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A MULTIPLICATIVE PROPERTY OF QUANTUM FLAG MINORS

PH. CALDERO

ABSTRACT. We study the multiplicative properties of the quantum dual canonical basis $\mathcal{B}^*$ associated to a semisimple complex Lie group G. We provide a subset D of $\mathcal{B}^*$ such that the following property holds: if two elements b, b' in $\mathcal{B}^*$ q-commute and if one of these elements is in D, then the product bb' is in $\mathcal{B}^*$ up to a power of q, where q is the quantum parameter. If G is SL_n, then D is the set of so-called quantum flag minors and we obtain a generalization of a result of Leclerc, Nazarov and Thibon.

0. INTRODUCTION

0.1. Let G be a semisimple complex Lie group and fix a maximal unipotent subgroup U^- of G. Let $\mathfrak{g}$ and $\mathfrak{n}^-$ be respectively the Lie algebras of G and U^-. G. Lusztig and M. Kashiwara have constructed the so-called canonical basis $\mathcal{B}$ of the enveloping algebra $U(\mathfrak{n}^-)$ of $\mathfrak{n}^-$, which has properties of compatibility with standard filtrations.

Let $\mathbb{C}[U^-]$ be the $\mathbb{C}$-algebra of regular functions on U^-. Then the action of U^- on itself by left multiplication provides an action of $U(\mathfrak{n}^-)$ on $\mathbb{C}[U^-]$ by differential operators. Now, consider the pairing $U(\mathfrak{n}^-)\times\mathbb{C}[U^-]\to\mathbb{C}$, $\delta\times f\mapsto\delta(f)(\mathrm{e})$, where e is the identity of U^-. Then this pairing provides the so-called dual canonical basis $\mathcal{B}^*$ of $\mathbb{C}[U^-]$. This article is concerned with some multiplicative properties of this basis.

0.2. Let q be an indeterminate and let $U_q(\mathfrak{n}^-)$, $\mathbb{C}_q[U^-]$ be respectively the quantum analogue of the classical objects $U(\mathfrak{n}^-)$ and $\mathbb{C}[U^-]$. We still denote by $\mathcal{B}$, resp. $\mathcal{B}^*$, the canonical basis, resp. the dual canonical basis, of $U_q(\mathfrak{n}^-)$, resp. $\mathbb{C}_q[U^-]$. We say that two elements b and b' of $\mathcal{B}^*$ q-commute if $bb'=q^m b'b$, for an integer m. We say that they are multiplicative if $q^n bb'$ belongs to $\mathcal{B}^*$ for an integer n. It is known (see [20]) that if two elements are multiplicative, then they q-commute. The converse was believed to be true until Bernard Leclerc found counterexamples ([12]).

Suppose that two elements b and b' of the dual canonical basis q-commute. Now we discuss in which cases they are known to be multiplicative:

1) For all b, b', if $\mathfrak{g}$ is of type A_n, $n\leq 3$, B_2 ([1]; see also [4]).

2) If $\mathfrak{g}$ is of type A_n and b is a small quantum minor ([20]).

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3) If $\mathfrak{g}$ is of type A_n and b, b' are quantum flag minors ([11]).

0.3. We present the results of this article. Let W be the Weyl group of $\mathfrak{g}$ and let w_0 be its longest element. For each reduced decomposition $\tilde{w}_0$ of w_0 we have constructed in [4] a subalgebra $A_{\tilde{w}_0}$ of $\mathbb{C}_q[U^-]$ such that

1) the space $A_{\tilde{w}_0}$ is generated by a part of $\mathcal{B}^*$,
2) every two elements in $\mathcal{B}^* \cap A_{\tilde{w}_0}$ are multiplicative,
3) the algebras $A_{\tilde{w}_0}$ and $\mathbb{C}_q[U^-]$ are equal up to localization.

They are called adapted algebras associated to a reduced decomposition of w_0. They are connected to the more general theory of cluster algebras; [8], see 5.3.

The main result of the article is the following:

Theorem. *Let $\tilde{w}_0$ be a reduced decomposition corresponding to an orientation of the Coxeter graph of $\mathfrak{g}$. Let b and b' be two q-commuting elements of the dual canonical basis. If b is in $A_{\tilde{w}_0}$, then b and b' are multiplicative.*

Note that this result makes sense for the simply-laced case only. But as shown in section 5.2 below, it can be extended to the general case.

As a particular case, we obtain:

Corollary. *Let $\mathfrak{g}$ be of type A_n. Let b be a quantum flag minor and let b' be any element of $\mathcal{B}^*$ which q-commutes with b, then b and b' are multiplicative.*

We refer to [11] for motivations of this result. Actually, it provides a criterion of irreducibility for modules on the affine Hecke algebra of type A which are induced by the so-called evaluation modules.

We sketch the proof of the theorem. If two elements b and b' of the dual canonical basis q-commute, then the only property we have to obtain in order to prove that b and b' are multiplicative is

$$q^n bb' \in b'' + q\mathcal{L}^*, \tag{0.3.1}$$

where n is an integer, b'' is in $\mathcal{B}^*$ and where $\mathcal{L}^*$ is the $\mathbb{Z}[q]$-lattice generated by $\mathcal{B}^*$. The natural question is: How do we control the powers of q in the multiplications of elements of the dual canonical basis? The control of these powers is based on two main ideas:

1) Kashiwara proved that bases of integrable modules of the quantized enveloping algebra $U_q(\mathfrak{g})$ crystalize at $q = 0$ and, are compatible with the tensor product. To be more precise, let P^+ be the semigroup of integral dominant weights and let $\overline{b}$, $\overline{b}'$ be the corresponding elements in the crystal bases $\mathcal{B}(\lambda)$ and $\mathcal{B}(\lambda')$ of the integrable modules of highest weight, respectively, λ and λ' in P^+. We can assert that if $\overline{b} \otimes \overline{b}'$ belongs to the connected component of the crystal $\mathcal{B}(\lambda) \otimes \mathcal{B}(\lambda')$ corresponding to $\mathcal{B}(\lambda + \lambda')$, then (0.3.1) holds. We can easily check whether the property $\overline{b} \otimes \overline{b}' \in \mathcal{B}(\lambda + \lambda')$ holds or not with Littelmann's path model of the crystal basis ([14], [15]) by comparing chains of elements of the Weyl group for the Bruhat ordering.

2) The quiver approach of the algebra $\mathbb{C}_q[U^-]$ enables us to interpret powers of q which appear in multiplications in terms of dim $\mathrm{Hom}(M, N)$ and dim $\mathrm{Ext}^1(M, N)$, where M and N are representations of a quiver. An important tool is that the map dim $\mathrm{Hom}(?, M)$ is increasing for the so-called degeneration ordering; see [3].

1. Notations

1.1. Let $\mathfrak{g}$ be a semisimple Lie $\mathbb{C}$-algebra of rank n with Cartan matrix $A = (a_{ij})$. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the triangular decomposition and set $N := \dim \mathfrak{n}$. Let $\{\alpha_i\}_i$ be a basis of the root system R resulting from this decomposition and let R^+ be the set of positive roots. Let P be the weight $\mathbb{Z}$-lattice generated by the fundamental weights ϖ_i, $i \in I := \{1, 2, \ldots, n\}$ and set $P^+ := \sum_i \mathbb{N}\varpi_i$. Let W be the Weyl group, generated by the reflections corresponding to the simple roots $s_i := s_{\alpha_i}$, with longest element w_0. We denote by $\langle\, , \rangle$ the W-invariant form on P.

1.2. In this section we define the quantized enveloping algebra of $\mathfrak{g}$ and the properties of its Poincaré-Birkhoff-Witt basis. We refer to [6] for proofs and details.

Let q be an indeterminate. Let $U_q(\mathfrak{g})$ be the quantized enveloping Hopf $\mathbb{Q}(q)$-algebra as defined in [6]. Let $U_q(\mathfrak{n})$, resp. $U_q(\mathfrak{n}^-)$, be the upper, resp. lower, "nilpotent" subalgebra of $U_q(\mathfrak{g})$. The algebra $U_q(\mathfrak{n})$, resp. $U_q(\mathfrak{n}^-)$, is generated by E_i, resp. F_i, $1 \le i \le n$ with quantum Serre relations. For all λ in $Q := \bigoplus_i \mathbb{Z}\alpha_i$, let K_λ be the corresponding element in the algebra $U_q^0 = \mathbb{Q}(q)[Q]$ of the torus of $U_q(\mathfrak{g})$.

For all $\mu \in Q$, let $U_q(\mathfrak{n})_\mu$ be the subspace of $U_q(\mathfrak{n})$ generated by the products $E_{i_1}^{n_1} \ldots E_{i_k}^{n_k}$ such that $\sum_l n_l \alpha_l = \mu$. An element X of $U_q(\mathfrak{n})_\mu$ will be called an (homogeneous) element of weight μ. We set $\mathrm{wt}(X) := \mu$.

Recall the triangular decomposition $U_q(\mathfrak{g}) = U_q(\mathfrak{n}^-) \otimes U_q^0 \otimes U_q(\mathfrak{n})$. We define the following subalgebras of $U_q(\mathfrak{g})$:

$$U_q(\mathfrak{b}) = U_q(\mathfrak{n}) \otimes U_q^0, \qquad U_q(\mathfrak{b}^-) = U_q(\mathfrak{n}^-) \otimes U_q^0.$$

As in [23], [17], we introduce Lusztig's automorphisms T_i, $1 \le i \le n$, which define a braid action on $U_q(\mathfrak{g})$ by

$$T_i(E_i) = -F_i K_{\alpha_i}, \tag{1.2.1}$$

$$T_i(E_j) = \sum_{s=0}^{-a_{ij}} (-1)^{-a_{ij}-s} q_{\alpha_i}^{a_{ij}+s} E_i^{(s)} E_j E_i^{(-a_{ij}-s)},\ 1 \le i, j \le n,\ i \ne j,$$

$$T_i(F_i) = -K_{-\alpha_i} E_i, \tag{1.2.2}$$

$$T_i(F_j) = (-1)^{-a_{ij}-s} \sum_{s=0}^{-a_{ij}} q_{\alpha_i}^{-a_{ij}-s} F_i^{(-a_{ij}-s)} F_j F_i^{(s)},\ 1 \le i, j \le n,\ i \ne j,$$

$$T_i(K_{\alpha_j}) = K_{s_i(\alpha_j)},\ 1 \le i, j \le n, \tag{1.2.3}$$

where $E_\alpha^{(k)} = \frac{1}{[k]_{q_\alpha}!} E_\alpha^k$, $[k]_{q_\alpha}! = [k]_{q_\alpha}[k-1]_{q_\alpha} \ldots [1]_{q_\alpha}$, $[k]_{q_\alpha} = \frac{q_\alpha^n - q_\alpha^{-n}}{q_\alpha - q_\alpha^{-1}}$, $q_\alpha = q^{\frac{(\alpha,\alpha)}{2}}$. Fix a reduced decomposition $\tilde{w}_0 = s_{i_1} s_{i_2} \ldots s_{i_N}$ of w_0. For each k, $1 \le k \le N$, set $\beta_k := s_{i_1} \ldots s_{i_{k-1}}(\alpha_k)$. It is well known ([18]) that $\{\beta_k, 1 \le k \le N\}$ is the set of positive roots and that

$$\beta_1 < \beta_2 < \ldots < \beta_N$$

defines a so-called convex ordering on R^+. This ordering will identify the semigroup $\mathbb{Z}_{\ge 0}^{R^+}$ with the semigroup $\mathbb{Z}_{\ge 0}^N$. In the sequel, we denote by $\{e_k, 1 \le k \le N\}$ the natural basis of this semigroup.

For all k, define $E_{\beta_k}^{\tilde{w}_0} = E_{\beta_k} = T_{i_1} \ldots T_{i_{k-1}}(E_{\alpha_{i_k}})$. For all $\mathbf{m} = (m_i) \in \mathbb{Z}_{\ge 0}^N$, set $E^{\tilde{w}_0}(\mathbf{m}) = E(\mathbf{m}) := E_{\beta_1}^{(m_1)} \ldots E_{\beta_N}^{(m_N)}$. It is known that $\{E(\mathbf{m}), \mathbf{m} \in \mathbb{Z}_{\ge 0}^N\}$ is a basis of $U_q(\mathfrak{n})$ called the Poincaré-Birkhoff-Witt basis, in short PBW-basis, associated to the reduced decomposition $\tilde{w}_0$. In the same way, we can define the PBW-basis $\{F(\mathbf{m}),\, \mathbf{m} \in \mathbb{Z}_{\geq 0}^N\}$ of $U_q(\mathfrak{n}^-)$.

In the sequel, we call a (left) factor of $\tilde{w}_0$ a reduced decomposition $\tilde{w} = s_{i_1} \dots s_{i_k}$, $1 \leq k \leq N$. Conversely, we say that $\tilde{w}_0$ is a (right) completion of $\tilde{w}$.

Let w be in the Weyl group and let $\tilde{w} = s_{i_1} \dots s_{i_k}$ be a reduced decomposition of w which is completed to $\tilde{w}_0 = s_{i_1} s_{i_2} \dots s_{i_N}$. Let $U_q(\mathfrak{n}_{\tilde{w}})$ be the $\mathbb{Q}(q)$-space generated by the $E^{\tilde{w}_0}(\mathbf{m})$ such that $m_i = 0$ for $i > k$. By [7, 2.3], $U_q(\mathfrak{n}_{\tilde{w}})$ is a subalgebra of $U_q(\mathfrak{n})$ which depends only on w and not on the reduced decomposition $\tilde{w}$. In the sequel, we shall denote it simply by $U_q(\mathfrak{n}_w)$.

Recall the following theorem, [6, 1.7]:

Theorem. *Fix a reduced decomposition $\tilde{w}_0$ of w_0 and set*

$$\mathcal{F}_{\mathbf{m}}^{\tilde{w}_0}(U_q(\mathfrak{n})) = \bigoplus_{\mathbf{n} \prec \mathbf{m}} \mathbb{Q}(q) E^{\tilde{w}_0}(\mathbf{n}), \qquad \mathbf{m} \in \mathbb{Z}_{\geq 0}^N,$$

where $\prec$ is the right lexicographical ordering of $\mathbb{Z}_{\geq 0}^N$. Then, the spaces $\mathcal{F}_{\mathbf{m}}^{\tilde{w}_0}(U_q(\mathfrak{n}))$, $\mathbf{m} \in \mathbb{Z}_{\geq 0}^N$, define a $\mathbb{Z}_{\geq 0}^N$-filtration of $U_q(\mathfrak{n})$. The associated graded algebra $Gr^{\tilde{w}_0}(U_q(\mathfrak{n}))$ is generated by $Gr^{\tilde{w}_0}(E_\alpha)$, $\alpha \in R^+$ with the relations

$$Gr^{\tilde{w}_0}(E_\alpha) Gr^{\tilde{w}_0}(E_\beta) = q^{\langle \alpha, \beta \rangle} Gr^{\tilde{w}_0}(E_\beta) Gr^{\tilde{w}_0}(E_\alpha), \qquad \alpha < \beta.$$

As in [11, 4.2], we define the bilinear forms $d^{\tilde{w}_0} = d$ and $c^{\tilde{w}_0} = c$ on $\mathbb{Z}_{\geq 0}^N \times \mathbb{Z}_{\geq 0}^N$ such that

$$\mathrm{Gr}(E(\mathbf{m}))\mathrm{Gr}(E(\mathbf{n})) \in q^{-d(\mathbf{m},\mathbf{n})}(1 + q\mathbb{Z}[q])\mathrm{Gr}(E(\mathbf{m}+\mathbf{n})),$$
$$\mathrm{Gr}(E(\mathbf{m}))\mathrm{Gr}(E(\mathbf{n})) = q^{c(\mathbf{n},\mathbf{m})}\mathrm{Gr}(E(\mathbf{n}))\mathrm{Gr}(E(\mathbf{m})).$$

To be more precise, we have

$$d(\mathbf{m}, \mathbf{n}) = \sum_{i>j} \langle \beta_i, \beta_j \rangle m_i n_j + \frac{1}{2} \sum_i \langle \beta_i, \beta_i \rangle m_i n_i.$$

We define the $\mathbb{Z}[q]$-lattice $\mathcal{L}$ generated by the $\{E^{\tilde{w}_0}(\mathbf{m}),\, \mathbf{m} = (m_i) \in \mathbb{Z}_{\geq 0}^N\}$. From [16, Proposition 2.3], this lattice does not depend on the choice of $\tilde{w}_0$.

1.3. There exists (see [24]) a unique nondegenerate Hopf pairing (,) on $U_q(\mathfrak{b}) \times U_q(\mathfrak{b}^-)$ such that

$$(u^+,\, u_1^- u_2^-) = (\Delta(u^+),\, u_1^- \otimes u_2^-)\,, \qquad u^+ \in U_q(\mathfrak{b});\ u_1^-, u_2^- \in U_q(\mathfrak{b}^-),$$
$$(u_1^+ u_2^+,\, u^-) = (u_2^+ \otimes u_1^+,\, \Delta(u^-))\,, \qquad u^- \in U_q(\mathfrak{b}^-);\ u_1^+, u_2^+ \in U_q(\mathfrak{b}),$$
$$(K_\lambda,\, K_\mu) = q^{-(\lambda,\mu)}\,, \qquad \lambda, \mu \in Q,$$
$$(K_\lambda,\, F_i) = 0\,, \qquad \lambda \in Q\,,\ 1 \leq i \leq n,$$
$$(E_i,\, K_\lambda) = 0\,, \qquad \lambda \in Q\,,\ 1 \leq i \leq n,$$
$$(E_i,\, F_i) = \frac{1}{1 - q_{\alpha_i}^{-2}} \delta_{ij}\,, \qquad 1 \leq i, j \leq n,$$

where Δ is the comultiplication of the Hopf algebra $U_q(\mathfrak{g})$. Here, (,) extends to $(U_q(\mathfrak{b}) \otimes U_q(\mathfrak{b})) \times (U_q(\mathfrak{b}^-) \otimes U_q(\mathfrak{b}^-))$ by the natural formula

$$(u_1^+ \otimes u_2^+, u_1^- \otimes u_2^-) = (u_1^+, u_1^-)(u_2^+, u_2^-).$$

Let $\{E(\mathbf{m})^*,\, \mathbf{m} \in \mathbb{Z}_{\geq 0}^N\}$ be the dual basis of $\{F(\mathbf{m}),\, \mathbf{m} \in \mathbb{Z}_{\geq 0}^N\}$ for this pairing.

From [13], we claim that:

Claim. For all $\mathbf{m}$ in $\mathbb{Z}_{\geq 0}^N$ there exists a function $f_{\mathbf{m}}(q)$ in $\mathbb{Q}(q)$ such that $E(\mathbf{m})^* = f_{\mathbf{m}}(q)E(\mathbf{m})$ and $f_{\mathbf{m}}(0) = 1$. Moreover, this function is an eigenvector for the $\mathbb{Q}$-automorphism of $\mathbb{Q}(q)$ defined by $q \mapsto q^{-1}$.

We set

$$\mathcal{L}^* = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{\geq 0}^N} \mathbb{Z}[q]E(\mathbf{m})^*.$$

1.4. In this section, we give some results about Lusztig's canonical basis $\mathcal{B}$ and its dual $\mathcal{B}^*$.

First, for λ in P^+, let $V_q(\lambda)$ be the simple $U_q(\mathfrak{g})$-module with highest weight λ. Choose a highest weight vector v_λ. It is known that $V_q(\lambda)$ verifies the Weyl character formula. For all w in W, let $v_{w\lambda}$ be a (nonzero) extremal vector of weight $w\lambda$ and let $V_{q,w}(\lambda) := U_q(\mathfrak{n}).v_{w\lambda}$ be the Demazure module.

Let $\mathcal{B}$ be Lusztig's canonical basis of $U_q(\mathfrak{n}^-)$ ([16]), which coincides with Kashiwara's global basis ([9]). It verifies the following property:

Theorem. *Fix λ in P^+, and let $\mathcal{B}(\lambda) := \{b \in \mathcal{B}, bv_\lambda \neq 0\}$. Then the set $\mathcal{B}(\lambda).v_\lambda$ is a basis of $V_q(\lambda)$. Moreover, for all w in W there exists a unique subset $\mathcal{B}_w$ of $\mathcal{B}$, which does not depend on λ and such that $(\mathcal{B}(\lambda) \cap \mathcal{B}_w).v_\lambda$ is a basis of $V_{q,w}(\lambda)$.*

Remark. In the sequel, if no confusion occurs, we shall identify $\mathcal{B}(\lambda)$ with $\mathcal{B}(\lambda).v_\lambda$.

Let $\mathcal{B}^* \subset U_q(\mathfrak{n})$ be the dual basis in $U_q(\mathfrak{n})$, i.e., $(b^*, b') = \delta_{b,b'}$. Note that this basis is not really canonical since it depends on the choice of a Hopf pairing (,). Nethertheless, we shall call it the dual canonical basis.

Let η be the $\mathbb{Q}$-automorphism of $U_q(\mathfrak{g})$ such that $\eta(E_i) = E_i$, $\eta(F_i) = F_i$, and $\eta(q) = q^{-1}$. Let σ be the $\mathbb{Q}(q)$-anti-automorphism of $U_q(\mathfrak{g})$ such that $\sigma(E_i) = E_i$, $\sigma(F_i) = F_i$. We can now give a characterization of $\mathcal{B}^*$; see [11, Proposition 16]. This characterization will give rise to Lusztig's parametrization of the dual canonical basis $\mathcal{B}^*$. It depends on the choice of a reduced decomposition of w_0.

Proposition. *Fix a reduced decomposition $\tilde{w}_0$ of w_0. Then, for each $\mathbf{m}$ in $\mathbb{Z}_{\geq 0}^N$, there exists a unique homogeneous element $X := B^{\tilde{w}_0}(\mathbf{m})^* = B(\mathbf{m})^*$ in $U_q(\mathfrak{n})$ such that*

$$\sigma\eta(X) = (-1)^{\mathrm{tr}(X)} q^{\langle \mathrm{wt}(X), \mathrm{wt}(X)\rangle/2} q_X X, \quad X \in E(\mathbf{m})^* + q\mathcal{L}^*,$$

where $\mathrm{wt}(X) = \sum_i k_i\alpha_i$ is the weight of X, $q_X = \prod_i q_{\alpha_i}^{k_i}$, and $\mathrm{tr}(X) = \sum_i k_i$.

Remark. First note that the proposition implies that $\mathcal{L}$, resp. $\mathcal{L}^*$, is the $\mathbb{Z}[q]$-lattice generated by $\mathcal{B}$, resp. $\mathcal{B}^*$. Note also that the eigenvalue of $X = B(\mathbf{m})^*$ for $\sigma\eta$ only depends on the weight of X. Now, it can be easily seen that the first condition in the proposition can be replaced by "X is an eigenvector for $\sigma\eta$".

We denote by $B^{\tilde{w}_0}(\mathbf{m}) = B(\mathbf{m})$ the corresponding element in the canonical basis $\mathcal{B}$. For w in W and λ in P^+, we set $\mathcal{B}_w^* := \{b^* \in \mathcal{B}^*, b \in \mathcal{B}_w\}$, and $\mathcal{B}(\lambda)^* := \{b^* \in \mathcal{B}^*, b \in \mathcal{B}(\lambda)\}$.

2. Preliminary results on the dual canonical basis

2.1. In this section, we obtain a property of triangularity for the decomposition matrix of the dual canonical basis in a general PBW basis.

Proposition. *Let $\tilde{w}_0$ be a reduced decomposition of w_0. Then, for all α in R^+, the element $(E_\alpha^{\tilde{w}_0})^*$ of the dual PBW-basis belongs to $\mathcal{B}^*$.*

Proof. By Proposition 1.4 and Remark 1.4, it is enough to prove that $(E_\alpha^{\tilde{w}_0})^*$ is an eigenvector for $\sigma\eta$. By Claim 1.3, it is sufficient to prove that this is true for $E_\alpha^{\tilde{w}_0}$.

Set $\tilde{w}_0 = s_{i_1} \dots s_{i_N}$. We have $E_\alpha^{\tilde{w}_0} = T_{i_1} \dots T_{i_{p-1}}(E_{i_p})$ for p such that $\alpha = \beta_p$. By (1.2.1)–(1.2.3), we obtain that for all homogeneous elements X in $U_q(\mathfrak{n})$ of weight μ, we have $\sigma\eta T_i(X) = (-q_{\alpha_i})^{\mathrm{tr}(s_i\mu-\mu)} T_i(\sigma\eta(X))$. Recall that E_{i_p} is a homogeneous eigenvector for $\sigma\eta$. By induction, this is also true for $E_\alpha^{\tilde{w}_0}$.

The following corollary generalizes [16, Theorem 9.13 (a)].

Corollary. *Let $\tilde{w}_0$ be a reduced decomposition of w_0. Then, for all $\mathbf{m}$ in $\mathbb{Z}^N_{\geq 0}$, we have the following homogeneous decompositions:*

(i) $\sigma\eta(E^{\tilde{w}_0}(\mathbf{m})) = e^{\mathbf{m}}_{\mathbf{m}} E^{\tilde{w}_0}(\mathbf{m}) + \sum_{\mathbf{n}\prec\mathbf{m}} e^{\mathbf{n}}_{\mathbf{m}} E^{\tilde{w}_0}(\mathbf{n})$, $e^{\mathbf{n}}_{\mathbf{m}} \in \mathbb{Z}[q, q^{-1}]$,
(ii) $B^{\tilde{w}_0}(\mathbf{m}) = F^{\tilde{w}_0}(\mathbf{m}) + \sum_{\mathbf{m}\prec\mathbf{n}} d^{\mathbf{n}}_{\mathbf{m}} F^{\tilde{w}_0}(\mathbf{n})$, $d^{\mathbf{n}}_{\mathbf{m}} \in q\mathbb{Z}[q]$,
(iii) $B^{\tilde{w}_0}(\mathbf{m})^* = E^{\tilde{w}_0}(\mathbf{m})^* + \sum_{\mathbf{n}\prec\mathbf{m}} c^{\mathbf{n}}_{\mathbf{m}} E^{\tilde{w}_0}(\mathbf{n})^*$, $c^{\mathbf{n}}_{\mathbf{m}} \in q\mathbb{Z}[q]$.

Proof. Let $\mathbf{m}$ be in $\mathbb{Z}^N_{\geq 0}$ and set $\sigma\eta(E(\mathbf{m})) = \sum_{\mathbf{n}} e^{\mathbf{n}}_{\mathbf{m}} E(\mathbf{n})$, $e^{\mathbf{n}}_{\mathbf{m}} \in \mathbb{Z}[q, q^{-1}]$. Up to a multiplicative scalar, we have from the previous proposition

$$\begin{aligned}\sigma\eta(E(\mathbf{m})) &= \sigma\eta(E_{\beta_1}^{(m_1)} \dots E_{\beta_N}^{(m_N)}) \\ &= \sigma\eta(E_{\beta_N}^{(m_N)}) \dots \sigma\eta(E_{\beta_1}^{(m_1)}) = E_{\beta_N}^{(m_N)} \dots E_{\beta_1}^{(m_1)}.\end{aligned}$$

Hence, by Theorem 1.2, $\mathrm{Gr}^{\tilde{w}_0}(\sigma\eta(E(\mathbf{m}))) = \mathrm{Gr}^{\tilde{w}_0}(E(\mathbf{m}))$, up to a multiplicative scalar. This implies (i).

By duality, it is sufficient to prove (iii). The coefficients $c^{\mathbf{n}}_{\mathbf{m}}$ are in $q\mathbb{Z}[q]$ by Proposition 1.4. Now, the property of triangularity comes from Proposition 1.4 and (i).

Remark. In the previous corollary, the lexicographical ordering $\prec$ can be replaced by a coarser ordering: the one generated by $\mathbf{m} \leq e_k + e_{k'}$, $1 \leq k < k' \leq N \Leftrightarrow E(\mathbf{m})$ is a term of the PBW decomposition of $E_{\beta_k} E_{\beta_{k'}} - q^{\langle \beta_{k'}, \beta_k \rangle} E_{\beta_{k'}} E_{\beta_k}$.

2.2. The previous corollary implies the compatibility of the dual canonical basis with the space $U_q(\mathfrak{n}_w)$, $w \in W$.

Proposition. *Let w be in W. Then, $U_q(\mathfrak{n}_w)$ is generated as a space by a part of $\mathcal{B}^*$.*

Proof. Let $\tilde{w}$ be a reduced decomposition of w and let $\tilde{w}_0$ be a reduced decomposition of w_0 which completes $\tilde{w}$. Fix an element of the PBW-basis which belongs to $U_q(\mathfrak{n}_w)$. Then, by 1.2, all smaller elements of the PBW-basis, with the same weight, belong to $U_q(\mathfrak{n}_w)$. By Corollary 2.1, this implies the proposition.

3. Quantum flag minors

3.1. For all λ in P^+, the weights of $\mathcal{B}(\lambda)^*$ are the $\lambda - \mu$, where μ runs over the weights of $V_q(\lambda)$ (with multiplicity). The quantum flag minors (see [10], [2, 4.2]), are elements of $\mathcal{B}(\varpi_i)^*$, $1 \leq i \leq n$, which correspond to extremal vectors. To be more precise, let w be in W and let $\tilde{w} = s_{i_1} \dots s_{i_k}$ be a reduced decomposition of w. There exists a unique element in $\mathcal{B}(\varpi_{i_k})^*$ with weight $(Id - w)(\varpi_{i_k})$. Denote by

$\Delta^*_{\tilde w}$ this element. Then, $\Delta^*_{\tilde w}$ is a quantum flag minor and each quantum flag minor can be written in this way.

The following proposition generalizes some properties of the q-center of $U_q(\mathfrak{n}) = U_q(\mathfrak{n}_{w_0})$ proved in [4, Proposition 3.2] to $U_q(\mathfrak{n}_w)$.

Proposition. *Fix an element w in W and let $\tilde w = s_{i_1} \dots s_{i_k}$ be a reduced decomposition of w. Then:*

(i) *Let X_μ be an element of weight μ in $U_q(\mathfrak{n}_w)$; $\Delta^*_{\tilde w} X_\mu = q^{\langle (Id+w)\varpi_{i_k}, \mu\rangle} X_\mu \Delta^*_{\tilde w}$.*

(ii) *Let b^*_μ be an element of weight μ in $\mathcal{B}^*_w$; $q^{\langle \varpi_{i_k}, \mu\rangle} b^*_\mu \Delta^*_{\tilde w} \in \mathcal{B}^*$ mod $q\mathcal{L}^*$.*

Proof. We prove (i). Set $\tilde w' = s_{i_1} \dots s_{i_{k'}}$, for $k' \le k$. From [5, (3.3.2)], we obtain

$$\Delta^*_{\tilde w}\Delta^*_{\tilde w'} = q^{-\langle (Id-w)\varpi_{i_k} - 2\varpi_{i_k}, \mu'\rangle} \Delta^*_{\tilde w'}\Delta^*_{\tilde w},$$

where $\mu' := (Id - w')\varpi_{i_{k'}}$ is the weight of $\Delta^*_{\tilde w'}$ and $k' < k$. Hence,

$$\Delta^*_{\tilde w}\Delta^*_{\tilde w'} = q^{\langle (Id+w)\varpi_{i_k}, \mu'\rangle} \Delta^*_{\tilde w'}\Delta^*_{\tilde w}.$$

This formula holds for $k' = k$ because $\langle (Id + w)\varpi_{i_k}, (Id - w)\varpi_{i_k}\rangle = \langle \varpi_{i_k}, \varpi_{i_k}\rangle - \langle w\varpi_{i_k}, w\varpi_{i_k}\rangle = 0$ by W-invariance. Now (i) comes from the fact that the division ring generated by the $\Delta^*_{\tilde w'}$, $1 \le k' \le k$, is the division ring of $U_q(\mathfrak{n}_w)$, [5, Corollary 3.2].

The proof of (ii) is a straightforward generalization of [4, Proposition 3.2]. We sketch the proof. Let b_μ and $\Delta_{\tilde w}$ be the elements in $\mathcal{B}$ corresponding respectively to b^*_μ and $\Delta^*_{\tilde w}$ and suppose that $b_\mu \in \mathcal{B}(\lambda)$, $\lambda \in P^+$. Then, $\Delta_{\tilde w} \otimes b_\mu \in \mathcal{B}(\varpi_{i_k}) \otimes \mathcal{B}(\lambda)$ and this element corresponds to an element of the crystal basis at $q = 0$. We know that b_μ is in $\mathcal{B}_w$. Hence, by Littelmann's path model, we can associate to b_μ a chain of elements in W which are lower than w for the Bruhat ordering. Moreover, the chain associated to $\Delta_{\tilde w}$ is reduced to w. So, both chains can be compared for the Bruhat ordering. This implies that $\Delta_{\tilde w} \otimes b_\mu \in \mathcal{B}(\varpi_{i_k} + \lambda)$ at the crystal level. This implies (ii) by [11, Proposition 33].

3.2. By [7, Theorem 3.2] and Proposition 2.2, we have:

Lemma. *Let $\tilde w_0 = s_{i_1} \dots s_{i_N}$ be a reduced decomposition of w_0 and let $w = s_{i_1} \dots s_{i_k}$, $1 \le k \le N$. Let $\mathbf{m}$ be in $\mathbb{Z}^N_{\ge 0}$ such that $m_{k'} = 0$ for $k' > k$. Then $B^{\tilde w_0}(\mathbf{m})$ is in $\mathcal{B}_w$.*

Note that this lemma can be directly proved by applying the Berenstein-Zelevinsky formula ([2, Theorem 3.7]) for the transition map between the Lusztig parametrization and the string parametrization of the dual canonical basis. Indeed, by [9, Theorem 12.4], the elements of $\mathcal{B}^*_w$ are characterized by a string parametrization.

We can now prove the key proposition:

Proposition. *Let $\tilde w_0 = s_{i_1} \dots s_{i_N}$ be a reduced decomposition of w_0 and let $w = s_{i_1} \dots s_{i_k}$, $1 \le k \le N$. Fix $\mathbf{m}$ in $\mathbb{Z}^N_{\ge 0}$. Then*

$$q^{d(\mathbf{n}_{\tilde w}, \mathbf{m})} \Delta^*_{\tilde w} E^{\tilde w_0}(\mathbf{m})^* \in E^{\tilde w_0}(\mathbf{n}_{\tilde w} + \mathbf{m})^* + q\mathcal{L}^*,$$

where $\mathbf{n}_{\tilde w}$ is such that $\Delta^*_{\tilde w} = B^{\tilde w_0}(\mathbf{n}_{\tilde w})^*$.

Proof. We first suppose that $\mathbf{m}$ is in $\mathbb{Z}^k_{\ge 0} \times \{0\}^{N-k}$. Then, by Proposition 3.1 and the previous lemma, we have

$$q^{\langle \nu - \varpi_{i_k}, \mu\rangle} \Delta^*_{\tilde w} B^{\tilde w_0}(\mathbf{m})^* \in \mathcal{B}^* + q\mathcal{L}^*,$$

where ν and μ are respectively the weights of $\Delta^*_{\tilde w}$ and $B^{\tilde w_0}(\mathbf{m})^*$.

Recall the notation of 1.2. By [11, 4.2], we have

$$d(\mathbf{n}_{\tilde{w}}, \mathbf{m}) + d(\mathbf{m}, \mathbf{n}_{\tilde{w}}) = \langle \nu, \mu \rangle, \qquad d(\mathbf{n}_{\tilde{w}}, \mathbf{m}) - d(\mathbf{m}, \mathbf{n}_{\tilde{w}}) = c(\mathbf{n}_{\tilde{w}}, \mathbf{m}).$$

Using Proposition 3.1, we obtain $d(\mathbf{n}_{\tilde{w}}, \mathbf{m}) = \langle \nu - \varpi_{i_k}, \mu \rangle$. Hence, there exists $\mathbf{m}'$ in $\mathbb{Z}_{\geq 0}^N$ such that

$$q^{d(\mathbf{n}_{\tilde{w}}, \mathbf{m})} \Delta_{\tilde{w}}^* B^{\tilde{w}_0}(\mathbf{m})^* \in E^{\tilde{w}_0}(\mathbf{m}')^* + \sum_{\mathbf{n}} q\mathbb{Z}[q] E^{\tilde{w}_0}(\mathbf{n})^*.$$

Using Theorem 1.2 and Claim 1.3, we obtain $\mathbf{m}' = \mathbf{n}_{\tilde{w}} + \mathbf{m}$. By Proposition 2.2, this implies

$$q^{d(\mathbf{n}_{\tilde{w}}, \mathbf{m})} \Delta_{\tilde{w}}^* E^{\tilde{w}_0}(\mathbf{m})^* \in E^{\tilde{w}_0}(\mathbf{n}_{\tilde{w}} + \mathbf{m})^* + q\mathcal{L}^*.$$

We can now study the general case. Let $\mathbf{m}$ be in $\mathbb{Z}_{\geq 0}^N$ and decompose $\mathbf{m} = \mathbf{n} + \mathbf{p}$ with $\mathbf{n} \in \mathbb{Z}_{\geq 0}^k \times \{0\}^{N-k}$ and $p_l = 0$ for $l \leq k$. We have $q^{d(\mathbf{n}_{\tilde{w}}, \mathbf{m})} \Delta_{\tilde{w}}^* E(\mathbf{m})^* = q^{d(\mathbf{n}_{\tilde{w}}, \mathbf{n})} \Delta_{\tilde{w}}^* E(\mathbf{n})^* E(\mathbf{p})^* \in (E(\mathbf{n}_{\tilde{w}} + \mathbf{n})^* + \sum q\mathbb{Z}[q] E(\mathbf{r})^*) E(\mathbf{p})^*$, where $\mathbf{r}$ runs over $\mathbb{Z}_{\geq 0}^k \times \{0\}^{N-k}$. Hence, $q^{d(\mathbf{n}_{\tilde{w}}, \mathbf{m})} \Delta_{\tilde{w}}^* E(\mathbf{m})^* \in E(\mathbf{n}_{\tilde{w}} + \mathbf{n} + \mathbf{p})^* + \sum q\mathbb{Z}[q] E(\mathbf{r} + \mathbf{p})^* \subset E(\mathbf{n}_{\tilde{w}} + \mathbf{m})^* + q\mathcal{L}^*$. This ends the proof.

4. Quiver orientations and quantum flag minors

According to [4], Proposition 3.2 is almost what we need if we want to test the Berenstein-Zelevinsky conjecture when one element is a quantum flag minor. In fact, we have to prove an analogue of this proposition where $E^{\tilde{w}_0}(\mathbf{m})^*$ is replaced by $B^{\tilde{w}_0}(\mathbf{m})^*$. This can be obtained if we prove some increasing property of the linear form $d^{\tilde{w}_0}(\mathbf{n}_{\tilde{w}}, ?)$. By results of Reineke [20] and Bongartz [3], it is possible to prove those properties by the quiver approach ([21]) of quantum groups.

4.1. This section refers to [22] for notation and definitions. We suppose here that the graph Γ of $\mathfrak{g}$ is simply laced. Fix an orientation $\overrightarrow{\Gamma}$ of Γ and let $\mathrm{Mod} k\overrightarrow{\Gamma}$ be the category of k-representations of the quiver $\overrightarrow{\Gamma}$, where k is an algebraically closed field. We also denote by $\overline{\mathrm{Mod}} k\overrightarrow{\Gamma}$ the set of isomorphism classes in the category $\mathrm{Mod} k\overrightarrow{\Gamma}$.

For M, N in $\overline{\mathrm{Mod}} k\overrightarrow{\Gamma}$, set

$$\varepsilon(M, N) = \dim_k \mathrm{Hom}(M, N), \qquad \zeta(M, N) = \dim_k \mathrm{Ext}^1(M, N).$$

Let $\overline{\mathrm{Ind}} k\overrightarrow{\Gamma}$ be the set of isomorphism classes of indecomposable modules of $\mathrm{Mod} k\overrightarrow{\Gamma}$. Let τ: $\overline{\mathrm{Ind}} k\overrightarrow{\Gamma} \to \overline{\mathrm{Ind}} k\overrightarrow{\Gamma} \cup \{0\}$ be the Auslander-Reiten translation, with the convention that $\tau(M) = 0$ if M is a projective module. Recall ([22]) that

$$\zeta(M, N) = \varepsilon(N, \tau(M)), \qquad M, N \in \overline{\mathrm{Ind}} k\overrightarrow{\Gamma}. \tag{4.1.1}$$

By the work of Ringel [21], the algebra $U_q(\mathfrak{n})$ can be realized as a deformation of the Hall algebra associated to $\overrightarrow{\Gamma}$. In particular, for a special reduced decomposition $\tilde{w}_0(\overrightarrow{\Gamma})$ of w_0, the PBW-basis of $U_q(\mathfrak{n})$ can be naturally parametrized by $\overline{\mathrm{Mod}} k\overrightarrow{\Gamma}$. Let $R(w_0)$ be the set of $(i_1, \ldots, i_N)$ in $\{1, \ldots, n\}^N$ such that $w_0 = s_{i_1} \ldots s_{i_N}$. We consider the set of "commutation classes" $\overline{R(w_0)}$ corresponding to the relation

$$(i_1, \ldots, i_k, i_{k+1}, \ldots, i_N) \equiv (i_1, \ldots, i_{k+1}, i_k, \ldots, i_N) \text{ if } s_{i_k} s_{i_{k+1}} = s_{i_{k+1}} s_{i_k}.$$

In this section, a reduced decomposition of w_0 will mean an element of $\overline{R(w_0)}$. The following lemma can be found in [16, par. 4].

Lemma. *Let $\overrightarrow{\Gamma}$ be an orientation of Γ and let i be a sink of $\overrightarrow{\Gamma}$. Then we can choose an N-uple $(i_1, \dots, i_N)$ in the class $\tilde{w}_0(\overrightarrow{\Gamma}) \in \overline{R(w_0)}$ such that $i_1 = i$. Let $\overrightarrow{\Gamma}_i$ be the oriented graph obtained from $\overrightarrow{\Gamma}$ by reversing the arrows which go to i, we have $(i_2, \dots, i_N, i^*) \in \tilde{w}_0(\overrightarrow{\Gamma}_i)$, where i^* is such that $w_0(\alpha_i) = -\alpha_{i^*}$.*

In the proposition that follows, we give a recollection of results which can be found in [20] and [16]. It describes the link between the $\overline{\text{Mod}}k\overrightarrow{\Gamma}$-parametrization and the Lusztig parametrization associated to $\tilde{w}_0(\overrightarrow{\Gamma})$.

Proposition. *Fix an orientation $\overrightarrow{\Gamma}$ of Γ. Then there exists a $\mathbb{Z}_{\geq 0}$-linear bijection $\iota\colon \overline{Mod}k\overrightarrow{\Gamma} \to \mathbb{Z}_{\geq 0}^N$ such that*

(i) *if M is indecomposable, then $\iota(M) = e_k$, for k in $\{1, \dots, N\}$,*
(ii) *if M is indecomposable projective and if $\iota(M) = e_k$, then k is minimal in $\{k', i_{k'} = i_k\}$,*
(iii) *if M is indecomposable nonprojective and if $\iota(M) = e_k$, then $\iota(\tau(M)) = e_{k'}$, where $k' < k$ is maximal such that $i_{k'} = i_k$,*
(iv) *$d(\iota(M), \iota(N)) = \varepsilon(N, M) - \zeta(M, N)$.*

4.2. In the following sections, we shall replace the lexicographical ordering $\prec$ by the coarser ordering discussed in Remark 2.1. We still denote it by $\prec$. If the reduced decomposition $\tilde{w}_0$ is associated to a graph orientation, then this ordering is the Ext-ordering on $\overline{\text{Mod}}k\overrightarrow{\Gamma} \simeq \mathbb{Z}_{\geq 0}^N$ ([3]), which is also the degeneration ordering; [3, Corollary 4.2, Proposition 3.2]. We are now ready to prove the expected "increasing property" of $d^{\tilde{w}_0}(\mathbf{n}_{\tilde{w}}, ?)$.

Proposition. *Let $\tilde{w}_0(\overrightarrow{\Gamma}) = s_{i_1} s_{i_2} \dots s_{i_N}$ be the reduced decomposition of w_0 associated to $\overrightarrow{\Gamma}$ and let $\tilde{w} = s_{i_1} s_{i_2} \dots s_{i_k}$ be a (left) subword of $\tilde{w}_0(\overrightarrow{\Gamma})$. Let $\mathbf{n}_{\tilde{w}}$ be as in 3.1. Then the form $d(\mathbf{n}_{\tilde{w}}, ?)$ is increasing for $\prec$, i.e., $\mathbf{m} \preceq \mathbf{n} \Rightarrow d(\mathbf{n}_{\tilde{w}}, \mathbf{m}) \leq d(\mathbf{n}_{\tilde{w}}, \mathbf{n})$.*

Proof. By [5, 3.2.2], we have $\mathbf{n}_{\tilde{w}} = \sum e_l$ where l runs over the set $\{l,\ 1 \leq l \leq k,\ i_l = i_k\}$. From Proposition 4.1, we obtain that

$$d(\mathbf{n}_{\tilde{w}}, ?) = \sum_{i \geq 0} \varepsilon(?, \tau^i(M_k)) - \sum_{i \geq 0} \zeta(\tau^i(M_k), ?),$$

where M_k is the indecomposable module corresponding to e_k. By (4.1.1), we obtain after elimination that $d(\mathbf{n}_{\tilde{w}}, ?) = \varepsilon(?, M_k)$, which is increasing by [3, Proposition 3.2].

4.3. Now, the natural question is to know if every quantum flag minor can be associated to a graph decomposition $\overrightarrow{\Gamma}$. More precisely, fix u in W and a reduced decomposition $\tilde{u}$ of u. Is there an orientation $\overrightarrow{\Gamma}$ of Γ and a (left) subword $\tilde{w}$ of $\tilde{w}_0 = \tilde{w}_0(\overrightarrow{\Gamma})$ such that $\Delta^*_{\tilde{u}} = \Delta^*_{\tilde{w}}$? This is true for the A_n case, but not in general.

Claim. Let $\mathfrak{g}$ be of type A_n. Then every quantum flag minor can be realized as $B^{\tilde{w}_0}(\mathbf{n}_{\tilde{w}})^*$ where $\tilde{w}$ is a factor of a reduced decomposition $\tilde{w}_0$ associated to an orientation of Γ.

Proof. It is known that for all k, $1 \leq k \leq n$, the set of quantum minors in $\mathcal{B}(\varpi_k)^*$ is naturally indexed by a set of rows $I = \{i_1 < i_2 < \ldots < i_k\}$. We generally denote it by $\Delta_q(I,J)$ where $J=\{1,\dots,k\}$ is the set of its columns. This (flag) quantum minor corresponds to an extremal vector in $V_q(\varpi_k)$ associated to

$$w \in s_{(i_1-1)}\dots s_1 s_{(i_2-1)}\dots s_1 \dots s_{(i_k-1)}\dots s_1 W^k,$$

where $W^k := \{u \in W,\, u\varpi_k = \varpi_k\}$.

Now, it is well known that $\tilde{w}_0 := s_1s_2s_1s_3s_2s_1\dots s_ns_{n-1}\dots s_2s_1$ is a reduced decomposition associated to the equioriented quiver $\overrightarrow{\Gamma}^0 : n \to n-1 \to \dots \to 1$. Let $\mathcal{O}$ be the set of orientations of the graph Γ. Define the map r_i, $1 \le i \le n$, from $\mathcal{O}\cup\emptyset$ to itself such that $r_i(\emptyset)=\emptyset$, $r_i(\overrightarrow{\Gamma}) = \overrightarrow{\Gamma}_i$ if i is a sink of $\overrightarrow{\Gamma}$ and $r_i(\overrightarrow{\Gamma})=\emptyset$ if not. Define $r_i^j = r_i\dots r_j$ if $1\le i\le j\le n$ and $r_i^j = Id : \overrightarrow{\Gamma} \mapsto \overrightarrow{\Gamma}$ if not. Set $\overrightarrow{\Gamma}^I = r_{i_k}^n r_{i_{k-1}}^{n-1}\dots r_{i_1}^{n-k+1} r_1^{n-k}\dots r_1^1(\overrightarrow{\Gamma}^0)$. Then, by a straightforward verification using Lemma 4.1, we obtain that $\overrightarrow{\Gamma}^I$ is in $\mathcal{O}$ and that $\tilde{w}^I := s_{(i_1-1)}\dots s_1 s_{(i_2-1)}\dots s_1\dots s_{(i_k-1)}\dots s_1$ is a (left) subword of $\tilde{w}_0(\overrightarrow{\Gamma}^I)$.

Remark. The claim is not true if we take $\mathfrak{g}$ of type D_4. With the standard notation, the quantum flag minor associated to the reduced decomposition $\tilde{w} = s_2s_1s_3s_2$ cannot be realized from a quiver orientation.

5. Adapted algebras associated to a quiver orientation

5.1. We first recall some basic facts on adapted algebras associated to a reduced decomposition of w_0; see [4]. Fix a reduced decomposition $\tilde{w}_0$ of w_0 and let $\tilde{W}$ be the set of left subwords of $\tilde{w}_0$. Then, the quantum flag minors $\Delta^*_{\tilde{w}}$, $\tilde{w}\in\tilde{W}$, q-commute. Moreover, they generate a $\mathbb{Q}(q)$-algebra $A_{\tilde{w}_0}$ such that:

1) $A_{\tilde{w}_0}$ is a q-polynomial algebra of GK-dimension $N =$ Card $\tilde{W}$.

2) As a space, $A_{\tilde{w}_0}$ is generated by a part of the dual canonical basis, namely the monomials in the $\Delta^*_{\tilde{w}}$, $\tilde{w}\in\tilde{W}$, up to a power of q.

Now, we reach the main theorem:

Theorem (simply laced case)**.** *Fix an orientation $\overrightarrow{\Gamma}$ of Γ and set $\tilde{w}_0 = \tilde{w}_0(\overrightarrow{\Gamma})$. Suppose that b^* in $A_{\tilde{w}_0}\cap\mathcal{B}^*$ and $(b')^*$ in $\mathcal{B}^*$ q-commute, then they are multiplicative, i.e., their product is an element of the dual canonical basis up to a power of q.*

Proof. Set $\tilde{w}_0 = s_{i_1}s_{i_2}\dots s_{i_N}$. The elements $b^* = B^{\tilde{w}_0}(\mathbf{m})^*$ and $(b')^* = B^{\tilde{w}_0}(\mathbf{m}')^*$ q-commute. Hence, in order to prove that they are multiplicative, it is sufficient (see [11, 5.1]) to prove that

$$q^{d^{\tilde{w}_0}(\mathbf{m},\mathbf{m}')} B^{\tilde{w}_0}(\mathbf{m})^* B^{\tilde{w}_0}(\mathbf{m}')^* \in B^{\tilde{w}_0}(\mathbf{m}+\mathbf{m}')^* + q\mathcal{L}^*. \tag{5.1.1}$$

Recall that, by Corollary 2.2,

$$B^{\tilde{w}_0}(\mathbf{m}')^* = E^{\tilde{w}_0}(\mathbf{m}') + \sum_{\mathbf{n}\prec\mathbf{m}'} c^{\mathbf{n}}_{\mathbf{m}'} E^{\tilde{w}_0}(\mathbf{n})^*, \tag{5.1.2}$$

with $c^{\mathbf{n}}_{\mathbf{m}'} \in q\mathbb{Z}[q]$. Moreover, $B^{\tilde{w}_0}(\mathbf{m})^*$ is a product of $\Delta^*_{\tilde{w}}$, where $\tilde{w}$ runs over a subset $\tilde{W}_0$ of $\tilde{W}$. This implies that $d^{\tilde{w}_0}(\mathbf{m},?) = \sum d^{\tilde{w}_0}(\mathbf{n}_{\tilde{w}},?)$, where $\tilde{w}$ runs over $\tilde{W}_0$ with multiplicity. By Proposition 4.2, this asserts that the form $d^{\tilde{w}_0}(\mathbf{m},?)$ is increasing for $\prec$. Hence, (5.1.1) is a consequence of Proposition 3.2 and (5.1.2).

Remark. It is likely that the theorem works for all reduced decompositions of w_0. In fact, it would result from a generalization of Proposition 4.2 for general $\tilde{w}_0$.

By Claim 4.3, we deduce the following corollary which generalizes a result of [11]:

Corollary. *Let $\mathfrak{g}$ be of type A_n. Let b^* and $(b')^*$ be q-commuting elements in $\mathcal{B}^*$. If b^* is a quantum flag minor, then b^* and $(b')^*$ are multiplicative.*

5.2. In this section, we sketch the proof for a generalization of Theorem 5.1 in the nonsimply laced case. We recall some facts on Cartan datum associated to quotient graphs; see also [17, par. 14] and [19, par. 7].

Let w be in W and let $\tilde{w} = s_{i_1} \dots s_{i_k}$ be a reduced decomposition of w.

For $\alpha = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$, we set $\mathbf{n}_\alpha := \mathbf{n}_{\tilde{w}}$, with $\mathbf{n}_{\tilde{w}}$ as in Proposition 3.2.

Let Γ be a Coxeter graph of type A-D-E and let a be an automorphism of the graph Γ. Suppose that this automorphism is admissible, i.e., there is no edge joining two vertices of the same a-orbit. Let I be the set of vertices of Γ and $\overline{I}$ be the set of a-orbits in I. Then we can define a Cartan datum on $\overline{I}$ in the following way.

If $\overline{u} \neq \overline{v}$, then $(\alpha_{\overline{u}}, \alpha_{\overline{v}})$ is equal to -1 times the number of edges joining a vertex of $\overline{u}$ to a vertex of $\overline{v}$.

$(\alpha_{\overline{u}}, \alpha_{\overline{u}})$ is equal to 2 times the number of vertices in $\overline{u}$.

It is known that the Cartan datum of a graph of type B-C-F-G can be realized in this way from a graph of type A-D-E and an admissible automorphism and we denote by $\overline{\Gamma} = (\Gamma, a)$ the quotient graph. Let Q_Γ, resp. $Q_{\overline{\Gamma}}$, be the $\mathbb{Z}$-lattice generated by the roots corresponding to Γ, resp. $\overline{\Gamma}$, endowed with its natural bilinear form. Fix an orientation $\overrightarrow{\Gamma}$ of the graph Γ which is compatible with the automorphism a. This provides an orientation of the quotient graph. Now, the following properties are easily checked:

The rule $\alpha_{\overline{u}} \mapsto \sum_{u \in \overline{u}} \alpha_u$ defines a linear map from $Q_{\overline{\Gamma}}$ to Q_Γ which is compatible with the bilinear forms.

The reduced decomposition of w_0 associated to $\overrightarrow{\Gamma}$ can be factorized in the following way:

$$\tilde{w}_0 = (\prod_{u_1 \in \overline{u}_1} s_{u_1}) \dots (\prod_{u_P \in \overline{u}_P} s_{u_P}).$$

This provides the reduced decomposition of the longest element $\overline{w}_0$ of the Weyl group associated to the orientation of the quotient graph $\overline{\Gamma}$: $\tilde{\overline{w}}_0 = s_{\overline{u}_1} \dots s_{\overline{u}_P}$. We fix both reduced decompositions in the sequel.

We can naturally identify the set of positive roots $\overline{R}^+$ with the set of a-orbits in R^+.

We define a linear map $\phi : \mathbb{Z}_{\geq 0}^{\overline{R}^+} \to \mathbb{Z}_{\geq 0}^{R^+}$ by $e_{\overline{\alpha}} \mapsto \sum_{\alpha \in \overline{\alpha}} e_\alpha$. We have $\phi(\mathbf{n}_{\overline{\alpha}}) = \sum_{\alpha \in \overline{\alpha}} \mathbf{n}_\alpha$ and $d(\mathbf{n}_{\overline{\alpha}}, \mathbf{m}) = d(\phi(\mathbf{n}_{\overline{\alpha}}), \phi(\mathbf{m}))$.

This last property provides the increasing property that we need in order to generalize Theorem 5.1 for the reduced decomposition associated to the orientation of the quotient graph:

Theorem (nonsimply laced case)**.** *Let $\mathfrak{g}$ be a Lie algebra of type $\overline{\Gamma} = (\Gamma, a)$. Fix an orientation of the quotient graph $\overline{\Gamma}$. Let $\tilde{\overline{w}}_0$ be the reduced decomposition of the longest element of the Weyl group of $\mathfrak{g}$ associated to the orientation of $\overline{\Gamma}$. Suppose that b^* in $A_{\tilde{\overline{w}}_0} \cap \mathcal{B}^*$ and $(b')^*$ in $\mathcal{B}^*$ q-commute, then they are multiplicative.*

5.3. In order to understand a "multiplicative" description of the dual canonical basis, S. Fomin and A. Zelevinsky have defined the so-called cluster algebras ([8]). Roughly speaking, cluster algebras are algebras equipped with a distinguished set of generators called cluster variables and this set is divided into a union of subsets called clusters. The cluster variables verify the so-called exchange relations. For each symmetrizable Cartan datum, Fomin and Zelevinsky associate a cluster algebra.

As a conclusion, we would like to highlight the link between the results of 5.1 and the theory of cluster algebras. This follows some discussions with A. Zelevinsky. One of the most amazing facts in the Fomin-Zelevinsky theory is that many algebras, such as $\mathbb{C}[U^-]$, $\mathbb{C}[G]$, encountered in the representation theory of semi-simple Lie groups seem to be realized in the algebraic framework of cluster algebras. In particular, the subalgebras $A_{\tilde{w}_0}$ should specialize at $q = 1$ onto a subalgebra of $\mathbb{C}[U^-]$ generated by a cluster. The connection to our problem is the following: it is reasonable to think that if b and b' are q-commuting elements of the dual canonical basis, and if b is specialized at $q = 1$ onto a cluster variable, then b and b' are multiplicative. What we proved in 5.1 is a particular case of this conjecture when the cluster is associated to some reduced word.

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Institut Girard Desargues, Université Claude Bernard – Lyon 1, 69622 Villeurbanne Cedex, France

E-mail address: `caldero@desargues.univ-lyon1.fr`