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CANONICAL BASES AND QUIVER VARIETIES

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ABSTRACT. We prove the existence of canonical bases in the K-theory of quiver varieties. This existence was conjectured by Lusztig.

1. INTRODUCTION

Lusztig proposed in [16] to construct a signed basis of the equivariant K-theory of a quiver variety. As in [14], this signed basis should be characterized by an involution and a metric. He suggested a formula for the involution and the metric and he conjectured the existence of the signed basis. This signed basis should also satisfy some positivity property, related, hopefully, to the positivity of the structural constants of the product and the coproduct of the modified quantum algebra in the canonical basis, for all simply laced types. The main purpose of this paper is to give a precise definition of this signed basis and to prove its existence. It was conjectured in [24] that the K-theory of the quiver variety, with the action of the quantized enveloping algebra of affine type defined in [20] (see also [23] for the type A case), is isomorphic to the "maximal integrable module" introduced by Kashiwara in [8]. This module has a canonical basis; see *loc. cit.* The conjectures in [9, §13] suggest that Kashiwara's canonical basis and the geometric one are related; see Remark 7.2.2.

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2. THE ALGEBRA $\mathbf{U}$

2.1. Let $\mathfrak{g}$ be a simple, simply laced, complex Lie algebra. Let $(a_{ij})_{i,j\in I}$ be the Cartan matrix. The quantum loop algebra associated to $\mathfrak{g}$ is the $\mathbb{Q}(q)$-algebra $\mathbf{U}'$ generated by $\mathbf{x}^\pm_{ir}$, $\mathbf{k}^\pm_{is}$, $\mathbf{k}^{\pm 1}_i = \mathbf{k}^\pm_{i0}$ ($i \in I$, $r \in \mathbb{Z}$, $s \in \pm\mathbb{N}^\times$) modulo the following defining relations:

$$\mathbf{k}_i\mathbf{k}_i^{-1} = 1 = \mathbf{k}_i^{-1}\mathbf{k}_i, \quad [\mathbf{k}^\pm_{i,\pm r}, \mathbf{k}^\varepsilon_{j,\varepsilon s}] = 0,$$
$$\mathbf{k}_i\mathbf{x}^\pm_{jr}\mathbf{k}_i^{-1} = q^{\pm a_{ij}}\mathbf{x}^\pm_{jr},$$
$$(w - q^{\pm a_{ji}}z)\,\mathbf{k}^\varepsilon_j(w)\,\mathbf{x}^\pm_i(z) = (q^{\pm a_{ji}}w - z)\,\mathbf{x}^\pm_i(z)\,\mathbf{k}^\varepsilon_j(w),$$
$$(z - q^{\pm a_{ij}}w)\mathbf{x}^\pm_i(z)\mathbf{x}^\pm_j(w) = (q^{\pm a_{ij}}z - w)\mathbf{x}^\pm_j(w)\mathbf{x}^\pm_i(z),$$
$$[\mathbf{x}^+_{ir}, \mathbf{x}^-_{js}] = \delta_{ij}\frac{\mathbf{k}^+_{i,r+s} - \mathbf{k}^-_{i,r+s}}{q - q^{-1}},$$

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and

$$\sum_w \sum_{p=0}^{m} (-1)^p \begin{bmatrix} m \\ p \end{bmatrix} \mathbf{x}^{\pm}_{ir_{w(1)}} \mathbf{x}^{\pm}_{ir_{w(2)}} \cdots \mathbf{x}^{\pm}_{ir_{w(p)}} \mathbf{x}^{\pm}_{js} \mathbf{x}^{\pm}_{ir_{w(p+1)}} \cdots \mathbf{x}^{\pm}_{ir_{w(m)}} = 0,$$

where $i \neq j$, $m = 1 - a_{ij}$, $r_1, ..., r_m \in \mathbb{Z}$, and $w \in S_m$. We have set $[n] = q^{1-n} + q^{3-n} + ... + q^{n-1}$ if $n \geq 0$, $[n]! = [n][n-1]...[2]$, and

$$\begin{bmatrix} m \\ p \end{bmatrix} = \frac{[m]!}{[p]![m-p]!}.$$

We have also set $\varepsilon = +$ or $-$, and

$$\mathbf{k}^{\pm}_i(z) = \sum_{r \geq 0} \mathbf{k}^{\pm}_{i,\pm r} z^{\mp r}, \quad \mathbf{x}^{\pm}_i(z) = \sum_{r \in \mathbb{Z}} \mathbf{x}^{\pm}_{ir}\, z^{\mp r}.$$

2.2. Put $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$. Consider the $\mathbb{A}$-subalgebra $\mathbf{U} \subset \mathbf{U}'$ generated by the quantum divided powers $(\mathbf{x}^{\pm}_{ir})^{(n)} = (\mathbf{x}^{\pm}_{ir})^n/[n]!$, the Cartan elements $\mathbf{k}^{\pm 1}_i$, and the coefficients of the series

$$\sum_{s \geq 0} \mathbf{p}_{i,\pm s}\, z^s = \exp\Big(\sum_{s \geq 1} \frac{\mathbf{h}_{i,\pm s}}{[s]} z^s\Big),$$

where the elements $\mathbf{h}_{is}$ are such that

$$\mathbf{k}^{\pm}_i(z) = \mathbf{k}^{\pm 1}_i \exp\Big(\pm(q - q^{-1}) \sum_{s \geq 1} \mathbf{h}_{i,\pm s} z^{\mp s}\Big).$$

We see that $\mathbf{U}$ coincides with the $\mathbb{A}$-subalgebra generated by the elements $(\mathbf{e}_i)^n/[n]!$, $(\mathbf{f}_i)^n/[n]!$, and $\mathbf{k}^{\pm 1}_i$, $i \in I \cup \{0\}$, where $\mathbf{e}_i, \mathbf{f}_i, \mathbf{k}^{\pm 1}_i$ are the Kac-Moody generators; see [3, Proposition 2.2 and 2.6].

2.3. Let Δ be the coproduct of $\mathbf{U}'$ defined in terms of the Kac-Moody generators as follows:

$$\Delta(\mathbf{e}_i) = \mathbf{e}_i \otimes 1 + \mathbf{k}_i \otimes \mathbf{e}_i, \quad \Delta(\mathbf{f}_i) = \mathbf{f}_i \otimes \mathbf{k}^{-1}_i + 1 \otimes \mathbf{f}_i, \quad \Delta(\mathbf{k}_i) = \mathbf{k}_i \otimes \mathbf{k}_i.$$

Let τ, ψ, S be the anti-automorphisms of $\mathbf{U}'$ such that

$$\tau(\mathbf{e}_i) = \mathbf{f}_i, \quad \tau(\mathbf{f}_i) = \mathbf{e}_i, \quad \tau(\mathbf{k}_i) = \mathbf{k}^{-1}_i, \quad \tau(q) = q^{-1},$$

$$\psi(\mathbf{e}_i) = q\mathbf{k}_i\mathbf{f}_i, \quad \psi(\mathbf{f}_i) = q\mathbf{k}^{-1}_i\mathbf{e}_i, \quad \psi(\mathbf{k}_i) = \mathbf{k}_i, \quad \psi(q) = q,$$

$$S(\mathbf{e}_i) = -\mathbf{e}_i\mathbf{k}^{-1}_i, \quad S(\mathbf{f}_i) = -\mathbf{k}_i\mathbf{f}_i, \quad S(\mathbf{k}_i) = \mathbf{k}^{-1}_i, \quad S(q) = q.$$

The map S is the antipode. Let $x \mapsto \bar{x}$ be the algebra automorphism of $\mathbf{U}'$ such that

$$\bar{\mathbf{e}}_i = \mathbf{e}_i, \quad \bar{\mathbf{f}}_i = \mathbf{f}_i, \quad \bar{\mathbf{k}}_i = \mathbf{k}^{-1}_i, \quad \bar{q} = q^{-1}.$$

2.4. Let $\dot{\mathbf{U}}'$ be the modified algebra of $\mathbf{U}'$, and let $\dot{\mathbf{U}}$ be the corresponding $\mathbb{A}$-form. Let $\eta_\lambda \in \dot{\mathbf{U}}$ be the idempotent denoted by 1_λ in [13, §23.1].

3. The braid group

3.1. Let P, Q, be the integral weight lattice, and the root lattice of $\mathfrak{g}$. Let ω_i, α_i, $i \in I$, be the fundamental weights and the simple roots. Let $Q^+ \subset Q$, $P^+ \subset P$ be the subsemigroups generated by the simple roots and the fundamental weights. We set $\rho = \sum_{i\in I} \omega_i$. Let a_i, $i \in I$, be the positive integers such that the element $\theta = \sum_{i\in I} c_i\alpha_i \in Q^+$ is the highest root. The integer $\mathbf{c} = 1 + \sum_i c_i$ is the Coxeter number of $\mathfrak{g}$.

Let δ be the smallest positive imaginary root of the corresponding affine root system. Recall that the affine root α_0 is $\delta - \theta$. We set $\hat{P} = P \oplus \mathbb{Z}\delta$.

Let W be the Weyl group of $\mathfrak{g}$. Let $w_0 \in W$ be the longest element. The extended affine Weyl group is the semi-direct product $\tilde{W} = W \ltimes P$. For any element $w \in \tilde{W}$ let $l(w)$ be the length of w. Let $s_i \in \tilde{W}$, $i \in I \cup \{0\}$, be the affine simple reflexions. The affine Weyl group is the normal subgroup $\hat{W} \subset \tilde{W}$ generated by the elements s_i, $i \in I \cup \{0\}$. Let Γ be the quotient group $\tilde{W}/\hat{W}$. It is identified with a group of diagram automorphisms of the extended Dynkin diagram of $\mathfrak{g}$. In particular, Γ acts on $\mathbf{U}$, $\tilde{W}$ in the obvious way.

Let B_W, $B_{\tilde{W}}$ be the braid groups of W, $\tilde{W}$. The group $B_{\tilde{W}}$ is generated by elements T_w, $w \in \tilde{W}$, with the relation $T_w T_{w'} = T_{ww'}$ whenever $l(ww') = l(w) + l(w')$. The group B_W is the subgroup generated by the elements T_w, $w \in W$. For simplicity we set $T_i = T_{s_i}$ for any $i \in I \cup \{0\}$, and $\theta_i = T_{\omega_i}$ for any $i \in I$. The group $B_{\tilde{W}}$ acts on $\mathbf{U}$ by algebra automorphisms. Let T_i be the operator denoted by $T''_{i,1}$ in [13, §37.1.3]. If $i \neq j$, we have

$$T_i(\mathbf{e}_j) = \sum_{s=0}^{-a_{ij}} (-1)^s q^{-s} \mathbf{e}_i^{(-a_{ij}-s)} \mathbf{e}_j \mathbf{e}_i^{(s)}, \quad T_i(\mathbf{e}_i) = -\mathbf{f}_i \mathbf{k}_i,$$

$$T_i(\mathbf{f}_j) = \sum_{s=0}^{-a_{ij}} (-1)^s q^{s} \mathbf{f}_i^{(s)} \mathbf{f}_j \mathbf{f}_i^{(-a_{ij}-s)}, \quad T_i(\mathbf{f}_i) = -\mathbf{k}_i^{-1} \mathbf{e}_i.$$

We also have $T_i(\mathbf{k}_j) = \mathbf{k}_j \mathbf{k}_i^{-a_{ij}}$ for all i, j.

For a future use, we introduce the following notation:

– let σ be the automorphism of $B_{\tilde{W}}$ such that $\sigma(T_w) = T_{w^{-1}}^{-1}$ for all $w \in \tilde{W}$;

– for any $\alpha \in Q$, let $\mathbf{U}_\alpha \subset \mathbf{U}$ be the subset of the elements x such that $\mathbf{k}_i \, x \, \mathbf{k}_i^{-1} = q^{(\alpha,\alpha_i)} x$ for all i;

– for any $i \in I$ let $\underline{i} \in I$ be the unique element such that $w_0(\alpha_i) = -\alpha_{\underline{i}}$;

– let $(,) : P \times P \to \mathbb{Q}$ be the pairing such that $(\omega_i, \alpha_j) = \delta_{ij}$;

– for any $\alpha \in Q$ we set $|\alpha|^2 = \sum_i (\omega_i, \alpha)^2$.

3.2. Let $\gamma s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression for the element $\omega_i \in \tilde{W}$. Set

$$\gamma_i = \sum_{\ell=1}^{k} \gamma s_{i_1} \cdots s_{i_{\ell-1}} (\alpha_{i_\ell}) \in \hat{P}.$$

Lemma. *We have* $(\gamma_i, \alpha_i) = -\mathbf{c}$.

Proof. Let $\Delta_\pm \subset \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ be the sets of positive and negative roots. Let $\hat{\Delta}_\pm \subset \Delta + \mathbb{Z}\delta$ be the sets of positive and negative affine roots. We put $\Delta = \Delta_+ \sqcup \Delta_-$,

$\hat{\Delta} = \hat{\Delta}_+ \sqcup \hat{\Delta}_-$ and $\hat{\Delta}(\omega_i) = \hat{\Delta}_+ \cap \omega_i(\hat{\Delta}_-)$. Then,

$$\gamma_i = \sum_{\beta \in \hat{\Delta}(\omega_i)} \beta.$$

Recall that

$$\hat{\Delta}_+ = \Delta_+ \cup \bigcup_{n \geq 1} (n\delta + \Delta), \quad \hat{\Delta}_- = \Delta_- \cup \bigcup_{n \geq 1} (-n\delta + \Delta),$$

and that $\omega_i(\alpha) = \alpha - (\omega_i, \alpha)\delta$ for all affine root α. Thus,

$$\hat{\Delta}(\omega_i) = \{\alpha - (n - a_i)\delta \,|\, \alpha \in \Delta_-, \, a_i > n \geq 0\},$$

where we set $a_i = -(\omega_i, \alpha)$. Thus,

$$\gamma_i = \sum_{\alpha \in \Delta_-} a_i \big(\alpha + \frac{1 + a_i}{2}\delta\big).$$

Let κ be the Killing form. We get

$$\begin{aligned} (\gamma_i, \alpha_i) &= -\textstyle\sum_{\alpha \in \Delta_+} (\omega_i, \alpha) \cdot (\alpha_i, \alpha) \\ &= -\kappa(\omega_i, \alpha_i)/2 \\ &= -\mathbf{c}; \end{aligned}$$

see [6, Exercise 6.2]. □

We fix the Drinfeld generators of $\mathbf{U}$ in such a way that

$$\text{(3.2.1)} \qquad \mathbf{x}^-_{ir} = o_i^r \theta_i^r(\mathbf{f}_i), \quad \mathbf{x}^+_{ir} = o_i^r \theta_i^{-r}(\mathbf{e}_i),$$

where $o_i = \pm 1$ and $o_i + o_j = 0$ if $a_{ij} < 0$; see [2, Definition 4.6]. Note that there is exactly two choices for the map $i \mapsto o_i$. A case-by-case computation shows that the integer $o_i o_{\underline{i}}$ does not depend on i: it is equal to $(-1)^{\mathbf{c}}$.

Proposition.

(1) *There are unique* $\mathbb{A}$*-algebra automorphisms* $A, B : \mathbf{U} \to \mathbf{U}$ *such that*

$$A(\mathbf{x}^\pm_{ir}) = -q^{\mp 1}\mathbf{x}^\pm_{ir}, \quad B(\mathbf{x}^+_{ir}) = -\mathbf{x}^+_{ir}\mathbf{k}_i, \quad B(\mathbf{x}^-_{ir}) = -\mathbf{k}_i^{-1}\mathbf{x}^-_{ir}.$$

(2) *We have* $\tau(\mathbf{x}^\pm_{ir}) = \mathbf{x}^\mp_{i,-r}$, $\tau(\mathbf{k}^\pm_{ir}) = \mathbf{k}^\mp_{i,-r}$.

(3) *We have* $\psi(\mathbf{x}^\pm_{ir}) = q^{-r\mathbf{c}} T_{w_0} A(\mathbf{x}^\pm_{\underline{i},-r})$, $\overline{\mathbf{x}^\pm_{ir}} = q^{r\mathbf{c}} T_{w_0} B(\mathbf{x}^\mp_{\underline{i}r})$.

Proof. Claim 2 is known; see [2]. Claim 1 is a consequence of the identities 3. Let us prove 3. Let $\overline{T}_i$, ${}^\psi T_i$ be the automorphisms of the algebra $\mathbf{U}$ such that $\psi(T_i(x)) = {}^\psi T_i(\psi(x))$, $\overline{T}_i(x) = \overline{T_i(\overline{x})}$ for all $x \in \mathbf{U}$. By [13, §37] we have $\overline{T}_i = T''_{i,-1}$. A case-by-case computation also gives ${}^\psi T_i = T''_{i,-1}$. If $x \in \mathbf{U}_\alpha$, $\alpha \in Q$, we have

$$T''_{i,-1}(x) = (-q)^{-(\alpha,\alpha_i)} T_i^{-1}(x)$$

for all i; see [13, §37]. Thus,

$$\begin{aligned} {}^\psi\theta_i(x) &= \overline{\theta}_i(x) \\ &= (-q)^{-(\alpha,\beta_i)}\sigma(\theta_i)(x) \\ &= (-q)^{(\alpha,\gamma_i)}\sigma(\theta_i)(x), \end{aligned}$$

where $\beta_i = \alpha_{i_k} + s_{i_k}(\alpha_{i_{k-1}}) + \cdots + s_{i_k} \cdots s_{i_2}(\alpha_{i_1})$. Note that $(\alpha, \gamma_i) = -(\alpha, \beta_i)$ since $\gamma_i = -\omega_i(\beta_i)$. Since the weight ω_i is dominant, we have $T_{w_0} T_{\omega_i} = T_{-\omega_{\underline{i}}} T_{w_0}$, i.e., $T_{w_0}\theta_i T_{w_0}^{-1} = \sigma(\theta_{\underline{i}})^{-1}$. Recall that

$$T_{w_0}(\mathbf{e}_i) = -\mathbf{f}_{\underline{i}}\mathbf{k}_{\underline{i}}, \quad T_{w_0}(\mathbf{f}_i) = -\mathbf{k}_{\underline{i}}^{-1}\mathbf{e}_{\underline{i}}, \quad T_{w_0}(\mathbf{k}_i) = \mathbf{k}_{\underline{i}}^{-1}, \quad \forall i \neq 0.$$

Note that $\theta_i(\mathbf{k}_i) = \mathbf{k}_i$; see [2]. Using (3.2.1) we get

$$\begin{aligned}\overline{(\mathbf{x}_{ir}^+)} &= o_i^r(-q)^{r\mathbf{c}}\sigma(\theta_i)^{-r}(\mathbf{e}_i)\\ &= -o_i^r(-q)^{r\mathbf{c}}T_{w_0}\theta_{\underline{i}}^r(\mathbf{k}_{\underline{i}}^{-1}\mathbf{f}_{\underline{i}})\\ &= -q^{r\mathbf{c}}T_{w_0}(\mathbf{k}_{\underline{i}}^{-1}\mathbf{x}_{\underline{i}r}^-).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\psi(\mathbf{x}_{ir}^+) &= o_i^r(-q)^{-r\mathbf{c}}\sigma(\theta_i)^{-r}(q\mathbf{k}_i\mathbf{f}_i)\\ &= -o_i^r(-q)^{-r\mathbf{c}}T_{w_0}\theta_{\underline{i}}^r(q^{-1}\mathbf{e}_{\underline{i}})\\ &= -q^{-1-r\mathbf{c}}T_{w_0}(\mathbf{x}_{\underline{i},-r}^+).\end{aligned}$$

The case of $\mathbf{x}_{ir}^-$ is identical. □

4. Reminder on quiver varieties

4.1. Let the couple (J, H) denote the quiver such that J is the set of vertices, H is the set of arrows. If $h \in H$, let $h', h'' \in J$ be the incoming and the outgoing vertex of h. Let $\overline{h}$ denote the arrow opposite to h. We will consider the following cases:

- $\Pi = (I, H)$ where I is as in 2.1 and H is such that there are $2\delta_{ij} - a_{ij}$ arrows from i to j for all i, j. Then, let $\Omega \subset H$ be any set such that $H = \Omega \sqcup \bar{\Omega}$. Let n_{ij} (resp. $\bar{n}_{ij}$) be the number of arrows in Ω (resp. $\bar{\Omega}$) from i to j. Note that $n_{ij} = \bar{n}_{ji}$.
- Fix a set I^1 with a bijection $I \xrightarrow{\sim} I^1$, $i \mapsto i^1$. The quiver $\Pi^e = (I^e, H^e)$ is such that $I^e = I \sqcup I^1$, $H^e = H \sqcup \{i \to i^1, i^1 \to i \,|\, i \in I\}$.

4.2. Fix $V = \bigoplus_{i\in I}\mathbb{C}^{a_i}$, $W = \bigoplus_{i\in I}\mathbb{C}^{\ell_i}$.

Conventions. Fix $(m_i) \in \mathbb{Z}^I$. Hereafter, let μ, λ, α denote elements in P, P^+, Q^+ respectively such that $\mu = \sum_i m_i\omega_i$, $\lambda = \sum_i \ell_i\omega_i$, $\alpha = \sum_i a_i\alpha_i$. The dimension of the graded vector space V is identified with the root α while the dimension of W is identified with the weight λ.

The space

$$M_{\lambda\alpha} = \bigoplus_{h\in H} M_{a_{h'}a_{h''}}(\mathbb{C}) \oplus \bigoplus_{i\in I}\big(M_{a_i\ell_i}(\mathbb{C}) \oplus M_{\ell_i a_i}(\mathbb{C})\big)$$

is identified with the set of representations of the quiver Π^e on $V \oplus W$. For any $(B, p, q) \in M_{\lambda\alpha}$ let B_h be the component of the element $B \in \mathrm{Hom}\,(V_{h''}, V_{h'})$ and set

$$m_{\lambda\alpha}(B, p, q) = \sum_{h\in H}\varepsilon(h)B_hB_{\overline{h}} + pq \in \bigoplus_i \mathrm{Hom}\,(V_i, V_i),$$

where ε is a function $\varepsilon : H \to \mathbb{C}^\times$ such that $\varepsilon(h) + \varepsilon(\overline{h}) = 0$. Put $G_\lambda = \prod_i \mathrm{GL}_{\ell_i}$, $G_\alpha = \prod_i \mathrm{GL}_{a_i}$. The group $\mathbb{C}^\times \times G_\lambda \times G_\alpha$ acts on $M_{\lambda\alpha}$ by

$$(z, g_\lambda, g_\alpha)\cdot(B, p, q) = (zg_\alpha Bg_\alpha^{-1}, zg_\alpha pg_\lambda^{-1}, zg_\lambda qg_\alpha^{-1}).$$

Following [19] , we consider the varieties

$$Q_{\lambda\alpha}^{(\mu)} = \mathrm{Proj}\,\big(\textstyle\bigoplus_{n\geq 0} A_n^{(\mu)}\big) \quad \text{and} \quad N_{\lambda\alpha} = m_{\lambda\alpha}^{-1}(0)/\!\!/G_\alpha,$$

where $/\!\!/$ is the categorical quotient,

$$A_n^{(\mu)} = \big\{f \in \mathbb{C}[m_{\lambda\alpha}^{-1}(0)] \,\big|\, f\big(g_\alpha\cdot(B, p, q)\big) = \chi_\mu(g_\alpha)^{-n}f(B, p, q)\big\},$$

and $\chi_\mu(g_\alpha) = \prod_i \mathrm{Det}\,(g_{a_i})^{m_i}$. The obvious projection $\pi_{\lambda\alpha} : Q^{(\mu)}_{\lambda\alpha} \to N_{\lambda\alpha}$ is a projective map. If μ, μ' are such that $m_i, m'_i > 0$ for all i, or $m_i, m'_i < 0$ for all i, then the varieties $Q^{(\mu)}_{\lambda\alpha}$, $Q^{(\mu')}_{\lambda\alpha}$ are canonically isomorphic. There is an open subset $m^{-1}_{\lambda\alpha}(0)^{(\mu)} \subset m^{-1}_{\lambda\alpha}(0)$ whose points are called μ-semistable, such that there is a good quotient of $m^{-1}_{\lambda\alpha}(0)^{(\mu)}$ by the group G_α and we have

$$m^{-1}_{\lambda\alpha}(0)^{(\mu)} /\!\!/ G_\alpha = Q^{(\mu)}_{\lambda\alpha}; \tag{4.2.1}$$

see [18, §1.7] for instance. Moreover, if μ is a regular weight, then (4.2.1) is a geometric quotient and the variety $Q^{(\mu)}_{\lambda\alpha}$ is smooth, see [21, Proposition 2.6]. If μ is regular dominant, i.e., if $m_i > 0$ for all i, we set $Q_{\lambda\alpha} = Q^{(\mu)}_{\lambda\alpha}$.

Conventions. Hereafter, we assume that $(\mu, \alpha) \neq 0$ for any root α.

4.3. Put $d_{\lambda\alpha} = \dim Q_{\lambda\alpha}$. It is known that $d_{\lambda\alpha} = (\alpha, 2\lambda - \alpha)$. If $\alpha \geq \beta$, the extension by zero of representations of the quiver gives a closed embedding $N_{\lambda\beta} \hookrightarrow N_{\lambda\alpha}$. For any α, α', we consider the fiber product

$$Z_{\lambda\alpha\alpha'} = Q_{\lambda\alpha} \times_\pi Q_{\lambda\alpha'}.$$

If $\alpha' = \alpha + n\alpha_i$, $n > 0$, let $X_{\lambda\alpha\alpha'} \subset Z_{\lambda\alpha\alpha'}$ be the set of pairs (x, x') which are the G_α-orbits of Π^e-modules y, y' in $M_{\lambda\alpha}$, $M_{\lambda\alpha'}$ with y a subrepresentation of y'. If $\alpha' = \alpha - n\alpha_i$, put $X_{\lambda\alpha\alpha'} = \phi\big(X_{\lambda\alpha'\alpha}\big) \subset Z_{\lambda\alpha\alpha'}$, where ϕ is the automorphism of $Q_\lambda \times Q_\lambda$ taking an element (x, y) to (y, x). The variety $X_{\lambda\alpha\alpha'}$ is smooth; see [20, §5.3]. Consider the following varieties:

$$N_\lambda = \bigcup_\alpha N_{\lambda\alpha}, \quad Q_\lambda = \bigsqcup_\alpha Q_{\lambda\alpha}, \quad Z_\lambda = \bigsqcup_{\alpha,\alpha'} Z_{\lambda\alpha\alpha'},$$
$$X_\lambda = \bigsqcup_{\alpha,\alpha'} X_{\lambda\alpha\alpha'}, \quad F_\lambda = \bigsqcup_\alpha F_{\lambda\alpha},$$

where α, α' take all the possible values in Q^+ and $F_{\lambda\alpha} = \pi^{-1}_{\lambda\alpha}(0)$.

4.4. For any complex algebraic linear group G, and any quasi-projective G-variety X let $\mathbf{K}^G(X)$ be the Grothendieck group of G-equivariant coherent sheaves on X. We put $\mathbf{R}^G = \mathbf{K}^G(\textit{point})$. Let $\mathbf{X}^G \subset \mathbf{R}^G$ be the set of the simple modules. If the G-equivariant sheaf $\mathcal{E}$ is locally free, let $\wedge^i\mathcal{E}$ be its i-th wedge power, and $\bigwedge_\mathcal{E}$ its maximal wedge power. Note that $\bigwedge_\mathcal{E}$ is still defined, in the obvious way, whenever $\mathcal{E}$ is a G-equivariant complex on X.

Conventions. Hereafter, let $f_*, f^*, \otimes$, denote the derived functors $Rf_*, Lf^*, \otimes^L$ when they exist. Here $\otimes$ is the tensor product of coherent sheaves. We use the same notation for a sheaf and its class in the Grothendieck group.

4.5. Set $\tilde{G}_\lambda = G_\lambda \times \mathbb{C}^\times$. Let q denote also the character of the group $\mathbb{C}^\times$ such that $z \mapsto z$. The canonical bundle of the variety $Q_{\lambda\alpha}$ is

$$\Omega_{Q_{\lambda\alpha}} = q^{-d_{\lambda\alpha}}; \tag{4.5.1}$$

see [24, §6.4] for instance. Let V_i, W_i be the vectorial representations of the groups GL_{a_i}, GL_{ℓ_i}. Consider the following elements in $\mathbf{R}^{\tilde{G}_\lambda \times G_\alpha}$:

$$F_i^+ = q^{-1}W_i - q^{-2}V_i + q^{-1}\sum_{a_{ij}=-1} V_j, \quad F_i^- = -V_i, \quad F_i = F_i^+ + F_i^-.$$

The group $\tilde{G}_\lambda$ acts on the variety $Q^{(\mu)}_{\lambda\alpha}$. If E is a $\tilde{G}_\lambda \times G_\alpha$-module, let $E^{(\mu)} = m^{-1}_{\lambda\alpha}(0)^{(\mu)} \times_{G_\alpha} E$ be the induced $\tilde{G}_\lambda$-bundle on $Q^{(\mu)}_{\lambda\alpha}$. There is a unique ring homomorphism

$$\mathbf{R}^{\tilde{G}_\lambda \times G_\alpha} \to \mathbf{K}^{\tilde{G}_\lambda}(Q^{(\mu)}_{\lambda\alpha})$$

such that $E \mapsto E^{(\mu)}$ for all E. If μ is dominant, we set $\mathcal{V}_i = V_i^{(\mu)}$ and similarly for $\mathcal{W}_i$, $\mathcal{F}_i^{\pm}$, $\mathcal{F}_i$. We also set $\mathcal{V} = \bigoplus_i \mathcal{V}_i$, $\mathcal{W} = \bigoplus_i \mathcal{W}_i$.

Conventions. The restriction to $Q^{(\mu)}_{\lambda\alpha}$ of a sheaf $\mathcal{E}$ on $Q^{(\mu)}_\lambda$ is denoted by $\mathcal{E}_\alpha$. For simplicity we set $\mathcal{F}_{i;\alpha} = (\mathcal{F}_i)_\alpha$, etc.

4.6. Consider the map

$$\dagger \,:\, M_{\lambda\alpha} \to M_{\lambda\alpha},\ (B,p,q) \mapsto (B,p,q)^\dagger = (-\varepsilon^t B, -{}^t q, {}^t p),$$

where the upperscript t stands for the transpose map. Note that $\dagger$ does not commute to the action of the group $\tilde{G}_\lambda \times G_\alpha$. Let $\dagger$ be the group automorphism of $\tilde{G}_\lambda \times G_\alpha$ such that

$$\dagger \,:\, (z, g_\lambda, g_\alpha) \mapsto (z, g_\lambda, g_\alpha)^\dagger = (z, {}^t g_\lambda^{-1}, {}^t g_\alpha^{-1}).$$

Then $(g \cdot x)^\dagger = g^\dagger \cdot x^\dagger$ for all $g \in \tilde{G}_\lambda \times G_\alpha$, $x \in M_{\lambda\alpha}$. The induced map $\dagger \,:\, Q^{(\mu)}_{\lambda\alpha} \to Q^{(-\mu)}_{\lambda\alpha}$ is an isomorphism of algebraic varieties. Let

$$\dagger \,:\, \mathbf{R}^{\tilde{G}_\lambda \times G_\alpha} \to \mathbf{R}^{\tilde{G}_\lambda \times G_\alpha},\ E \mapsto E^\dagger$$

be the ring automorphism induced by the group automorphism $\dagger$. For any element $E \in \mathbf{R}^{\tilde{G}_\lambda \times G_\alpha}$, let $(E^{(\mu)})^\dagger \in \mathbf{K}^{\tilde{G}_\lambda}(Q^{(\mu)}_{\lambda\alpha})$ be the pull-back of $E^{(-\mu)} \in \mathbf{K}^{\tilde{G}_\lambda}(Q^{(-\mu)}_{\lambda\alpha})$ by the automorphism $\dagger$. We have

$$(E^{(\mu)})^\dagger = (E^\dagger)^{(\mu)}. \tag{4.6.1}$$

For any $w \in W$ we set $w * \alpha = \lambda - w(\lambda) + w(\alpha)$. The element $w * \alpha$ depends on the weight λ. However, since λ is fixed in the whole paper the notation $w * \alpha$ should not cause any confusion. There is a $\tilde{G}_\lambda$-equivariant isomorphism of algebraic varieties $S_w \,:\, Q^{(\mu)}_{\lambda\alpha} \to Q^{(w(\mu))}_{\lambda, w*\alpha}$ for each w, such that

$$S_i^2 = 1 \quad \text{and} \quad S_{ww'} = S_w S_{w'} \text{ if } l(ww') = l(w) + l(w');$$

see [15], [18], [21] (for simplicity we set $S_i = S_{s_i}$, where s_i is the simple reflexion with respect to the root α_i). The precise definition of S_w is given in the proof of Lemma 4.6. Consider the composed map $\omega = S_{w_0}\,\dagger$. This choice is motivated by [16] and [21, Theorem 11.7]. The map ω is an isomorphism of algebraic varieties $Q_{\lambda\alpha} \xrightarrow{\sim} Q_{\lambda, w_0 * \alpha}$.

Lemma.

(1) *We have* $\omega^*(\mathcal{F}_i) = -q^{\mathbf{c}} \mathcal{F}^\dagger_{\underline{i}}$, $\omega^*(\mathcal{W}_i) = \mathcal{W}_i^\dagger$ *and*

$$\sum_j [a_{ij}]\big(\omega^*(\mathcal{V}_j) + q^{\mathbf{c}} \mathcal{V}^\dagger_{\underline{j}}\big) = \mathcal{W}_i^\dagger + q^{\mathbf{c}} \mathcal{W}^\dagger_{\underline{i}}.$$

(2) *We have* $\omega^2 = Id$.

(3) *We have* $\omega(F_{\lambda\alpha}) = F_{\lambda, w_0 * \alpha}$.

Proof. We use the construction of the operator S_w given in [18]; see also [15]. We recall it briefly. Set $\alpha' = s_i * \alpha$, $\mu' = s_i(\mu)$, $\mu' = \sum_i m'_i \omega_i$. First, we assume that $m_i < 0$. Then $m'_i > 0$. Following [15, §3.2], let

$$Z_i^\mu \subset m_{\lambda\alpha}^{-1}(0)^{(\mu)} \times m_{\lambda\alpha'}^{-1}(0)^{(\mu')}$$

be the set of pairs (x, x'), where $x = (B, p, q)$, $x' = (B', p', q')$ are such that

– the sequence of $\tilde{G}_\lambda \times G_\alpha$-modules

$$0 \longrightarrow q^{-2} V_i' \xrightarrow{a(x')} q^{-1} W_i \oplus q^{-1} \bigoplus_{a_{ij}=-1} V_j \xrightarrow{b(x)} V_i \longrightarrow 0$$

such that $a(x') = (q'_i, B'_h)$, $b(x) = p_i + \varepsilon_{\bar{h}} B_h$ is exact,

– we have $a(x)b(x) - a(x')b(x') = 0$,

– we have $B_h = B'_h$ if $h', h'' \neq i$, and $p_j = p'_j$, $q_j = q'_j$ if $j \neq i$.

Note that $a(x')$ is injective and $b(x)$ is surjective; see [18, Lemma 38]. Thus Z_i^μ is a closed subset of $m_{\lambda\alpha}^{-1}(0)^{(\mu)} \times m_{\lambda\alpha'}^{-1}(0)^{(\mu')}$. Consider the group $G_{\alpha\alpha'} = GL_{a_i} \times GL_{a'_i} \times \prod_{j \neq i} GL_{a_j}$. The categorical quotient

$$Q_i^\mu = Z_i^\mu /\!\!/ G_{\alpha\alpha'},$$

is a smooth variety. Moreover, the obvious projections are isomorphisms of algebraic varieties

$$p_{1,\alpha}^{(\mu)} : Q_i^\mu \xrightarrow{\sim} Q_{\lambda\alpha}^{(\mu)}, \quad p_{2,\alpha'}^{(\mu')} : Q_i^\mu \xrightarrow{\sim} Q_{\lambda\alpha'}^{(\mu')};$$

see [18, Proposition 40]. The group $\tilde{G}_\lambda$ acts in the obvious way on Q_i^μ, making the maps $p_{1,\alpha}^{(\mu)}$, $p_{2,\alpha'}^{(\mu')}$ equivariant. By construction, for any $i \neq j$ we have

$$(4.6.2) \qquad \begin{aligned} (p_{1\alpha}^{(\mu)})^*(F_i^{(\mu)} + q^{-2} V_i^{(\mu)}) &= (p_{2\alpha'}^{(\mu')})^*(q^{-2} V_i^{(\mu')}), \\ (p_{1\alpha}^{(\mu)})^*(V_j^{(\mu)}) &= (p_{2\alpha'}^{(\mu')})^*(V_j^{(\mu')}). \end{aligned}$$

We set (recall that $m_i < 0$)

$$S_i = p_{2\alpha'}^{(\mu')} (p_{1\alpha}^{(\mu)})^{-1} : Q_{\lambda\alpha}^{(\mu)} \to Q_{\lambda\alpha'}^{(\mu')}.$$

If $m_i > 0$, we set

$$S_i = p_{1\alpha'}^{(\mu')} (p_{2\alpha}^{(\mu)})^{-1} : Q_{\lambda\alpha}^{(\mu)} \to Q_{\lambda\alpha'}^{(\mu')}.$$

Using (4.6.2) we get, if $m_i < 0$ and $i \neq j$,

$$S_i^*(V_i^{(\mu')}) = q^2 F_i^{(\mu)} + V_i^{(\mu)}, \quad S_i^*(V_j^{(\mu')}) = V_j^{(\mu)}.$$

Note that the map S_i commutes to the action of the group $\tilde{G}_\lambda$. Thus,

$$S_i^*(W_j^{(\mu)}) = W_j^{(\mu')}$$

for all j. Set $\varepsilon_i = +1$ if $m_i > 0$, $\varepsilon_i = -1$ if $m_i < 0$. A case-by-case analysis gives the following equalities in $\mathbf{K}^{\tilde{G}_\lambda}(Q_\lambda^{(\mu')})$:

$$S_i^*(F_j^{(\mu)}) = \begin{cases} -q^{2\varepsilon_i} F_j^{(\mu')} & \text{if} \quad i = j, \\ F_j^{(\mu')} & \text{if} \quad a_{ij} = 0, \\ F_j^{(\mu')} + q^{\varepsilon_i} F_i^{(\mu')} & \text{if} \quad a_{ij} = -1. \end{cases}$$

The general formula is

$$(4.6.3) \qquad S_i^*(F_j^{(\mu)}) = F_j^{(\mu')} - q^{\varepsilon_i} [a_{ij}] F_i^{(\mu')}.$$

We now assume that the weight μ is dominant. Thus, $F_j^{(\mu)} = \mathcal{F}_j$. Fix an element w in the Weyl group. Let us prove that

$$w(\alpha_i) = \alpha_j \Rightarrow S_w^*(\mathcal{F}_j) = q^{a(w,i)} F_i^{(w^{-1}(\mu))}, \tag{4.6.4}$$

where

$$a(w,i) = \frac{1}{2} \sum_{\alpha \in \Delta_+ \cap w^{-1}\Delta_-} (\alpha_i, \alpha)^2.$$

We may assume that $l(w) > 0$ and that (4.6.4) holds for any x with $l(x) < l(w)$. Fix $k \in I$ such that $w(\alpha_k) \in -Q^+$. Let $\langle s_i, s_k \rangle$ be the subgroup generated by s_i, s_k. Let x be the element of minimal length in the set $w\langle s_i, s_k\rangle$. Then, $x(\alpha_i), x(\alpha_k) \in Q^+$. One of the following two cases holds; see [12, Proof of Proposition 1.8].

– Either $a_{ik} = 0$, $w = xs_k$, $l(w) = l(x) + 1$. Then, $x(\alpha_i) = \alpha_j$. Using (4.6.4) for x, and (4.6.3), we get

$$S_w^*(\mathcal{F}_j) = q^{a(x,i)} S_k^*(F_i^{(x^{-1}(\mu))}) = q^{a(x,i)} F_i^{(\mu)}.$$

Using the identity

$$\Delta_+ \cap w^{-1}\Delta_- = s_k(\Delta_+ \cap x^{-1}\Delta_-) \cup \{\alpha_k\}$$

we get also $a(w,i) = a(x,i)$. Thus (4.6.4) holds.

– Either $a_{ik} = -1$, $w = xs_is_k$, $l(w) = l(x) + 2$. Then, $x(\alpha_k) = \alpha_j$. Using (4.6.4) for x we get

$$S_w^*(\mathcal{F}_j) = q^{a(x,k)} S_k^* S_i^*(F_k^{(x^{-1}(\mu))}).$$

We are reduced to the A_2 case. Set $\nu = x^{-1}(\mu)$. We have $w^{-1}(\mu) = s_ks_i(\nu) < s_i(\nu) < \nu$. A direct computation using (4.6.3) gives

$$S_k^* S_i^*(F_k^{(\nu)}) = q F_i^{(s_ks_i(\nu))}.$$

Using the identity

$$\Delta_+ \cap w^{-1}\Delta_- = s_ks_i(\Delta_+ \cap x^{-1}\Delta_-) \cup \{\alpha_k, \alpha_i + \alpha_k\}$$

we get also $a(w,i) = a(x,k) + 1$. Thus (4.6.4) holds.

Setting $w, i, j \to w_0 s_{\underline{i}}, \underline{i}, i$ in (4.6.4) and using the formula for $S_{\underline{i}}^*$, we get $S_{w_0}^*(\mathcal{F}_i) = -q^{a(w_0 s_{\underline{i}}, \underline{i})+2} F_{\underline{i}}^{(w_0(\mu))}$. Thus

$$\omega^*(\mathcal{F}_i) = -q^{a(w_0 s_{\underline{i}}, \underline{i})+2} \mathcal{F}_{\underline{i}}^\dagger; \tag{4.6.5}$$

see (4.6.1). Moreover, we have (see 3.2)

$$\begin{aligned} a(w_0 s_{\underline{i}}, \underline{i}) &= \tfrac{1}{2} \textstyle\sum_{\alpha \in \Delta_+ \setminus \{\alpha_{\underline{i}}\}} (\alpha_{\underline{i}}, \alpha)^2 \\ &= \tfrac{1}{4} \kappa(\alpha_{\underline{i}}, \alpha_{\underline{i}}) - 2 \\ &= \mathbf{c} - 2. \end{aligned}$$

By definition we have $q\mathcal{F}_i = \mathcal{W}_i - \sum_j [a_{ij}] \mathcal{V}_j$. Using (4.6.1), (4.6.5) and the equality $a_{ij} = a_{\underline{ij}}$ we get the identity

$$\sum_j [a_{ij}] \omega^*(\mathcal{V}_j) = \mathcal{W}_i^\dagger + q^{\mathbf{c}} \mathcal{W}_{\underline{i}}^\dagger - q^{\mathbf{c}} \sum_j [a_{ij}] \mathcal{V}_{\underline{j}}^\dagger.$$

Claim 1 is proved.

From the definition of the operator S_i we get $\dagger S_i = S_i \dagger$ for all i. Claim 2 follows immediately.

We now prove Claim 3. Using Claim 2 it is sufficient to prove that $\omega(F_{\lambda\alpha}) \subseteq F_{\lambda,w_0*\alpha}$. Assume that $\alpha' = s_i * \alpha$, $\mu' = s_i(\mu)$ as above. It is sufficient to prove that $S_i(F^{(\mu)}_{\lambda\alpha}) \subseteq F^{(\mu')}_{\lambda\alpha'}$. By [10, §1.3] the ring of G_α-invariant polynomials on $m^{-1}_{\lambda\alpha}(0)$ is generated by the following two types of functions:

(i) $\mathrm{tr}_{V_j}(B_{h_1}B_{h_2}\cdots B_{h_n})$ for any sequence $h_1, h_2, ..., h_n \in H$ such that $j = h'_1$, $h''_1 = h'_2$,..., $h''_{n-1} = h'_n$, $h''_n = j$,

(ii) $\varphi(q_j B_{h_1}B_{h_2}\cdots B_{h_n}p_k)$ for any sequence $h_1, h_2, ..., h_n \in H$ such that $j = h'_1$, $h''_1 = h'_2$,..., $h''_{n-1} = h'_n$, $h''_n = k$, and any linear form φ on $\mathrm{Hom}\,(W_k, W_j)$.

We may assume that $m_i < 0$. Fix an element (x, x') in Z^μ_i. Set $x = (B_h, p_j, q_j)$, $x' = (B'_h, p'_j, q'_j)$. In particular, we have

$$B_h = B'_h \text{ if } h', h'' \neq i; \quad B_{h_1}B_{h_2} = B'_{h_1}B'_{h_2} \text{ if } h''_1 = h'_2 = i.$$

Thus any function of type (i) coincides on x and x'. We have also

$$q_j = q'_j,\ p_j = p'_j \text{ if } j \neq i; \quad q_i p_i = q'_i p'_i;$$

$$q_i B_h = q'_i B'_h \text{ if } h' = i; \quad B_h p_i = B'_h p'_i \text{ if } h'' = i.$$

Thus, any function of type (ii) coincide on x and x'. In particular, $x \in F^{(\mu)}_{\lambda\alpha}$ iff $x' \in F^{(\mu')}_{\lambda\alpha'}$. We are done. □

Remark. The dual of the $\tilde{G}_\lambda$-bundle $E^{(\mu)}$ on $Q^{(\mu)}_\lambda$ is $(E^*)^{(\mu)}$, where E^* is the dual module, obtained by composing the $\tilde{G}_\lambda$-action by the group automorphism $(z, g_\lambda, g_\alpha) \mapsto (z^{-1}, {}^t g^{-1}_\lambda, {}^t g^{-1}_\alpha)$. Note that, in the particular case where $E = V_i$, W_i we have $\mathcal{V}^\dagger_i = \mathcal{V}^*_i$, $\mathcal{W}^\dagger_i = \mathcal{W}^*_i$.

Conventions. Put $1_{\lambda\alpha} = \mathcal{O}_{F_{\lambda\alpha}}$, $1'_{\lambda\alpha} = \mathcal{O}_{Q_{\lambda\alpha}}$. It is convenient to also set $1_\lambda = 1_{\lambda 0}$, $1'_\lambda = 1'_{\lambda 0}$. To simplify the notation we put $\nu = w_0 * 0$.

5. The involution on the convolution algebra

5.1. Given smooth quasi-projective G-varieties X_1, X_2, X_3, consider the projection $p_{ab} : X_1 \times X_2 \times X_3 \to X_a \times X_b$ for all $1 \leq a, b \leq 3$, $a \neq b$. Fix closed subvarieties $Z_{ab} \subset X_a \times X_b$ such that the restriction of p_{13} to $p^{-1}_{12}Z_{12} \cap p^{-1}_{23}Z_{23}$ is proper and maps to Z_{13}. The convolution product is the map

$$\star : \quad \mathbf{K}^G(Z_{12}) \times \mathbf{K}^G(Z_{23}) \to \mathbf{K}^G(Z_{13}), \quad (\mathcal{E}, \mathcal{F}) \mapsto p_{13\,*}\big((p^*_{12}\mathcal{E}) \otimes (p^*_{23}\mathcal{F})\big).$$

If $Z_{12} = Z_{23} = Z_{13} = Z$, the map $\star$ endows $\mathbf{K}^G(Z)$ with the structure of an $\mathbf{R}^G$-algebra. See [4] for more details.

5.2. Let $\mathbb{D}_{X_a}$ be the Serre-Grothendieck duality operator on $\mathbf{K}^G(X_a)$. Assume that X_a is connected. Let Ω_{X_a} be the canonical bundle of X_a, and let $\mathcal{O}_{X_a}$ be the structural sheaf. We have

$$\mathbb{D}_{X_a}(\mathcal{E}) = (-1)^{\dim X_a}\mathcal{E}^* \otimes \Omega_{X_a}$$

for any G-equivariant locally free sheaf $\mathcal{E}$ on X_a. Assume that there is a character q of the group G such that $\Omega_{X_a} = q^{-\dim X_a}$ for all a. Consider the operator $D_{Z_{ab}} = q^{d_{ab}}\mathbb{D}_{Z_{ab}}$, where $d_{ab} = (\dim X_a + \dim X_b)/2$. Recall that the automorphism $\phi : X_a \times X_b \to X_b \times X_a$ is the flip.

Lemma. *Fix* $x \in \mathbf{K}^G(Z_{12})$, $y \in \mathbf{K}^G(Z_{23})$.

(1) $\phi^*(x\star y) = \phi^*(y)\star\phi^*(x)$, $D_{Z_{12}}(x)\star D_{Z_{23}}(y) = D_{Z_{13}}(x\star y)$, $\phi^* D_{Z_{ab}} = D_{Z_{ba}}\phi^*$.

(2) *If* $Z_{12} = Z_{23} = Z_\lambda$, *then* $(\omega\times\omega)^*(x)\star(\omega\times\omega)^*(y) = (\omega\times\omega)^*(x\star y)$.
(3) *If* $Z_{12} = Z_\lambda$, $Z_{23} = Q_\lambda$ *or* F_λ, *then* $(\omega\times\omega)^*(x)\star\omega^*(y) = \omega^*(x\star y)$.

See [14] for more details.

5.3. We consider the maps γ_λ, γ'_λ, Γ_λ, ζ_λ on $\mathbf{K}^{\tilde G_\lambda}(F_\lambda)$, $\mathbf{K}^{\tilde G_\lambda}(Q_\lambda)$, $\mathbf{K}^{\tilde G_\lambda}(Z_\lambda)$ such that

$$\gamma_\lambda = \bigoplus\nolimits_\alpha q^{d_{\lambda\alpha}/2}\omega^*\mathbb{D}_{F_{\lambda\alpha}},\quad \gamma'_\lambda = \bigoplus\nolimits_\alpha q^{3d_{\lambda\alpha}/2}\omega^*\mathbb{D}_{Q_{\lambda\alpha}},$$
$$\Gamma_\lambda = \bigoplus\nolimits_{\alpha,\alpha'}(\omega\times\omega)^* D_{Z_{\lambda\alpha\alpha'}},\quad \zeta_\lambda = (\omega\times\omega)^*\phi^*$$

(see Lemma 4.6.3 for γ_λ). Let

$$^- : \mathbf{R}^{\tilde G_\lambda}\to\mathbf{R}^{\tilde G_\lambda},\ V\mapsto\bar V$$

be the ring automorphism induced by the group automorphism $\tilde G_\lambda\to\tilde G_\lambda$, $(z,g_\lambda)\mapsto(z^{-1},g_\lambda)$. By Lemma 4.6.1 the operators ω^*, ζ_λ are $\dagger$-semilinear automorphisms of $\mathbf{R}^{\tilde G_\lambda}$-modules, and γ_λ, γ'_λ, Γ_λ are $^-$-semilinear. Let $\kappa : F_\lambda\hookrightarrow Q_\lambda$ be the closed embedding.

Lemma. *The following identities hold:*
(1) $\omega^*\mathbb{D}_{F_\lambda} = \mathbb{D}_{F_\lambda}\omega^*$, $\omega^*\mathbb{D}_{Q_\lambda} = \mathbb{D}_{Q_\lambda}\omega^*$, $(\omega\times\omega)^*\mathbb{D}_{Z_\lambda} = \mathbb{D}_{Z_\lambda}(\omega\times\omega)^*$,
(2) $\kappa_*\omega^* = \omega^*\kappa_*$, $(\omega\times\omega)^*\phi^* = \phi^*(\omega\times\omega)^*$, $(\kappa_*\times\kappa_*)\phi^* = \phi^*(\kappa_*\times\kappa_*)$,
(3) $\gamma_\lambda(u\star x) = \Gamma_\lambda(u)\star\gamma_\lambda(x)$, *for any* $x\in\mathbf{K}^{\tilde G_\lambda}(F_\lambda)$, $u\in\mathbf{K}^{\tilde G_\lambda}(Z_\lambda)$,
(4) $\gamma'_\lambda(u\star x) = q^{d_{\lambda\alpha}-d_{\lambda\alpha'}}\Gamma_\lambda(u)\star\gamma'_\lambda(x)$, *for any* $x\in\mathbf{K}^{\tilde G_\lambda}(Q_{\lambda\alpha'})$, $u\in\mathbf{K}^{\tilde G_\lambda}(Z_{\lambda\alpha\alpha'})$.

5.4. Let $\mathbf{A}_{\lambda\alpha\alpha'}$ be the quotient of the $\mathbf{R}^{\tilde G_\lambda}$-module $\mathbf{K}^{\tilde G_\lambda}(Z_{\lambda\alpha\alpha'})$ by its torsion submodule. We set $\mathbf{A}_\lambda = \bigoplus_{\alpha,\alpha'}\mathbf{A}_{\lambda\alpha\alpha'}$. Setting $Z_{12} = Z_{23} = Z_\lambda$ in 5.1, we get an associative product on the space $\mathbf{A}_\lambda$. The rings $\mathbf{R}^{\mathbb{C}^\times}$, $\mathbb{A}$ are identified as in 4.5. An $\mathbb{A}$-algebra homomorphism $\Phi_\lambda : \mathbf{U}\to\mathbf{A}_\lambda$ is given in [20]. In this subsection we fix a particular normalization for Φ_λ. Let $\delta : Q_\lambda\hookrightarrow Q_\lambda\times Q_\lambda$ be the diagonal embedding, and let $p, p' : Q_\lambda\times Q_\lambda\to Q_\lambda$ be the first and the second projection. Let $f_{i;\alpha}$, $f^\pm_{i;\alpha}$, $v_{i;\alpha}$ be the ranks of $\mathcal{F}_{i;\alpha}$, $\mathcal{F}^\pm_{i;\alpha}$, $\mathcal{V}_{i;\alpha}$. We have

$$f_{i;\alpha} = (\alpha_i, \lambda-\alpha),\quad f^-_{i;\alpha} = -v_{i;\alpha} = -(\omega_i,\alpha).$$

Set $t_\alpha = (\alpha, 2\lambda-\alpha)/2 + |\alpha|^2$, and

$$r^+_{i;\alpha} = (\lambda,\alpha_i) - \big(\omega_i - \textstyle\sum_j n_{ij}\omega_j,\alpha\big),\quad r^-_{i;\alpha} = -\big(\omega_i - \textstyle\sum_j \bar n_{ij}\omega_j,\alpha\big).$$

Let $1_{\lambda\alpha\alpha'}\in\mathbf{A}_\lambda$ be the class of the structural sheaf of $X_{\lambda\alpha\alpha'}$. For any $r\in\mathbb{Z}$ we put

$$x^+_{ir} = q^{(1-\mathbf{c})r}\sum_{\alpha'=\alpha+\alpha_i}(-1)^{r^+_{i;\alpha'}}(q^{-1}{\textstyle\bigwedge}^{-1}_{\mathcal{V}}\boxtimes{\textstyle\bigwedge}_{\mathcal{V}})^{r+f^-_{i;\alpha'}}\otimes p'^*{\textstyle\bigwedge}^{-1}_{\mathcal{F}^+_i}\otimes{\textstyle\bigwedge}^{t_{\alpha'}-t_\alpha}_{\mathcal{W}}\otimes 1_{\lambda\alpha\alpha'},$$

$$x^-_{ir} = q^{(1-\mathbf{c})r}\sum_{\alpha'=\alpha-\alpha_i}(-1)^{r^-_{i;\alpha'}}(q^{-1}{\textstyle\bigwedge}_{\mathcal{V}}\boxtimes{\textstyle\bigwedge}^{-1}_{\mathcal{V}})^{r+f^+_{i;\alpha'}}\otimes p'^*{\textstyle\bigwedge}^{-1}_{\mathcal{F}^-_i}\otimes{\textstyle\bigwedge}^{t_{\alpha'}-t_\alpha}_{\mathcal{W}}\otimes 1_{\lambda\alpha\alpha'}.$$

Also, let $k^\pm_i(z)$ be the expansion at $z=\infty$ or 0 of

$$\delta_*\sum_\alpha q^{f_{i;\alpha}}\big(\textstyle\sum_{r\ge 0}(-q^{-\mathbf{c}}/z)^r\wedge^r\mathcal{F}_{i;\alpha}\big)\otimes\big(\textstyle\sum_{r\ge 0}(-q^{2-\mathbf{c}}/z)^r\wedge^r\mathcal{F}_{i;\alpha}\big)^{-1}.$$

The map Φ_λ takes $\mathbf{x}^\pm_{ir}$ to $x^\pm_{ir}$, and $\mathbf{k}^\pm_{ir}$ to $k^\pm_{ir}$. For future use, we mention the following.

Lemma. *For any $n > 0$ we have*

$$(x_{i0}^{+})^{(n)} = \pm \sum_{\alpha'=\alpha+n\alpha_i} (q^{-n}{\textstyle\bigwedge}_{\mathcal{V}}^{-1} \boxtimes {\textstyle\bigwedge}_{\mathcal{V}})^{f_{i;\alpha'}^{-}} \otimes p'^{*}{\textstyle\bigwedge}_{\mathcal{F}_i^{+}}^{-n} \otimes {\textstyle\bigwedge}_{\mathcal{W}}^{t_{\alpha'}-t_{\alpha}} \otimes 1_{\lambda\alpha\alpha'},$$

and similarly for $(x_{i0}^{-})^{(n)}$.

Proof. By the same argument as in [20, §11.1-3] it is enough to check this relation for type A_1. In this case, using the faithful representation introduced in [22] the formula follows from a direct computation: the formula for the action of the operator $(x_{i0}^{+})^{(n)}$ may be found in [20, Lemma 12.1.1], its relation with the identity above is proved as in [22]. □

Conventions. Hereafter, we may omit the maps $\delta_*, p^*, {p'}^*, \otimes$, hoping that it causes no confusion.

5.5. Let $H_\lambda \subseteq G_\lambda$ be any closed subgroup. Set $\tilde{H}_\lambda = H_\lambda \times \mathbb{C}^\times$. For simplicity we set $\mathbf{W}_{H_\lambda,\alpha} = \mathbf{K}^{\tilde{H}_\lambda}(F_{\lambda\alpha})$, $\mathbf{W}'_{H_\lambda,\alpha} = \mathbf{K}^{\tilde{H}_\lambda}(Q_{\lambda\alpha})$, $\mathbf{W}_{H_\lambda} = \bigoplus_\alpha \mathbf{W}_{H_\lambda,\alpha}$, $\mathbf{W}'_{H_\lambda} = \bigoplus_\alpha \mathbf{W}'_{H_\lambda,\alpha}$. Taking $Z_{12} = Z_\lambda$, $X_3 = \{point\}$, and $Z_{13} = Z_{23} = Q_\lambda$ or F_λ in 5.1 we get a left $\mathbf{U}$-action on the $\mathbf{R}^{\tilde{H}_\lambda}$-modules $\mathbf{W}_{H_\lambda}$, $\mathbf{W}'_{H_\lambda}$ such that

$$(u, x) \mapsto u \cdot x = \Phi_\lambda(u) \star x.$$

Taking $Z_{23} = Z_\lambda$, $X_1 = \{point\}$, and $Z_{12} = Z_{13} = Q_\lambda$ or F_λ we get a right $\mathbf{U}$-action on $\mathbf{W}_{H_\lambda}$, $\mathbf{W}'_{H_\lambda}$ such that $x \cdot u = x \star \Phi_\lambda(u) = \phi^*\Phi_\lambda(u) \star x$; see Lemma 5.2. We fix a maximal torus $T_\lambda \subset G_\lambda$.

Lemma.

(1) *The $\mathbf{R}^{\tilde{H}_\lambda}$-modules $\mathbf{W}_{H_\lambda}$, $\mathbf{W}'_{H_\lambda}$ are free of finite type, and we have $\mathbf{W}_{H_\lambda} = \mathbf{W}_{G_\lambda} \otimes_{\mathbf{R}_\lambda} \mathbf{R}^{\tilde{H}_\lambda}$, $\mathbf{W}'_{H_\lambda} = \mathbf{W}'_{G_\lambda} \otimes_{\mathbf{R}_\lambda} \mathbf{R}^{\tilde{H}_\lambda}$. Moreover, there is a canonical action of the Weyl group of G_λ on $\mathbf{W}_{T_\lambda}$, $\mathbf{W}'_{T_\lambda}$ such that the forgetful map identifies $\mathbf{W}_{G_\lambda}$, $\mathbf{W}'_{G_\lambda}$ with the subspaces of invariant elements in $\mathbf{W}_{T_\lambda}$, $\mathbf{W}'_{T_\lambda}$.*

(2) *We have $\mathbf{W}_{H_\lambda} = \mathbf{U} \cdot (\mathbf{R}^{H_\lambda} \otimes 1_\lambda)$.*

(3) *We have $\mathbf{W}_{G_\lambda} = \mathbf{U} \cdot 1_\lambda$.*

Proof. Claim 1 is proved in [20, Theorem 7.3.5]. See also [4, Chapter 5]. Claim 2 is proved as in [20, Proposition 12.3.2]. Let us prove that, if $H_\lambda = G_\lambda$, then $\mathbf{R}^{G_\lambda} \otimes 1_\lambda \in \mathbf{U} \cdot 1_\lambda$. By construction $\mathbf{k}_i^{\pm}(z) \cdot 1_\lambda$ is the expansion of

$$q^{f_{i;0}} \big(\textstyle\sum_{r\geq 0}(-1/qz)^r \wedge^r W_i\big) \otimes \big(\sum_{r\geq 0}(-q/z)^r \wedge^r W_i\big)^{-1}$$

in $\mathbf{R}^{\tilde{G}_\lambda}[[z^{-1}]]$. Thus, the elements $\mathbf{p}_{i,s} \in \mathbf{R}^{\mathrm{GL}_{\ell_i}}$ are the elementary symmetric polynomials or zero. Claim 3 follows. □

Conventions. Although most of our constructions are meaningful for any closed subgroup $H_\lambda \subseteq G_\lambda$, hereafter H_λ will be either G_λ or T_λ. For simplicity, if $H_\lambda = G_\lambda$, we put $\mathbf{W}_\lambda = \mathbf{W}_{G_\lambda}$, $\mathbf{W}'_\lambda = \mathbf{W}'_{G_\lambda}$, $\mathbf{R}_\lambda = \mathbf{R}^{\tilde{G}_\lambda}$, $\mathbf{X}_\lambda = \mathbf{X}^{G_\lambda}$.

5.6. The same construction as in 5.5 yields also a left and a right action of $\mathbf{A}_\lambda$ on $\mathbf{W}_{H_\lambda}$, $\mathbf{W}'_{H_\lambda}$. Since $\mathbf{W}_{H_\lambda}$, $\mathbf{W}'_{H_\lambda}$ are integrable left $\mathbf{U}$-modules, they admit a left $\mathbf{R}^{\tilde{H}_\lambda}$-linear action of the group B_W. We normalize this action in such a way that the element $T_i \in B_W$ acts as the operator $T''_{i,1}$ in [13, §5.2.1]. Similarly, let $\check{T}_w$ be the left action of the element $T_w \in B_W$ associated to the right $\mathbf{U}$-action.

Proposition.

(1) *There is a unique action of the group* B_W *on* $\mathbf{A}_\lambda$ *by* $\mathbf{R}^{\tilde{H}_\lambda}$*-algebra automorphisms such that*

$$T_w(x \star y) = T_w(x) \star T_w(y), \quad \forall x \in \mathbf{A}_\lambda, y \in \mathbf{W}_{H_\lambda} \text{ or } \mathbf{W}'_{H_\lambda},\ w \in W.$$

(2) *We have* $T_w\Phi_\lambda = \Phi_\lambda T_w$*, and* $\check{T}_w(y \star x) = \check{T}_w(y) \star \sigma(T_w)(x)$ *for any* x, y *as above.*

(3) *There is an invertible element* $r_\lambda \in \mathbf{R}^{\tilde{H}_\lambda}$ *such that* $T_{w_0}(1'_{\lambda\nu}) = r_\lambda \otimes 1'_\lambda$, $T_{w_0}(1_{\lambda\nu}) = r_\lambda \otimes 1_\lambda$.

(4) *There is an invertible element* $s_\lambda \in \mathbf{R}^{\tilde{H}_\lambda}$ *such that* $T_{w_0}(1'_\lambda) = s_\lambda \otimes 1'_{\lambda\nu}$, $T_{w_0}(1_\lambda) = s_\lambda \otimes 1_{\lambda\nu}$.

(5) *There is an invertible element* $\vartheta \in \mathbb{A}$ *such that* $r_\lambda \otimes 1_{\lambda\nu} = \vartheta \otimes \bigwedge_{\mathcal{W}}^{t_\nu} \otimes \bigotimes_i \bigwedge_{\mathcal{V}_{i;\nu}}^{-v_{i;\nu}}$*. Moreover,* $r_\lambda s_\lambda = (-q)^{(\rho,\nu)}$ *and* $\vartheta = \pm q^{|\nu|^2/2+\mathbf{c}(\lambda,\lambda)/2}$.

Proof. Let us prove Claim 1. We fix elements $i \in I$, $\lambda \in P^+$. Let $\mathbf{U}'_i \subset \mathbf{U}'$ be the subalgebra generated by $\mathbf{e}_i$, $\mathbf{f}_i$, $\mathbf{k}_i^{\pm 1}$. The proof uses an element τ'_i introduced in [17]. The element τ'_i belongs to a ring completion of $\mathbf{U}'_i$, is invertible, and satisfies the identity

$$\tau'_i \cdot x \cdot {\tau'_i}^{-1} = T'_{i,1}(x), \quad \forall x \in \mathbf{U}.$$

We recall the construction of τ'_i, following [5].

For any $\mathbb{Q}(q)$-vector space V we set $V^* = \mathrm{Hom}_{\mathbb{Q}(q)}(V, \mathbb{Q}(q))$. Let $\mathbf{R}'_i \subset {\mathbf{U}'_i}^*$ be the $\mathbb{Q}(q)$-space spanned by the matrix elements of the finite dimensional $\mathbf{U}'_i$-modules. It is a Hopf algebra. Let $\sum x_0 \otimes x_1$ denote the image of the element $x \in \mathbf{R}'_i$ by the coproduct. The space ${\mathbf{R}'_i}^*$ is a ring such that $(f \cdot g)(x) = \sum f(x_0)\, g(x_1)$ for all $x \in \mathbf{R}'_i$. The canonical map $\mathbf{U}'_i \to {\mathbf{R}'_i}^*$ is a ring homomorphism. An integrable $\mathbf{U}'$-module V restricts to an integrable $\mathbf{U}'_i$-module via the canonical embedding $\mathbf{U}'_i \subset \mathbf{U}'$. It is also an $\mathbf{R}'_i$-comodule for the co-action $V \to \mathbf{R}'_i \otimes V$, $v \mapsto \sum v_1 \otimes v_2$ such that $x \cdot v = \sum v_1(x)\, v_2$ for all $x \in \mathbf{U}'_i$. Therefore, the ring ${\mathbf{R}'_i}^*$ acts on V by $f \cdot v = \sum f(v_1)\, v_2$. This action restricts to the original $\mathbf{U}'_i$-action via the map $\mathbf{U}'_i \to {\mathbf{R}'_i}^*$. Let $\tau_i \in {\mathbf{R}'_i}^*$ be the element denoted by t in [5, §1.6]. It is invertible. Let t_i be the operator on V taking v to $\tau_i \cdot v$. It is invertible and the inverse takes v to $\tau_i^{-1} \cdot v$. We have

$$t_i(v) = T''_{i,-1}(v), \qquad t_i x t_i^{-1}(v) = T''_{i,-1}(x) \cdot v,$$

for all $x \in \mathbf{U}'$ and $v \in V$, by [5, §2]. Therefore,

$$\tau_i \cdot x \cdot {\tau_i}^{-1} = T''_{i,-1}(x), \quad \forall x \in \mathbf{U}',$$

where we take the product in the ring ${\mathbf{R}'_i}^*$ in the left-hand side.

For any $n \in \mathbb{N}$ let $\Lambda_i(n)$ be the simple $\mathbf{U}'_i$-module with highest weight $n\omega_i$. Let $\mathbf{R}'_{in} \subset \mathbf{R}'_i$ be the subspace spanned by matrix elements of the module $\bigoplus_{n' \leq n} \Lambda_i(n')$. It is a subcoalgebra. Let $\mathbf{I}'_{in} \subset \mathbf{U}'_i$ be the annihilator of $\bigoplus_{n' \leq n} \Lambda_i(n')$. It is a two-sided ideal. The canonical map $\mathbf{U}'_i \to {\mathbf{R}'_i}^*$ factorizes through an isomorphism $\mathbf{U}'_i/\mathbf{I}'_{in} \xrightarrow{\sim} (\mathbf{R}'_{in})^*$. Since $\mathbf{R}'_i = \varinjlim_n \mathbf{R}'_{in}$, we get a $\mathbb{Q}(q)$-algebra isomorphism

$$\varprojlim_n (\mathbf{U}'_i/\mathbf{I}'_{in}) \xrightarrow{\sim} {\mathbf{R}'_i}^*.$$

For each n we choose $\tau_{in} \in \mathbf{U}'_i$ such that

$$\tau_{in} - \tau_i \in \varprojlim_{n' \geq n} (\mathbf{I}'_{in}/\mathbf{I}'_{in'}).$$

Since $\mathbf{U}'_i$ embeds in $\mathbf{U}'$, the space $\mathbf{A}_\lambda \otimes_{\mathbb{A}} \mathbb{Q}(q)$ is a $\mathbf{U}'_i \otimes_{\mathbb{A}} \mathbf{R}^{\tilde{H}_\lambda}$-bimodule of finite type over $\mathbf{R}^{\tilde{H}_\lambda} \otimes_{\mathbb{A}} \mathbb{Q}(q)$. In particular, there is an integer n such that the ideal $\mathbf{I}'_{in}$ acts trivially on $\mathbf{A}_\lambda \otimes_{\mathbb{A}} \mathbb{Q}(q)$. Fix such an integer n. Then the operator $T''_{i,-1}$ acts on $\mathbf{A}_\lambda \otimes_{\mathbb{A}} \mathbb{Q}(q)$ via the conjugation by the element $\Phi_\lambda(\tau_{in}) \in \mathbf{A}_\lambda \otimes_{\mathbb{A}} \mathbb{Q}(q)$. Moreover, the formulas in [5, §2] imply that the left and right product by $\Phi_\lambda(\tau_{in})$, $\Phi_\lambda({\tau_{in}}^{-1})$ preserves the subspace $\mathbf{A}_\lambda \subset \mathbf{A}_\lambda \otimes_{\mathbb{A}} \mathbb{Q}(q)$.

Recall that T_i acts as Lusztig's operator $T''_{i,1}$. The element τ'_i yielding the action of $T''_{i,1}$ on $\mathbf{A}_\lambda$ can be constructed as τ_i, using the identity

$$T''_{i,1}(x) = (-q)^{(\alpha,\alpha_i)}(T''_{i,-1})^{-1}(x)$$

for any $x \in \mathbf{U}_\alpha$; see [13, §37.2.4].

Recall that N_λ is a cone over the point 0 equal to the class of the trivial representation, and that the fixed points subset $(N_\lambda)^{\mathbb{C}^\times}$ is reduced to $\{0\}$. Hence $A_\lambda \otimes_{\mathbb{A}} \mathbb{Q}(q)$ coincides with the tensor product $(\mathbf{W}_{G_\lambda} \otimes_{\mathbf{R}^{\tilde{G}_\lambda}} \mathbf{W}_{G_\lambda}) \otimes_{\mathbb{A}} \mathbb{Q}(q)$ by the Kunneth isomorphism; see [4, 5.6] and [20, §7]. Since A_λ is torsion free over $\mathbb{A}$, it embeds in $A_\lambda \otimes_{\mathbb{A}} \mathbb{Q}(q)$. Then, a standard argument implies that $\mathbf{W}_{G_\lambda}$ is a faithful $\mathbf{A}_\lambda$-module; see [4, §5] for more details. For the same reason $\mathbf{W}'_{H_\lambda}$ is also a faithful $\mathbf{A}_\lambda$-module. The unicity in Claim 1 follows.

Claim 2 is obvious from the previous construction. Claims 3 and 4 are obvious since T_{w_0} is an invertible $\mathbf{R}^{\tilde{H}_\lambda}$-linear homomorphism from $\mathbf{W}'_{H_\lambda,0}$ to $\mathbf{W}'_{H_\lambda,\nu}$, and both $\mathbf{R}^{\tilde{H}_\lambda}$-modules are free of rank one, generated by $1'_\lambda, 1'_{\lambda\nu}$ respectively.

Let us prove Claim 5. Part two of the claim follows from [11, 5.4(a) and Corollary 5.9] (note that the formula for s' in [11, 5.4(a)] should be replaced by $s' = \sum_{i,j} a_{ij}t_it_j/2 - \sum_i t_i(1+d_i)$). Let us prove Part one. Consider an element $w \in W$ such that $l(w) > 0$. Fix $i \in I$ such that $l(s_iw) = l(w) - 1$. Put $w' = s_iw$. Set $\alpha' = w' * \nu$, $\alpha = w * \nu$, $n = (ww_0(\lambda), \alpha_i)$. Thus $n > 0$ and $\alpha = \alpha' - n\alpha_i$. We have

$$T_i(1'_{\lambda\alpha'}) = \mathbf{e}_i^{(n)}(1'_{\lambda\alpha'});$$

see [11, Lemma 5.6]. Let $r_w \in \mathbf{R}^{\tilde{H}_\lambda}$ be the unique element such that $T_w(1'_{\lambda\nu}) = r_w 1'_{\lambda\alpha}$. The varieties $Q_{\lambda\alpha}$, $Q_{\lambda\alpha'}$ are reduced to a point. Thus, using Lemma 5.4 we get

$$T_w(1'_{\lambda\nu}) = \pm\big(q^{-n}{\textstyle\bigwedge}_{\mathcal{V}_\alpha}^{-1}{\textstyle\bigwedge}_{\mathcal{V}_{\alpha'}}\big)^{f^-_{i;\alpha'}}{\textstyle\bigwedge}^{-n}_{\mathcal{F}^+_{i;\alpha'}}{\textstyle\bigwedge}_{\mathcal{W}}^{t_{\alpha'}-t_\alpha}(r_{w'}\,1'_{\lambda\alpha}).$$

The classes of the $\tilde{H}_\lambda$-equivariant sheaves $\mathcal{V}_{i;\alpha}, \mathcal{V}_{i;\alpha'}, \mathcal{F}^+_{i;\alpha'}$ are identified with elements of $\mathbf{R}^{\tilde{H}_\lambda}$ in the obvious way. First, assume that we have

$$\text{(5.6.1)} \qquad {\textstyle\bigwedge}_{\mathcal{F}^+_{i;\alpha'}} = {\textstyle\bigwedge}_{\mathcal{V}_{i;\alpha}} \in \mathbf{R}^{\tilde{H}_\lambda}.$$

Then

$$r_w = \pm q^s {\textstyle\bigwedge}_{\mathcal{V}_{i;\alpha'}}^{-v_{i;\alpha'}}{\textstyle\bigwedge}_{\mathcal{V}_{i;\alpha}}^{v_{i;\alpha}}{\textstyle\bigwedge}_{\mathcal{W}}^{t_{\alpha'}-t_\alpha} r_{w'}, \text{ where } s = (\alpha'-\alpha,\omega_i)(\alpha',\omega_i).$$

By induction on $l(w)$ we get

$$\text{(5.6.2)} \qquad r_w \in \pm q^{\mathbb{Z}}{\textstyle\bigwedge}_{\mathcal{W}}^{t_\nu - t_\alpha}\prod_j {\textstyle\bigwedge}_{\mathcal{V}_{j;\nu}}^{-v_{j;\nu}}{\textstyle\bigwedge}_{\mathcal{V}_{j;\alpha}}^{v_{j;\alpha}}.$$

Setting $w = w_0$ in (5.6.2) we get $r_\lambda = \vartheta \bigwedge_{\mathcal{W}}^{t_\nu} \prod_i \bigwedge_{\mathcal{V}_{i;\nu}}^{-v_{i;\nu}}$, with $\vartheta \in \pm q^{\mathbb{Z}}$. A direct computation gives $\vartheta \in \pm q^t$, where

$$\begin{aligned} t &= |\nu|^2/2 + \sum_{\alpha \in \Delta_+} (\lambda, \alpha)^2/2 \\ &= |\nu|^2/2 + \mathbf{c}(\lambda, \lambda)/2; \end{aligned}$$

see [6, Exercise 6.2].

Finally, we prove (5.6.1). We have an isomorphism of $\tilde{G}_\lambda$-varieties $Q_{\lambda\alpha}^{(\mu)} \to Q_{\lambda\alpha'}^{(\mu')}$; see the proof of Lemma 4.6 and the notation therein. This isomorphism takes $V_i^{(\mu)}$ to $(F_i^+)^{(\mu')}$. Assume that μ is regular dominant, so that $Q_{\lambda\alpha}^{(\mu)} = Q_{\lambda\alpha}$ and $V_i^{(\mu)} = \mathcal{V}_{i;\alpha}$. Since the $\tilde{G}_\lambda$-variety $Q_{\lambda\alpha'}$ is reduced to a point, it is canonically isomorphic to $Q_{\lambda\alpha'}^{(\mu')}$, and the isomorphism takes $\mathcal{F}_{i;\alpha'}^+$ to $(F_i^+)^{(\mu')}$. □

5.7. For each i, α we consider the elements

$$\begin{aligned} g_{i;\alpha} &= -1 + (\mathbf{c}-1) f^+_{\underline{i};w_0 * \alpha} + f^+_{i;\alpha} - \mathbf{c} f^-_{\underline{i};w_0*\alpha} \in \mathbb{Z}, \\ h_{i;\alpha} &= r^+_{\underline{i};w_0*\alpha} + d_{\lambda, w_0*\alpha - \alpha_{\underline{i}}, w_0*\alpha} + r^-_{i;\alpha} \in \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

Conventions. The elements $r^{\pm}_{i;\alpha}$ depend on the choice of the orientation Ω. Hereafter we assume that $n_{ij} = n_{\underline{ij}}$ for all i, j if $\mathbf{c}$ is even, and $n_{ij} = \bar{n}_{\underline{ij}}$ for all i, j if $\mathbf{c}$ is odd (i.e., if $\mathfrak{g}$ is of type A_{2n}). The existence of such an orientation is checked case-by-case.

Using the convention above, a direct computation gives

$$g_{i;\alpha+\alpha_j} + g_{j;\alpha} = g_{j;\alpha+\alpha_i} + g_{i;\alpha}, \quad h_{i;\alpha+\alpha_j} + h_{j;\alpha} = h_{j;\alpha+\alpha_i} + h_{i;\alpha}$$

for all i, j. Thus there are unique quadratic maps $x : Q \to \mathbb{Z}$, $y : Q \to \mathbb{Z}/2\mathbb{Z}$ such that

$$(5.7.1) \qquad \begin{aligned} & x(\nu) = (\mathbf{c}-1)|\nu|^2 - \mathbf{c}(\lambda,\lambda), \quad x(\alpha+\alpha_i) - x(\alpha) = \mathbf{c} v_{i;\nu} - g_{i;\alpha}, \\ & y(\nu) = 0, \quad y(\alpha+\alpha_i) - y(\alpha) = h_{i;\alpha}, \end{aligned}$$

for all α, i. Put $\xi(\alpha) = q^{x(\alpha)}$, $\varrho(\alpha) = (-1)^{y(\alpha)}$, $t_{\alpha\alpha'} = t_{w_0*\alpha'} - t_{\alpha'} - t_{w_0*\alpha} + t_\alpha$. Consider the element

$$c_{\lambda\alpha} = \varrho(\alpha)\xi(\alpha) q^{-\mathbf{c}|\alpha|^2} \bigwedge_{\mathcal{W}}^{t_{0\alpha}} \otimes \bigotimes_i \left(\bigwedge_{\mathcal{V}_{i;\alpha}}^{2v_{i;\nu}} \otimes \bigwedge_{\mathcal{V}^*_{i;\alpha} \otimes \omega^* \mathcal{V}^*_{\underline{i};w_0*\alpha}} \right) \otimes r_\lambda \in \mathbf{W}'_{H_\lambda,\alpha}.$$

Put $c_\lambda = \sum_\alpha c_{\lambda\alpha}$.

Lemma.

(1) *If $\alpha' = \alpha - \alpha_j$, then the restriction of $c_\lambda^{-1} \boxtimes c_\lambda$ to $X_{\lambda\alpha\alpha'}$ is*

$$(-1)^{h_{j;\alpha'}} q^{g_{j;\alpha'}} (\textstyle\bigwedge_{\mathcal{V}} \boxtimes \bigwedge_{\mathcal{V}}^{-1})^{-f^+_{j;\alpha'} - f^+_{\underline{j};w_0*\alpha'}} \bigwedge_{\mathcal{W}}^{t_{\alpha\alpha'}} \otimes p'^* \bigwedge_{\mathcal{F}_j^- + q^{\mathbf{c}} \omega^* \mathcal{F}_{\underline{j}}^{-*}} \otimes 1_{\lambda\alpha\alpha'}.$$

(2) *We have $\omega^*(c_{\lambda\alpha}) = c_{\lambda, w_0*\alpha}$, $c_{\lambda\nu} = r_\lambda^{-1} \otimes 1'_{\lambda\nu}$.*

Proof. Fix α, α' such that $\alpha' = \alpha - \alpha_j$. For any i we consider the following elements in $\mathbf{W}'_{H_\lambda}$:

$$\mathcal{U}_i = q^{\mathbf{c}} \omega^*(\mathcal{V}^*_{\underline{i}}) + \mathcal{V}_i, \quad c'_\lambda = \textstyle\bigotimes_i \bigwedge_{q^{\mathbf{c}} \mathcal{V}_i^* \otimes \omega^* \mathcal{V}^*_{\underline{i}}}.$$

The rank of $\mathcal{U}_{i;\alpha}$ is $v_{i;\alpha} + v_{\underline{i};w_0*\alpha} = v_{\underline{i};\nu}$. From Lemma 4.6.1 we have

$$\sum_j [a_{ij}] \mathcal{U}_{j;\alpha} = q^{\mathbf{c}} \mathcal{W}_{\underline{i};\alpha} + \mathcal{W}_{i;\alpha}.$$

The quantum Cartan matrix (i.e., the $I \times I$-matrix whose (i,j)-th entry is $[a_{ij}]$) is invertible over $\mathbb{Q}(q)$. Thus, for any α, α', i we get

$$(1_{\lambda\alpha} \boxtimes \mathcal{U}_{i;\alpha'})|_{X_{\lambda\alpha\alpha'}} = (\mathcal{U}_{i;\alpha} \boxtimes 1_{\lambda\alpha'})|_{X_{\lambda\alpha\alpha'}} \in \mathbf{K}^{\tilde{H}_\lambda}(X_{\lambda\alpha\alpha'}). \tag{5.7.2}$$

We have $c'_\lambda = \bigotimes_i \bigwedge_{\mathcal{V}_i^* \otimes \mathcal{U}_i}$ and $\mathcal{F}_i^- = -\mathcal{V}_i$. Thus, using (5.7.2) we get

$$({c'}_\lambda^{-1} \boxtimes c'_\lambda)|_{X_{\lambda\alpha\alpha'}} = (\textstyle\bigwedge_{\mathcal{V}_\alpha} \boxtimes \bigwedge_{\mathcal{V}_{\alpha'}}^{-1})^{v_{j;\nu}}|_{X_{\lambda\alpha\alpha'}} \otimes {p'}^* \bigwedge_{\mathcal{F}_j^- + q^{\mathbf{c}} \omega^* \mathcal{F}_{\underline{j}}^{-*}}.$$

Note that

$$c_{\lambda\alpha} = \xi(\alpha)\varrho(\alpha) r_\lambda \otimes c'_{\lambda\alpha} \otimes \textstyle\bigotimes_i \bigwedge^{2v_{i;\nu}}_{q^{-\mathbf{c}/2}\mathcal{V}_{i;\alpha}} \otimes \bigwedge_{\mathcal{W}}^{t_{0\alpha}}.$$

Thus Claim 1 follows, using (5.7.1) and the identity $v_{j;\nu} = f^+_{j;\alpha'} + f^+_{\underline{j};w_0*\alpha'}$. We prove Claim 2. By definition of c'_λ we have $\omega^*(c'_{\lambda\alpha}) = c'_{\lambda,w_0*\alpha}$ and $c'_{\lambda 0} = 1'_\lambda$. Thus, using Proposition 5.6.4 we get

$$c_{\lambda\nu} = \vartheta^{-2} r_\lambda \otimes \textstyle\bigwedge_{\mathcal{W}}^{-2t_\nu} \otimes \bigotimes_i \bigwedge^{2v_{i;\nu}}_{\mathcal{V}_{i;\nu}} = r_\lambda^{-1} \otimes 1'_{\lambda\nu}.$$

Since $Q_{\lambda\nu}$ is a point, we identify the equivariant sheaf $\bigwedge_{\mathcal{V}_{i;\nu}}$ with an element in $\mathbf{R}^{\tilde{H}_\lambda}$ in the obvious way. Using (5.7.2) we get

$$\begin{aligned}\textstyle\bigotimes_i \bigwedge^{2v_{i;\nu}}_{q^{-\mathbf{c}/2}\omega^*(\mathcal{V}_{i;\alpha}) + q^{\mathbf{c}/2}\mathcal{V}^*_{\underline{i};w_0*\alpha}} &= \left(\textstyle\bigotimes_i \bigwedge^{2v_{\underline{i};\nu}}_{q^{-\mathbf{c}/2}\omega^*(\mathcal{V}_{i;0}) + q^{\mathbf{c}/2}\mathcal{V}^*_{\underline{i};\nu}}\right) \otimes 1'_{\lambda,w_0*\alpha} \\ &= \vartheta^{-2} q^{|\nu|^2\mathbf{c}} r_\lambda^2 \otimes \textstyle\bigwedge_{\mathcal{W}}^{-2t_\nu} \otimes 1'_{\lambda,w_0*\alpha}.\end{aligned}$$

Thus, $\omega^*(r_\lambda) = \vartheta^2 q^{-|\nu|^2\mathbf{c}} r_\lambda^{-1}$ and

$$\begin{aligned}\omega^*\left(r_\lambda \otimes \textstyle\bigwedge_{\mathcal{W}}^{-t_\nu} \otimes \bigotimes_i \bigwedge^{2v_{i;\nu}}_{q^{-\mathbf{c}/2}\mathcal{V}_{i;\alpha}}\right) &= \vartheta^2 q^{-|\nu|^2\mathbf{c}} r_\lambda^{-1} \otimes \textstyle\bigwedge_{\mathcal{W}}^{t_\nu} \otimes \bigotimes_i \bigwedge^{2v_{i;\nu}}_{q^{-\mathbf{c}/2}\omega^*\mathcal{V}_{i;\alpha}} \\ &= r_\lambda \otimes \textstyle\bigwedge_{\mathcal{W}}^{-t_\nu} \otimes \bigotimes_i \bigwedge^{2v_{i;\nu}}_{q^{-\mathbf{c}/2}\mathcal{V}_{\underline{i};w_0*\alpha}}.\end{aligned}$$

A direct computation (see the Appendix) shows that $\xi(\alpha) = \xi(w_0 * \alpha)$, $\varrho(\alpha) = \varrho(w_0 * \alpha)$ for all α. We are done. □

Let C_λ be the $\mathbf{R}_\lambda$-linear automorphism of $\mathbf{A}_\lambda$ such that

$$C_\lambda(x) = x \otimes (c_\lambda \boxtimes c_\lambda^{-1}).$$

Proposition.

(1) *The map $C_\lambda\Gamma_\lambda$ is an algebra involution of $\mathbf{A}_\lambda$ such that $C_\lambda\Gamma_\lambda(q) = q^{-1}$, $C_\lambda\Gamma_\lambda(x^\pm_{ir}) = q^{r\mathbf{c}} x^\mp_{\underline{i}r}$.*
(2) *The map $C_\lambda\zeta_\lambda$ is an algebra anti-involution of $\mathbf{A}_\lambda$ such that $C_\lambda\zeta_\lambda(q) = q$, $C_\lambda\zeta_\lambda(x^\pm_{ir}) = q^{-r\mathbf{c}} x^\pm_{\underline{i},-r}$.*

Proof. The variety $X_{\lambda\alpha\alpha'}$ is smooth of dimension $d_{\lambda\alpha\alpha'} := (d_{\lambda\alpha} + d_{\lambda\alpha'})/2$. Let $\Omega_{\lambda\alpha\alpha'}$ be its canonical bundle. If $\alpha' = \alpha + \alpha_i$, using (4.5.1), Lemma 4.6.1 and Remark 4.6 we get

$$\Omega_{\lambda\alpha\alpha'} = q^{f_{i;\alpha'} - d_{\lambda\alpha'}} (q^{-1} \textstyle\bigwedge_{\mathcal{V}_\alpha}^{-1} \boxtimes \bigwedge_{\mathcal{V}_{\alpha'}})^{f_{i;\alpha'}} \otimes {p'}^* \bigwedge_{\mathcal{F}_{i;\alpha'}}^{-1}, \tag{5.7.3}$$

$$d_{\lambda\alpha\alpha'} - d_{\lambda\alpha'} + f_{i;\alpha'} = -1, \tag{5.7.4}$$

$$(\omega \times \omega)^* (\textstyle\bigwedge_{\mathcal{V}_{i;\alpha}} \boxtimes \bigwedge_{\mathcal{V}_{i;\alpha'}}^{-1}) = q^{\mathbf{c}} \bigwedge_{\mathcal{V}_{\underline{i};w_0*\alpha}} \boxtimes \bigwedge^{-1}_{\mathcal{V}_{\underline{i};w_0*\alpha'}}. \tag{5.7.5}$$

Using (5.7.3) and (5.7.4) we get

$$\begin{aligned}D_{Z_\lambda}(x^+_{ir}) &= \sum_{\alpha'=\alpha+\alpha_i}(-1)^{r^+_{i;\alpha'}+d_{\lambda\alpha\alpha'}}q^{r\mathbf{c}-r+d_{\lambda\alpha\alpha'}}(q^{-1}\bigwedge\nolimits_{\mathcal{V}}^{-1}\boxtimes\bigwedge\nolimits_{\mathcal{V}})^{-r-f^-_{i;\alpha'}}\\ &\quad\otimes p'^*\bigwedge\nolimits_{\mathcal{F}^+_i}\otimes\bigwedge\nolimits_{\mathcal{W}}^{t_\alpha-t_{\alpha'}}\otimes\Omega_{\lambda\alpha\alpha'}\\ &= \sum_{\alpha'=\alpha+\alpha_i}(-1)^{r^+_{i;\alpha'}+d_{\lambda\alpha\alpha'}}q^{r\mathbf{c}-r-1}(q^{-1}\bigwedge\nolimits_{\mathcal{V}}^{-1}\boxtimes\bigwedge\nolimits_{\mathcal{V}})^{-r+f^+_{i;\alpha'}}\\ &\quad\otimes p'^*\bigwedge\nolimits_{\mathcal{F}^-_i}^{-1}\otimes\bigwedge\nolimits_{\mathcal{W}}^{t_\alpha-t_{\alpha'}}\otimes 1_{\lambda\alpha\alpha'}.\end{aligned}$$

Thus, using (5.7.5) we get

$$\begin{aligned}\Gamma_\lambda(x^+_{ir}) &= \sum_{\alpha'=\alpha-\alpha_{\underline{i}}}(-1)^{e_{\underline{i};\alpha'}}q^{-1-r+\mathbf{c}f^+_{i;w_0*\alpha'}}(q\bigwedge\nolimits_{\mathcal{V}}\boxtimes\bigwedge\nolimits_{\mathcal{V}}^{-1})^{r-f^+_{i;w_0*\alpha'}}\\ &\quad\otimes p'^*\omega^*\bigwedge\nolimits_{\mathcal{F}^-_i}^{-1}\otimes\bigwedge\nolimits_{\mathcal{W}}^{t_{w_0*\alpha'}-t_{w_0*\alpha}}\otimes 1_{\lambda\alpha\alpha'}\\ &= x^-_{\underline{i}r}\otimes\sum_{\alpha,\alpha'}(-1)^{h_{\underline{i};\alpha'}}q^{r\mathbf{c}-1+(\mathbf{c}-2)f^+_{i;w_0*\alpha'}}(q^{-1}\bigwedge\nolimits_{\mathcal{V}_\alpha}\boxtimes\bigwedge\nolimits_{\mathcal{V}_{\alpha'}}^{-1})^{-f^+_{\underline{i};\alpha'}-f^+_{i;w_0*\alpha'}}\\ &\quad\otimes\bigwedge\nolimits_{\mathcal{W}}^{t_{\alpha\alpha'}}\otimes p'^*\bigwedge\nolimits_{\mathcal{F}^-_{\underline{i}}+\omega^*\mathcal{F}^{-*}_i}\\ &= q^{r\mathbf{c}}x^-_{\underline{i}r}\otimes\sum_{\alpha,\alpha'}(-1)^{h_{\underline{i};\alpha'}}q^{g_{\underline{i};\alpha'}}(\bigwedge\nolimits_{\mathcal{V}_\alpha}\boxtimes\bigwedge\nolimits_{\mathcal{V}_{\alpha'}}^{-1})^{-f^+_{\underline{i};\alpha'}-f^+_{i;w_0*\alpha'}}\\ &\quad\otimes\bigwedge\nolimits_{\mathcal{W}}^{t_{\alpha\alpha'}}\otimes p'^*\bigwedge\nolimits_{\mathcal{F}^-_{\underline{i}}+q^{\mathbf{c}}\omega^*\mathcal{F}^{-*}_i},\end{aligned}$$

where $e_{\underline{i};\alpha'}=r^+_{i;w_0*\alpha'}+d_{\lambda,w_0*\alpha,w_0*\alpha'}$, and $g_{i;\alpha}$, $h_{i;\alpha}$ are as at the beginning of 5.7. Using Lemma 5.7.1 we get

$$C_\lambda\Gamma_\lambda(x^+_{ir})=q^{r\mathbf{c}}x^-_{\underline{i}r}.$$

Using Lemmas 5.2.1, 5.3.1 and 5.3.2 we get

$$\zeta_\lambda=\Gamma_\lambda\phi^*D_{Z_\lambda}=\phi^*D_{Z_\lambda}\Gamma_\lambda,\quad C_\lambda\phi^*D_{Z_\lambda}=\phi^*D_{Z_\lambda}C_\lambda.$$

Thus, $(C_\lambda\Gamma_\lambda)^2=(C_\lambda\zeta_\lambda)^2$. Using Lemma 5.7.2 we get

$$(C_\lambda\Gamma_\lambda)^2=(C_\lambda\zeta_\lambda)^2=\mathrm{Id}. \tag{5.7.6}$$

Thus, $C_\lambda\Gamma_\lambda(x^-_{ir})=q^{r\mathbf{c}}x^+_{\underline{i}r}$ either. Recall that $\phi^*D_{Z_\lambda}\Phi_\lambda=\Phi_\lambda\tau$; see [24, Lemma 6.5]. Then Claim 2 follows from Proposition 3.2.2 and (5.7.6). □

5.8. Let $A_\lambda, B_\lambda:\mathbf{A}_\lambda\to\mathbf{A}_\lambda$ be the $\mathbf{R}_\lambda$-algebra automorphisms such that

$$A_\lambda(x)=(-q)^{(\rho,\alpha-\alpha')}x,\quad B_\lambda(x)=(-q)^{(\rho,\alpha-\alpha')}q^{(\alpha'-\alpha,2\lambda-\alpha'-\alpha)/2}x,$$

for any element $x\in\mathbf{A}_{\lambda\alpha\alpha'}$. Then

$$\Phi_\lambda A=A_\lambda\Phi_\lambda,\quad \Phi_\lambda B=B_\lambda\Phi_\lambda. \tag{5.8.1}$$

We consider the automorphisms β_{Z_λ}, ψ_{Z_λ} of the ring $\mathbf{A}_\lambda$ such that

$$\beta_{Z_\lambda}=T_{w_0}B_\lambda C_\lambda\Gamma_\lambda,\quad \psi_{Z_\lambda}=T_{w_0}A_\lambda C_\lambda\zeta_\lambda. \tag{5.8.2}$$

Corollary.

(1) *The map β_{Z_λ} is $^-$-semilinear and the map ψ_{Z_λ} is †-semilinear. Moreover, we have $\beta^2_{Z_\lambda}=\psi^2_{Z_\lambda}=Id$.*

(2) *For any $u\in\mathbf{U}$ we have $\Phi_\lambda(\bar u)=\beta_{Z_\lambda}\Phi_\lambda(u)$, $\Phi_\lambda\psi(u)=\psi_{Z_\lambda}\Phi_\lambda(u)$.*

Proof. From Proposition 5.7.2 the map $C_\lambda\zeta_\lambda$ is an antihomomorphism of $\mathbf{A}_\lambda$ such that $q \mapsto q$, $x^\pm_{i0} \mapsto x^\pm_{\underline{i}0}$ for all i. Thus, using [13, §37] we get

$$A_\lambda C_\lambda\zeta_\lambda = C_\lambda\zeta_\lambda A_\lambda, \quad T_{w_0}A_\lambda^{-1} = A_\lambda T_{w_0}, \quad T_{w_0}C_\lambda\zeta_\lambda = C_\lambda\zeta_\lambda T_{w_0}^{-1}. \tag{5.8.3}$$

Thus ψ_{Z_λ} is an idempotent. From Proposition 5.7.1 and [13, §37] we get also

$$B_\lambda C_\lambda\Gamma_\lambda T_{w_0} = C_\lambda\Gamma_\lambda T_{w_0}B_\lambda^{-1}, \quad T_{w_0}C_\lambda\Gamma_\lambda = C_\lambda\Gamma_\lambda T_{w_0}^{-1}. \tag{5.8.4}$$

Thus β_{Z_λ} is an idempotent. Claim 2 is immediate. □

6. The metric and the involution on standard modules

6.1. Set $a_{\lambda\alpha} = (-q)^{(\rho,\alpha)}$, $b_{\lambda\alpha} = (-q)^{(\rho,\alpha-2\lambda)}q^{-d_{\lambda\alpha}/2}$. Let a_λ, b_λ be the automorphisms of the $\mathbf{R}^{\tilde{H}_\lambda}$-module $\mathbf{W}_{H_\lambda}$ (resp. of $\mathbf{W}'_{H_\lambda}$) such that $a_\lambda(x) = a_{\lambda\alpha}x$, $b_\lambda(x) = b_{\lambda\alpha}x$ for any element $x \in \mathbf{W}_{\lambda\alpha}$ (resp. $x \in \mathbf{W}'_{\lambda\alpha}$). Using (5.8.1) we get

$$b_\lambda(u \cdot x) = B(u) \cdot b_\lambda(x), \quad a_\lambda(u \cdot x) = A(u) \cdot a_\lambda(x).$$

Let β_λ, β'_λ be the automorphisms of $\mathbf{W}_{H_\lambda}$, $\mathbf{W}'_{H_\lambda}$ respectively such that

$$\beta_\lambda = T_{w_0}b_\lambda c_\lambda\gamma_\lambda, \quad \beta'_\lambda = T_{w_0}b_\lambda c_\lambda\gamma'_\lambda.$$

Proposition.

(1) *We have* $\beta_\lambda(u \star x) = \beta_{Z_\lambda}(u) \star \beta_\lambda(x)$ *for any* $u \in \mathbf{A}_\lambda$, $x \in \mathbf{W}_{H_\lambda}$.
(2) *We have* $\beta'_\lambda(u \star x) = q^{d_{\lambda\alpha}-d_{\lambda\alpha'}}\beta_{Z_\lambda}(u) \star \beta'_\lambda(x)$, *for any* $u \in \mathbf{K}^{\tilde{H}_\lambda}(Z_{\lambda\alpha\alpha'})$, $x \in \mathbf{W}'_{H_\lambda,\alpha'}$.
(3) *The maps* β_λ, β'_λ *are* $^-$*-semilinear. Moreover, we have* $\beta_\lambda^2 = Id$, ${\beta'_\lambda}^2 = Id$.
(4) *We have* $\beta_\lambda(1_\lambda) = 1_\lambda$, $\beta'_\lambda(1'_\lambda) = 1'_\lambda$.

Proof. Claim 1 follows from Lemma 5.3.3. Claim 2 follows from Lemma 5.3.4. Claim 3 follows from Corollary 5.8.1 and the equality $F_{\lambda\alpha} = Z_{\lambda\alpha 0}$. Using Lemma 5.7.2 we get

$$\gamma_\lambda(1_\lambda) = 1_{\lambda\nu}, \quad \gamma'_\lambda(1'_\lambda) = 1'_{\lambda\nu}, \quad b_{\lambda\nu} = 1, \quad c_{\lambda\nu} = r_\lambda^{-1}1'_{\lambda\nu}.$$

Thus Claim 4 follows from Proposition 5.6.3. □

Remark. For any closed subgroup $H'_\lambda \subset H_\lambda$, the forgetful maps $\mathbf{W}_{H_\lambda} \to \mathbf{W}_{H'_\lambda}$, $\mathbf{W}'_{H_\lambda} \to \mathbf{W}'_{H'_\lambda}$ commute with the involutions β_λ, β'_λ.

6.2. For any $\mathbb{A}$-module M, let $\hat{M}$ be the set of formal series in q^{-1} with coefficients in M. We get (see 5.5)

$$\hat{\mathbf{W}}_{H_\lambda} = \mathbf{W}_{H_\lambda} \otimes_{\mathbf{R}^{\tilde{H}_\lambda}} \hat{\mathbf{R}}^{\tilde{H}_\lambda}, \quad \mathbf{W}'_{H_\lambda} = \mathbf{W}'_{H_\lambda} \otimes_{\mathbf{R}^{\tilde{H}_\lambda}} \hat{\mathbf{R}}^{\tilde{H}_\lambda}.$$

Recall that if $\lambda = \lambda_1 + \lambda_2$ in P^+, then the direct sum of representations of the quiver Π^e gives an embedding $\varpi : Q_{\lambda_1} \times Q_{\lambda_2} \hookrightarrow Q_\lambda$. Fix a pair of ring isomorphisms

$$\mathbf{R}^{T_{\lambda_1}} \simeq \mathbb{Z}[x_1^{\pm 1}, ..., x_{\ell_1}^{\pm 1}], \quad \mathbf{R}^{T_{\lambda_2}} \simeq \mathbb{Z}[y_1^{\pm 1}, ..., y_{\ell_2}^{\pm 1}].$$

We have $\mathbf{R}^{T_\lambda} \simeq \mathbf{R}^{T_{\lambda_1}} \otimes \mathbf{R}^{T_{\lambda_2}}$. Set

$$\hat{\mathbf{R}}_{\lambda_1/\lambda_2} = \mathbb{Z}[[q^{-1}, y_i/x_j; i, j]] \otimes_{\mathbb{Z}[q^{-1}, y_i/x_j; i,j]} \mathbf{R}^{\tilde{T}_\lambda}.$$

where $1 \leq i \leq \ell_1$ and $1 \leq j \leq \ell_2$. Recall that κ is the closed embedding $F_\lambda \hookrightarrow Q_\lambda$.

Lemma.

(1) *The direct image map* κ_* *is an isomorphism* $\hat{\mathbf{W}}_{H_\lambda} \xrightarrow{\sim} \hat{\mathbf{W}}'_{H_\lambda}$.

(2) *Assume that* $\lambda = \lambda_1 + \lambda_2$ *in* P^+. *Then, there is a unique isomorphism of* $\hat{\mathbf{R}}_{\lambda_1/\lambda_2} \otimes \mathbf{U}$*-modules*

$$\varpi_{\lambda_1/\lambda_2} : \hat{\mathbf{R}}_{\lambda_1/\lambda_2} \otimes_{\mathbf{R}^{\tilde{T}_\lambda}} (\mathbf{W}_{T_{\lambda_1}} \otimes_{\mathbb{A}} \mathbf{W}_{T_{\lambda_2}}) \xrightarrow{\sim} \hat{\mathbf{R}}_{\lambda_1/\lambda_2} \otimes_{\mathbf{R}^{\tilde{T}_\lambda}} \mathbf{W}_{T_\lambda}$$

such that $1_{\lambda_1} \otimes 1_{\lambda_2} \mapsto 1_\lambda$.

Proof. Let first prove Claim 1. Assume that $H_\lambda = T_\lambda$. We set $\ell = \sum_i \ell_i$. There is an isomorphism of rings $\mathbf{R}^{T_\lambda} \simeq \mathbb{Z}[z_1^{\pm 1}, ..., z_\ell^{\pm 1}]$. Fix $\mathbf{R}^{\tilde{T}_\lambda}$-bases in $\mathbf{W}_\lambda$, $\mathbf{W}'_\lambda$. By Thomason's concentration theorem in equivariant K-theory and by [20, Proposition 4.2.2], the determinant of the map κ_* in those bases belongs to the set

$$\left(\mathbf{R}^{\tilde{T}_\lambda}\right)^\times \cdot \prod_k (1 - q^{n_k} z_{i_k}/z_{j_k})$$

for some $i_k, j_k \in [1, \ell]$, $n_k \in \mathbb{Z} \setminus \{0\}$. We can assume that $n_k < 0$ for all k. Thus this determinant is invertible in the ring $\hat{\mathbf{R}}^{\tilde{T}_\lambda}$. The case of a general group H_λ follows from Lemma 5.5. Let us prove Claim 2. In [24, Proposition 7.10.(v)] we define an embedding of $\mathbf{R}^{T_\lambda} \otimes \mathbf{U}$-modules

$$\Delta_W : \mathbf{W}_{T_\lambda} \to \mathbf{W}_{T_{\lambda_1}} \otimes_{\mathbb{A}} \mathbf{W}_{T_{\lambda_2}}.$$

By [24, Theorem 7.12] the map Δ_W is an isomorphism whenever q, x_j, y_i are specialized to nonzero complex numbers such that $y_i/x_j \notin q^{1+\mathbb{N}}$ for all i, j. Hence it yields an isomorphism of $\hat{\mathbf{R}}_{\lambda_1/\lambda_2} \otimes \mathbf{U}$-modules

$$\hat{\mathbf{R}}_{\lambda_1/\lambda_2} \otimes_{\mathbf{R}^{\tilde{T}_\lambda}} (\mathbf{W}_{T_{\lambda_1}} \otimes_{\mathbb{A}} \mathbf{W}_{T_{\lambda_2}}) \xrightarrow{\sim} \hat{\mathbf{R}}_{\lambda_1/\lambda_2} \otimes_{\mathbf{R}^{\tilde{T}_\lambda}} \mathbf{W}_{T_\lambda}.$$

The unicity follows from Lemma 5.5. □

6.3. Let a be the map from Q_λ to the point. We consider the pairing of $\mathbf{R}^{\tilde{H}_\lambda}$-modules

$$(\ :\ ) : \mathbf{W}_{H_\lambda} \times \mathbf{W}'_{H_\lambda} \to \mathbf{R}^{\tilde{H}_\lambda}$$

given by $(x : y) = a_*(x \otimes y)$, where $\otimes$ is the tor-product relative to the smooth variety Q_λ. The pairing $(\ :\ )$ is perfect; see [20]. Note that, since $\mathbf{W}'_{H_\lambda}$ is a free $\mathbb{A}$-module, there is an embedding $\mathbf{W}'_{H_\lambda} \subset \hat{\mathbf{W}}'_{H_\lambda}$. Let us consider the pairings

$$(\,||\,) : \mathbf{W}_{H_\lambda} \times \mathbf{W}'_{H_\lambda} \to \mathbf{R}^{\tilde{H}_\lambda},$$

$$(\,|\,) : \mathbf{W}_{H_\lambda} \times \mathbf{W}_{H_\lambda} \to \mathbf{R}^{\tilde{H}_\lambda}, \quad (\,|\,)' : \mathbf{W}'_{H_\lambda} \times \mathbf{W}'_{H_\lambda} \to \hat{\mathbf{R}}^{\tilde{H}_\lambda}$$

such that

$$(x||y) = (c_\lambda^{-1} a_\lambda x : \omega^* T_{w_0}^{-1}(y)), \quad (x|y) = (x||\kappa_*(y)), \quad (x|y)' = (\kappa_*^{-1}(x)||y);$$

see Lemma 6.2.1. Let $\partial : \mathbf{R}^{\tilde{H}_\lambda} \to \mathbb{A}$ be the group homomorphism such that $\partial(q) = q$, and $\partial(V) = 0$ if V is a nontrivial simple H_λ-module.

Proposition.

(1) *We have* $(x|y) = (y|x)^\dagger$.
(2) *We have* $(x1_\lambda|y1_\lambda) = xy^\dagger$, *for all* $x, y \in \mathbf{R}^{\tilde{H}_\lambda}$.
(3) *We have* $(u \cdot x|y) = (x|\psi(u) \cdot y)$.
(4) *The pairing* $(\,|\,)$ *is uniquely determined by conditions* 2 *and* 3.
(5) *We have* $(\beta_\lambda(x)||y) = \overline{(x||\beta'_\lambda(y))}$.
(6) *The pairing of* $\mathbb{A}$*-modules* $\partial(\,||\,)$ *is perfect.*
(7) *Claims* 1 *and* 2 *hold for the pairing* $(\,|\,)'$ *also.*

Proof. First, note that $a_{\lambda 0} = 1$. By Lemma 5.7.2 we have $c_{\lambda 0} = (r_\lambda^\dagger)^{-1} 1'_\lambda$. By Proposition 5.6.3 we have $\omega^* T_{w_0}^{-1}(1_\lambda) = (r_\lambda^\dagger)^{-1} 1_\lambda$. Thus

$$\begin{aligned}(x1_\lambda | y1_\lambda) &= (c_{\lambda 0}^{-1} a_{\lambda 0} x 1_\lambda : \omega^* T_{w_0}^{-1}(y 1_\lambda)),\\ &= (x1_\lambda : y^\dagger 1_\lambda),\\ &= xy^\dagger.\end{aligned}$$

Claim 2 is proved. Fix $u \in \mathbf{U}$, $x \in \mathbf{W}_{H_\lambda,\alpha}$, $y \in \mathbf{W}'_{H_\lambda,\alpha}$. For all $w \in W$ let $1_{\lambda,w*0} \boxtimes x \in \mathbf{K}^{\tilde{H}_\lambda}(Z_{\lambda,w*0,\alpha})$ be the obvious element. We have

$$(1_{\lambda,w*0} \boxtimes x) \star y = (x : y) 1_{\lambda,w*0}.$$

Thus, the associativity of $\star$ gives

$$(x \cdot u : y) 1_\lambda = (1_\lambda \boxtimes x) \star \Phi_\lambda(u) \star y = (x : u \cdot y) 1_\lambda.$$

Thus we get

$$(u \cdot x : y) = (x : \phi^* \Phi_\lambda(u) \star y). \tag{6.3.0}$$

Assume now that $x \in \mathbf{W}_{H_\lambda,\alpha'}$, $y \in \mathbf{W}_{H_\lambda,\alpha}$. Using (6.3.0), Lemma 5.2.3, Proposition 5.6.1, (5.7.6) and (5.8.3) we get

$$\begin{aligned}(u \cdot x | y) &= (c_{\lambda\alpha}^{-1} a_{\lambda\alpha} \Phi_\lambda(u) \star x : \omega^* T_{w_0}^{-1} \kappa_*(y))\\ &= (c_{\lambda\alpha'}^{-1} a_{\lambda\alpha'} x : \phi^* C_\lambda^{-1} A_\lambda \Phi_\lambda(u) \star \omega^* T_{w_0}^{-1} \kappa_*(y))\\ &= (x | T_{w_0} \zeta_\lambda C_\lambda^{-1} A_\lambda \Phi_\lambda(u) \star y)\\ &= (x | T_{w_0} A_\lambda C_\lambda \zeta_\lambda \Phi_\lambda(u) \star y)\\ &= (x | \psi_{Z_\lambda} \Phi_\lambda(u) \star y).\end{aligned}$$

Then, apply Corollary 5.8.2. Claim 3 is proved. Claim 4 follows from Lemma 5.5.2. Assume now that $x \in \mathbf{W}_{H_\lambda,\alpha}$, $y \in \mathbf{W}'_{H_\lambda,\alpha}$ as above. For all w we have

$$1_{\lambda,w*0} \star (1_{\lambda,w*0} \boxtimes x) = x.$$

Fix u_λ such that $\check{T}_{w_0}(1_{\lambda\nu}) = u_\lambda 1_\lambda$. Using Proposition 5.6 we get

$$u_\lambda 1_\lambda \star T_{w_0}^{-1}(1_{\lambda\nu} \boxtimes x) = \check{T}_{w_0}(x), \quad T_{w_0}^{-1}(1_{\lambda\nu} \boxtimes x) \star T_{w_0}^{-1}(y) = s_\lambda^{-1}(x : y) 1_\lambda,$$

i.e.,

$$\begin{aligned}&T_{w_0}^{-1}(1_{\lambda\nu} \boxtimes x) = u_\lambda^{-1} 1_\lambda \boxtimes \check{T}_{w_0}(x),\\ &(1_\lambda \boxtimes \check{T}_{w_0}(x)) \star T_{w_0}^{-1}(y) = u_\lambda s_\lambda^{-1}(x : y) 1_\lambda.\end{aligned} \tag{6.3.1}$$

This yields

$$(\check{T}_{w_0}(x) : T_{w_0}^{-1}(y)) = u_\lambda s_\lambda^{-1}(x : y). \tag{6.3.2}$$

Claim 5 is analoguous to [14, Lemma 12.15]. First, using (4.5.1) one gets

$$\overline{(x : \mathbb{D}_{Q_{\lambda\alpha}}(y))}^\dagger = q^{d_{\lambda\alpha}} (\mathbb{D}_{F_{\lambda\alpha}}(x) : y). \tag{6.3.3}$$

Note that

$$b_{\lambda,w_0*\alpha} a_{\lambda\alpha} = q^{-d_{\lambda\alpha}/2}. \tag{6.3.4}$$

Using (6.3.0), Lemma 5.7.2, (6.3.4) and (6.3.3) we get

$$\begin{aligned}\overline{(x||\beta'_\lambda(y))} &= q^{-3d_{\lambda\alpha}/2}\overline{(c_{\lambda\alpha}^{-1}a_{\lambda\alpha}x : c_{\lambda\alpha}b_{\lambda,w_0*\alpha}\mathbb{D}_{Q_{\lambda\alpha}}(y))} \\ &= q^{-d_{\lambda\alpha}}\overline{(x : \mathbb{D}_{Q_{\lambda\alpha}}(y))} \\ &= (\mathbb{D}_{F_{\lambda\alpha}}(x) : y)^\dagger.\end{aligned}$$

Using Lemma 5.7.2 and (6.3.1) we get

$$T_{w_0}^{-1}C_\lambda\zeta_\lambda(x \boxtimes 1_\lambda) = T_{w_0}^{-1}C_\lambda(1_{\lambda\nu} \boxtimes \omega^*x) = u_\lambda^{-1}r_\lambda^{-1}1_\lambda \boxtimes \check{T}_{w_0}c_\lambda^{-1}\omega^*x.$$

The same argument as for (6.3.1) gives

$$T_{w_0}(x \boxtimes 1_\lambda) = s_\lambda^{-1}T_{w_0}(x) \boxtimes 1_{\lambda\nu}.$$

Recall that ω is $\dagger$-linear, $(r_\lambda s_\lambda)^\dagger = r_\lambda s_\lambda$, and $c_{\lambda 0} = (r_\lambda^\dagger)^{-1}1'_\lambda$. Thus we get

$$C_\lambda\zeta_\lambda T_{w_0}(x \boxtimes 1_\lambda) = r_\lambda^{-1}s_\lambda^{-1}(1_\lambda \boxtimes c_\lambda^{-1}\omega^*T_{w_0}x).$$

Thus, (5.8.3) gives $\check{T}_{w_0}c_\lambda^{-1}\omega^*(x) = u_\lambda s_\lambda^{-1}c_\lambda^{-1}\omega^*T_{w_0}(x)$, i.e.,

$$T_{w_0}c_\lambda\omega^*(x) = c_\lambda\omega^*\check{T}_{w_0}(u_\lambda^{-1}s_\lambda x) \tag{6.3.5}$$

for all $x \in \mathbf{W}_{H_\lambda}$. Using (6.3.4), (6.3.5), (6.3.2) we get

$$\begin{aligned}(\beta_\lambda(x)||y) &= q^{d_{\lambda\alpha}/2}(c_{\lambda\alpha}^{-1}a_{\lambda\alpha}T_{w_0}b_{\lambda,w_0*\alpha}c_{\lambda,w_0*\alpha}\omega^*\mathbb{D}_{F_{\lambda\alpha}}(x) : \omega^*T_{w_0}^{-1}(y)) \\ &= (T_{w_0}c_{\lambda,w_0*\alpha}\omega^*\mathbb{D}_{F_{\lambda\alpha}}(x) : c_{\lambda\alpha}^{-1}\omega^*T_{w_0}^{-1}(y)) \\ &= (u_\lambda^{-1}s_\lambda\check{T}_{w_0}\mathbb{D}_{F_{\lambda\alpha}}(x) : T_{w_0}^{-1}(y))^\dagger \\ &= (\mathbb{D}_{F_{\lambda\alpha}}(x) : y)^\dagger.\end{aligned}$$

Claim 5 is proved. Claim 1 follows from (6.3.2), (6.3.5) and Lemma 5.7.2. Indeed

$$\begin{aligned}(x|y) &= (c_\lambda^{-1}a_\lambda x : \omega^*T_{w_0}^{-1}\kappa_*(y)) \\ &= (u_\lambda s_\lambda^{-1}\check{T}_{w_0}^{-1}\omega^*c_\lambda^{-1}a_\lambda\kappa_*(x) : y)^\dagger \\ &= (\omega^*c_\lambda^{-1}T_{w_0}^{-1}a_\lambda\kappa_*(x) : y)^\dagger \\ &= (y|x)^\dagger.\end{aligned}$$

Claim 6 follows from the Schur Lemma and the fact that $(\,:\,)$ is a perfect pairing of $\mathbf{R}^{\tilde{H}_\lambda}$-modules. ☐

Remarks.

(1) The pairings $(\,|\,)$, $(\,|\,)'$ are obviously compatible with the forgetful maps; see Remark 6.1.

(2) The $\mathbf{U}$-module $\mathbf{W}'_{H_\lambda}$ has the following algebraic interpretation : let $\mathbf{W}^*_{H_\lambda}$ be $\mathbf{W}'_{H_\lambda}$ with the new action of $\mathbf{U}$, denoted by $\diamond$, such that $u \diamond x = \phi^*\Phi_\lambda S(u) \star x$, where S is the antipode. Then, the $\mathbf{U}$-module $\mathbf{W}^*_{H_\lambda}$ is the right dual of $\mathbf{W}_{H_\lambda}$.

7. Construction of the signed basis

7.1. Following Lusztig we consider the sets

$$\mathcal{B}'_{H_\lambda} = \{\mathbf{b} \in \mathbf{W}'_{H_\lambda} \,|\, \beta'_\lambda(\mathbf{b}) = \mathbf{b},\, \partial(\mathbf{b}|\mathbf{b})' \in 1 + q^{-1}\mathbb{Z}[[q^{-1}]]\},$$

$$\mathcal{B}_{H_\lambda} = \{\mathbf{b} \in \mathbf{W}_{H_\lambda} \,|\, \beta_\lambda(\mathbf{b}) = \mathbf{b},\, \partial(\mathbf{b}|\mathbf{b}) \in 1 + q^{-1}\mathbb{Z}[q^{-1}]\}.$$

We also set $\mathcal{B}'_\lambda = \mathcal{B}'_{G_\lambda}$, $\mathcal{B}_\lambda = \mathcal{B}_{G_\lambda}$.

Proposition.

(1) *If the subset* $\mathbf{B} \subset \mathcal{B}_{H_\lambda}$ *satisfies*
- $\mathbf{B}$ *is a basis of the* $\mathbb{A}$*-module* $\mathbf{W}_{H_\lambda}$,
- *for any elements* $\mathbf{b}, \mathbf{b}' \in \mathbf{B}$ *we have* $\partial(\mathbf{b}|\mathbf{b}') \in \delta_{\mathbf{b},\mathbf{b}'} + q^{-1}\mathbb{Z}[q^{-1}]$,

then $\mathcal{B}_{H_\lambda} = \pm\mathbf{B}$. *A similar statement holds for* $\mathcal{B}'_{H_\lambda}$.

(2) *We have* $x1_\lambda, xr_\lambda 1_{\lambda\nu} \in \mathcal{B}_{H_\lambda}$, *and* $x1'_\lambda, xr_\lambda 1'_{\lambda\nu} \in \mathcal{B}'_{H_\lambda}$, *for any* $x \in \mathbf{X}^{H_\lambda}$.

Proof. Claim 1 is standard; see [14, §12.20] for instance. We reproduce a proof here for the convenience of the reader. Fix an element $\mathbf{b} \in \mathcal{B}_{H_\lambda}$. Set $\mathbf{b} = \sum_i p_i \mathbf{b}_i$ where $\mathbf{b}_i \in \mathbf{B}$ and $p_i \in \mathbb{A}$. Fix $n \in \mathbb{Z}$ such that $p_i \in q^n\mathbb{Z}[q^{-1}]$ for all i and $p_i \notin q^{n-1}\mathbb{Z}[q^{-1}]$ for some i. For all i let $p_{in} \in \mathbb{Z}$ be such that $p_i \in p_{in}q^n + q^{n-1}\mathbb{Z}[q^{-1}]$. Then, $\sum_i p_{in}^2 > 0$. Thus,

$$\partial(\mathbf{b}|\mathbf{b}) \in q^{2n}\textstyle\sum_i p_{in}^2 + q^{2n-1}\mathbb{Z}[q^{-1}].$$

On the other hand, we have $\partial(\mathbf{b}|\mathbf{b}) \in 1 + q^{-1}\mathbb{Z}[q^{-1}]$. It follows that $n = 0$ and $\sum_i p_{in}^2 = 1$. Since $\beta_\lambda(\mathbf{b}) = \mathbf{b}$ and $\beta_\lambda(\mathbf{b}_i) = \mathbf{b}_i$ for all i, we must have $\bar{p}_i = p_i$ for all i. Hence $p_i \in \mathbb{Z}$ for all i. Then $\sum_i p_i^2 = 1$. Thus $\mathbf{b} \in \pm\mathbf{B}$. Let us prove Claim 2. By Proposition 6.1.4 and 6.3.2 we have $x1_\lambda \in \mathcal{B}_{H_\lambda}$, $x1'_\lambda \in \mathcal{B}'_{H_\lambda}$. Hence, using [11] we get $T_{w_0}^{-1}(x1_\lambda) \in \mathcal{B}_{H_\lambda}$, $T_{w_0}^{-1}(x1'_\lambda) \in \mathcal{B}'_{H_\lambda}$. Finally, Proposition 5.6.3 gives $T_{w_0}^{-1}(x1_\lambda) = xr_\lambda 1_{\lambda\nu}$. We are done. □

Remark. In general, $1_{\lambda\alpha} \notin \mathcal{B}_{H_\lambda}$.

7.2. For any $\lambda \in P^+$ let $V(\lambda)$ be Kashiwara's maximal integrable module. By definition, $V(\lambda)$ is the free $\mathbb{A}$-module with the action of the algebra $\mathbf{U}$ such that there is a weight vector v_λ of weight λ which generates $V(\lambda)$ and satisfies the following defining relations:

(7.2.1)
$$\mathbf{U}_\alpha(v_\lambda) = 0 \text{ for any } \alpha \in Q \setminus \{0\} \text{ s.t. } (\alpha, \lambda) \geq 0,$$
$$\mathbf{f}_i^{1+\ell_i}(v_\lambda) = 0 \text{ if } i \neq 0, \quad \mathbf{e}_0^{1+(\theta,\lambda)}(v_\lambda) = 0;$$

see [9, §5.1]. It is proved in [8] that the module $V(\lambda)$ admits a global basis. Let $\mathbf{B}(\lambda)$ be this basis. The element v_λ belongs to $\mathbf{B}(\lambda)$. Let $^- : V(\lambda) \to V(\lambda)$ be the unique $\mathbb{A}$-antilinear map such that $\bar{\mathbf{b}} = \mathbf{b}$ for all elements $\mathbf{b} \in \mathbf{B}(\lambda)$. It is conjectured in [24, Remark 7.19] that there is an isomorphism of $\mathbf{U}$-modules $V(\lambda) \to \mathbf{W}_\lambda$ such that $v_\lambda \mapsto 1_\lambda$. First, we consider the case $\lambda = \omega_i$. Let $W(\omega_i)'$ be the fundamental simple finite dimensional $\mathbf{U}'$-module associated to the weight ω_i; see [9, (5.7)], [1, §1.3]. Let $W(\omega_i) \subset W(\omega_i)'$ be the corresponding $\mathbb{A}$-form For any $\mathbf{U}$-module M and any formal variable z, let M_z be the representation of $\mathbf{U}$ on the space $M[z^{\pm 1}]$ such that $(\mathbf{x}_{jr}^{\pm})^{(n)} \mapsto (\mathbf{x}_{jr}^{\pm})^{(n)} \otimes z^r$, $\mathbf{k}_{jr}^{\pm} \mapsto \mathbf{k}_{jr}^{\pm} \otimes z^r$. Fix a weight vector $w_{\omega_i} \in W(\omega_i)$ of weight ω_i. By [9, Theorem 5.15.$(viii)$] there is a unique isomorphism of $\mathbf{U}$-modules

$$V(\omega_i) \xrightarrow{\sim} W(\omega_i)_z \tag{7.2.2}$$

such that $v_{\omega_i} \mapsto w_{\omega_i}$. The product by z is an automorphism of $\mathbf{U}$-modules. It preserves the basis $\mathbf{B}(\omega_i)$. There is a unique basis $\mathbf{B}^0(\omega_i)$ of $W(\omega_i)$ such that the map (7.2.2) takes $\mathbf{B}(\omega_i)$ to $\bigsqcup_{n\in\mathbb{Z}} z^n\mathbf{B}^0(\omega_i)$; see [9, Theorem 5.15.$(iii)$].

Since the group G_{ω_i} is isomorphic to $\mathbb{C}^\times$, we identify $\mathbf{R}^{G_{\omega_i}}$ with $\mathbb{Z}[z_i^{\pm 1}]$ in the usual way.

Theorem A.

(1) *There is a unique element* $a_i \in \mathbb{Q}(q)^\times$ *and a unique isomorphism of* $\mathbf{U}$*-modules* $\phi : V(\omega_i) \xrightarrow{\sim} \mathbf{W}_{\omega_i}$ *such that* $v_{\omega_i} \mapsto 1_{\omega_i}$ *and the multiplication by* z *is mapped to the multiplication by* $a_i z_i$*. Moreover,* $a_i = \pm 1$.[1]

(2) *Assume that* $\langle \,|\, \rangle : V(\omega_i) \times V(\omega_i) \to \mathbb{A}$ *is a symmetric perfect pairing of* $\mathbb{A}$*-modules such that* $\langle z^n v_{\omega_i} | z^m v_{\omega_i} \rangle = \delta_{n,m}$ *and* $\langle u \cdot x | y \rangle = \langle x | \psi(u) \cdot y \rangle$*. Then*

$$\pm \mathbf{B}(\omega_i) = \{\mathbf{b} \in V(\omega_i) \,|\, \bar{\mathbf{b}} = \mathbf{b},\ \langle \mathbf{b}|\mathbf{b}\rangle \in 1 + q^{-1}\mathbb{Z}[q^{-1}]\}.$$

Moreover, $\langle \mathbf{b}|\mathbf{b}'\rangle \in q^{-1}\mathbb{Z}[q^{-1}]$ *if* $\mathbf{b}, \mathbf{b}' \in \mathbf{B}(\omega_i)$ *and* $\mathbf{b} \neq \mathbf{b}'$.

(3) $\mathcal{B}_{G_{\omega_i}} = \pm \phi(\mathbf{B}(\omega_i))$*. It is a signed basis of* $\mathbf{W}_{\omega_i}$.

Proof. Let us prove Claim 1. We identify $\mathbf{K}^{\mathbb{C}^\times}(F_{\omega_i})$ with the specialization of the $\mathbf{U}$-module $\mathbf{W}_{\omega_i}$ at the maximal ideal of $\mathbf{R}^{G_{\omega_i}}$ associated to $1 \in G_{\omega_i}$. There is a unique element $a_i \in \mathbb{Q}(q)^\times$ and a unique isomorphism of $\mathbf{U}'$-modules $\mathbb{Q}(q) \otimes_{\mathbb{A}} \mathbf{K}^{\mathbb{C}^\times}(F_{\omega_i}) \xrightarrow{\sim} W(\omega_i)'_{a_i}$ such that $1_{\omega_i} \mapsto w_{\omega_i}$, since both $\mathbf{U}'$-modules are simple; see [20]. Since the $\mathbf{U}$-modules $\mathbf{K}^{\mathbb{C}^\times}(F_{\omega_i})$, $W(\omega_i)_{a_i}$ are cyclicly generated by 1_{ω_i}, w_{ω_i}, we get an isomorphism $\mathbf{K}^{\mathbb{C}^\times}(F_{\omega_i}) \xrightarrow{\sim} W(\omega_i)_{a_i}$ such that $1_{\omega_i} \mapsto w_{\omega_i}$. The identification of the group G_{ω_i} with $\mathbb{C}^\times$ is such that for any $(B,p,q) \in M_{\omega_i \alpha}$ and any $g_{\omega_i} \in G_{\omega_i}$ we have

$$(1, g_{\omega_i}, 1) \cdot (B,p,q) = (B, g_{\omega_i}^{-1} p, g_{\omega_i} q).$$

Since $p_j = 0$, $q_j = 0$ if $j \neq i$ we have

$$(1, g_{\omega_i}, 1) \cdot (B,p,q) = (1, 1, g_\alpha) \cdot (B,p,q)$$

for $g_\alpha = (g_{a_j})_j$ with $g_{a_j} = g_{\omega_i}^{-1} \mathrm{Id}_{\mathbb{C}^{a_j}}$. Then the group G_{ω_i} acts trivially on F_{ω_i} and the natural isomorphism of $\mathbb{A}[z_i^{\pm 1}]$-modules

$$\mathbf{W}_{\omega_i} = \mathbf{K}^{G_{\omega_i} \times \mathbb{C}^\times}(F_{\omega_i}) \xrightarrow{\sim} \mathbf{K}^{\mathbb{C}^\times}(F_{\omega_i})[z_i^{\pm 1}]$$

takes $\mathcal{V}_j$ to $\mathcal{V}_j \otimes z_i$, and $\mathcal{W}_j$ to $\mathcal{W}_j \otimes z_i$. In particular, we have

$$\textstyle\bigwedge_{\mathcal{V}_\alpha} \mapsto \bigwedge_{\mathcal{V}_\alpha} \otimes z_i^{(\rho,\alpha)},\ \bigwedge_{\mathcal{F}^+_{j;\alpha}} \mapsto \bigwedge_{\mathcal{F}^+_{j;\alpha}} \otimes z_i^{f^+_{j;\alpha}},\ \bigwedge_{\mathcal{F}^-_{j;\alpha}} \mapsto \bigwedge_{\mathcal{F}^-_{j;\alpha}} \otimes z_i^{f^-_{j;\alpha}}$$

and $\mathbf{x}^\pm_{jr} \mapsto \mathbf{x}^\pm_{jr} \otimes z_i^r$, $\mathbf{k}^\pm_{jr} \mapsto \mathbf{k}^\pm_{jr} \otimes z_i^r$, because $t_{\alpha+\alpha_j} - t_\alpha = f^+_{j;\alpha} - f^-_{j;\alpha}$. Hence $\mathbf{W}_{\omega_i} \simeq (W(\omega_i)_{a_i})_{z_i} \simeq W(\omega_i)_{a_i z_i}$.

The map ϕ takes the involution $v \mapsto \bar{v}$ on $V(\omega_i)$ to the involution β_{ω_i} on $\mathbf{W}_{\omega_i}$ since both $\mathbf{U}$-modules are cyclic and the involutions are compatible with $u \mapsto \bar{u}$ on $\mathbf{U}$. The map β_{ω_i} is z_i linear by Corollary 5.8.1, the map $v \mapsto \bar{v}$ on $V(\omega_i)$ is z linear since product by z preserves $\mathbf{B}(\omega_i)$. Hence $\bar{a}_i = a_i$. The j-th Drinfeld polynomial of $\mathbf{K}^{\mathbb{C}^\times}(F_{\omega_i})$ is $P_j(t) = (t - q^{-\mathbf{c}})^{\delta_{ij}}$. Hence the j-th Drinfeld polynomial of $W(\omega_i)$ is $P_j(t) = (t - a_i^{-1} q^{-\mathbf{c}})^{\delta_{ij}}$. An easy computation shows that the elements $\mathbf{h}_{i,\pm 1}$ act on the vector $w_{\omega_i} \in W(\omega_i)$ as follows: $\mathbf{h}_{i,\pm 1}(w_{\omega_i}) = a_i^{\pm 1} q^{\pm \mathbf{c}} w_{\omega_i}$. Since $\mathbf{h}_{i,\pm 1}$ belongs to $\mathbf{U}$, the element $a_i q^{-\mathbf{c}}$ belongs to $\mathbb{A}$ and is invertible. Thus $a_i = \pm 1$ because $a_i \in \pm q^{\mathbb{Z}}$ and $a_i = \bar{a}_i$.

Let us prove Claim 2. We first recall some well-known fact. Let $\dot{\mathbf{B}}$ be the canonical basis of $\dot{\mathbf{U}}$; see [13, §25.2]. By [8, §8] there is a subset $\mathbf{I}(\lambda) \subset \dot{\mathbf{B}}$ such that the space $I(\lambda) = \bigoplus_{\mathbf{b} \in \mathbf{I}(\lambda)} \mathbb{A}\,\mathbf{b} \subset \dot{\mathbf{U}} \eta_\lambda$ is a left $\dot{\mathbf{U}}$-submodule, and such that there is

[1] H. Nakajima remarked that Theorem A.1 was not stated correctly in a previous version of the paper. He also mentioned to us that the isomorphism of $\mathbf{U}$-modules $V(\omega_i) \xrightarrow{\sim} \mathbf{W}_{\omega_i}$ was known to him.

a unique isomorphism of $\dot{\mathbf{U}}$-modules $\dot{\mathbf{U}}\eta_\lambda/I(\lambda) \to V(\lambda)$ which takes η_λ to v_λ. Since the $\mathbf{U}$-module $V(\lambda)$ is integrable, Kashiwara's modified operators $\tilde{e}_j$, $\tilde{f}_j$, $j \in I\cup\{0\}$, act on $V(\lambda)$. Let $L(\lambda) \subset V(\lambda)$ be the $\mathbb{Z}[q^{-1}]$-lattice linearly spanned by $\mathbf{B}(\lambda)$. It is stable by the operators $\tilde{e}_j$, $\tilde{f}_j$; see [8, Proposition 9.1], and contains the element v_λ. The induced operators on the quotient $L(\lambda)/q^{-1}L(\lambda)$ are still denoted by $\tilde{e}_j$, $\tilde{f}_j$. Let $\pi : L(\lambda) \to L(\lambda)/q^{-1}L(\lambda)$ be the projection. We set $B(\lambda) = \pi\big(\mathbf{B}(\lambda)\big)$. It is known that $\tilde{e}_j$, $\tilde{f}_j$ take $B(\lambda)$ to $B(\lambda) \sqcup \{0\}$.

We now assume that $\lambda = \omega_i$. Then the operators $\tilde{e}_j$, $\tilde{f}_j$ are z-linear. Let $L^0(\omega_i)$ be the $\mathbb{Z}[q^{-1}]$-module spanned by $\mathbf{B}^0(\omega_i)$. Let $B^0(\omega_i)$ be the projection of $\mathbf{B}^0(\omega_i)$ in $L^0(\omega_i)/q^{-1}L^0(\omega_i)$. There is an isomorphism of crystals

$$\big(B^0(\omega_i), L^0(\omega_i)\big) \simeq \big(B(\omega_i), L(\omega_i)\big)/(z-1).$$

Any element in $B^0(\omega_i)$ can be reached at w_{ω_i} after applying a monomial in the operators $\tilde{e}_j$, $j \in I \cup \{0\}$; see [1, Lemma 1.5.(1) and (2)] and [9, Proposition 5.4.(i)]. Thus any element in $B(\omega_i)$ can be reached at $\{z^m v_{\omega_i}\,;\, m \in \mathbb{Z}\}$ after applying a monomial to the operators $\tilde{e}_j$, $j \in I \cup \{0\}$.

Set $L(\omega_i)^\infty = \bigcup_{k\geq 0} L(\omega_i)^k$, where

$$L(\omega_i)^k = \sum_{\ell\leq k}\sum_{j_1,\dots j_\ell} \mathbb{Z}[q^{-1}, z^{\pm 1}]\, \tilde{f}_{j_1}\cdots\tilde{f}_{j_\ell}(v_{\omega_i}).$$

We claim that

$$L(\omega_i) = L(\omega_i)^\infty + q^{-1}L(\omega_i), \tag{7.2.3}$$

$$\langle L(\omega_i)\,|\,L(\omega_i)\rangle \subseteq \mathbb{Z}[q^{-1}], \quad \langle \tilde{f}_j(x)|y\rangle \in \langle x|\tilde{e}_j(y)\rangle + q^{-1}\mathbb{Z}[q^{-1}], \tag{7.2.4}$$

$\forall j \in I \cup \{0\}$, $\forall x, y \in L(\omega_i)$. Claim (7.2.3) is obvious. To prove (7.2.4) we use the following lemma, whose proof is given after the proof of the proposition.

Lemma. *Fix $j \in I \cup \{0\}$. For any $x \in V(\omega_i)$ fix elements $x_r \in V(\omega_i)$, $r \in [0,t]$, such that $x = \sum_{r=0}^t \mathbf{f}_j^{(r)}(x_r)$ and $\mathbf{e}_j(x_r) = 0$.*

1. *If $x \in L(\omega_i)$, then $x_r \in L(\omega_i)$.*
2. *If $x \in \mathbf{B}(\omega_i)$, then there is $r_0 \in [0,t]$ such that $x_{r_0} \in \mathbf{B}(\omega_i) + q^{-1}L(\omega_i)$ and $x_r \in q^{-1}L(\omega_i)$ if $r \neq r_0$.*
3. *Fix $\lambda, \mu \in P$. Assume that $x \in L(\omega_i)_\lambda$, and $\langle x\,|\,L(\omega_i)\rangle \subseteq \mathbb{Z}[q^{-1}]$. Fix $y \in L(\omega_i)_\mu$, with $\mu = \lambda - \alpha_j$ and fix elements $y_s \in V(\omega_i)$, $s \in [0,u]$, such that $y = \sum_{s=0}^u \mathbf{f}_j^{(s)}(y_s)$ and $\mathbf{e}_j(y_s) = 0$. Then $\langle x_r\,|\,y_s\rangle \in \mathbb{Z}[q^{-1}]$ for all r, s.*

The $\mathbf{U}$-module $V(\omega_i)$ is endowed with its $\hat{P}$-gradation; see [9]. For any $\hat{\mu} \in \hat{P}$ let $V(\omega_i)_{\hat{\mu}} \subset V(\omega_i)$, $L(\omega_i)_{\hat{\mu}} = L(\omega_i) \cap V(\omega_i)_{\hat{\mu}}$ and $L(\omega_i)^k_{\hat{\mu}} = L(\omega_i)^k \cap V(\omega_i)_{\hat{\mu}}$ be the corresponding weight subspaces. Then

$$\langle V(\omega_i)_{\hat{\mu}_1}\,|\,V(\omega_i)_{\hat{\mu}_2}\rangle \neq 0 \Rightarrow \hat{\mu}_1 - \hat{\mu}_2 \in \mathbb{Z}\delta,$$

$$L(\omega_i)^k_{\hat{\mu}} = \textstyle\sum_j \tilde{f}_j\big(L(\omega_i)^{k-1}_{\hat{\mu}+\alpha_j}\big), \tag{7.2.5}$$

$$\tilde{e}_j\big(L(\omega_i)_{\hat{\mu}}\big) \subseteq L(\omega_i)_{\hat{\mu}+\alpha_j}.$$

We first prove by induction on k that

$$\langle L(\omega_i)^k\,|\,L(\omega_i)\rangle \subseteq \mathbb{Z}[q^{-1}], \tag{7.2.6}$$

$$\langle \tilde{f}_j(x)|y\rangle \in \langle x|\tilde{e}_j(y)\rangle + q^{-1}\mathbb{Z}[q^{-1}], \tag{7.2.7}$$

$\forall j \in I \cup \{0\}$, $\forall x \in L(\omega_i)^k$, $\forall y \in L(\omega_i)$. We have $L(\omega_i)^0 = \mathbb{Z}[q^{-1}, z^{\pm 1}]v_{\omega_i}$, and $L(\omega_i)_{\omega_i+n\delta} = \mathbb{Z}[q^{-1}]z^n v_{\omega_i}$. Thus, (7.2.6) for $k=0$ reduces to

$$\langle \mathbb{Z}[q^{-1}]z^m v_{\omega_i} \,|\, \mathbb{Z}[q^{-1}]z^n v_{\omega_i}\rangle \subseteq \mathbb{Z}[q^{-1}],$$

which follows from $\langle z^m v_{\omega_i} \,|\, z^n v_{\omega_i}\rangle = \delta_{mn}$. Similarly, (7.2.7) for $k=0$ reduces to

$$\langle \tilde{f}_j(z^n v_{\omega_i})|y\rangle \in \langle z^n v_{\omega_i}|\tilde{e}_j(y)\rangle + q^{-1}\mathbb{Z}[q^{-1}]. \tag{7.2.8}$$

This is obvious if $j=0$ because $\langle \tilde{f}_j(z^n v_{\omega_i})|y\rangle \neq 0$ or $\langle z^n v_{\omega_i}|\tilde{e}_j(y)\rangle \neq 0$ implies that $y \in \bigoplus_n V(\omega_i)_{\omega_i+\theta+n\delta}$, and $V(\omega_i)_{\omega_i+\theta+n\delta} = \{0\}$ for all n; see [9, Proposition 5.14.(i)] for instance. If $j \neq 0$, (7.2.8) is proved as follows. Since $\mathbf{e}_j(z^n v_{\omega_i}) = 0$, we have $\tilde{f}_j(z^n v_{\omega_i}) = \mathbf{f}_j(z^n v_{\omega_i})$. Fix elements $y_s \in V(\omega_i)$, $s \in [0,u]$, as in Lemma 7.2.3. Then, $y_s \in L(\omega_i)$ by Lemma 7.2.1. We must show that

$$\langle \mathbf{f}_j(z^n v_{\omega_i}) \,|\, \textstyle\sum_{s\geq 0} \mathbf{f}_j^{(s)}(y_s)\rangle \in \langle z^n v_{\omega_i} \,|\, \textstyle\sum_{s\geq 1} \mathbf{f}_j^{(s-1)}(y_s)\rangle + q^{-1}\mathbb{Z}[q^{-1}].$$

The computation in [13, Proposition 19.1.3] gives the result, since $\langle z^n v_{\omega_i} \,|\, y_s\rangle \in \mathbb{Z}[q^{-1}]$ for all s, by (7.2.6) for $k=0$. We may therefore assume that (7.2.6), (7.2.7) are already known for $k-1$ with $k>0$. Using (7.2.5) we see that (7.2.6) for k follows from (7.2.6), (7.2.7) for $k-1$. Finally, (7.2.7) for k is proved as in [13, Proposition 19.1.3] using (7.2.6) for k, and Lemma 7.2.3.

From (7.2.6) we get

$$\langle L(\omega_i)^\infty \,|\, L(\omega_i)\rangle \subseteq \mathbb{Z}[q^{-1}].$$

On the other hand, there is a positive integer a such that

$$\langle L(\omega_i) \,|\, L(\omega_i)\rangle \subseteq q^a\mathbb{Z}[q^{-1}]$$

because $\mathbf{B}^0(\omega_i)$ is finite. Using (7.2.3) yields

$$\langle L(\omega_i) \,|\, L(\omega_i)\rangle \subseteq \mathbb{Z}[q^{-1}].$$

Similarly, given $x, y \in L(\omega_i)$ we fix $x^\infty \in L(\omega_i)^\infty$ such that $x - x^\infty \in q^{-1}L(\omega_i)$; see (7.2.3). Then (7.2.7) yields

$$\langle \tilde{f}_j(x^\infty)|y\rangle \in \langle x^\infty|\tilde{e}_j(y)\rangle + q^{-1}\mathbb{Z}[q^{-1}].$$

Thus

$$\begin{aligned}\langle \tilde{f}_j(x)|y\rangle \in \langle \tilde{f}_j(x^\infty)|y\rangle + q^{-1}\mathbb{Z}[q^{-1}] &= \langle x^\infty|\tilde{e}_j(y)\rangle + q^{-1}\mathbb{Z}[q^{-1}] \\ &= \langle x|\tilde{e}_j(y)\rangle + q^{-1}\mathbb{Z}[q^{-1}].\end{aligned}$$

We have proved (7.2.4).

Then, Claim 2 is proved as in [13, Lemma 19.1.4], using (7.2.3), (7.2.4) and Proposition 7.1.1. More precisely, for any element $\mathbf{b} \in \mathbf{B}(\omega_i)$, let $\ell(\mathbf{b})$ be the smallest $k \geq 0$ such that $\mathbf{b} \in L(\omega_i)^k + q^{-1}L(\omega_i)$; see (7.2.3). For any $\mathbf{b}, \mathbf{b}' \in \mathbf{B}(\omega_i)$ we prove by induction on $\ell(\mathbf{b})$ that

$$\langle \mathbf{b} \,|\, \mathbf{b}'\rangle \in q^{-1}\mathbb{Z}[q^{-1}] \text{ if } \mathbf{b} \neq \mathbf{b}', \tag{7.2.9}$$

$$\langle \mathbf{b} \,|\, \mathbf{b}\rangle \in 1 + q^{-1}\mathbb{Z}[q^{-1}]. \tag{7.2.10}$$

If $\ell(\mathbf{b}) = 0$, then $\mathbf{b} = z^n v_{\omega_i}$ for some $n \in \mathbb{Z}$. Thus, if $\langle \mathbf{b} \,|\, \mathbf{b}'\rangle \neq 0$, then $\mathbf{b}' = z^m v_{\omega_i}$ for some m and both statements are obvious. Fix $k>0$. Assume that (7.2.9), (7.2.10) hold for any $\mathbf{b}, \mathbf{b}'$ such that $\ell(\mathbf{b}) < k$. Fix $\mathbf{b}, \mathbf{b}'$ such that $\ell(\mathbf{b}) = k$.

By (7.2.3) there is an integer $j \in I \cup \{0\}$ and an element $\mathbf{b}_1 \in \mathbf{B}(\omega_i)$ such that $\tilde{f}_j(\mathbf{b}_1) \in \mathbf{b} + q^{-1}L(\omega_i)$ and $\ell(\mathbf{b}_1) = k-1$. Using (7.2.4) we get

$$\langle \mathbf{b} \,|\, \mathbf{b}' \rangle \in \langle \tilde{f}_j(\mathbf{b}_1) \,|\, \mathbf{b}' \rangle + q^{-1}\mathbb{Z}[q^{-1}] = \langle \mathbf{b}_1 \,|\, \tilde{e}_j(\mathbf{b}') \rangle + q^{-1}\mathbb{Z}[q^{-1}].$$

We have $\tilde{e}_j(\mathbf{b}) \in \mathbf{b}_1 + q^{-1}L(\omega_i)$. Thus,

$$\langle \mathbf{b} \,|\, \mathbf{b} \rangle \in \langle \mathbf{b}_1 \,|\, \mathbf{b}_1 \rangle + q^{-1}\mathbb{Z}[q^{-1}].$$

Hence (7.2.10) for k follows from (7.2.10) for $k-1$. If $\mathbf{b} \neq \mathbf{b}'$, either $\tilde{e}_j(\mathbf{b}') \in q^{-1}L(\omega_i)$. Then

$$\langle \mathbf{b} \,|\, \mathbf{b}' \rangle \in q^{-1}\mathbb{Z}[q^{-1}]$$

by (7.2.4), or there is an element $\mathbf{b}_1' \in \mathbf{B}(\omega_i)$ such that $\tilde{e}_j(\mathbf{b}') \in \mathbf{b}_1' + q^{-1}L(\omega_i)$. In the last case $\mathbf{b}_1 \neq \mathbf{b}_1'$ (or by applying $\tilde{f}_j$ we would get $\mathbf{b} \in \mathbf{b}' + q^{-1}L(\omega_i)$, and thus $\mathbf{b} = \mathbf{b}'$). Hence (7.2.9) for k follows from (7.2.9) for $k-1$. Finally, Claim 2 follows from (7.2.9), (7.2.10) and Proposition 7.1.1.

Claim 3 is obvious from Claim 1 and Claim 2. □

Proof of Lemma 7.2. Claims 1 and 2 generalize [13, Lemma 18.2.2] to the nonhighest weight module case. The proof follows [13, Lemma 18.2.2]. Note that we only use Claim 1; Claim 2 is given for the sake of completness.

We prove Claim 1 by induction on t. It is obvious if $t=0$. Since

$$\tilde{e}_j(x) = \sum_{r=0}^{t-1} \mathbf{f}_j^{(r)}(x_{r+1}), \quad \tilde{e}_j(L(\omega_i)) \subseteq L(\omega_i),$$

we get $\sum_{r=0}^{t-1} \mathbf{f}_j^{(r)}(x_{r+1}) \in L(\omega_i)$. The induction hypothesis for $t-1$ gives $x_{r+1} \in L(\omega_i)$ for all $r \in [0, t-1]$. Since $\mathbf{e}_j(x_{r+1}) = 0$, we have $\mathbf{f}_j^{(r+1)}(x_{r+1}) = \tilde{f}_j^{r+1}(x_{r+1})$. Since $\tilde{f}_j(L(\omega_i)) \subset L(\omega_i)$, we get $\mathbf{f}_j^{(r+1)}(x_{r+1}) \in L(\omega_i)$. Using $x \in L(\omega_i)$, $\mathbf{f}_j^{(r)}(x_r) \in L(\omega_i)$ for all $r \in [1, t]$ we get $x_0 \in L(\omega_i)$.

We prove Claim 2 by induction on t. It is obvious if $t = 0$. Since $x \in \mathbf{B}(\omega_i)$, we have $\tilde{e}_j(x) \in \mathbf{B}(\omega_i) + q^{-1}L(\omega_i)$ or $\tilde{e}_j(x) \in q^{-1}L(\omega_i)$. In the second case we get, using Claim 1, $x_r \in q^{-1}L(\omega_i)$ for all $r \in [1, t]$. Thus $x_0 \in x + q^{-1}L(\omega_i)$ and we are done. Consider now the first case. Using the induction hypothesis for $t-1$ we get an integer $r_0 \in [0, t-1]$ such that $x_{r_0+1} \in \mathbf{B}(\omega_i) + q^{-1}L(\omega_i)$ and $x_{r+1} \in q^{-1}L(\omega_i)$ for all $r \in [0, t-1] \setminus \{r_0\}$. Thus $\tilde{e}_j(x) \in \tilde{f}_j^{r_0}(x_{r_0+1}) + q^{-1}L(\omega_i)$. Hence

$$x \in \tilde{f}_j \tilde{e}_j(x) + q^{-1}L(\omega_i) = \tilde{f}_j^{r_0+1}(x_{r_0+1}) + q^{-1}L(\omega_i).$$

Hence, necessarily $x_0 \in q^{-1}L(\omega_i)$. We are done.

We prove Claim 3. To simplify we set $x_r = y_s = 0$ if $r > t$, $s > u$. For any $a, b \geq 0$ we have (see [13, Proposition 19.1.3])

$$\langle \mathbf{f}_j^{(a)}(x_r) \,|\, \mathbf{f}_j^{(b)}(y_s) \rangle = \delta_{a,b} \delta_{r+1,s} C_{a,s} \langle x_r \,|\, y_s \rangle,$$

where

$$C_{a,s} = q^{a^2 - a(\mu + s\alpha_j, \alpha_j)} \begin{bmatrix} (\mu + s\alpha_j, \alpha_j) \\ a \end{bmatrix}.$$

From Claim 1 we have $y_s \in L(\omega_i)_{\mu + s\alpha_j}$ for all s. Since $\mathbf{e}_j(y_s) = 0$ we have $\mathbf{f}_j^{(s-1)}(y_s) = \tilde{f}_j^{s-1}(y_s)$. In particular, $\mathbf{f}_j^{(s-1)}(y_s) \in L(\omega_i)_{\mu+\alpha_j}$. Thus $\langle x \,|\, \mathbf{f}_j^{(s-1)}(y_s) \rangle \in$

$\mathbb{Z}[q^{-1}]$. On the other hand,

$$\langle x \,|\, \mathbf{f}_j^{(s-1)}(y_s)\rangle = \sum_{r=0}^{t} \langle \mathbf{f}_j^{(r)}(x_r) \,|\, \mathbf{f}_j^{(s-1)}(y_s)\rangle = C_{s-1,s}\langle x_{s-1} \,|\, y_s\rangle.$$

Now

$$C_{s-1,s} = q^{(s-1)^2-(s-1)(\mu+s\alpha_j,\alpha_j)} \begin{bmatrix} (\mu+s\alpha_j,\alpha_j) \\ s-1 \end{bmatrix} \in 1+q^{-1}\mathbb{Z}[q^{-1}].$$

Thus $\langle x_{s-1} \,|\, y_s\rangle \in \mathbb{Z}[q^{-1}]$ for all $s \geq 1$. If $s \neq r+1$, then $\langle x_r \,|\, y_s\rangle = 0$ since x_r, y_s have different weights. □

Here is the main result of the paper.

Theorem B.

(1) *The sets $\mathcal{B}_{T_\lambda}$, $\mathcal{B}'_{T_\lambda}$ are signed bases of $\mathbf{W}_{T_\lambda}$, $\mathbf{W}'_{T_\lambda}$. Moreover, for any $\mathbf{b}, \mathbf{b}' \in \mathcal{B}_{T_\lambda}$ (resp. $\mathbf{b}, \mathbf{b}' \in \mathcal{B}'_{T_\lambda}$) we have*

$$\partial(\mathbf{b}|\mathbf{b}') \in \delta_{\mathbf{b},\mathbf{b}'} + q^{-1}\mathbb{Z}[q^{-1}] \quad (\text{resp. } \partial(\mathbf{b}|\mathbf{b}')' \in \delta_{\mathbf{b},\mathbf{b}'} + q^{-1}\mathbb{Z}[[q^{-1}]]).$$

(2) *The signed bases $\mathcal{B}_{T_\lambda}$, $\mathcal{B}'_{T_\lambda}$ are dual with respect to the pairing $\partial(\,||\,)$ in the following sense: for all $\mathbf{b} \in \mathcal{B}_{T_\lambda}$ there is a unique element $\mathbf{b}' \in \mathcal{B}'_{T_\lambda}$ such that $\partial(\mathbf{b}||\mathbf{b}') = 1$, and we have $\partial(\mathbf{b}||\mathbf{b}'') = 0$ whenever $\mathbf{b}'' \neq \pm\mathbf{b}'$.*

Proof. Fix a decomposition $\lambda = \sum_{k=1}^{\ell} \omega_{i_k}$. We set $v_{i_1,\dots,i_\ell} = v_{\omega_1} \otimes \cdots \otimes v_{\omega_\ell}$. By [9, §8] the $\mathbf{U}$-submodule

$$\mathbf{N} := \mathbf{U} \cdot (\mathbf{R}^{T_\lambda} \otimes v_{i_1,\dots,i_\ell}) \subset \bigotimes_k V(\omega_{i_k})$$

admits a unique involution c^{nor} such that

$$c^{\mathrm{nor}}(v_{i_1,\dots,i_\ell}) = v_{i_1,\dots,i_\ell}, \quad c^{\mathrm{nor}}(u \cdot v_{i_1,\dots,i_\ell}) = \bar{u} \cdot v_{i_1,\dots,i_\ell}, \quad \forall u \in \mathbf{U}.$$

Set $\mathbf{R}^{T_{\omega_{i_k}}} = \mathbb{Z}[z_k^{\pm 1}]$ for all k. Also, set

$$\hat{\mathbf{R}}_{i_1/\dots/i_\ell} = \mathbb{Z}[[q^{-1}, z_{k+1}/z_k; k]] \otimes_{\mathbb{Z}[q^{-1}, z_{k+1}/z_k; k]} \mathbf{R}^{\tilde{T}_\lambda},$$

where k takes all possible values in $[1,\ell]$. The tensor product

$$\hat{\bigotimes}_k \mathbf{W}_{T_{\omega_{i_k}}} := \hat{\mathbf{R}}_{i_1/\dots/i_\ell} \otimes_{\mathbf{R}^{\tilde{T}_\lambda}} \bigotimes_k \mathbf{W}_{T_{\omega_{i_k}}}$$

is endowed with the unique pairing of $\hat{\mathbf{R}}_{i_1/\dots/i_\ell}$-modules such that

$$(\bigotimes_k x_k \,|\, \bigotimes_k y_k)_{i_1/\dots/i_\ell} = \bigotimes_k (x_k \,|\, y_k), \quad \forall x_k, y_k \in \mathbf{W}_{T_{\omega_{i_k}}},$$

and the pairing $(\,|\,)$ on each factor is as in 6.3. As in 7.1 let $\partial(\,|\,)_{i_1/\dots/i_\ell}$ be the corresponding pairing

$$\hat{\bigotimes}_k \mathbf{W}_{T_{\omega_{i_k}}} \times \hat{\bigotimes}_k \mathbf{W}_{T_{\omega_{i_k}}} \to \mathbb{Z}((q^{-1})).$$

We have an isomorphism of $\mathbf{U}$-modules $\mathbf{W}_{T_{\omega_i}} \simeq V(\omega_i)$ such that $1_{\omega_i} \mapsto v_{\omega_i}$; see Theorem 7.2.A.1. By Theorem 7.2.A.2 we have

$$\pm \bigotimes_k \mathbf{B}(\omega_{i_k}) + \hat{\bigotimes}_k q^{-1} L(\omega_{i_k}) = \{\mathbf{b} \in \hat{\bigotimes}_k \mathbf{W}_{T_{\omega_{i_k}}} \,|\, \partial(\mathbf{b}|\mathbf{b})_{i_1/\dots/i_\ell} = 1 + q^{-1}\mathbb{Z}[[q^{-1}]]\}.$$

Thus, by [9, Theorem 8.5 and Proposition 8.6] the set

$$\{\mathbf{b} \in \mathbf{N} \,|\, c^{\mathrm{nor}}(\mathbf{b}) = \mathbf{b},\, \partial(\mathbf{b}|\mathbf{b})_{i_1/\dots/i_\ell} = 1 + q^{-1}\mathbb{Z}[q^{-1}]\}$$

is a signed basis of $\mathbf{N}$. Iterating $(\ell-1)$-times the map in Lemma 6.2.2 we get an isomorphism of $\mathbf{U}$-modules

$$\varpi_{i_1/\dots/i_\ell} : \hat{\bigotimes}_k \mathbf{W}_{T_{\omega_{i_k}}} \xrightarrow{\sim} \hat{\mathbf{R}}_{i_1/\dots/i_\ell} \otimes_{\mathbf{R}^{\tilde{T}_\lambda}} \mathbf{W}_{T_\lambda}, \quad v_{\omega_1} \otimes \cdots v_{\omega_\ell} \mapsto 1_\lambda.$$

The $\mathbf{U}$-module $\mathbf{W}_{T_\lambda}$ is generated by $\mathbf{R}^{T_\lambda} \otimes 1_\lambda$ by Lemma 5.5.2. Thus $\varpi_{i_1/\dots/i_\ell}(\mathbf{N}) = \mathbf{W}_{T_\lambda}$. By [7] the pairing $(\,|\,)_{i_1/\dots/i_\ell}$ still satisfies Proposition 6.3.3 with 1_λ instead of $v_{i_1,\dots,i_\ell}$. It is easy to see that Proposition 6.3.2 also holds in this setting. Thus, by Proposition 6.3.4 the pairings $(\,|\,)_{i_1/\dots/i_\ell}$ and $(\,|\,)$ on $\mathbf{N}$ and $\mathbf{W}_{T_\lambda}$ coincide. Moreover, Proposition 6.1 and Corollary 5.8.2 give $\varpi_{i_1/\dots/i_\ell} c^{\mathrm{nor}} = \beta_\lambda \varpi_{i_1/\dots/i_\ell}$. Thus,

$$\begin{aligned}\varpi_{i_1/\dots/i_\ell}\big(\{\mathbf{b} \in \mathbf{N} \,|\, c^{\mathrm{nor}}(\mathbf{b}) = \mathbf{b},\, \partial(\mathbf{b}|\mathbf{b})_{i_1/\dots/i_\ell} = 1 + q^{-1}\mathbb{Z}[q^{-1}]\}\big) \\ = \{\mathbf{b} \in \mathbf{W}_{T_\lambda} \,|\, \beta_\lambda(\mathbf{b}) = \mathbf{b},\, \partial(\mathbf{b}|\mathbf{b}) \in 1 + q^{-1}\mathbb{Z}[q^{-1}]\}.\end{aligned}$$

In particular, $\mathcal{B}_{T_\lambda}$ is a signed basis of $\mathbf{W}_{T_\lambda}$ such that

$$\partial(\mathbf{b}|\mathbf{b}') \in \delta_{\mathbf{b},\mathbf{b}'} + q^{-1}\mathbb{Z}[q^{-1}] \tag{7.2.11}$$

for all $\mathbf{b}, \mathbf{b}' \in \mathcal{B}_{T_\lambda}$. By Proposition 6.3.6 there is a signed basis $\mathcal{B}^*_{T_\lambda} \subset \mathbf{W}'_{T_\lambda}$ dual to $\mathcal{B}_{T_\lambda}$ with respect to the pairing $\partial(\,||\,)$. Let $\mathbf{b}^* \in \mathcal{B}^*_{T_\lambda}$ be the element dual to $\mathbf{b} \in \mathcal{B}_{T_\lambda}$. Let $E : \mathbf{W}_{T_\lambda} \to \mathbf{W}'_{T_\lambda}$ be the unique $\mathbb{A}$-modules isomorphism such that $E(\mathbf{b}) = \mathbf{b}^*$ for all $\mathbf{b} \in \mathcal{B}_{T_\lambda}$. Using (7.2.11) we get

$$(\kappa_* - E)\big(\textstyle\bigoplus_{\mathbf{b}\in\mathcal{B}_{T_\lambda}} \mathbb{Z}[q^{-1}]\mathbf{b}\big) \subset \textstyle\bigoplus_{\mathbf{b}\in\mathcal{B}_{T_\lambda}} q^{-1}\mathbb{Z}[q^{-1}]\mathbf{b}^*.$$

Thus, the map $\kappa_* : \hat{\mathbf{W}}_{T_\lambda} \to \hat{\mathbf{W}}'_{T_\lambda}$ is invertible (see also Lemma 6.2.1) and we have

$$(\kappa_*)^{-1} = \sum_{n\geq 0} (-1)^n E^{-1}\big((\kappa_* - E)E^{-1}\big)^n.$$

In particular, we get

$$\partial\big((\kappa_*)^{-1}(\mathbf{b}^*)|\mathbf{b}'^*\big) \in \delta_{\mathbf{b},\mathbf{b}'} + q^{-1}\mathbb{Z}[[q^{-1}]],$$

i.e.,

$$\partial(\mathbf{b}^*|\mathbf{b}'^*)' \in \delta_{\mathbf{b},\mathbf{b}'} + q^{-1}\mathbb{Z}[[q^{-1}]],$$

for all $\mathbf{b}, \mathbf{b}' \in \mathcal{B}_{T_\lambda}$. Using Proposition 6.3.5 we get also $\beta'_\lambda(\mathbf{b}^*) = \mathbf{b}^*$ for all $\mathbf{b} \in \mathcal{B}_{T_\lambda}$. Thus $\mathcal{B}^*_{T_\lambda} \subset \mathcal{B}'_{T_\lambda}$. Then, apply Proposition 7.1.1. We are done. □

Remarks.

(1) It is easy to see that $\mathbf{X}^{T_\lambda}$ is a subgroup of the multiplicative group of $\mathbf{R}^{\tilde{T}_\lambda}$ such that $\mathbf{X}^{T_\lambda}\,\mathcal{B}_{T_\lambda} = \mathcal{B}_{T_\lambda}$. Thus there is a subset $\mathbf{B}^0_{T_\lambda} \subset \mathcal{B}_{T_\lambda}$ which is an $\mathbf{R}^{\tilde{T}_\lambda}$-basis of $\mathbf{W}_{T_\lambda}$. In particular, for any maximal ideal $I \subset \mathbf{R}^{\tilde{T}_\lambda}$ the set $\mathbf{B}^0_{T_\lambda} \otimes 1$ is a $\mathbb{Z}$-basis of $\mathbf{W}_{T_\lambda} \otimes_{\mathbf{R}^{\tilde{T}_\lambda}} (\mathbf{R}^{\tilde{T}_\lambda}/I)$. If $\dagger(I) = I$, then the involution β_{T_λ} and the metric $\partial(\,|\,)$ descend to $\mathbf{W}_{T_\lambda} \otimes_{\mathbf{R}^{\tilde{T}_\lambda}} (\mathbf{R}^{\tilde{T}_\lambda}/I)$. It is not clear if $\pm\mathbf{B}^0_{T_\lambda} \otimes 1$ admits a similar characterization as $\mathcal{B}_{T_\lambda}$ in 7.1.

(2) Probably the sets $\mathcal{B}_\lambda$, $\mathcal{B}'_\lambda$ are signed bases of $\mathbf{W}_\lambda$, $\mathbf{W}'_\lambda$. The conjectures in [9, §13] and the previous theorem suggest that Kashiwara's canonical basis of $V(\lambda)$ coincide with $\mathcal{B}_\lambda$, $\mathcal{B}'_\lambda$, up to signs.

7.3. We do not assume any more that $\mathfrak{g}$ is simply laced. Fix $i \in I$. The fundamental module $W(\omega_i)$ is as in [9]. Let $W(\omega_i)[z^{\pm 1}]$ be the affinized module; see [9, §4.2]. Set $d_i = \max(1, (\alpha_i, \alpha_i)/2)$. Then $V(\omega_i)$ is isomorphic to the $\mathbf{U}$-submodule $W(\omega_i)[z^{\pm d_i}] \subset W(\omega_i)[z^{\pm 1}]$; see [9, Theorem 5.15.$(viii)$]. Set $z_i = z^{d_i}\mathrm{Id} : V(\omega_i) \to V(\omega_i)$. Let us mention the following fact, which is not used in the paper.

Proposition. *For any $\mathfrak{g}$ (not necessarily simply laced) and any $i \in I$ there is a unique pairing of $\mathbb{A}$-modules $\langle\,|\,\rangle : V(\omega_i) \times V(\omega_i) \to \mathbb{A}$ such that*

$$\langle z_i^n v_{\omega_i} | z_i^m v_{\omega_i}\rangle = \delta_{n,m}, \quad \langle u \cdot x | y\rangle = \langle x | \psi(u) \cdot y\rangle.$$

This pairing is perfect and symmetric.

Proof. The proof is similar to the proof of [13, Proposition 19.1.2]. By [9, Proposition 5.14.(iii)] the space $V(\omega_i)_{\hat{\mu}}$ is finite-dimensional for any $\hat{\mu} \in \hat{P}$. Set

$$V(\omega_i)^* = \bigoplus_{\hat{\mu}\in\hat{P}} \mathrm{Hom}\,(V(\omega_i)_{\hat{\mu}}, \mathbb{A}).$$

Since ψ is an anti-automorphism, there is a unique $\mathbf{U}$-module structure on $V(\omega_i)^*$ such that

$$(u \cdot f)(x) = f(\psi(u) \cdot x), \quad \forall u \in \mathbf{U}, \forall x \in V(\omega_i). \tag{7.3.1}$$

The $\mathbf{U}$-module $V(\omega_i)^*$ is endowed with the $\hat{P}$-grading such that

$$V(\omega_i)^*_{\mu+n\delta} = \mathrm{Hom}\,(V(\omega_i)_{\mu+n\delta}, \mathbb{A}).$$

Recall that $V(\omega_i)_{\omega_i} = \mathbb{A} \cdot v_{\omega_i}$. Let $f_{\omega_i} \in V(\omega_i)^*$ be the unique linear form such that $f_{\omega_i}(v_{\omega_i}) = 1$, and $f_{\omega_i}(v) = 0$ for all $v \in V(\omega_i)_{\hat{\mu}}$ with $\hat{\mu} \neq \omega_i$. Hence, $f_{\omega_i} \in V(\omega_i)^*_{\omega_i}$. We must prove that there is a unique morphism of $\mathbf{U}$-modules $V(\omega_i) \to V(\omega_i)^*$ which takes v_{ω_i} to f_{ω_i}, and that it is invertible. The spaces $V(\omega_i)_{\hat{\mu}}$, $V(\omega_i)^*_{\hat{\mu}}$ have the same dimension for all $\hat{\mu}$. Thus the set of the weights $\mu \in P$ such that $V(\omega_i)^*_{\mu+n\delta} \neq \{0\}$ for some $n \in \mathbb{Z}$ is contained in $\omega_i - \sum_{j\in I} \mathbb{N}\,\alpha_j$, since this is true for $V(\omega_i)$. Hence f_{ω_i} is an extremal vector of weight ω_i; see [9, Theorem 5.3]. By the universal property of $V(\omega_i)$, there is a unique morphism of $\mathbf{U}$-modules $\phi : V(\omega_i) \to \mathbf{U} \cdot f_{\omega_i} \subseteq V(\omega_i)^*$ which takes v_{ω_i} to f_{ω_i}. Moreover, we have $\phi(V(\omega_i)_{\hat{\mu}}) \subseteq V(\omega_i)^*_{\hat{\mu}}$ for all $\hat{\mu} \in \hat{P}$. Since $V(\omega_i)_{\hat{\mu}}$ is finite dimensional it is sufficient to prove that the map ϕ is injective. Let the operator z_i acts on $V(\omega_i)^*$ by

$$(z_i \cdot f)(x) = f(z_i^{-1} \cdot x), \quad \forall x \in V(\omega_i), \forall f \in V(\omega_i)^*.$$

Then ϕ commutes to z_i. Since $W(\omega_i) \simeq V(\omega_i)/(z_i - 1)V(\omega_i)$, the map ϕ induces a nonzero morphism of $\mathbf{U}$-modules $W(\omega_i) \to W(\omega_i)$. It is injective since $W(\omega_i)$ is simple. Thus ϕ is injective.

The pairing is symmetric because it is unique and $\psi^2 = \mathrm{Id}$. □

8. Example

We assume that $\Pi = (\{1\}, \emptyset)$. We set $\lambda = \ell\omega_1$, $\alpha = a\alpha_1$. To simplify, we omit the subscripts 1 and we set $Q_{\ell a} = Q_{\lambda\alpha}$, $d_{\ell a} = d_{\lambda\alpha}$, etc. Set

$$\tilde{x}_r^+ = \sum_{a'=a+1} (-1)^{\ell-a'} q^{-2r-a'} (\textstyle\bigwedge_{\mathcal{V}}^{-r-\ell+a'} \boxtimes \bigwedge_{\mathcal{V}}^{r+\ell-a}) \otimes \bigwedge_{\mathcal{W}}^{-1} \otimes 1_{\ell a a'},$$

$$\tilde{x}_r^- = \sum_{a'=a-1} (-1)^{a'} q^{-2r+a'} (\textstyle\bigwedge_{\mathcal{V}}^{r-a'} \boxtimes \bigwedge_{\mathcal{V}}^{-r+a}) \otimes 1_{\ell a a'}.$$

We have $Q_{\ell a} \neq \emptyset$ if and only if $0 \leq a \leq \ell$. More precisely, $F_{\ell a}$ is smooth and isomorphic the Grassmanian of a-dimensional subspaces in $\mathbb{C}^\ell$, and $Q_{\ell a} = T^*F_{\ell a}$. An element in $Q_{\ell a}$ may be viewed as a couple (V, u), where $V \subseteq \mathbb{C}^\ell$ is an a-dimensional subspace, and $u \in \mathrm{End}\,(\mathbb{C}^\ell)$ is a nilpotent map such that $\mathrm{Im}\, u \subseteq V \subseteq \mathrm{Ker}\, u$. The element $z \in \mathbb{C}^*$ acts on $T^*F_{\ell a}$ by multiplication by the scalar z^2 along the fibers. The automorphism $\omega : Q_{\ell a} \to Q_{\ell,\ell-a}$ takes the pair (V, u) to $(V^\perp, {}^t u)$, where $V^\perp \subseteq \mathbb{C}^\ell$ is the subspace orthogonal to V, with respect to the canonical scalar product on $\mathbb{C}^\ell$. Let $\mathcal{E}'_a$ be the tautological rank a vector bundle on $T^*F_{\ell a}$. Let $\mathcal{E}_a$ be the restriction of $\mathcal{E}'_a$ to $F_{\ell a}$. Set $\mathcal{Q}'_a = \mathcal{W}_a/\mathcal{E}'_a$, $\mathcal{Q}_a = \mathcal{W}_a/\mathcal{E}_a$. Set $\mathcal{E} = \bigoplus_a \mathcal{E}_a$, $\mathcal{E}' = \bigoplus_a \mathcal{E}'_a$, etc. We have

$$\mathcal{V} = q\mathcal{E}', \quad \tilde{x}^+_r = \textstyle\bigwedge^{-\ell}_{\mathcal{E}'} x^+_r \bigwedge^{-\ell}_{\mathcal{W}} \bigwedge^{\ell}_{\mathcal{E}'}, \quad \tilde{x}^-_r = \bigwedge^{-\ell}_{\mathcal{E}'} x^-_r \bigwedge^{\ell}_{\mathcal{W}} \bigwedge^{\ell}_{\mathcal{E}'}.$$

Hereafter, we omit the operators κ_* and $\otimes$. The following lemma is immediate; see, for instance, [22].

Lemma.

(1) *We have* $\tilde{x}^+_0(1_{\ell a}) = [\ell - a + 1]1_{\ell,a-1}$, $\tilde{x}^-_0(1_{\ell a}) = [a+1]1_{\ell,a+1}$.
(2) *We have* $1_{\ell a} = \sum_{i=0}^{a(\ell-a)} (-1)^i q^{-2i} \wedge^i \left(\mathcal{Q}'_a \mathcal{E}'^*_a\right)$.

A direct computation gives

Proposition.

(1) *We have* $\mathcal{B}'_1 = \mathcal{B}_1 = \pm\{x1_{10}, x\bigwedge^{-1}_{\mathcal{W}} 1_{11}; x \in \mathbf{X}_1\}$.
(2) *We have* $\mathcal{B}'_2 = \pm\{x1'_{20}, x\bigwedge^{-2}_{\mathcal{W}} 1'_{21}, q^{-1}x\bigwedge^{-2}_{\mathcal{W}} \mathcal{Q}'_{21}, x\bigwedge^{-4}_{\mathcal{W}} 1'_{22}; x \in \bigwedge^2_{\mathcal{E}'_2} \otimes \mathbf{X}_2\}$, $\mathcal{B}_2 = \pm\{x1_{20}, x\bigwedge^{-2}_{\mathcal{W}} 1_{21}, q^{-1}x\bigwedge^{-2}_{\mathcal{W}} \mathcal{E}_{21}, x\bigwedge^{-4}_{\mathcal{W}} 1_{22}; x \in \bigwedge^2_{\mathcal{E}_2} \otimes \mathbf{X}_2\}$.

Appendix

We check that $x(w_0 * \alpha) = x(\alpha)$ and $y(w_0 * \alpha) = y(\alpha)$, for all $\alpha \in Q$, where $x : Q \to \mathbb{Z}$ and $y : Q \to \mathbb{Z}/2\mathbb{Z}$ are the quadratic maps that satisfy (5.7.1). We give the proof for the map x; the case of the map y is left to the reader. Set

$$\alpha = \sum_j a_j \alpha_j, \quad \lambda = \sum_j \ell_j \omega_j, \quad \nu = \lambda - w_0(\lambda) = \sum_j k_j \alpha_j.$$

Write $x(\alpha) = Q(\alpha) + L(\alpha) + a$, where Q is a quadratic form, L is a linear form and $a \in \mathbb{Z}$. Put

$$Q(\alpha) = \sum_{i,j} q_{ij} a_i a_j, \quad L(\alpha) = \sum_j b_j a_j.$$

A direct computation gives $g_{i;\alpha} = c_i + \sum_j c_{ij} a_j$ with

$$c_i = -1 + (2 - \mathbf{c})\ell_i + (2\mathbf{c} - 1)k_i, \quad c_{ij} = (\mathbf{c} - 2)a_{ij} + \delta_{ij}(2 - 2\mathbf{c}).$$

Using the relation $x(\alpha + \alpha_i) - x(\alpha) = \mathbf{c}k_i - g_{i;\alpha}$, we get

$$\begin{aligned} b_i &= \mathbf{c}k_i - c_i + \frac{1}{2}c_{ii} = (\mathbf{c} - 2)\ell_i + (1 - \mathbf{c})k_i, \\ q_{ij} &= -\frac{1}{2}c_{ij} = (1 - \frac{\mathbf{c}}{2})a_{ij} + \delta_{ij}(\mathbf{c} - 1); \end{aligned} \tag{1}$$

that is,

$$Q(\alpha) = (\mathbf{c} - 1)|\alpha|^2 + (1 - \frac{\mathbf{c}}{2})(\alpha, \alpha), \quad L(\alpha) = (\mathbf{c} - 2)(\lambda, \alpha) + (1 - \mathbf{c}) \sum_j k_j a_j. \tag{2}$$

The identity $Q(\nu) + L(\nu) + a = x(\nu) = (\mathbf{c} - 1)|\nu|^2 - \mathbf{c}(\lambda, \lambda)$, gives

$$a = -\mathbf{c}(\lambda,\lambda) + (\frac{\mathbf{c}}{2} - 1)(\nu,\nu) - (\mathbf{c}-2)(\lambda,\nu) + (\mathbf{c}-1)|\nu|^2 = (\mathbf{c}-1)|\nu|^2 - \mathbf{c}(\lambda,\lambda),$$

since $2(\lambda, \nu) = (\nu, \nu)$. In particular,

$$a = x(0) = x(\nu).$$

Now we want to check that $x(w_0 * \alpha) = x(\alpha)$. Since $x(\nu) = x(0)$, we have $Q(\nu) + L(\nu) = 0$. Moreover, (2) gives $Q(\alpha) = Q(w_0(\alpha))$. Thus it is enough to prove that

$$L(w_0(\alpha)) - L(\alpha) = 2\sum_{i,j} k_i q_{ij} a_{\underline{j}}.$$

By (2), the left-hand side is equal to

$$(2-\mathbf{c})(\nu,\alpha) + (\mathbf{c}-1)\sum_j k_j(a_j + a_{\underline{j}}).$$

By (1), the right-hand side is equal to

$$(2-\mathbf{c})(\nu,\alpha) + 2(\mathbf{c}-1)\sum_j k_j a_{\underline{j}}.$$

Since $w_0(\nu) = -\nu$, we have $k_j = k_{\underline{j}}$. We are done.

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