta \mp \alpha_i) > 0$ for some $i \in I$ and $s \in \mathbf{Z}$. Then $\mu \pm \alpha_i - s\delta \in \Omega(V)$.*

Proof. Recall that if M is a finite-dimensional $\mathbf{U}_q(\mathfrak{sl}_2)$ module and $Km = q^k m$ for some $m \in M$, with $k > 0$, then $Fm \neq 0$. Consider a $\mathbf{U}_q(\mathfrak{sl}_2)$-module $\mathbf{U}^\mp_{i,s} V_\mu$. Since $C^s K_i^{\mp 1}$ acts on V_μ by q^k with $k = (\mu \,|\, s\delta \mp \alpha_i)$ positive, it follows that $x^\pm_{i,-s} V_\mu \neq 0$. Since $x^\pm_{i,-s} V_\mu \subset V_{\mu \pm \alpha_i - s\delta}$, the result follows. □

3.4. Now we will prove the main result of this section.

Theorem 5. *Let $V = \bigoplus_{\nu \in \widehat{P}} V_\nu$ be an irreducible integrable $\widehat{\mathbf{U}}_q$-module of type 1 and assume that $\dim V_\nu < \infty$ for all $\nu \in \widehat{P}$ and let $r \in \mathbf{Z}$ be such that $Cv = q^r v$ for all $v \in V$. Suppose that $\dim V > 1$.*

(i) *If $r > 0$, then there exists $\lambda \in \widehat{P}^+$ such that $V \cong V(\lambda)$.*
(ii) *If $r = 0$ then, there exists an n-tuple of polynomials $\boldsymbol{\pi}$ with constant term 1 and $d \in \mathbf{Z}$ such that V is isomorphic to an irreducible component of $L(V(\boldsymbol{\pi}); d)$.*
(iii) *If $r < 0$, then there exists $\lambda \in \widehat{P}^+$ such that $V \cong V(\lambda)^\#$.*

Proof. By Proposition 3.2 we can choose $\lambda \in \widehat{P} \in \Omega(V)$ such that $\lambda + \eta$ is not a weight of V for all $\eta \in Q^+$ and $(\lambda \,|\, \alpha_i) \geq 0$ for all $i \in I$. Given $\eta = \sum_{i \in I} k_i \alpha_i \in Q^+$, set $\operatorname{ht} \eta := \sum_{i \in I} k_i$.

(i) Assume that $r > 0$. Then there exists $m \in \mathbf{N}$ such that $V_{\lambda + s\delta} = 0$ for all $s > m$. Indeed, otherwise we can choose $s > 0$ such that $\lambda + s\delta \in \Omega(V)$ and $(\lambda + s\delta \,|\, s\delta - \alpha_i) = rs - (\lambda \,|\, \alpha_i) > 0$ for all $i \in I$. It follows from Lemma 3.3 that $\lambda + \alpha_i$ is a weight of V for all $i \in I$, which contradicts the choice of λ. Next, we prove that the $\mathbf{U}_q(\mathfrak{g})$-module $M = \widehat{\mathbf{U}}^+_q(\ll) \widehat{\mathbf{U}}^+_q(0) \widehat{\mathbf{U}}^+_q(\gg) V_{\lambda + m\delta}$ is finite-dimensional. Since

$$\widehat{\mathbf{U}}^+_q(0) \widehat{\mathbf{U}}^+_q(\gg) \subset \widehat{\mathbf{U}}^+_q(\gg) \widehat{\mathbf{U}}^+_q(0),$$

it suffices, by the choice of m, to prove that $\widehat{\mathbf{U}}^+_q(\ll) \widehat{\mathbf{U}}^+_q(\gg) V_{\lambda + m\delta}$ is finite-dimensional.

Let us prove first that

$$\widehat{\mathbf{U}}^+_q(\gg) V_{\lambda + m\delta} = \mathbf{U}^+_q(\mathfrak{g}) V_{\lambda + m\delta}, \tag{3.1}$$

where $\mathbf{U}^+_q(\mathfrak{g})$ is the subalgebra of $\mathbf{U}_q(\mathfrak{g})$ generated by the $x^+_{i,0} : i \in I$. First, suppose that $x^+_{i,s} V_{\lambda + m\delta} \neq 0$ for some $i \in I$ and $s \in \mathbf{N}^+$. Then $\lambda + (m+s)\delta + \alpha_i$ is a weight of V. Since $(\lambda + (m+s)\delta + \alpha_i \,|\, \alpha_i) = (\lambda \,|\, \alpha_i) + 2d_i > 0$, it follows from Lemma 3.3 that $\lambda + (m+s)\delta \in \Omega(V)$ which is a contradiction. Then using induction on $\operatorname{ht} \eta$ we conclude that $\lambda + \eta + (m+s)\delta$, $\eta \in Q^+$ is not a weight of V for all $s > 0$, whence

$$\widehat{\mathbf{U}}^+_q(\gg) V_{\lambda + m\delta} = \mathbf{U}^+_q(\mathfrak{g}) V_{\lambda + m\delta} \subset \mathbf{U}_q(\mathfrak{g}) V_{\lambda + m\delta}.$$

Yet $\mathbf{U}_q(\mathfrak{g})V_{\lambda+m\delta}$ is finite-dimensional by Corollary 1.7 and (3.1) is proved.

In particular, there exists a finite set $\{\eta_k\} \subset Q^+$ such that $\mathbf{U}_q^+(\mathfrak{g})V_{\lambda+m\delta} \subset \bigoplus_k V_{\mu_k}$ where $\mu_k = \lambda + \eta_k + m\delta$ and that $M \subset \bigoplus_k \widehat{\mathbf{U}}_q^+(\ll)V_{\mu_k}$. Since $\dim \widehat{\mathbf{U}}_q^+(\ll)_{-\eta+s\delta} < \infty$ for all $\eta \in Q^+$ and $s \in \mathbf{N}$, to prove that M is finite-dimensional, it is now sufficient to prove that

$$\widehat{\mathbf{U}}_q^+(\ll)_{-\eta+s\delta}V_{\mu_k} = 0 \tag{3.2}$$

for all but finitely many $\eta \in Q^+$ and $s \in \mathbf{N}^+$, which is an immediate consequence of the following two assertions:

1°. The set $\{\eta \in Q^+ : \mu_k - \eta + l\delta \in \Omega(M) \text{ for some } l > 0\}$ is finite.

2°. For every $\eta \in Q^+$, $M_{\mu_k-\eta+l\delta} = 0$ for l sufficiently large.

In order to prove 1°, assume it to be false and observe that the weights of M are all of the form $\mu_k - \eta + l\delta : \eta \in Q^+$. Since M is an integrable $\mathbf{U}_q(\mathfrak{g})$-module, it follows (cf. 1.9) that $\Omega(M)$ is W-invariant. Since the set of the μ_k is finite, as is the group W, and P, δ are preserved by W, we can always choose $\eta \in Q^+$ such that $\mu_k - \eta + l\delta$ is a weight and the W-orbit of $\mu_k - \eta + l\delta$ contains an element which is not of the form $\mu_r - \eta' + l\delta$ for some r and $\eta' \in Q^+$, which is a contradiction.

To prove 2° we proceed by induction on $\operatorname{ht}\eta$. If $\operatorname{ht}\eta = 1$, that is $\eta = \alpha_j$ for some $j \in I$, and $\mu_k - \eta + l\delta \in \Omega(M)$ for infinitely many l, choose l large enough so that $\mu_k - \alpha_j + l\delta \in \Omega(M)$ and $(\mu_k - \eta + l\delta \,|\, (l-1)\delta - \alpha_j) = r(l-1) - (\mu_k \,|\, \alpha_j) + 2d_j > 0$. Applying Lemma 3.3 we conclude that $\mu_k + \delta$ is a weight of M, which is a contradiction. For the inductive step, suppose that $\eta \in Q^+$ with $\operatorname{ht}\eta > 1$. Then there exists $j \in I$ such that $\eta - \alpha_j \in Q^+$. Furthermore, by the induction hypothesis there exists N such that $\mu_k - (\eta - \alpha_i) + l\delta \notin \Omega(M)$ for all $l > N$. Choose $s > 0$ so that $\mu_k - \eta + l\delta \in \Omega(M)$ whilst $(\mu_k - \eta + l\delta \,|\, (s+1)\delta - \alpha_j) = r(s+1) - (\mu_k \,|\, \alpha_j) + (\eta \,|\, \alpha_j) > 0$, where $l = N + s + 1$. Then Lemma 3.3 yields $\mu_k - (\eta - \alpha_j) + (N+1)\delta \in \Omega(M)$, which is a contradiction by the choice of N.

Since V is simple, it follows from Proposition 1.2 that $\Omega(V) \subset \Omega(M) - \widehat{Q}^+$. Further, any $\mu \in \widehat{P}$ with $(\mu \,|\, \delta) > 0$ is $\widehat{W}$ conjugate to an element in $\widehat{P}^+$. Since V is integrable, $\Omega(V)$ is $\widehat{W}$-invariant hence $\Omega(V)$ is contained in a finite union of cones of the form $\mu - \widehat{Q}^+$ where $\mu \in \widehat{P}^+$. In order to complete the proof of (i), it only remains to apply Proposition 1.7.

(ii) If $r = 0$, then $\Omega(V)$ is contained in P^e. We prove first that there exists $\mu \in \Omega(V)$ such that $x_{i,m}^+ V_\mu = 0$ for all $i \in I$, $m \in \mathbf{Z}$.

Suppose that $x_{i,m}^+ V_\lambda \neq 0$ for some $i \in I$ and $m \in \mathbf{Z}$. Set $\mu = \lambda + \alpha_i + m\delta$. Suppose further that $x_{j,s}^+ . V_\mu \neq 0$ for some $j \in I$, $s \in \mathbf{Z}$. Observe that, if $i \neq j \in I$, then either $a_{ij} + 2 > 0$ or $a_{ji} + 2 > 0$ and obviously $a_{ii} + 2 > 0$. Thus we may assume, without loss of generality, that $a_{ij} + 2 > 0$. Then $(\lambda + \alpha_i + \alpha_j \,|\, \alpha_i) = (\lambda \,|\, \alpha_i) + d_i(a_{ij} + 2) > 0$ and so by Lemma 3.3 we conclude that $\lambda + \alpha_j \in \Omega(V)$, which is a contradiction by the choice of λ.

Thus, we have proved that $V = \widehat{\mathbf{U}}_q v_0$ where v_0 satisfies

$$x_{i,r}^+ . v_0 = 0, \qquad \forall\, i \in I, \quad r \in \mathbf{Z}.$$

Since $\mathbf{U}_q^e = \mathbf{U}_q(<)\mathbf{U}_q^e(0)\mathbf{U}_q(>)$ it follows from standard arguments that $\mathbf{U}_q^e(0)v_0$ must be an irreducible $\mathbf{U}_q^e(0)$-module. Since $\mathbf{U}_q^e(0)$ is a $\mathbf{Z}$-graded commutative algebra, it follows that the irreducible graded representations must just be the quotient of $\mathbf{U}_q^e(0)$ by a maximal graded ideal $\mathfrak{M}$ of $\mathbf{U}_q^e(0)$, which annihilates v_0.

This proves that V is an ℓ-highest weight module of $\mathbf{U}_q^e$ and (ii) now follows from Corollary 3.1.

(iii) This case is similar to the first one. □

3.5. Assume that the bilinear form on $\widehat{\mathfrak{h}}^*$ is normalized in such a way that its values on $\widehat{P}$ are rational (for, it is sufficient to assign a rational value to $(\omega_0 \,|\, \omega_0)$). Let $M = \bigoplus_{\nu \in \widehat{P}} M_\nu$ be a $\widehat{\mathbf{U}}_q$-module of type 1. We say, following [20, 21], that M is a bounded module if $(\nu \,|\, \nu) \le N$ for some $N \in \mathbf{Z}$ and for all $\nu \in \Omega(M)$. In particular, if M is simple, then this upper bound is attained (cf. [20, 7.2]), that is, there exists $\lambda \in \Omega(M)$ such that $(\nu \,|\, \nu) \le (\lambda \,|\, \lambda)$ for all $\nu \in \Omega(M)$. We call such a λ maximal.

Observe that a bounded module is necessarily integrable. Indeed, if both μ and $\mu + n\alpha_i$ are weights of M, then $(\mu + n\alpha_i \,|\, \mu + n\alpha_i) = (\mu \,|\, \mu) + 2n(\mu \,|\, \alpha_i) + n^2(\alpha_i \,|\, \alpha_i) \le N$ which imposes a bound on $|n|$. It is shown in [21] that a simple bounded $\widehat{\mathfrak{g}}$-module is admissible, being necessarily of one of the types described in [4, 6].

We will now establish a similar result for $\widehat{\mathbf{U}}_q$-modules.

Proposition. *Let V be a simple bounded $\widehat{\mathbf{U}}_q$-module and suppose that λ is its maximal weight and that $\dim V \neq 1$.*

(i) *If $(\lambda \,|\, \delta) > 0$, then $V \cong V(\mu)$, where $\mu \in \widehat{W}\lambda \cap \widehat{P}^+$.*
(ii) *If $(\lambda \,|\, \delta) = 0$, then V is isomorphic to a simple submodule of $L(V(\boldsymbol{\pi}); d)$ for some n-tuple $\boldsymbol{\pi}$ of polynomials with constant term 1 and $d \in \mathbf{Z}$.*
(iii) *If $(\lambda \,|\, \delta) < 0$, then V is isomorphic to $V(\mu)^{\#}$, where $\mu \in \widehat{W}(-\lambda) \cap \widehat{P}^+$.*

In particular, a simple bounded $\widehat{\mathbf{U}}_q$-module is admissible.

Proof. Since the form $(\cdot \,|\, \cdot)$ is $\widehat{W}$-invariant and V is integrable, any element of the $\widehat{W}$-orbit of λ is also a maximal weight of V.

(i) Suppose that $(\lambda \,|\, \delta) > 0$. Then the W-orbit of λ contains $\mu \in \widehat{P}^+$. Since μ is maximal by the above remark, it follows that $\mu + \alpha_i$ is not a weight of V for all $i \in \hat{I}$. Indeed, $(\mu + \alpha_i \,|\, \mu + \alpha_i) = (\mu \,|\, \mu) + 2(\mu \,|\, \alpha_i) + (\alpha_i \,|\, \alpha_i) > (\mu \,|\, \mu)$, which contradicts the maximality of μ. Therefore, $E_i V_\mu = 0$ for all $i \in \hat{I}$, that is V is a highest weight module of highest weight μ. Since V is simple, the claim follows immediately from Proposition 1.7.

(ii) Suppose that $(\lambda \,|\, \delta) = 0$. Then $\lambda \in P^e$ and so $\widehat{W}\lambda \cap \widehat{P}^+$ is empty. However, since W is a finite group, we can always conjugate λ to some $\mu \in P^e$ such that $(\mu \,|\, \alpha_i) \ge 0$ for all $i \in I$. Since $(\mu + \alpha_i + s\delta \,|\, \mu + \alpha_i + s\delta) = (\mu \,|\, \mu) + 2(\mu \,|\, \alpha_i) + (\alpha_i \,|\, \alpha_i) > (\mu \,|\, \mu)$ and μ is a maximal weight, we conclude that $\mu + \alpha_i + s\delta$ is not a weight of V for all $i \in I$ and $s \in \mathbf{Z}$. It follows that $x_{i,s}^+ V_\mu = 0$ for all $i \in I$ and $s \in \mathbf{Z}$. The rest of the argument repeats that of the proof of the second part of Theorem 5.

(iii) The argument repeats that of (i). □

4. Characters of quantum loop modules in $\mathfrak{sl}_{n+1}$ case

4.1. In the classical case, we get the following (cf. [14]). Let $V = \bigotimes_{i=1}^k V(\lambda_i)^{\otimes m_i m}$ be a finite-dimensional $L\mathfrak{g} = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}]$-module with evaluation parameters $a_{i,r}\zeta^s$ where $\lambda_i \in P^+$ and are distinct, $a_{i,r} \in \mathbf{C}^\times$, $r = 1, \ldots, m_i$, $s = 0, \ldots, m-1$

and $a_{i,r}/a_{i',r'}$ is not an mth root of unity. Let $L(V)$ be the corresponding loop $\widehat{\mathfrak{g}}$-module. Then the formula of [14, Theorem 4.4] yields

$$\dim L^s(V)_{\nu+r\delta} = \frac{1}{m}\sum_{d|m}\varphi_{r-s}(d)\dim V^{1/d}_{\nu/d}, \tag{4.1}$$

where $V^{1/d} := \bigotimes_{i=1}^k V(\lambda_i)^{\otimes m_i m/d}$ and

$$\varphi_k(d) = \varphi(d)\,\frac{\mu(d/\gcd(d,k))}{\varphi(d/\gcd(d,k))},$$

where φ is the Euler function and μ is the classical Möbius function, $\mu(k) = 0$ if k is divisible by a square and $\mu(k) = (-1)^l$ if k is a product of l distinct primes. Alternatively, set $V_0 = \bigotimes_{i=1}^k V(\lambda_i)^{\otimes m_i}$. Then

$$\dim L^s(V)_{\nu+r\delta} = \frac{1}{m}\sum_{d|m}\varphi_{r-s}(d)\dim(V_0)^{\otimes m/d}_{\nu/d}. \tag{4.2}$$

4.2. Retain the notations of Section 2 and define, for any d dividing m,

$$\boldsymbol{\pi}^{1/d} = (\pi_1^{1/d}, \dots, \pi_n^{1/d}),$$

where $\pi_i^{1/d}(u) = \prod_{j=0}^{m/d-1}\pi_i^0(\zeta^{jd}u)$. In particular, if $d = 1$, then $\boldsymbol{\pi}^1 = \boldsymbol{\pi}$. It follows immediately from Theorem 3 and Lemma 2.4 that $V(\boldsymbol{\pi}^{1/d})$ is isomorphic to $V(\boldsymbol{\pi}^0)\otimes V(\boldsymbol{\pi}^0_{\zeta^d})\otimes\cdots\otimes V(\boldsymbol{\pi}^0_{\zeta^{m-d}})$. Then we conjecture the following quantum analogue of (4.1) and (4.2)

Conjecture. *Let $\nu \in P$, $r \in \mathbf{Z}$ and $s = 0, \dots, m-1$. Then*

$$\begin{aligned}\dim L^s(V(\boldsymbol{\pi}))_{\nu+r\delta} &= \frac{1}{m}\sum_{d|m}\varphi_{r-s}(d)\dim V(\boldsymbol{\pi}^{1/d})_{\nu/d}\\ &= \frac{1}{m}\sum_{d|m}\varphi_{r-s}(d)\dim(V(\boldsymbol{\pi}^0)^{\otimes m/d})_{\nu/d}.\end{aligned}$$

In the rest of this section we prove this conjecture in some special case for $\mathfrak{g} \cong \mathfrak{sl}_{n+1}$.

4.3. Recall that $V(\boldsymbol{\pi})$ is a direct sum of eigenspaces of $\eta_{\boldsymbol{\pi}^0}$ and that the eigenvalues of the latter are mth roots of unity. Let $V(\boldsymbol{\pi})^{(k)} \subset V(\boldsymbol{\pi})$ be the eigenspace of $\eta_{\boldsymbol{\pi}^0}$ corresponding to the eigenvalue ζ^k. Define, for all $v \in V(\boldsymbol{\pi})$ and for all $s \in \mathbf{Z}$,

$$\Pi_s(v) = \frac{1}{m}\sum_{k=0}^{m-1}\zeta^{-ks}\eta^k_{\boldsymbol{\pi}^0}(v).$$

Evidently, Π_s depends only on the residue class of $s \pmod m$. Observe that the Π_s are $\mathbf{U}_q(\mathfrak{g})$-module endomorphisms of $V(\boldsymbol{\pi})$. Furthermore, for all $v \in V(\boldsymbol{\pi})$ and for all $r, s \in \mathbf{Z}$, define

$$\widehat{\Pi}_s(v\otimes t^r) = \Pi_{s-r}(v)\otimes t^r.$$

Lemma. (i) *The $\Pi_s : s = 0, \dots, m-1$ are orthogonal projectors onto $V(\boldsymbol{\pi})^{(s)}$.*

(ii) *The $\widehat{\Pi}_s : s = 0, \dots, m-1$ are $\mathbf{U}_q^e$-module endomorphisms of $L(V(\boldsymbol{\pi}))$ and orthogonal projectors onto its simple $\mathbf{U}_q^e$ submodules $L^s(V(\boldsymbol{\pi}))$.*

Proof. For the first part, we have, for all $v \in V(\boldsymbol{\pi})$,

$$\eta_{\boldsymbol{\pi}^0}(\Pi_s(v)) = \frac{1}{m}\sum_{k=0}^{m-1} \zeta^{-ks}\eta_{\boldsymbol{\pi}^0}^{k+1}(v) = \frac{1}{m}\sum_{k=1}^{m} \zeta^{-(k-1)s}\eta_{\boldsymbol{\pi}^0}^{k}(v) = \zeta^s\Pi_s(v),$$

whence $\Pi_s(v) \in V(\boldsymbol{\pi})^{(s)}$. Furthermore, write $u = \Pi_s(v)$. Then $\eta_{\boldsymbol{\pi}^0}^k(u) = \zeta^{ks}u$, and so

$$\Pi_{s'}\Pi_s(v) = \frac{1}{m}\sum_{k=0}^{m-1}\zeta^{k(s-s')}u.$$

If $s - s' = 0 \pmod m$, then the sum in the right-hand side equals m, whence $\Pi_s^2 = \Pi_s$. Otherwise, the sum equals $(\zeta^{m(s-s')}-1)/(\zeta^{(s-s')}-1) = 0$, that is, $\Pi_{s'}\circ\Pi_s = 0$ if $s \neq s' \pmod m$.

For the second part, take $x \in (\mathbf{U}_q)_k$, $v \in V(\boldsymbol{\pi})$, $r \in \mathbf{Z}$. Then

$$\begin{aligned}\widehat{\Pi}_s(x(v\otimes t^r)) = \widehat{\Pi}_s(x.v\otimes t^{r+k}) &= \frac{1}{m}\sum_{j=0}^{m-1}\zeta^{j(r+k-s)}\eta_{\boldsymbol{\pi}^0}^j(x.v)\otimes t^{r+k}\\ &= \frac{1}{m}\sum_{j=0}^{m-1}\zeta^{j(r-s)}x.\eta_{\boldsymbol{\pi}^0}^j(v)\otimes t^{r+k} = x.\widehat{\Pi}_s(v\otimes t^r).\end{aligned}$$

It follows that $\widehat{\Pi}_s$ commutes with the action of $\mathbf{U}_q^e$. Furthermore,

$$\begin{aligned}\eta(\widehat{\Pi}_s(v\otimes t^r)) &= \zeta^r\eta_{\boldsymbol{\pi}^0}\Pi_{s-r}(v)\otimes t^r = \zeta^s\Pi_{s-r}(v)\otimes t^r\\ &= \zeta^s\widehat{\Pi}_s(v\otimes t^r),\end{aligned}$$

whence the image of $\widehat{\Pi}_s$ is contained in $L^s(V(\boldsymbol{\pi}))$. Finally,

$$\widehat{\Pi}_s'\widehat{\Pi}_s(v\otimes t^r) = \Pi_{s'-r}\Pi_{s-r}(v)\otimes t^r.$$

It follows immediately from the first part that $\widehat{\Pi}_s^2 = \widehat{\Pi}_s$ whilst $\widehat{\Pi}_{s'}\circ\widehat{\Pi}_s = 0$ if $s \neq s'$ $\pmod m$. □

Observe that $\eta_{\boldsymbol{\pi}^0}$ preserves weight spaces of $V(\boldsymbol{\pi})$.

Corollary. *Given $\nu \in P$, set $V(\boldsymbol{\pi})_\nu^{(k)} := V(\boldsymbol{\pi})_\nu \cap V(\boldsymbol{\pi})^{(k)}$. Then*

$$\dim L^s(V(\boldsymbol{\pi}))_{\nu+r\delta} = \dim V(\boldsymbol{\pi})_\nu^{(k)},$$

where $k = s - r \pmod m$.

Proof. This follows immediately from Lemma 4.3. □

4.4. Throughout the rest of this section $\mathfrak{g}$ is assumed to be isomorphic to $\mathfrak{sl}_{n+1}$. It will be convenient to identify the set $\hat{I}$ with $\mathbf{Z}/(n+1)\mathbf{Z}$ in the sense that $i+k : i \in \hat{I}$, $k \in \mathbf{Z}$ is understood as $i + k \pmod{n+1}$. Let V be the quantum analogue of the natural representation of $\mathfrak{g}$. Explicitly, V is an $(n+1)$-dimensional vector space over $\mathbf{C}(q)$ with a basis $v_0, \dots, v_n$, the action of the generators $E_i, F_i, K_i : i \in I$ of $\mathbf{U}_q(\mathfrak{g})$ being given by

$$E_iv_j = \delta_{i,j}v_{j-1}, \quad F_iv_j = \delta_{i,j+1}v_{j+1}, \quad K_iv_j = q^{\delta_{j+1,i}-\delta_{j,i}}v_j, \qquad i \in I, \quad j \in \hat{I}.$$

Given $a \in \mathbf{C}(q)^\times$, we can endow V with a structure of a $\mathbf{U}_q$-module which we denote by $V(a)$ by setting

$$E_0v_j = \delta_{j,0}av_n, \qquad F_0v_j = \delta_{j,n}a^{-1}\,v_0, \qquad K_0v_j = q^{\delta_{j+1,0}-\delta_{j,0}}v_j, \qquad j \in \hat{I}.$$

This module corresponds to the n-tuple of polynomials $\boldsymbol{\pi} = (1 - au, 1, \dots, 1)$.

Let $a, b \in \mathbf{C}(q)^\times$ such that $b/a \neq q^{\mathbf{Z}}$. Then, by [5], $V(a) \otimes V(b)$ is irreducible as a $\mathbf{U}_q$-module and $V(a) \otimes V(b) \cong V(b) \otimes V(a)$. A computation analogous to that of [7, 5.4] and based on a simple observation that $V^{\otimes 2} \cong V(2\varpi_1) \oplus V(\varpi_2)$ (where $\varpi_2 = 0$ if $n = 1$) as a $\mathbf{U}_q(\mathfrak{g})$-module, allows one to obtain explicit formulae for the isomorphism $I_z : V(a) \otimes V(b) \to V(b) \otimes V(a)$ preserving the highest weight vector

$$\begin{aligned} &I_z(v_i \otimes v_i) = v_i \otimes v_i, \qquad i \in \hat{I}, \\ (4.3)\quad &I_z(v_i \otimes v_j) = \left(\frac{1-q^2}{z-q^2}\right) v_i \otimes v_j + \left(\frac{q(z-1)}{z-q^2}\right) v_j \otimes v_i, \\ &I_z(v_j \otimes v_i) = \left(\frac{q(z-1)}{z-q^2}\right) v_i \otimes v_j + \left(\frac{(1-q^2)z}{z-q^2}\right) v_j \otimes v_i, \qquad 0 \le i < j \le n \end{aligned}$$

where $z = b/a$. In particular, we may always assume, without loss of generality, that $a = 1$. More generally, take $a_1, \ldots, a_m \in \mathbf{C}(q)^\times$ such that $a_i/a_j \neq q^{\mathbf{Z}}$. Then $I_{a_{i+1}/a_i, i} := \mathrm{id}^{\otimes i-1} \otimes I_{a_{i+1}/a_i} \otimes \mathrm{id}^{\otimes m-i-1}$ gives an isomorphism of $\mathbf{U}_q$-modules which permutes the ith and the $(i+1)$th factors in the tensor product $V(a_1) \otimes \cdots \otimes V(a_m)$. The latter is isomorphic to $V(\boldsymbol{\pi})$ where $\boldsymbol{\pi} = (\prod_{i=1}^s (1 - a_i u), 1, \ldots, 1)$.

Now, suppose that $a_i = \zeta^{i-1}$, where ζ is an mth primitive root of unity. Then we are in the situation of Lemma 2.4 and $\boldsymbol{\pi}^0 = (1-u, 1, \ldots, 1)$. It follows that the isomorphism $\tau_{\boldsymbol{\pi}^0}$ can be constructed explicitly as $\mathbf{I}_m = I_{\zeta^{m-1}, m-1} \circ I_{\zeta^{m-2}, m-2} \circ \cdots \circ I_{\zeta,1}$. Furthermore, since $V(\zeta^i)$ is simple as a $\mathbf{U}_q(\mathfrak{g})$-module, $\phi_{\boldsymbol{\pi}^0_{\zeta^i}, \zeta^{-1}} = \mathrm{id}$ as a map of vector spaces. Therefore, $\mathbf{I}_m$ can be identified with the map $\eta_{\boldsymbol{\pi}^0}$ defined in 2.6.

4.5. Recall that $\mathbf{C}(q)$ is the quotient field of $\mathbf{C}[q]$ and define a subring A of $\mathbf{C}(q)$ by $A := \{f/g \,:\, f, g \in \mathbf{C}[q],\, g(0) \neq 0\}$. Then A is a local ring, the unique maximal ideal being qA. Evidently, if $f/g \in A$, then $f/g = f(0)/g(0) \pmod{qA}$. In particular, $A/qA \cong \mathbf{C}$. Observe also that, given an A-module M, we get $\overline{M} := M/qM \cong M \otimes_A A/qA$. Given $m \in M$, we denote its canonical image in $\overline{M}$ by $\bar{m}$.

Let $\mathbf{B}(\boldsymbol{\pi})$ be the set of all tensor products of the form $v_{i_m} \otimes \cdots \otimes v_{i_1} \,:\, i_r \in \hat{I}$ and set $\mathcal{L}(\boldsymbol{\pi}) = \bigoplus_{\mathbf{v} \in \mathbf{B}(\boldsymbol{\pi})} A\mathbf{v}$. It follows from (4.3) that $\eta_{\boldsymbol{\pi}^0}$ maps $\mathcal{L}(\boldsymbol{\pi})$ into itself and, in particular, preserves $q\mathcal{L}(\boldsymbol{\pi})$. Furthermore, if $\nu \in P$ is a weight of $V(\boldsymbol{\pi})$, set $\mathcal{L}(\boldsymbol{\pi})_\nu := \mathcal{L}(\boldsymbol{\pi}) \cap V(\boldsymbol{\pi})_\nu$. Since each $\mathbf{v} \in \mathbf{B}(\boldsymbol{\pi})$ is a weight vector, it follows that $\eta_{\boldsymbol{\pi}^0}$ preserves $\mathcal{L}(\boldsymbol{\pi})_\nu$. Set $\overline{\mathcal{L}(\boldsymbol{\pi})}_\nu := \overline{\mathcal{L}(\boldsymbol{\pi})_\nu}$. Then $\dim_{\mathbf{C}} \overline{\mathcal{L}(\boldsymbol{\pi})}_\nu = \operatorname{rank}_A \mathcal{L}(\boldsymbol{\pi})_\nu = \dim_{\mathbf{C}(q)} V(\boldsymbol{\pi})_\nu$.

Lemma. *Given* $\mathbf{v} = v_{i_m} \otimes \cdots \otimes v_{i_1} \in \mathbf{B}(\boldsymbol{\pi})$*, define*

$$\operatorname{desc}(\mathbf{v}) = \{r \,:\, i_{r+1} < i_r,\, 1 \le r < m\}$$

and let $\operatorname{Maj}(\mathbf{v})$ *be the sum of elements of* $\operatorname{desc}(\mathbf{v})$ *if the latter is nonempty and zero otherwise. Then*

$$\eta_{\boldsymbol{\pi}^0}(\mathbf{v}) = \zeta^{\operatorname{Maj}(\mathbf{v})}\mathbf{v} \pmod{q\mathcal{L}(\boldsymbol{\pi})}.$$

Proof. Observe that (4.3) yields, for all $r = 1, \ldots, m-1$,

$$I_{\zeta^{m-r}, m-r}(\mathbf{v}) = \begin{cases} \mathbf{v}, & i_{r+1} \ge i_r \\ \zeta^{r-m}\mathbf{v}, & i_{r+1} < i_r \end{cases} \pmod{q\mathcal{L}(\boldsymbol{\pi})},$$

It follows immediately from the definition of $\operatorname{desc}(\mathbf{v})$ that $I_{\zeta^{m-r},m-r}(\mathbf{v}) = \zeta^r \mathbf{v} \pmod{q\mathcal{L}(\boldsymbol{\pi})}$ provided that $r \in \operatorname{desc}(\mathbf{v})$ and $I_{\zeta^{m-r},m-r}(\mathbf{v}) = \mathbf{v} \pmod{q\mathcal{L}(\boldsymbol{\pi})}$ otherwise. The result is now immediate since $\eta_{\boldsymbol{\pi}^0}(q\mathcal{L}(\boldsymbol{\pi})) \subset q\mathcal{L}(\boldsymbol{\pi})$. □

4.6. Denote by $\bar{\eta}_{\boldsymbol{\pi}^0}$ the map of $\mathbf{C}$-vector space $\overline{\mathcal{L}(\boldsymbol{\pi})}$ into itself which obtains canonically from $\eta_{\boldsymbol{\pi}^0}$. By the above Lemma, the eigenvalues of $\bar{\eta}_{\boldsymbol{\pi}^0}$ are mth roots of unity.

Lemma. *Let $\overline{\mathcal{L}(\boldsymbol{\pi})}^{(k)}$ be the eigenspace of $\bar{\eta}_{\boldsymbol{\pi}^0}$ corresponding to the eigenvalue ζ^k. Then, for all $k = 0, \ldots, m-1$,*

$$\dim_{\mathbf{C}(q)} V(\boldsymbol{\pi})^{(k)} = \dim_{\mathbf{C}} \overline{\mathcal{L}(\boldsymbol{\pi})}^{(k)}.$$

Moreover, if we set $\overline{\mathcal{L}(\boldsymbol{\pi})}^{(k)}_{\nu} := \overline{\mathcal{L}(\boldsymbol{\pi})}^{(k)} \cap \overline{\mathcal{L}(\boldsymbol{\pi})}_{\nu}$, then

$$\dim_{\mathbf{C}(q)} V(\boldsymbol{\pi})^{(k)}_{\nu} = \dim_{\mathbf{C}} \overline{\mathcal{L}(\boldsymbol{\pi})}^{(k)}_{\nu}.$$

Proof. Given $\mathbf{v} \in \mathbf{B}(\boldsymbol{\pi})$, set $\langle \mathbf{v} \rangle := \Pi_k(\mathbf{v})$ where $k = \operatorname{Maj}(\mathbf{v})$. Evidently, $\langle \mathbf{v} \rangle \in \mathcal{L}(\boldsymbol{\pi})$ and, moreover, $\langle \mathbf{v} \rangle = \mathbf{v} \pmod{q\mathcal{L}(\boldsymbol{\pi})}$ by Lemma 4.5. Since the canonical images of $\mathbf{v} \in \mathbf{B}(\boldsymbol{\pi})$ in $\overline{\mathcal{L}(\boldsymbol{\pi})}$ form a basis of $\overline{\mathcal{L}(\boldsymbol{\pi})}$ over $\mathbf{C}$, the $\langle \mathbf{v} \rangle \,:\, \mathbf{v} \in \mathbf{B}(\boldsymbol{\pi})$ generate $\mathcal{L}(\boldsymbol{\pi})$ as an A-module and are linearly independent over A by Nakayama's Lemma. Therefore, the $\langle \mathbf{v} \rangle \,:\, \mathbf{v} \in \mathbf{B}(\boldsymbol{\pi})$ form a basis of $V(\boldsymbol{\pi})$ over $\mathbf{C}(q)$. Furthermore, by Lemma 4.3(i), $\langle \mathbf{v} \rangle \in V(\boldsymbol{\pi})^{(k)}$. We conclude that $V(\boldsymbol{\pi})^{(k)}$ contains a linearly independent subset $\{\langle \mathbf{v} \rangle \,:\, \mathbf{v} \in \mathbf{B}(\boldsymbol{\pi}),\, \operatorname{Maj}(\mathbf{v}) = k \pmod m\}$, whose cardinality equals $\dim_{\mathbf{C}} \overline{\mathcal{L}(\boldsymbol{\pi})}^{(k)}$ by Lemma 4.5. Thus, $\dim_{\mathbf{C}(q)} V(\boldsymbol{\pi})^{(k)} \geq \dim_{\mathbf{C}} \overline{\mathcal{L}(\boldsymbol{\pi})}^{(k)}$. On the other hand, since $V(\boldsymbol{\pi})$ (respectively, $\overline{\mathcal{L}(\boldsymbol{\pi})}$) is a direct sum of eigenspaces of $\eta_{\boldsymbol{\pi}^0}$ (respectively, $\bar{\eta}_{\boldsymbol{\pi}^0}$), it follows that $\dim_{\mathbf{C}(q)} V(\boldsymbol{\pi}) = \sum_{k=0}^{m-1} \dim_{\mathbf{C}(q)} V(\boldsymbol{\pi})^{(k)} \geq \sum_{k=0}^{m-1} \dim_{\mathbf{C}} \overline{\mathcal{L}(\boldsymbol{\pi})}^{(k)} = \dim_{\mathbf{C}} \overline{\mathcal{L}(\boldsymbol{\pi})} = \dim_{\mathbf{C}(q)} V(\boldsymbol{\pi})$, whence the desired equality. The second assertion is immediate since all the $\mathbf{v} \in \mathbf{B}(\boldsymbol{\pi})$ are weight vectors. □

4.7. It follows immediately from Lemma 4.5 that

$$\dim_{\mathbf{C}} \overline{\mathcal{L}(\boldsymbol{\pi})}^{(k)}_{\nu} = \#\{\mathbf{v} \,:\, \mathbf{v} \in V(\boldsymbol{\pi})_{\nu},\, \operatorname{Maj}(\mathbf{v}) = k \pmod m\}.$$

The cardinality of the set which appears in the right-hand side was computed in [15]. Namely, there is a bijection between the set of weights of $V(\boldsymbol{\pi})$ (or $\mathcal{L}(\boldsymbol{\pi})$) and the set

$$\{(k_0, \ldots, k_n) \in \mathbf{N}^{n+1} \,:\, k_0 + \cdots + k_n = m\}.$$

Indeed, take $\mathbf{v} = v_{i_m} \otimes \cdots \otimes v_{i_1} \in \mathbf{B}(\boldsymbol{\pi})$ and set $k_i(\mathbf{v}) \,:\, i = 0, \ldots, n$ where $k_i(\mathbf{v}) = \#\{r \,:\, i_r = i\}$. Then $\mathbf{v}$ is of weight $\nu = \sum_{i=0}^{n} k_i(\varpi_{i+1} - \varpi_i) = \sum_{i=1}^{n} (k_{i-1} - k_i)\varpi_i$, where we set $\varpi_0 = \varpi_{n+1} = 0$. It follows from the definition that $k_0(\mathbf{v}) + \cdots + k_i(\mathbf{v}) = m$. One can easily check that $\mathbf{v}, \mathbf{v}' \in \mathbf{B}(\boldsymbol{\pi})$ are of the same weight if and only if $k_i(\mathbf{v}) = k_i(\mathbf{v}')$ for all $i = 0, \ldots, n$.

Suppose that ν corresponds to $(k_0, \ldots, k_n)$. Then

$$\dim_{\mathbf{C}} \overline{\mathcal{L}(\boldsymbol{\pi})}_{\nu} = \dim_{\mathbf{C}(q)} V(\boldsymbol{\pi})_{\nu} = \binom{m}{k_0, \ldots, k_n}$$

and by [15, A.1-A.6],

$$\dim_{\mathbf{C}} \overline{\mathcal{L}(\boldsymbol{\pi})}^{(k)}_{\nu} = \frac{1}{m} \sum_{d|m} \varphi_k(d) \binom{\frac{m}{d}}{\frac{k_0}{d}, \ldots, \frac{k_n}{d}} = \frac{1}{m} \sum_{d|m} \varphi_k(d) \dim_{\mathbf{C}(q)} V(\boldsymbol{\pi}^{1/d})_{\nu/d}.$$

Applying 4.3 and Lemma 4.6, we obtain the following

Proposition. *Assume that $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_{n+1}$. Let $\boldsymbol{\pi}^0 = (1-u, 1, \dots, 1)$ and $\boldsymbol{\pi} = (\prod_{j=0}^{m-1}(1-\zeta^{j-1}u), 1, \dots, 1)$, where ζ is an mth primitive root of unity. Then*

$$\dim_{\mathbf{C}(q)} L^s(V(\boldsymbol{\pi}))_{\nu+r\delta} = \frac{1}{m}\sum_{d|m}\varphi_{r-s}(d)\dim_{\mathbf{C}(q)} V(\boldsymbol{\pi}^{1/d})_{\nu/d}$$
$$= \frac{1}{m}\sum_{d|m}\varphi_{r-s}(d)\dim_{\mathbf{C}(q)}(V(\boldsymbol{\pi}^0)^{\otimes m/d})_{\nu/d},$$

for all $\nu \in P$, $r \in \mathbf{Z}$ and $s = 0, \dots, m-1$.

4.8. In the remainder of this section we will discuss the crystal basis theory of our modules $L(V(\boldsymbol{\pi}))$. One can easily see that $V(\zeta^i)$ does not a admit a crystal basis in the sense of Kashiwara. However, one can slightly modify the definition of a crystal basis so that it makes sense in our particular case.

Let M be an integrable $\widehat{\mathbf{U}}_q$-module or a finite-dimensional $\mathbf{U}_q$-module. Then, for $i \in \hat{I}$ fixed, the Kashiwara operators $\tilde{E}_i$, $\tilde{F}_i$ are defined in the following way (cf. [16, 2.2] or [24, 16.1]). Any $m \in M_\lambda$ can be written uniquely as $m = \sum_{s\geq 0, s+t\geq 0} F_i^{(s)}u_s$, where $F_i^{(s)} := F_i^s/[s]!$, $t = (\lambda\,|\,\alpha_i)$, $u_s \in \ker E_i \cap M_{\lambda+s\alpha_i}$ and equals zero for s sufficiently large. Then

$$\tilde{E}_i m = \sum_{s>0,\, s+t\geq 0} F_i^{(s-1)}u_s, \qquad \tilde{F}_i m = \sum_{s\geq 0,\, s+t\geq 0} F_i^{(s+1)}u_s.$$

Definition. Let $\mathcal{L}$ be a free A-submodule of M satisfying $M = \mathcal{L}\otimes_A \mathbf{C}(q)$ and $\mathcal{B}$ be a basis of $\overline{\mathcal{L}}$. Fix $\zeta \in \mathbf{C}^\times$. We say that $(\mathcal{L}, \mathcal{B})$ is a ζ-crystal basis of M if

(i) $\mathcal{L} = \bigoplus_\nu \mathcal{L}_\nu$, $\mathcal{B} = \coprod_\nu \mathcal{B}_\nu$, where $\mathcal{L}_\nu = \mathcal{L}\cap M_\nu$ and $\mathcal{B}_\nu = \mathcal{B}\cap\overline{\mathcal{L}}_\nu$.
(ii) $\mathcal{L}$ is stable by the $\tilde{E}_i$, $\tilde{F}_i$ for all $i \in \hat{I}$, hence the $\tilde{E}_i$, $\tilde{F}_i$ act on $\overline{\mathcal{L}}$.
(iii) $\tilde{E}_i\mathcal{B}, \tilde{F}_i\mathcal{B} \subset \zeta^{\mathbf{Z}\delta_{i,0}}\mathcal{B}\cup\{0\}$, for all $i \in \hat{I}$.
(vi) For $\mathbf{v}, \mathbf{v}' \in \mathcal{B}$ one has $\mathbf{v}' = \zeta^k\tilde{F}_i\mathbf{v}$ if and only if $\mathbf{v} = \zeta^{-k}\tilde{E}_i\mathbf{v}'$.

If $\zeta = 1$, the above definition reduces to the Kashiwara's definition of crystal bases (cf. [16, 2.3]).

The following lemma is an adaptation of [24, Lemma 20.1.2] for ζ-crystal bases.

Lemma. *Assume that $(\mathcal{L}, \mathcal{B})$ is a ζ-crystal basis of M. Let $m \in \mathcal{L}_\lambda$ and fix $i \in \hat{I}$. Write $m = \sum_{s\geq 0\, s+t\geq 0} F_i^{(s)}u_s$, where $u_s \in \ker E_i \cap M_{\lambda+s\alpha_i}$, and $u_s = 0$ if $s \gg 0$. Then*

(i) *For all $s \geq 0$ and $r \geq 0$, $F_i^{(r)}x_s \in \mathcal{L}$.*
(ii) *If $\bar{m} \in \mathcal{B}$, then there exists s_0 such that $u_s \in q\mathcal{L}$ if $s_0 \neq s$, $\bar{u}_{s_0} \in \zeta^{\mathbf{Z}\delta_{i,0}}\mathcal{B}$ and $m = F_i^{(s_0)}u_{s_0} \pmod{q\mathcal{L}}$.*

Proof. The argument is an obvious modification of that of [24, Lemma 20.1.2]. □

4.9. Retain the notation of 4.4.

Proposition. *Fix $\zeta \in \mathbf{C}^\times$ and let $\boldsymbol{\pi} = (\prod_{j=0}^{N-1}(1-\zeta^j u), 1, \dots, 1)$. Let $V(\boldsymbol{\pi})$ be the corresponding simple $\mathbf{U}_q$-module which is isomorphic to $V(1)\otimes\cdots\otimes V(\zeta^{N-1})$. Set $\mathbf{B}(\boldsymbol{\pi}) = \{v_{i_1}\otimes\cdots\otimes v_{i_N} : i_r \in \hat{I}\}$ and $\mathcal{L}(\boldsymbol{\pi}) = \bigoplus_{\mathbf{v}\in\mathbf{B}(\boldsymbol{\pi})} A\mathbf{v}$. Then the pair $(\mathcal{L}(\boldsymbol{\pi}), \mathbf{B}(\boldsymbol{\pi}))$ forms a ζ-crystal basis of $V(\boldsymbol{\pi})$. Moreover, the action of $\tilde{E}_i$, $\tilde{F}_i$*

on $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_N}$ *is given by*

$$\tilde{E}_i \mathbf{v} = \zeta^{(r-1)\delta_{i,0}} v_{i_1} \otimes \cdots \otimes \tilde{E}_i v_{i_r} \otimes \cdots \otimes v_{i_N} \pmod{q\mathcal{L}(\boldsymbol{\pi})},$$
$$\tilde{F}_i \mathbf{v} = \zeta^{-(s-1)\delta_{i,0}} v_{i_1} \otimes \cdots \otimes \tilde{F}_i v_{i_s} \otimes \cdots \otimes v_{i_N} \pmod{q\mathcal{L}(\boldsymbol{\pi})},$$

where r and s are determined by the standard Kashiwara rules for the tensor product of crystals (*cf.* [16, Theorem 1] *and* [17, 1.3]).

Proof. The property (i) of Definition 4.8 is obvious. The other three for $i \in I$ follow from the standard results of Kashiwara on crystal bases (cf., for example, [24, 20.2]). So, it only remains to prove (ii)–(iv) for $i = 0$.

The proof is basically an adaptation of the standard argument. Let $\mathbf{U}_0$ be the subalgebra of $\widehat{\mathbf{U}}_q$ generated by E_0, F_0 and $K_0^{\pm 1}$, which is isomorphic to $\mathbf{U}_q(\mathfrak{sl}_2)$. Throughout the rest of the proof we shall omit indices of the operators E_0 and F_0.

We use the following inductive argument. Set $M_k := V(1)\otimes V(\zeta)\otimes\cdots\otimes V(\zeta^{k-1})$. It is clear that (ii)–(iv) of Definition 4.8 hold for M_1 and that (iv) follows from (ii)–(iii), the second assertion of our proposition and Kashiwara's tensor product rules. Now, suppose that they hold for M_k and the pair $(\mathcal{L}_k, \mathbf{B}_k)$, where $\mathbf{B}_k = \{v_{i_1} \otimes \cdots \otimes v_{i_k} : i_r \in \hat{I}\}$ and $\mathcal{L}_k = \bigoplus_{\mathbf{b} \in \mathbf{B}_k} A\mathbf{b}$. Given $\mathbf{b} \in \mathbf{B}_k$ of weight ν, set $t = (\nu \,|\, \alpha_0)$ and write $\mathbf{b} = \sum_{s \geq 0, s+t \geq 0} F^{(s)} u_s$ as in 4.8.

Notice that $M_{k+1} \cong V(1) \otimes \phi_{\boldsymbol{\pi}_k,\zeta} M_k$, where $\boldsymbol{\pi}_k = (\prod_{j=0}^{k-1}(1-\zeta^j u), 1, \ldots, 1)$. For the inductive step we should prove first that, for all $i \in \hat{I}$ and for all $\mathbf{b} \in \mathbf{B}_k$, $\tilde{E}(v_i \otimes \mathbf{b}), \tilde{F}(v_i \otimes \mathbf{b}) \in \mathcal{L}_{k+1}$. Since the $v_i : i \neq 0, n$ span trivial $\mathbf{U}_0$-modules, we can write

$$v_i \otimes \mathbf{b} = \sum_{s \geq 0, s+t \geq 0} v_i \otimes F^{(s)} u_s = \sum_{s \geq 0, s+t \geq 0} F^{(s)}(v_i \otimes \zeta^s u_s).$$

Since $E(v_i \otimes u_s) = 0$ if $i \neq 0$, the claim follows by induction hypothesis. When it is easy to check that (iii) holds for $\mathbf{v} \in \mathbf{B}_{k+1}$ of the form $v_i \otimes \mathbf{b}$ where $\mathbf{b} \in \mathbf{B}_k$ and $i \neq 0, n$. Now, by Lemma 4.8, $\mathbf{b} = F^{(s_0)} u_{s_0} \pmod{q\mathcal{L}_k}$ for some $s_0 \geq 0$. It follows that $v_i \otimes \mathbf{b} = \zeta^{s_0} F^{(s_0)}(v_i \otimes u_{s_0})$. If s_0=0, then obviously $\tilde{E}(v_i \otimes \mathbf{b}) = 0$. Otherwise, $\tilde{E}(v_i \otimes \mathbf{b}) = \zeta^{s_0} F^{(s_0-1)}(v_i \otimes u_{s_0}) \pmod{q\mathcal{L}_{k+1}} = \zeta v_i \otimes \tilde{E}\mathbf{b} \pmod{q\mathcal{L}_{k+1}}$. Similarly, $\tilde{F}(v_i \otimes \mathbf{b}) = \zeta^{s_0} F^{(s_0+1)}(v_i \otimes \mathbf{b}) \pmod{q\mathcal{L}_{k+1}} = \zeta^{-1} v_i \otimes \tilde{F}\mathbf{b} \pmod{q\mathcal{L}_{k+1}}$, which proves the second assertion and (iv).

It remains to consider the case when $i = 0, n$. One can easily check that the weight vectors

$$X_s := v_n \otimes u_s, \qquad Y_s := v_n \otimes F u_s - \zeta q^{t+2s}[t+2s] v_0 \otimes u_s$$

are annihilated by E. Furthermore, for all $r > 0$,

$$\begin{aligned} F^{(r)} X_s &= \zeta^{-r} q^r v_n \otimes F^{(r)} u_s + \zeta^{-r+1} v_0 \otimes F^{(r-1)} u_s, \\ F^{(r)} Y_s &= \zeta^{-r} q^r [r+1] v_n \otimes F^{(r+1)} u_s + \zeta^{-r+1}([r] - q^{t+2s-r}[t+2s]) v_0 \otimes F^{(r)} u_s \\ &= \zeta^{-r} \frac{q^{2r}-1}{q^2-1} v_n \otimes F^{(r+1)} u_s + \zeta^{-r+1} q^r \frac{q^{2(t+2s-r)}-1}{q^2-1} v_0 \otimes F^{(r)} u_s. \end{aligned}$$

In particular, we have

$$F^{(s)}X_s = \zeta^{-s}q^s v_n \otimes F^{(s)}u_s + \zeta^{-s+1}v_0 \otimes F^{(s-1)}u_s,$$
$$F^{(s-1)}Y_s = \zeta^{-s+1}\,\frac{q^{2(s-1)}-1}{q^2-1}\,v_n \otimes F^{(s)}u_s + \zeta^{-s+2}q^s\,\frac{q^{2(t+s+1)}-1}{q^2-1}\,v_0 \otimes F^{(s-1)}u_s.$$

Fix $s > 0$. Evidently, $v_n \otimes F^{(s)}u_s$ and $v_0 \otimes F^{(s-1)}u_s$ are nonzero and linearly independent. Furthermore, the determinant of the matrix of coefficients of $F^{(s)}X_s$ and $F^{(s-1)}Y_s$ in the basis consisting of $v_n \otimes F^{(s)}u_s$ and $v_0 \otimes F^{(s-1)}u_s$ equals $-\zeta^{-2(s-1)} \pmod{qA}$, whence is a unit in A. Therefore, there exist $a_s, b_s \in A$ such that $v_n \otimes F^{(s)}u_s = a_sF^{(s)}X_s + b_sF^{(s-1)}Y_s$, $s > 0$. Thus, we can write

$$\begin{aligned} v_n \otimes \mathbf{b} &= \sum_{s\geq 0,\, s+t\geq 0} v_n \otimes F^{(s)}u_s = \sum_{s\geq 0,\, s+t\geq 0} a_sF^{(s)}X_s + \sum_{s>0,\, s+t\geq 0} b_sF^{(s-1)}Y_s \\ &= \sum_{s\geq 0,\, s+t+1\geq 0} F^{(s)}(a_sX_s + b_{s+1}Y_{s+1}) = \sum_{s\geq 0,\, s+t+1\geq 0} F^{(s)}U_s, \end{aligned}$$

which is the decomposition of $v_n \otimes \mathbf{b}$ of 4.8. It follows from the definitions of X_s, Y_s, Lemma 4.8 and the induction hypothesis that $\tilde{E}(v_n \otimes \mathbf{b}) = \sum_{s>0,\, s+t+1\geq 0} F^{(s-1)}U_s$ and $\tilde{F}(v_n \otimes \mathbf{b}) = \sum_{s\geq 0, s+t+1\geq 0} F^{(s+1)}U_s$ lie in $\mathcal{L}_{k+1}$. A similar argument shows that $\tilde{E}(v_0 \otimes \mathbf{b}), \tilde{F}(v_0 \otimes \mathbf{b}) \in \mathcal{L}_{k+1}$.

Now observe that $F^{(s)}X_s = \zeta^{-s+1}v_0 \otimes F^{(s-1)}u_s \pmod{q\mathcal{L}_{k+1}}$ whilst $F^{(s-1)}Y_s = \zeta^{-s+1}v_n \otimes F^{(s)}u_s \pmod{q\mathcal{L}_{k+1}}$, $s > 0$. Assume first that $\tilde{E}\mathbf{b} = 0 \pmod{q\mathcal{L}_k}$. Then $\mathbf{b} = u_0 \pmod{q\mathcal{L}_k}$ and so $v_n \otimes \mathbf{b} = X_0 \pmod{q\mathcal{L}_{k+1}}$. It follows that $\tilde{E}(v_n \otimes \mathbf{b}) = 0 = \tilde{E}v_n \otimes \mathbf{b} \pmod{q\mathcal{L}_{k+1}}$. On the other hand, $\tilde{F}(v_n \otimes \mathbf{b}) = v_0 \otimes u_s \pmod{q\mathcal{L}_{k+1}} = \tilde{F}v_n \otimes \mathbf{b}$. Suppose further that $\tilde{E}\mathbf{b} \notin q\mathcal{L}_k$. Then $\mathbf{b} = F^{(s)}u_s \pmod{q\mathcal{L}_k}$ for some $s > 0$. It follows that $v_n \otimes \mathbf{b} = \zeta^{s-1}F^{(s-1)}Y_s \pmod{q\mathcal{L}_{k+1}}$. If $s = 1$, then $\tilde{E}(v_n \otimes \mathbf{b}) = 0 = \tilde{E}v_n \otimes \mathbf{b} \pmod{q\mathcal{L}_{k+1}}$. Otherwise,

$$\tilde{E}(v_n \otimes \mathbf{b}) = \zeta^{s-1}F^{(s-2)}Y_s = \zeta v_n \otimes F^{(s-1)}u_s = \zeta v_n \otimes \tilde{E}\mathbf{b} \pmod{q\mathcal{L}_{k+1}}.$$

Similarly, $\tilde{F}(v_n \otimes \mathbf{b}) = \zeta^{s-1}F^{(s)}Y_s = \zeta^{-1}v_n \otimes \tilde{F}\mathbf{b} \pmod{q\mathcal{L}_{k+1}}$. We omit an analogous computation for $v_0 \otimes \mathbf{b}$. □

4.10. Suppose now that ζ is an mth primitive root of unity and retain the notation of 4.5 and 4.6.

Proposition. *Set* $\mathbf{B}(\boldsymbol{\pi})^{(k)} = \{\mathbf{v} \in \mathbf{B}(\boldsymbol{\pi}) : \mathrm{Maj}(\mathbf{v}) = k \pmod m\}$ *and define*

$$\widehat{\mathcal{L}^s(\boldsymbol{\pi})} = \bigoplus_{r\in\mathbf{Z}} \bigoplus_{\mathbf{v}\in\mathbf{B}(\boldsymbol{\pi})^{(s-r)}} A\langle\mathbf{v}\rangle \otimes t^r, \qquad \widehat{\mathbf{B}^s(\boldsymbol{\pi})} = \coprod_{r\in\mathbf{Z}} \mathbf{B}(\boldsymbol{\pi})^{(s-r)} \otimes t^n.$$

Then $(\widehat{\mathcal{L}^s(\boldsymbol{\pi})}, \widehat{\mathbf{B}^s(\boldsymbol{\pi})})$ *is a ζ-crystal basis of* $L^s(V(\boldsymbol{\pi}))$.

Proof. Set $\widehat{\mathcal{L}} = \mathcal{L}(\boldsymbol{\pi}) \otimes_A A[t, t^{-1}]$, $\widehat{\mathcal{B}} = \coprod_{r\in\mathbf{Z}} \mathbf{B}(\boldsymbol{\pi}) \otimes t^r$. One can easily check that the above proposition implies that $(\widehat{\mathcal{L}}, \widehat{\mathcal{B}})$ is a ζ-crystal basis of $L(V(\boldsymbol{\pi}))$. Set $\widehat{\mathcal{L}^s} = \widehat{\mathcal{L}} \cap L^s(V(\boldsymbol{\pi}))$. We claim that $\widehat{\mathcal{L}^s} = \widehat{\mathcal{L}^s(\boldsymbol{\pi})}$ defined above. Indeed, since $\langle\mathbf{v}\rangle$ lies in $\mathcal{L}(\boldsymbol{\pi})$ for all $\mathbf{v} \in \mathbf{B}(\boldsymbol{\pi})$, $\widehat{\mathcal{L}^s(\boldsymbol{\pi})} \subset \widehat{\mathcal{L}}$. Suppose further that $\mathbf{v} \in \mathbf{B}(\boldsymbol{\pi})^{(s-r)}$. Then $\eta(\langle\mathbf{v}\rangle \otimes t^r) = \zeta^r\eta_{\boldsymbol{\pi}^0}(\langle\mathbf{v}\rangle) \otimes t^r = \zeta^s\langle\mathbf{v}\rangle \otimes t^r$ (cf. the proof of Lemma 4.6), whence $\widehat{\mathcal{L}^s(\boldsymbol{\pi})} \subset L^s(V(\boldsymbol{\pi})) \cap \widehat{\mathcal{L}} = \widehat{\mathcal{L}^s}$ as an A submodule. Furthermore, both $\widehat{\mathcal{L}^s(\boldsymbol{\pi})}$ and $\widehat{\mathcal{L}^s}$ are direct sums of free weight A-submodules and $\widehat{\mathcal{L}^s(\boldsymbol{\pi})}_{\nu+r\delta} \subset \widehat{\mathcal{L}^s}_{\nu+r\delta}$ for all $\nu \in P$, $r \in \mathbf{Z}$. Observe that $\widehat{\mathcal{L}^s(\boldsymbol{\pi})}_{\nu+r\delta}$ is generated as an A-module by $\langle \mathbf{v} \rangle \otimes t^r$ where $\mathbf{v} \in \mathbf{B}(\boldsymbol{\pi})_\nu^{(s-r)}$. Since the images of these elements in $\widehat{\mathcal{L}^s}_{\nu+r\delta}/q\widehat{\mathcal{L}^s}_{\nu+r\delta}$ form a basis of that vector space by 4.6, it follows by Nakayama's lemma that they generate $\widehat{\mathcal{L}^s}$ as an A-module. Therefore, $\widehat{\mathcal{L}^s(\boldsymbol{\pi})} = \widehat{\mathcal{L}^s}$. The result now follows immediately from Proposition 4.9 and [15, Proposition 3.6 and Lemma A.1]. □

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