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ON INTEGRAL REPRESENTATIONS OF p-SOLVABLE GROUPS

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ABSTRACT. It is a long standing problem whether every irreducible representation of a finite group G can be realized over the ring of integers $\mathbb{Z}[\mu_g]$ of the $g = \exp(G)$-cyclotomic field $\mathbb{Q}(g)$. We present a result which combines and extends the previously known criteria.

1. INTRODUCTION

Let $\chi \in \mathrm{Irr}(G)$ be an irreducible (complex) character of the finite group G. By Brauer's celebrated theorem χ can be written in the g-th cyclotomic field $\mathbb{Q}(g) = \mathbb{Q}(\mu_g)$, $g = \exp(G)$ (the least common multiple of the orders of the elements of G) and μ_g a primitive g-th root of unity. It is a long standing conjecture, already due to Schur, whether there is even a matrix representation with entries in its ring of (algebraic) integers $\mathbb{Z}[\mu_g]$.

In 1992, Cliff, Ritter and Weiss [1] settled this for solvable groups G. Knapp and Schmid [2] presented a different approach leading to a slightly stronger result. Also, in [2] the question for nonsolvable groups is reduced (essentially) to quasi-simple groups and verified for the sporadic groups and some other (small) simple groups. The only infinite series of quasi-simple groups treated so far are the alternating groups A_n in [3] (but not their covering groups) and the 2-dimensional linear groups over the prime fields $\mathrm{SL}(2, p)$ in [4].

However, it seems that the present understanding of the complex representations of groups of Lie type does not allow a treatment of the general problem. Even for groups of Lie rank one the complete answer is not known.

Our main theorem summarizes and extends the previously known general results.

Let $K = \mathbb{Q}(\mu_g)$ and $R = R_K = \mathbb{Z}[\mu_g]$ be the ring of integers in K. Denote by h_K the class number of R (or K).

Theorem 1.1. *Let $\chi \in Irr(G)$. Suppose that the group G is p-solvable for all primes p dividing the greatest common divisor $(\chi(1), h_K)$. Then χ can be realized over R.*

Schur already showed that χ is realizable over the ring of integers R_K of a splitting field K of χ if the degree $\chi(1)$ is coprime to the class number h_K of K.

Theorem 1.1 is a consequence of a more general result proven in section 3 (Theorem 3.1).

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2. Some Preliminaries

Let N be a normal subgroup of the finite group G. Let $K \supseteq \mathbb{Q}(\mu_n)$, $n = \exp(N)$ be an algebraic number field containig the n-th cyclotomic field and $R = R_K$ the ring of integers in K.

A representation $\rho : N \to \mathrm{GL}_d(R)$ is called G-stable if for any $x \in G$ there is a (invertible) matrix $T_x \in \mathrm{GL}_d(R)$ such that $\rho(x^{-1}gx) = T_x^{-1}\rho(g)T_x$ for all $g \in N$. We recall one of the main results of [2]:

Theorem 2.1. *Suppose $\rho : N \to GL_d(R)$ is an absolutely irreducible G-stable representation affording the character χ. Assume that χ can be extended to an R-valued character χ_0 of G. Then there is a unique representation $\rho_0 : G \to GL_d(R)$ extending ρ and affording χ_0.*

Proof. For convenience, we sketch a proof. There is a unique cohomology class $\omega(\rho) \in \mathrm{H}^2(G/N, R^*)$ which vanishes precisely if ρ can be extended to a representation of G. The cohomology group $\mathrm{H}^2(G/N, R^*)$ fits into an exact sequence

$$\mathrm{Hom}(G/N, K^*/R^*) \to \mathrm{H}^2(G/N, R^*) \to \mathrm{H}^2(G/N, K^*).$$

Now use that K^*/R^* is torsionfree since R is integrally closed. So the Hom-group is trivial. But our assumptions on ρ yield that, by a result of Dade, the image of $\omega(\rho)$ in $\mathrm{H}^2(G/N, K^*)$ is also trivial; for details see [2]. □

In applying Theorem 2.1 the problem is to find G-stable representations ρ affording χ, or equivalently, finding G-stable R-free RN-lattices affording χ.

Let U be an RN-lattice affording χ. As an R-module, U is the direct sum of $d = \chi(1)$ (nonzero) ideals J_i of R. By a well-known result of Steinitz this rank d together with its Steinitz class

$$[U] = [\prod_i J_i] = \prod_i [J_i] \in \mathrm{Cl}(R)$$

in the (finite) ideal class group $\mathrm{Cl}(R)$ of the Dedekind domain R determine the R-isomorphism type of U. The lattice U is R-free if and only if $[U] = 1$.

At least in the case of p-groups N we can always find a G-stable R-free RN-lattice U affording χ. Let W be a KN-module affording χ. A proof of the following lemma can be found in [3], Lemma 1.

Lemma 2.2. *Let $U \subseteq V$ be full RN-lattices in W. There exists a fractional ideal J of R such that $JU \subseteq V$ and such that the order ideal $\mathrm{ord}(V/JU)$ is divisible only by (nonzero) prime ideals $\mathfrak{p}$ of R for which $V/\mathfrak{p}V$ is not an irreducible RG-module or for which $\mathfrak{p}$ is not a d-th power in the class group $Cl(R)$, where $d = dim_R U$.*

Corollary 2.3. *Let N be a normal p-subgroup of G and assume that χ is G-invariant. Then there is a G-stable R-free RN-lattice U in W.*

Proof. By Lemma 1 of [2] there exists a G-stable RN-lattice $\tilde{U}$ in W affording χ. Since N is a monomial group, the character $\chi = \lambda^N$ is induced from a linear character. Hence there is an R_0-free R_0N-lattice V_0 affording χ where R_0 is the ring of integers of the field $\mathbb{Q}(\lambda)$ obtained by adjoining the values of λ to the rationals. Let $V = V_0 \otimes_{R_0} R$. Then V is an R-free RN-lattice affording χ. By multiplication with a scalar we may assume $\tilde{U} \subseteq V$. Now we apply Lemma 2.2 and see that for a suitable ideal J and $U = J\tilde{U}$ the order ideal $\mathrm{ord}(V/U)$ is only divisible by powers of the principal ideal $(1 - \mu_n)R$, so the order ideal $\mathrm{ord}(V/U)$ is principal. Hence, since V is R-free and $[U] = [V][\mathrm{ord}(V/U)]$, U is an R-free RN-lattice in the genus of $\tilde{U}$, so it is also G-stable. □

3. Realization of Characters

The theorem stated in the introduction is a special case ($G = N$) of the following result.

Theorem 3.1. *Let $\chi_0 \in Irr(G)$ and let N be a normal subgroup of G. Let $n = \exp(N)$ and $K = K(\chi_0)$ be an algebraic number field containing a primitive n-th root of unity.*

Suppose that the restriction $\chi_{0|N} = \chi$ remains irreducible and that for all primes p dividing $(\chi(1), h_K)$ the group N is p-solvable. Then there is an R-representation ρ of N affording χ and each one has a unique extension

$$\rho_0 : G \to GL_{\chi_0(1)}(R)$$

affording χ_0 where R is the ring of integers in K.

Proof. Let $d = \chi(1)$. By [2], Lemma 1 there exists a G-stable RN-lattice U_1 affording χ. By Theorem 2.1 we have to find a G-stable R-free RN-lattice U affording χ. We argue by induction, first on the degree of χ and then on the order of the normal subgroup N of G and assume that a counterexample G is chosen such that, first $\chi(1) = \chi_0(1)$ and then $|N|$ is minimal. Then χ is faithful because $\mathrm{Ker}\chi$ is a normal subgroup of G since χ is G-invariant. So N is not abelian and the center $Z(N)$ is cyclic.

Let $M \leq N$ be a G-invariant subgroup of N and $\theta \in \mathrm{Irr}(M)$ a constituent of the restriction χ_M. Let $I = I(\theta)$ be the inertia group of θ on G and $I_N = I \cap N$ the inertia group of θ in N. Then for $g \in G$ the conjugated character θ^g is a constituent of χ hence also an N-conjugate of θ and we conclude that $G = NI$. Let $\psi \in \mathrm{Irr}(I_N)$ be the unique character of I_N such that $\psi^N = \chi$. Then just by considering the degrees of the characters involved we see that there is a character $\psi_0 \in \mathrm{Irr}(I)$ extending ψ and $\chi_0 = \psi_0^G$. If $I_N < N$, then $\psi(1) < \chi(1)$ and our hypotheses on N carry over to I_N and, by induction, there is an I-stable R-free RI_N lattice $\tilde{U}$ affording ψ. So

$$U = \tilde{U}^N = \tilde{U} \otimes_{RI_N} RN$$

fulfils the requirements. Hence we have $I_N = N$ and the restriction of χ to every G-invariant subgroup M of N is homogenous. In particular, every abelian G-invariant subgroup of N is cyclic and central in G.

Now let E be a minimal non-central G-invariant subgroup of N and let $Z = E \cap Z(N)$. So E/Z is a chief factor of G and characteristically simple. Thus E is either a p-group or a central product of isomorphic (non-abelian) quasi-simple groups. Further, it follows from P. Hall's characterisation of p-groups of symplectic type that in the p-group case either $p > 2$ and E is extraspecial of exponent p, or $p = 2$ and E is extraspecial or the central product of an extraspecial 2-group with a cyclic group of order 4.

Now we claim that $N = E$.

Assume that $E < N$ and let $\theta \in \mathrm{Irr}(E)$ be a constituent of the restriction χ_E. Then θ is not linear and, further, θ is G-invariant. Now we invoke the concept of representation groups as represented in [5]. Let $G(\theta)$ be a representation group of θ with respect to G, so we have a central extension

$$C \rightarrowtail G(\theta) \twoheadrightarrow G/E$$

where C is a cyclic group of order $\exp(E)$. We denote by

$$\widehat{G} = G(\theta) \curlywedge G$$

the fibre-product with amalgamated factor group G/E, yielding a group extension

$$C \times E \rightarrowtail \widehat{G} \twoheadrightarrow G/E.$$

We view θ as a character of $\mathrm{Ker}(\widehat{G} \to G(\theta)) = 1 \times E$. By [5], there is a unique R-valued character $\widehat{\theta} \in \mathrm{Irr}(\widehat{G})$ extending θ. By induction we get an R-free RE-lattice U affording θ. Hence, by Theorem 2.1, there is an R-free $R\widehat{G}$-lattice $\widehat{U}$ affording $\widehat{\theta}$.

There is an epimorphism $\phi : \widehat{G} \twoheadrightarrow G/E \twoheadrightarrow G/N$. Let $\widehat{N}$ be the kernel of ϕ. Then $\widehat{N} = N(\theta) \curlywedge N$ is also a fibre product of a representation group $N(\theta)$ of θ with respect to N. There is a unique character $\widehat{\chi}$ of $\widehat{N}$ such that $1 \times E \leq \mathrm{Ker}(\widehat{\chi})$ and, viewing χ as a character of $\widehat{N}$ with $C \times 1$ in its kernel,

$$\chi = \widehat{\theta}_{\widehat{N}} \otimes \widehat{\chi}.$$

By uniqueness, $\widehat{\chi}$ is $\widehat{G}$-stable. We may view $\widehat{\chi}$ as a character of $N(\theta)$. By [5], $\exp(N(\theta))$ is a divisor of $\exp(N)$. As $\widehat{\theta}(1) = \theta(1) > 1$ we have $\widehat{\chi}(1) < \chi(1)$ and we get inductively a $G(\theta)$-stable R-free $RN(\theta)$-lattice V affording $\widehat{\chi}$. Now it is obvious that

$$U = \widehat{U}_{\widehat{N}} \otimes V$$

is an R-free $R\widehat{N}$-lattice affording χ. Identify $\widehat{N}/(\text{Kernel of } \widehat{N} \text{ on } U)$ with N. Then we may view U as a G-stable R-free RN-lattice affording χ. Since G is a counterexample, we must have $E = N$.

If E is a p-group we apply Corollary 2.3.

If E is nonsolvable, then our assumptions force that $d = \chi(1)$ is coprime to the class number h_K and every element of the class group $\mathrm{Cl}(R)$ is a d-th power. So let $[U_1] = [J]^d = [J^d]$ for some ideal J of R. Then, since $[J^{-1}U_1] = [J]^{-d}[U_1] = 1$, we see that $U = J^{-1}U_1$ is a G-stable R-free RN-lattice affording χ. □

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