

**t -ANALOGS OF q -CHARACTERS OF KIRILLOV-RESHETIKHIN
 MODULES OF QUANTUM AFFINE ALGEBRAS**

HIRAKU NAKAJIMA

Dedicated to Professor Takushiro Ochiai on his sixtieth birthday

ABSTRACT. We prove the Kirillov-Reshetikhin conjecture concerning certain finite dimensional representations of a quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$ when $\widehat{\mathfrak{g}}$ is an untwisted affine Lie algebra of type ADE . We use t -analog of q -characters introduced by the author in an essential way.

1. INTRODUCTION

Let \mathfrak{g} be a complex simple Lie algebra of type ADE , $\mathbf{Lg} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ its loop algebra, and $\mathbf{U}_q(\mathbf{Lg})$ its quantum universal enveloping algebra, or the quantum loop algebra for short. It is a subquotient of the quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$, i.e., without central extension and degree operator. It is customary to define $\mathbf{U}_q(\mathbf{Lg})$ as an algebra over $\mathbb{Q}(q)$, but here we consider q as a nonzero complex number which is not a root of unity. Let α_i and Λ_i ($i \in I$) be simple roots and fundamental roots of \mathfrak{g} , where I is the index set. Let $(a_{ij})_{i,j \in I}$ be the Cartan matrix. By Drinfeld [3], Chari-Pressley [2], isomorphism classes of irreducible finite dimension representations of $\mathbf{U}_q(\mathbf{Lg})$ are parametrized by I -tuples of polynomials $P = (P_i(u))_{i \in I}$ with normalization $P_i(0) = 1$. They are called *Drinfeld polynomials*. Let us denote by $L(P)$ the corresponding irreducible representation.

For $i \in I$, $k \in \mathbb{Z}_{\geq 0}$, $a \in \mathbb{C}^*$, let

$$P_{k,a}^{(i)} = \left((P_{k,a}^{(i)})_j(u) \right)_{j \in I}; \quad (P_{k,a}^{(i)})_j(u) \stackrel{\text{def.}}{=} \begin{cases} \prod_{s=1}^k (1 - aq^{2s-2}u) & \text{if } j = i, \\ 1 & \text{otherwise.} \end{cases}$$

Let $W_{k,a}^{(i)} \stackrel{\text{def.}}{=} L(P_{k,a}^{(i)})$ be the irreducible finite dimensional representation with Drinfeld polynomial $P_{k,a}^{(i)}$. This particular class of representations are called *Kirillov-Reshetikhin modules*. They were introduced in [8] and have been studied intensively by a motivation from the so-called Bethe ansatz (see [11, §5.7] and the references therein). Our first goal is to prove the following recursion formula, called the T -system, which was conjectured by Kuniba-Nakanishi-Suzuki [10], and a convergence property:

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Theorem 1.1. (1) *There exists an exact sequence*

$$0 \rightarrow \bigotimes_{j:a_{ij}=-1} W_{k,aq^j}^{(j)} \rightarrow W_{k,a}^{(i)} \otimes W_{k,aq^2}^{(i)} \rightarrow W_{k+1,a}^{(i)} \otimes W_{k-1,aq^2}^{(i)} \rightarrow 0 \quad (k = 1, 2, \dots).$$

Moreover, the first and third terms are irreducible. Here we have the following convention: we introduce some ordering among j 's such that $a_{ij} = -1$ in order to define the tensor product of the first term. We set $W_{k,a}^{(i)}$ to be the trivial representation if $k = 0$. If \mathfrak{g} is of type A_1 , the first term of the exact sequence is understood as the trivial representation.

(2) *The normalized q -character of $W_{k,a}^{(i)}$, considered as a polynomial in $A_{i,a}^{-1}$, has a limit as a formal power series:*

$$\exists \lim_{k \rightarrow \infty} \frac{\chi_q(W_{k,a}^{(i)})}{Y_{i,a} Y_{i,aq^2} \dots Y_{i,aq^{2k-2}}} \in \mathbb{Z}[[A_{i,aq^m}^{-1}]]_{i \in I, m \in \mathbb{Z}_{>0}}.$$

The definition of χ_q will be recalled in §2. Taking the χ_q of the exact sequence, we get

$$\begin{aligned} \chi_q(W_{k,a}^{(i)})\chi_q(W_{k,aq^2}^{(i)}) &= \chi_q(W_{k+1,a}^{(i)})\chi_q(W_{k-1,aq^2}^{(i)}) \\ &+ \prod_{j:a_{ij}=-1} \chi_q(W_{k,aq^j}^{(j)}) \quad (k = 1, 2, \dots). \end{aligned}$$

This is the original T -system. The T -system is a recursion formula. In particular, we can determine all $\chi_q(W_{k,a}^{(i)})$ inductively, once we know $\chi_q(W_{1,a}^{(i)})$ for any $i \in I$. These representations $W_{1,a}^{(i)}$, i.e., Kirillov-Reshetikhin modules with $k = 1$, are called l -fundamental representations (or simply “fundamental representations”). They are basic building blocks in the representation theory of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$, since any irreducible representations are subquotients of their tensor products. If \mathfrak{g} is of type A_n , they are lifts of the representation of fundamental representations of $\mathbf{U}_q(\mathfrak{g})$. For general \mathfrak{g} , there is a combinatorial algorithm to compute them due to Frenkel-Mukhin [6]. But the algorithm is too complicated and the author’s computer program, written by C, did not give answers for two fundamental representations of E_8 so far. For type A_n, D_n see [16]. Even if we can compute $\chi_{q,t}(W_{1,a}^{(i)})$ for the special node i of E_8 , it is a polynomial consisting of 6899079264 monomials. It is practically impossible to compute even $\chi_{q,t}(W_{2,a}^{(i)})$ by the recursion.

However, we can deduce weaker information for $W_{k,a}^{(i)}$ as follows. The quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ contains the finite dimensional quantum enveloping algebra $\mathbf{U}_q(\mathfrak{g})$ as a subalgebra. Let Res be the functor sending $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ -modules to $\mathbf{U}_q(\mathfrak{g})$ -modules by restriction. We set $Q_k^{(i)} \stackrel{\text{def.}}{=} \text{Res } W_{k,a}^{(i)}$. It is known that it is independent of a .

Theorem 1.1 has the following specialization:

$$(1.2) \quad Q_k^{(i)} \otimes Q_k^{(i)} = \left(Q_{k+1}^{(i)} \otimes Q_{k-1}^{(i)} \right) \oplus \bigotimes_{j:a_{ij}=-1} Q_k^{(j)} \quad (k = 1, 2, \dots)$$

and

$$\exists \lim_{k \rightarrow \infty} e^{-k\Lambda_i} \chi(Q_k^{(i)}) \in \mathbb{Z}[[e^{-\alpha_i}]]_{i \in I}.$$

The equation (1.2) is called a Q -system. It also has a recursive structure, and we do not know the initial term. However, Hatayama, Kuniba, Okado, Takagi, and

Yamada [7] showed that the solution of the Q -system with the above convergence property, which is a sum of characters of representations of \mathfrak{g} , has the following explicit expression. (See also [11] for more general results in this direction.)

Corollary 1.3 (Kirillov-Reshetikhin conjecture). *Let $\mathcal{Q}_k^{(i)} \stackrel{\text{def.}}{=} e^{-k\Lambda_i} \chi(Q_k^{(i)})$ be the normalized character of $Q_k^{(i)}$. For a sequence $\nu = (\nu_k^{(i)})_{i \in I, k \in \mathbb{Z}_{>0}}$ such that all but finitely many $\nu_k^{(i)}$ are zero, we set $\mathcal{Q}^\nu \stackrel{\text{def.}}{=} \prod_{i,k} (\mathcal{Q}_k^{(i)})^{\nu_k^{(i)}}$. Then we have*

$$\mathcal{Q}^\nu \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha}) = \sum_{N=(N_k^{(i)})} \prod_{i \in I, k \in \mathbb{Z}_{>0}} \binom{P_k^{(i)}(\nu, N) + N_k^{(i)}}{N_k^{(i)}} e^{-kN_k^{(i)}\alpha_i},$$

$$P_k^{(i)}(\nu, N) \stackrel{\text{def.}}{=} \sum_{l=1}^{\infty} \nu_l^{(i)} \min(k, l) - \sum_{j \in I} a_{ij} \sum_{l \in \mathbb{Z}_{>0}} N_l^{(j)} \min(k, l),$$

where Δ_+ is the set of the positive roots, and $\binom{a}{b} = \Gamma(a + 1)/\Gamma(a - b + 1)\Gamma(b + 1)$.

Corollary 1.3 and equation (1.2) were stated in [8], but proofs were not provided. Then the paper became the starting point of the subject. See [11, 5.7] for the status of the conjecture in the last year.

For a geometric side, it gives us an explicit formula of the Euler number (but not of Betti numbers) of arbitrary quiver varieties of type ADE . The formula is slightly different from the specialization of an explicit formula for Betti numbers, which was conjectured by Lusztig [12] in connection with the fermionic formula. The difference is the definition of binomial coefficients. His (q -analog of) $\binom{a}{b}$ is set 0 if $b > a$. The equivalence between two formulae are not known so far. (See [11, Remark 1.3] for details.)

For the proofs of the above results, we use “ t -analogs of q -characters”, introduced by the author [14, 15] by using quiver varieties. They were defined for arbitrary representations of $\mathbf{U}_q(\mathbf{Lg})$ and there are combinatorial algorithms to compute them for arbitrary irreducible representations. Therefore, we can, in principle, compute many things, such as tensor product decompositions, etc. (The original q -characters were introduced and studied by Knight, Frenkel and Reshetikhin, and Frenkel and Mukhin [9, 5, 6].) A new point in this paper is that the algorithms are drastically simplified, when they are applied to Kirillov-Reshetikhin modules. In [15, §10] it was conjectured that a geometric reason for this simplification was *smallness* of certain natural morphisms between two graded quiver varieties. We do not prove this conjecture, but find that checking the smallness only for few cases is enough to derive the T -system.

It is highly desirable to have a characterization of the solution of T -system and its t -analog, similar to the result of Kuniba-Nakanishi-Tsuboi [11]. Such a result should have many applications to the representation theory of $\mathbf{U}_q(\mathbf{Lg})$, according to the importance of t -analogs of q -characters.

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2. REVIEW ON t -ANALOGS OF q -CHARACTERS

In this section we recall the theory of t -analogs of q -characters [14, 15], which we will use later. We give their properties, axiomized in [15], so that we do not need to go back to the original definitions. But in order to understand a flavor of the theory, we first briefly summarize results of [5, 6] and [13, 14, 15]. This part is expository. See [14] for a more detailed survey.

Let \mathbf{R} be the Grothendieck ring of the category of finite dimensional representations of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$. The class $\{L(P)\}_P$ of irreducible finite dimensional representations is a basis of \mathbf{R} . The q -character $\chi_q(V)$ [9, 5, 6] is the generating function of l -weight multiplicities of a representation V . Here an l -weight space of V is a simultaneous generalized eigenspace with respect to the commutative subalgebra of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ which corresponds to $\mathbf{U}(\mathfrak{h} \otimes \mathbb{C}[z, z^{-1}])$ at $q = 1$. (\mathfrak{h} is the Cartan subalgebra of \mathfrak{g} .) The corresponding eigenvalues, which we call l -weights, are parametrized by an I -tuple of rational functions $Q/R = (Q_i(u)/R_i(u))_{i \in I}$ with normalization $Q_i(0) = R_i(0) = 1$ [5]. We say an l -weight is l -dominant when $Q_i(u)/R_i(u)$ are polynomials for all i . These concepts are natural analogs of the corresponding concepts without “ l ” for finite dimensional representations of \mathfrak{g} . That is, weight spaces are simultaneous eigenspaces of \mathfrak{h} , weights are their eigenvalues. Irreducible representations are parametrized by dominant weights, and so on. There is a natural analog of the dominance order on weights for l -weights. It will be denoted by $<$.

As in the case of finite dimensional representations of \mathfrak{g} , the q -characters define a ring homomorphism (see [5] for the proof)

$$\chi_q: \mathbf{R} \rightarrow \mathcal{Y} \stackrel{\text{def.}}{=} \mathbb{Z}[Y_{i,a}, Y_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^*},$$

where \mathcal{Y} is the Laurent polynomial ring of uncountably many variables $Y_{i,a}$'s ($i \in I$, $a \in \mathbb{C}^*$). Moreover, this is injective. Then, Frenkel and Mukhin gave a characterization of the image of χ_q , which gives a combinatorial algorithm to compute $\chi_q(L(P))$ when $L(P)$ is an l -fundamental representation [6].

Around the same time, the author introduced another family of finite dimensional representations $\{M(P)\}$ of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$, also parametrized by Drinfeld polynomial P [13]. They are called *standard modules*. Those $M(P)$ and irreducible representations $L(P)$ naturally define classes in the t -analog of the representation ring $\mathbf{R}_t \stackrel{\text{def.}}{=} \mathbf{R} \otimes \mathbb{Z}[t, t^{-1}]$. We denote them by the same symbol. There is a bar involution $\overline{}$ on \mathbf{R}_t . Then a main result of [13] (although stated in a different form there) is the following characterization of the class $L(P)$:

$$(2.1) \quad \overline{L(P)} = L(P), \quad L(P) \in M(P) + \sum_{Q: Q < P} t^{-1} \mathbb{Z}[t^{-1}] M(Q).$$

This is very similar to definitions of Kazhdan-Lusztig bases of Hecke algebras, canonical bases of quantum enveloping algebras for type ADE , etc. As these bases, this characterization gives us an algorithm to compute the transition matrix between two bases $\{L(P)\}, \{M(P)\}$ of \mathbf{R}_t , once we compute $\overline{M(P)}$.

In order to compute $\overline{M(P)}$, the author introduced the t -analogs of q -characters, which gives a $\mathbb{Z}[t, t^{-1}]$ -linear homomorphism

$$\chi_{q,t}: \mathbf{R}_t \rightarrow \mathcal{Y}_t,$$

where $\mathcal{Y}_t \stackrel{\text{def.}}{=} \mathbb{Z}[t, t^{-1}, Y_{i,a}, Y_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^*}$. This is injective again and the image has a similar characterization as χ_q . In particular, we have a combinatorial algorithm

to compute them for l -fundamental representations. Although $\chi_{q,t}$ is not a ring homomorphism, we have a simple “twist” of multiplication so that we can compute $\chi_{q,t}(M(P))$ for arbitrary P , which are tensor products of l -fundamental representations [17]. The bar involution on $\chi_{q,t}$ is easily defined so that $\overline{M(P)}$ can be computed, if we know all values of $\chi_{q,t}(M(P))$. Combining with the above characterization of $L(P)$, we have a combinatorial algorithm to compute the transition matrix, although it is difficult to compute them explicitly, as we mentioned in the introduction. This is the end of our exposition.

Now we fix notations. We use symbols $Y_{i,a}$ following [5]. It corresponds to the character of the fundamental weight $y_i = e^{\Lambda_i}$ in the above analogy. Let \mathcal{M} be the set of monomials in \mathcal{Y} . For an I -tuple of rational functions $Q/R = (Q_i(u)/R_i(u))_{i \in I}$ with $Q_i(0) = R_i(0) = 1$, we set

$$e^{Q/R} \stackrel{\text{def.}}{=} \prod_{i \in I} \prod_{\alpha} \prod_{\beta} Y_{i,\alpha} Y_{i,\beta}^{-1},$$

where α (resp. β) runs roots of $Q_i(1/u) = 0$ (resp. $R_i(1/u) = 0$), i.e., $Q_i(u) = \prod_{\alpha} (1 - \alpha u)$ (resp. $R_i(u) = \prod_{\beta} (1 - \beta u)$). As a special case, an I -tuple of polynomials $P = (P_i(u))_{i \in I}$ defines $e^P = e^{P/1}$. In this way, the set \mathcal{M} of monomials are identified with the set of I -tuples of rational functions, and the set of l -dominant monomials are identified with the set of I -tuples of polynomials. A *monomial* in \mathcal{Y}_t means a monomial only in $Y_{i,a}^{\pm}$, containing no t 's. So they are the same as monomials in \mathcal{Y} . We consider $\mathbb{Z}[t, t^{-1}]$ as coefficients.

Let

$$A_{i,a} \stackrel{\text{def.}}{=} Y_{i,aq} Y_{i,aq^{-1}} \prod_{j:j \neq i} Y_{j,a}^{a_{ij}},$$

where a_{ij} is the (i, j) -entry of the Cartan matrix. (This is different from the original definition in [5] for general \mathfrak{g} . But it is the same when \mathfrak{g} is of type ADE .)

Definition 2.2. (1) For a monomial $m \in \mathcal{M}$, we define $u_{i,a}(m) \in \mathbb{Z}$ to be the degree in $Y_{i,a}$, i.e.,

$$m = \prod_{i,a} Y_{i,a}^{u_{i,a}(m)}.$$

(2) A monomial $m \in \mathcal{M}$ is said to be i -dominant if $u_{i,a}(m) \geq 0$ for all a . It is said to be l -dominant if it is i -dominant for all i .

(3) Let m, m' be monomials in \mathcal{M} . We say $m \leq m'$ if m/m' is a monomial in $A_{i,a}^{-1}$ ($i \in I, a \in \mathbb{C}^*$). Here a monomial in $A_{i,a}^{-1}$ means a product of nonnegative powers of $A_{i,a}^{-1}$. It does not contain any factors $A_{i,a}$. In such a case we define $v_{i,a}(m, m') \in \mathbb{Z}_{\geq 0}$ by

$$m = m' \prod_{i,a} A_{i,a}^{-v_{i,a}(m, m')}.$$

This is well defined since the q -analog of the Cartan matrix is invertible. We say that $m < m'$ if $m \leq m'$ and $m \neq m'$.

(4) For an i -dominant monomial $m \in \mathcal{M}$ we define

$$E_i(m) \stackrel{\text{def.}}{=} m \prod_a \sum_{r_a=0}^{u_{i,a}(m)} t^{r_a(u_{i,a}(m) - r_a)} \begin{bmatrix} u_{i,a}(m) \\ r_a \end{bmatrix}_t A_{i,aq}^{-r_a},$$

where $\begin{bmatrix} n \\ r \end{bmatrix}_t$ is the t -binomial coefficient. We call $E_i(m)$ an *expansion* at m .

(5) Let $\mathcal{K}_{t,i}$ be the $\mathbb{Z}[t, t^{-1}]$ -linear subspace of \mathcal{Y}_t generated by $E_i(m)$ with i -dominant monomials m . Let \mathcal{K}_t be the intersection $\bigcap_i \mathcal{K}_{t,i}$.

By the identification between monomials in \mathcal{M} and I -tuples of rational functions, we also apply the above definitions to the latter. For example, $Q/R \leq Q'/R'$ means $e^{Q/R} \leq e^{Q'/R'}$.

For $m \in \mathcal{M}$, we define $(\tilde{u}_{i,a}(m))_{i \in I, a \in \mathbb{C}^*}$ as the solution of

$$u_{i,a}(m) = \tilde{u}_{i,aq^{-1}}(m) + \tilde{u}_{i,aq}(m) - \sum_{j: a_{ij} = -1} \tilde{u}_{j,a}(m).$$

To solve the system, we may assume that $u_{i,a}(m) = 0$ unless a is a power of q . Then the above is a recursive system, since q is not a root of unity. So it has a unique solution such that $\tilde{u}_{i,q^s}(m) = 0$ for sufficiently small s . Note that $\tilde{u}_{i,a}(m)$ is nonzero for possibly infinitely many a 's, although $u_{i,a}(m)$ is not.

Suppose that l -dominant monomials m_{P^1} , m_{P^2} and monomials $m^1 \leq m_{P^1}$, $m^2 \leq m_{P^2}$ are given. We define an integer $d(m^1, m_{P^1}; m^2, m_{P^2})$ by

$$\begin{aligned} & d(m^1, m_{P^1}; m^2, m_{P^2}) \\ (2.3) \quad & \stackrel{\text{def.}}{=} \sum_{i,a} (v_{i,aq}(m^1, m_{P^1})u_{i,a}(m^2) + u_{i,aq}(m_{P^1})v_{i,a}(m^2, m_{P^2})) \\ & = \sum_{i,a} (u_{i,a}(m^1)v_{i,aq^{-1}}(m^2, m_{P^2}) + v_{i,a}(m^1, m_{P^1})u_{i,aq^{-1}}(m_{P^1})). \end{aligned}$$

We also define

$$\begin{aligned} (2.4) \quad & \tilde{d}(m^1, m^2) \stackrel{\text{def.}}{=} - \sum_{i,a} u_{i,aq}(m^1)\tilde{u}_{i,a}(m^2), \\ & \varepsilon(m^1, m^2) \stackrel{\text{def.}}{=} \tilde{d}(m^1, m^2) - \tilde{d}(m^2, m^1). \end{aligned}$$

Since $u_{i,a}(m^2) = 0$, except for finitely many a 's, this is well defined. Moreover, we have

$$(2.5) \quad \varepsilon(m^1, m^2) = d(m^1, m_{P^1}; m^2, m_{P^2}) - d(m^2, m_{P^2}; m^1, m_{P^1}) + \varepsilon(m_{P^1}, m_{P^2}).$$

We also write $\varepsilon(P^1, P^2)$ instead of $\varepsilon(m_{P^1}, m_{P^2})$.

We express the property of $\chi_{q,t}$ for its slightly modified version $\widetilde{\chi}_{q,t}$.

Theorem 2.6. (1) *The $\widetilde{\chi}_{q,t}$ of a standard module $M(P)$ has a form*

$$\widetilde{\chi}_{q,t}(M(P)) = e^P + \sum a_m(t)m,$$

where the summation runs over monomials $m < e^P$.

(2) *For each $i \in I$, $\widetilde{\chi}_{q,t}(M(P))$ can be expressed as a linear combination (over $\mathbb{Z}[t, t^{-1}]$) of $E_i(m)$ with i -dominant monomials m . Moreover, the image of $\widetilde{\chi}_{q,t}$ is exactly equal to \mathcal{K}_t .*

(3) *Suppose that two I -tuples of polynomials $P^1 = (P_i^1)$, $P^2 = (P_i^2)$ satisfy the following condition:*

$$(2.7) \quad \begin{aligned} & a/b \notin \{q^n \mid n \in \mathbb{Z}, n \geq 2\} \text{ for any pair } a, b \text{ with } P_i^1(1/a) = 0, \\ & P_j^2(1/b) = 0 \text{ (} i, j \in I \text{)}. \end{aligned}$$

Then we have

$$\widetilde{\chi}_{q,t}(M(P^1 P^2)) = \sum_{m^1, m^2} t^{2d(m^1, m_{P^1}; m^2, m_{P^2})} a_{m^1}(t) a_{m^2}(t) m^1 m^2,$$

where $\widetilde{\chi}_{q,t}(M(P^a)) = \sum_{m^a} a_{m^a}(t)m^a$ with $a = 1, 2$.

Moreover, properties (1), (2), and (3) uniquely determine $\widetilde{\chi}_{q,t}$.

Apart from the existence problem, one can consider the above properties (1), (2), and (3) as the definition of $\widetilde{\chi}_{q,t}$ (an axiomatic definition). We only use the above properties, and the reader can safely forget the original definition.

Let us explain briefly why the properties (1), (2), and (3) determine $\widetilde{\chi}_{q,t}$. First consider that the case $M(P)$ is an l -fundamental representation. (We have $M(P) = L(P)$ in this case.) Then one can determine $\widetilde{\chi}_{q,t}(M(P))$ starting from m_P and using property (2) inductively. The graph will never contain l -dominant monomials and will stop eventually. (The idea can be seen in the example below.) For general P , write $P = P^1 P^2 P^3 \dots$, so that each $M(P^\alpha)$ is an l -fundamental representation, and the condition (2.7) is met with respect to the ordering. Then we apply (3) successively to get $\widetilde{\chi}_{q,t}(M(P))$ from $\widetilde{\chi}_{q,t}(M(P^\alpha))$ with $\alpha = 1, 2, \dots$.

We now define $\chi_{q,t}(M(P))$ by

$$(2.8) \quad \chi_{q,t}(M(P)) = \sum_m t^{-d(m, m_P; m, m_P)} a_m(t)m \quad \text{if } \widetilde{\chi}_{q,t}(M(P)) = \sum_m a_m(t)m.$$

We define an involution $\bar{}$ on \mathfrak{Y}_t by $\bar{t} = t^{-1}$, $\overline{Y_{i,a}^\pm} = Y_{i,a}^\pm$. One can show (see [15, §3]) that we can define an involution $\bar{}$ on \mathbf{R}_t by

$$\chi_{q,t}(\overline{V}) = \overline{\chi_{q,t}(V)}.$$

(By the second statement of Theorem 2.6(2), it is enough to check that right-hand side is contained in \mathcal{K}_t , after multiplying a power of t . This can be checked.)

We attach to each standard module $M(P)$ an oriented colored graph as follows. (It is a slight modification of the graph in [5, 5.3].) The vertices are monomials in $\widetilde{\chi}_{q,t}(M_P)$. We draw a colored edge $\xrightarrow{i,a}$ from m_1 to m_2 if $m_2 = m_1 A_{i,a}^{-1}$. We also write the coefficients of the monomials in $\widetilde{\chi}_{q,t}(M(P))$. In fact, edges are determined from monomials on vertices. Now $E_i(m)$ in Theorem 2.6 gives edges with color i starting from m and subsequent monomials. Similarly, we also draw a graph for an irreducible representation $L(P)$.

Let us define matrices with entries in $\mathbb{Z}[t, t^{-1}]$:

$$c_{PQ}(t) \stackrel{\text{def.}}{=} \text{the coefficient of } m_Q \text{ in } \chi_{q,t}(M(P)),$$

$$L_{PQ}(t) \stackrel{\text{def.}}{=} \text{the coefficient of } m_Q \text{ in } \chi_{q,t}(L(P)),$$

$$Z_{PQ}(t) \text{ by } M(P) = \sum_Q Z_{PQ}(t)L(Q).$$

By geometric interpretations of these polynomials [15, §8], all of these have positivity, i.e., they are contained in $\mathbb{Z}_{\geq 0}[t, t^{-1}]$.

We define a new multiplication on \mathbf{R}_t by

$$m^1 * m^2 \stackrel{\text{def.}}{=} t^{\varepsilon(m^1, m^2)} m^1 m^2.$$

This is noncommutative. One can show (see [15, §3], where $*$ is denoted by $\tilde{*}$ there) that there exists a unique multiplication \otimes on \mathbf{R}_t satisfying

$$\chi_{q,t}(V_1 \otimes V_2) = \chi_{q,t}(V_1) * \chi_{q,t}(V_2).$$

When $t = 1$, this corresponds to the tensor product on \mathbf{R} . So there will be no fear of the confusion. Varagnolo and Vasserot define the multiplication in a geometric

way [18]. Their definition implies the positivity of structure constants; i.e., if we define $a_{PQ}^R(t)$ by

$$L(P) \otimes L(Q) = \sum_R a_{PQ}^R(t)L(R),$$

it has nonnegative coefficients $a_{PQ}^R(t) \in \mathbb{Z}_{\geq 0}[t, t^{-1}]$.

We close this section by giving an example of computation of $W_{2,a}^{(1)} = L(P_{2,a}^{(1)})$ for $\mathfrak{g} = A_2$.

Example 2.9. Let \mathfrak{g} be of type A_2 . We put a numbering $I = \{1, 2\}$.

(1) The graph of the l -fundamental representation $\widetilde{\chi}_{q,t}(W_{1,a}^{(1)}) = \widetilde{\chi}_{q,t}(M(P_{1,a}^{(1)}))$ is

$$Y_{1,a} \xrightarrow{1,aq} Y_{1,aq^2}^{-1} Y_{2,aq} \xrightarrow{2,aq^2} Y_{2,aq^3}^{-1}.$$

This can be checked by Theorem 2.6(1),(2). The graph of $\widetilde{\chi}_{q,t}(M(P_{1,aq^2}^{(1)}))$ is exactly the same if we replace a by aq^2 .

(2) By Theorem 2.6(3) and (2.8), we can compute the graph of $\chi_{q,t}(M(P_{2,a}^{(1)}))$ as

$$\begin{array}{ccccc} Y_{1,a} Y_{1,aq^2} & \xrightarrow{1,aq} & t^{-1} Y_{2,aq} & \xrightarrow{2,aq^2} & t^{-1} Y_{1,aq^2} Y_{2,aq^3}^{-1} \\ 1,aq^3 \downarrow & & & & 1,aq^3 \downarrow \\ Y_{1,a} Y_{1,aq^4}^{-1} Y_{2,aq^3} & \xrightarrow{1,aq} & Y_{1,aq^2}^{-1} Y_{1,aq^4}^{-1} Y_{2,aq} Y_{2,aq^3} & \xrightarrow{2,aq^2} & t^{-1} Y_{1,aq^4}^{-1} \\ 2,aq^4 \downarrow & & 2,aq^4 \downarrow & & \\ Y_{1,a} Y_{2,aq^5}^{-1} & \xrightarrow{1,aq} & Y_{1,aq^2}^{-1} Y_{2,aq} Y_{2,aq^5}^{-1} & \xrightarrow{2,aq^2} & Y_{2,aq^3}^{-1} Y_{2,aq^5}^{-1} \end{array}$$

(3) Finally, by applying the characterization (2.1), we find

$$t^{-\varepsilon(P_{1,a}^{(1)}, P_{1,aq^2}^{(1)})} W_{1,a}^{(1)} \otimes W_{1,aq^2}^{(1)} = M(P_{2,a}^{(1)}) = W_{2,a}^{(1)} + t^{-1} L(P_{1,aq}^{(2)}) = W_{2,a}^{(1)} + t^{-1} W_{1,aq}^{(2)}.$$

This is a special case of the T -system. The graph of $\chi_{q,t}(W_{2,a}^{(1)})$ is obtained simply from the above graph by erasing monomials with coefficients t^{-1} , i.e., $Y_{2,aq}$, $Y_{1,aq^2} Y_{2,aq^3}^{-1}$, Y_{1,aq^4}^{-1} . The resulting graph turns out to be given by the following rule without using (2.1): if one gets an l -dominant monomial in the expansion at m , then he/she does not add it to the graph and continues the expansion.

The result of this paper is proved by checking that we have the same structure for general $W_{k,a}^{(i)}$.

3. A STUDY OF RIGHT NEGATIVE MONOMIALS

We only consider monomials m whose factors are of the form Y_{i,aq^s} for a fixed $a \in \mathbb{C}^*$ hereafter.

Definition 3.1. (1) For a monomial $m \neq 1$, we define

$$r(m) = \max \{s \mid \text{there is a factor } Y_{i,aq^s} \text{ appearing in } m \text{ for some } i\}.$$

(2) Following [6], we say a monomial m is *right negative* if the factors $Y_{i,aq^{r(m)}}$ have nonpositive powers for all i . (It must be negative for at least one i by definition of $r(m)$.) The product of right negative monomials is right negative. An l -dominant monomial is not right negative.

In this section, we shall prove the following.

Theorem 3.2. (1) All monomials in $\chi_{q,t}(W_{k,a}^{(i)})$ are right negative except $m_{P_{k,a}^{(i)}}$ corresponding to the l -highest weight vector.

(2) Let m be a right negative monomial appearing in $\chi_{q,t}(W_{k,a}^{(i)})$ such that $r(m) \leq 2k$. Then it is

$$\begin{aligned}
 m &= \prod_{t=0}^{s-1} Y_{i,aq^{2t}} \cdot \prod_{t=s+1}^k \left(Y_{i,aq^{2t}}^{-1} \prod_{j:a_{ij}=-1} Y_{j,aq^{2t-1}} \right) \\
 &= Y_{i,a} \cdots Y_{i,aq^{2s-2}} Y_{i,aq^{2s+2}}^{-1} \cdots Y_{i,aq^{2k}}^{-1} \prod_{j:a_{ij}=-1} Y_{j,aq^{2s+1}} \cdots Y_{j,aq^{2k-1}}
 \end{aligned}$$

for some $s = 0, 1, \dots, k - 1$. Moreover, its coefficient is equal to 1.

(3) We have

$$t^{-\varepsilon(P_{k,a}^{(i)}, P_{1,aq^{2k}}^{(i)})} W_{k,a}^{(i)} \otimes W_{1,aq^{2k}}^{(i)} = W_{k+1,a}^{(i)} + t^{-1} L(P_{k-1,a}^{(i)} \prod_{j:a_{ij}=-1} P_{1,aq^{2k-1}}^{(j)})$$

for $k = 1, 2, 3, \dots$

Remark 3.3. (1) We do not have

$$L(P_{k-1,a}^{(i)} \prod_{j:a_{ij}=-1} P_{1,aq^{2k-1}}^{(j)}) = W_{k-1,a}^{(i)} \otimes \bigotimes_{j:a_{ij}=-1} W_{1,aq^{2k-1}}^{(j)}$$

in general. A counterexample can be found, e.g., $\mathfrak{g} = A_2$, $k = 2$. Therefore, Theorem 3.2(3) does not give us an inductive formula for $\chi_{q,t}(W_{k,a}^{(i)})$ unlike the T -system in Theorem 1.1.

(2) It may not be easy for a reader to understand the importance of the statements of Theorem 3.2(1),(2) at first sight. But they are very useful to study tensor product decompositions of $W_{k,a}^{(i)}$'s, as shown in the proof of Theorems 1.1, 3.2. As another application of Theorem 3.2(1), we can apply Frenkel and Mukhin's algorithm [6, §5.5] to compute $\chi_q(W_{k,a}^{(i)})$. (See also [15] for a geometric explanation.) This is because $\chi_{q,t}(W_{k,a}^{(i)})$ does not have l -dominant monomials other than $m_{P_{k,a}^{(i)}}$. This gives us an important application later. See Figure 1 for a part of the graph for $\chi_{q,t}(W_{k+1,a}^{(i)})$.

We shall prove the assertions (1) and (2) simultaneously by induction on k . The statement (3) will be proved during the induction argument, i.e., we do not need (3) for k during the proof. The assertion (1) is true for $k = 1$. (See [6, Lemma 6.5] or [15, Proposition 4.13] for a simple geometric proof.) The assertion (2) for $k = 1$ follows from Theorem 2.6(2). See also the proof of Theorem 3.2(2) below.

Let us assume the assertions (1) and (2) for k and prove them for $k + 1$. We consider

$$t^{-\varepsilon(P_{k,a}^{(i)}, P_{1,aq^{2k}}^{(i)})} \chi_{q,t}(W_{k,a}^{(i)} \otimes W_{1,aq^{2k}}^{(i)}) = Y_{i,a} Y_{i,aq^2} \cdots Y_{i,aq^{2k-2}} \cdot Y_{i,aq^{2k}} + \cdots$$

Lemma 3.4. Take monomials m and m' from $\chi_{q,t}(W_{k,a}^{(i)})$ and $\chi_{q,t}(W_{1,aq^{2k}}^{(i)})$ respectively such that mm' is not right negative.

(1) We have

$$m = Y_{i,a} \cdots Y_{i,aq^{2k-2}} \quad \text{or} \quad \prod_{t=0}^{s-1} Y_{i,aq^{2t}} \cdot \prod_{t=s+1}^k \left(Y_{i,aq^{2t}}^{-1} \prod_{j:a_{ij}=-1} Y_{j,aq^{2t-1}} \right),$$

$$m' = Y_{i,aq^{2k}}$$

for some $s = 0, 1, \dots, k-1$.

(2) The coefficient of mm' in $t^{-\varepsilon(P_{k,a}^{(i)}, P_{1,aq^{2k}}^{(i)})} \chi_{q,t}(W_{k,a}^{(i)} \otimes W_{1,aq^{2k}}^{(i)})$ is equal to 1 if $m = Y_{i,a} \cdots Y_{i,aq^{2k-2}}$ and t^{-1} otherwise.

Proof. Since mm' is not right negative, m or m' is not right negative. Suppose that m is not right negative. The induction hypothesis (1) for k implies $m = Y_{i,a} \cdots Y_{i,aq^{2k-2}}$. Moreover, if $m' \neq Y_{i,aq^{2k}}$, then m' is right negative and $r(m') \geq 2k+2$ by Theorem 2.6(2). Therefore, mm' becomes right negative, and we have a contradiction to the assumption. Thus we have $m' = Y_{i,aq^{2k}}$ in this case.

Next, suppose m' is not right negative. Then we have $m' = Y_{i,aq^{2k}}$. We may assume that m is right negative. Since mm' is not right negative, the factor $Y_{j,aq^{r(m)}}$ must be equal to $m'^{-1} = Y_{i,aq^{2k}}^{-1}$. By the induction hypothesis (2) for k , the monomial m must be of the form of the second case of the assertion of the lemma. Thus we have completed the proof of (1) of the lemma.

Let us compute the coefficients of mm' in $t^{-\varepsilon(P_{k,a}^{(i)}, P_{1,aq^{2k}}^{(i)})} \chi_{q,t}(W_{k,a}^{(i)} \otimes W_{1,aq^{2k}}^{(i)})$ when m, m' are as in the claim. If $mm' = Y_{i,a} \cdots Y_{i,aq^{2k-2}} Y_{i,aq^{2k}}$, then the coefficient is 1. So we assume m is the second case of the assertion of the claim. By the induction hypothesis (2) for k , the coefficient of m in $\chi_{q,t}(W_{k,a}^{(i)})$ is 1. The coefficient of m' in $\chi_{q,t}(W_{1,aq^{2k}}^{(i)})$ is also 1 by Theorem 2.6(2). We have

$$d(m, Y_{i,aq^{2k}}) = v_{i,aq^{2k+1}}(m, Y_{i,a} \cdots Y_{i,aq^{2k-2}}) = 0,$$

$$d(Y_{i,aq^{2k}}, m) = v_{i,aq^{2k-1}}(m, Y_{i,a} \cdots Y_{i,aq^{2k-2}}) = 1.$$

Therefore, the coefficient of mm' is equal to t^{-1} , where we have used (2.5). \square

Proof of Theorem 3.2(3). Let us define $a_P(t)$ by

$$t^{-\varepsilon(P_{k,a}^{(i)}, P_{1,aq^{2k}}^{(i)})} \chi_{q,t}(W_{k,a}^{(i)} \otimes W_{1,aq^{2k}}^{(i)}) = \sum_P a_P(t) \chi_{q,t}(L(P)),$$

where P runs all Drinfeld polynomials. Consider the coefficient of an l -dominant monomial m_Q in both sides. Since an l -dominant monomial is not right negative, Lemma 3.4 implies

$$m_Q = Y_{i,a} \cdots Y_{i,aq^{2k-2}} Y_{i,aq^{2k}} = m_{P_{k+1,a}^{(i)}}, \quad \text{or}$$

$$m_Q = Y_{i,a} \cdots Y_{i,aq^{2k-4}} \prod_{j:a_{ij}=-1} Y_{j,aq^{2k-1}} = m_{P_{k-1,a}^{(i)}} \prod_{j:a_{ij}=-1} m_{P_{1,aq^{2k-1}}^{(j)}}.$$

The latter is the case $s = k-1$ of Lemma 3.4(1). Note that the corresponding m_Q is not l -dominant for $s \neq k-1$. The coefficient of m_Q is 1 or t^{-1} respectively. On the other hand, from the right-hand side, it is equal to

$$\sum_P a_P(t) L_{PQ}(t).$$

Remember that we have $a_P(t), L_{PQ}(t) \in \mathbb{Z}_{\geq 0}[t, t^{-1}]$ and $L_{PQ}(t^{-1}) = L_{PQ}(t)$, $L_{PP}(t) = 1$. Therefore, if Q is not as above, we have $a_Q(t) = 0$. And if Q is as above, we have $a_Q(t) = 1$ or t^{-1} respectively. This shows Theorem 3.2(3). \square

Proof of Theorem 3.2(1). Take $\chi_{q,t}$ of both sides of Theorem 3.2(3). We have

$$\begin{aligned} &\chi_{q,t}(L(P_{k-1,a}^{(i)} \prod_{j:a_{ij}=-1} P_{1,aq^{2k-1}}^{(j)})) \\ &= Y_{i,a} \cdots Y_{i,aq^{2k-4}} \prod_{j:a_{ij}=-1} Y_{j,aq^{2k-1}} \\ &\quad + Y_{i,a} \cdots Y_{i,aq^{2k-6}} Y_{i,aq^{2k-2}}^{-1} \prod_{j:a_{ij}=-1} Y_{j,aq^{2k-3}} Y_{j,aq^{2k-1}} \\ &\quad + Y_{i,a} \cdots Y_{i,aq^{2k-8}} Y_{i,aq^{2k-4}}^{-1} Y_{i,aq^{2k-2}}^{-1} \prod_{j:a_{ij}=-1} Y_{j,aq^{2k-5}} Y_{j,aq^{2k-3}} Y_{j,aq^{2k-1}} + \cdots, \end{aligned}$$

where we have used the expansion (Theorem 2.6(2)). Therefore, any monomial m such that

- (1) it is not right negative,
- (2) it appears in $\chi_{q,t}$ of the left-hand side of Theorem 3.2(3),

which we just classified in Lemma 3.4, appears above, except the l -highest weight monomial. Thus $\chi_{q,t}(W_{k+1,a}^{(i)})$ does not contain monomials, which are not right negative, other than $m_{P_{k+1,a}^{(i)}}$. \square

Proof of Theorem 3.2(2). We may assume Theorem 3.2(1) for $k+1$ now. Therefore, we can apply Frenkel and Mukhin’s algorithm to compute $\chi_{q,t}(W_{k+1,a}^{(i)})$. But we will use a self-contained argument, using only the induction hypothesis. We want to classify all right negative monomials m with $r(m) \leq 2k + 2$.

Let us consider the graph Γ of $\chi_{q,t}(W_{k+1,a}^{(i)})$. Since m is not l -dominant, it must have an incoming arrow $m' \rightarrow m$. Noticing that if m' is also right negative (i.e., it is not the l -highest weight monomial), then we have $r(m') \leq r(m) \leq 2k + 2$. We can repeat this procedure until $m^{(N)} = (\cdots((m')')\cdots)'$ (N -times) becomes the l -highest weight monomial $Y_{i,a} \cdots Y_{i,aq^{2k}}$. So we study the graph Γ from $Y_{i,a} \cdots Y_{i,aq^{2k}}$ and forget monomials m with $r(m) > 2k + 2$. The first part of the graph Γ is

$$m^{(N)} = Y_{i,a} \cdots Y_{i,aq^{2k}} \xrightarrow{i,aq^{2k+1}} Y_{i,a} \cdots Y_{i,aq^{2k-2}} Y_{i,aq^{2k+2}}^{-1} \prod_{j:a_{ij}=-1} Y_{j,aq^{2k+1}}.$$

There are no other arrows from $Y_{i,a} \cdots Y_{i,aq^{2k}}$; other candidates, which are of the form

$$Y_{i,a} \cdots Y_{i,aq^{2s-2}} Y_{i,aq^{2s+4}} \cdots Y_{i,aq^{2k}} \prod_{j:a_{ij}=-1} Y_{j,aq^{2s+1}}$$

$$\begin{array}{c}
Y_{i,a} \cdots Y_{i,aq^{2k}} \\
\downarrow i, aq^{2k+1} \\
Y_{i,a} \cdots Y_{i,aq^{2k-2}} Y_{i,aq^{2k+2}}^{-1} \prod_j Y_{j,aq^{2k+1}} \xrightarrow{j, aq^{2k+2}} \\
\downarrow i, aq^{2k-1} \\
\vdots \\
\downarrow i, aq^3 \\
Y_{i,a} Y_{i,aq^4}^{-1} \cdots Y_{i,aq^{2k+2}}^{-1} \prod_j Y_{j,aq^3} \cdots Y_{j,aq^{2k+1}} \xrightarrow{j, aq^{2k+2}} \\
\downarrow i, aq \\
Y_{i,aq^2}^{-1} \cdots Y_{i,aq^{2k+2}}^{-1} \prod_j Y_{j,aq} \cdots Y_{j,aq^{2k+1}} \xrightarrow{j, aq^{2k+2}}
\end{array}$$

FIGURE 1. A part of the graph for $\chi_{q,t}(W_{k+1,a}^{(i)})$

are l -dominant, and they do not appear in $\chi_{q,t}(W_{k+1,a}^{(i)})$. Therefore, $m^{(N-1)}$ is the second monomial in the graph. We continue the graph:

$$\begin{array}{c}
m^{(N-1)} = Y_{i,a} \cdots Y_{i,aq^{2k-2}} Y_{i,aq^{2k+2}}^{-1} \prod_{j:a_{ij}=-1} Y_{j,aq^{2k+1}} \xrightarrow{j, aq^{2k+2}} \\
\downarrow i, aq^{2k-1} \\
Y_{i,a} \cdots Y_{i,aq^{2k-4}} Y_{i,aq^{2k}}^{-1} Y_{i,aq^{2k+2}}^{-1} \prod_{j:a_{ij}=-1} Y_{j,aq^{2k-1}} Y_{j,aq^{2k+1}}
\end{array}$$

We do not consider the incoming monomial of $\xrightarrow{j, aq^{2k+2}}$ since its largest exponent l is $2k+3$. There are no other outgoing arrows from $m^{(N-1)}$, i.e., $\xrightarrow{i, aq^{2s-1}}$ with $s=1, 2, \dots, k-1$. Such an arrow goes to a monomial, which leads to a contradiction with the induction hypothesis (2) for k , as one can easily check by considering $W_{k,a}^{(i)} \otimes W_{1,aq^{2k}}^{(i)}$. We continue the graph to get all

$$Y_{i,a} \cdots Y_{i,aq^{2s-2}} Y_{i,aq^{2s+2}}^{-1} \cdots Y_{i,aq^{2k+2}}^{-1} \prod_{j:a_{ij}=-1} Y_{j,aq^{2s+1}} \cdots Y_{j,aq^{2k+1}}$$

for $s=k, k-1, \dots, 0$. We cannot have an outgoing arrow $\xrightarrow{j, aq^{2t}}$ with $t=s, \dots, k$, again by a study of $\chi_{q,t}(W_{k,a}^{(i)} \otimes W_{1,aq^{2k}}^{(i)})$. We do not consider the incoming monomial of $\xrightarrow{j, aq^{2k+2}}$ as above. The last monomial is

$$Y_{i,aq^2}^{-1} \cdots Y_{i,aq^{2k+2}}^{-1} \prod_{j:a_{ij}=-1} Y_{j,aq} \cdots Y_{j,aq^{2k+1}}.$$

Then an outgoing arrow goes to a monomial such that its maximal exponent is greater than $2k+2$. Thus we obtain all the monomials satisfying the condition,

and get Theorem 3.2(2) for $k + 1$. (See Figure 1 for the part of the graph for $\chi_{q,t}(W_{k,a}^{(i)})$.) \square

Proof of Theorem 1.1(2). We have already computed a part of $\chi_{q,t}(W_{k+1,a}^{(i)})$ as Figure 1. The horizontal arrows, not written here, are multiplications by $A_{j,b}$'s for $(j,b) \neq (i, aq^{2s-1})$ ($s = 1, \dots, k + 1$). Compare with the graph for $\chi_{q,t}(W_{k,aq^2}^{(i)})$. If we multiply $Y_{i,a}$, the graph is exactly the same as the above graph except for the last line. This is because we do not touch the factor $Y_{i,a}$ until the last line. Therefore, we have

$$\chi_{q,t}(W_{k+1,a}^{(i)}) = Y_{i,a} \cdot \chi_{q,t}(W_{k,aq^2}^{(i)}) + Y_{i,a} \cdots Y_{i,aq^{2k}} \cdot E,$$

where E is a polynomial in $A_{i,b}^{-1}$ of degree greater than k . Here we assign $A_{i,b}^{-1}$ degree 1 for any i, b . This implies the convergence statement of Theorem 1.1(2). \square

4. PROOF OF THEOREM 1.1(1)

In this section, we shall complete the proof of Theorem 1.1. Our first goal is to show that $W_{k+1,a}^{(i)} \otimes W_{k-1,aq^2}^{(i)}$ is irreducible. We need the following.

Lemma 4.1. *Let $P = P_{k,a}^{(i)} P_{k,aq^2}^{(i)} = P_{k+1,a}^{(i)} P_{k-1,aq^2}^{(i)}$. Then $\chi_{q,t}(L(P))$ contains all monomials*

$$\begin{aligned} & \prod_{t=0}^{s-1} Y_{i,aq^{2t}} Y_{i,aq^{2t+2}} \cdot \prod_{t=s}^{k-1} Y_{j,aq^{2t+1}} \\ & = Y_{i,a} Y_{i,aq^2}^2 \cdots Y_{i,aq^{2s-2}}^2 Y_{i,aq^{2s}} \prod_{j:a_{ij}=-1} Y_{j,aq^{2s+1}} \cdots Y_{j,aq^{2k-1}} \end{aligned} \quad (s = 1, 2, \dots, k - 1)$$

with coefficients 1.

Proof. Let us consider the standard module $M(P)$ with Drinfeld polynomial P . We consider the first few terms of $\chi_{q,t}(M(P))$:

$$\begin{aligned} \chi_{q,t}(M(P)) &= Y_{i,a} Y_{i,aq^2}^2 \cdots Y_{i,aq^{2k-2}}^2 Y_{i,aq^{2k}} \\ &+ \sum_{s=1}^{k-1} (1+t^2)^{k-s} t^{2(s-k)} Y_{i,a} Y_{i,aq^2}^2 \cdots Y_{i,aq^{2s-2}}^2 Y_{i,aq^{2s}} \prod_{j:a_{ij}=-1} Y_{j,aq^{2s+1}} \cdots Y_{j,aq^{2k-1}} \\ &+ \cdots \end{aligned}$$

Note that $(1+t^2)^{k-s} t^{2(s-k)} \in \mathbb{Z}[t^{-1}]$ and the constant term is equal to 1. Then the characterization (2.1) gives us

$$\begin{aligned} \chi_{q,t}(L(P)) &= Y_{i,a} Y_{i,aq^2}^2 \cdots Y_{i,aq^{2k-2}}^2 Y_{i,aq^{2k}} \\ &+ \sum_{s=1}^{k-1} Y_{i,a} Y_{i,aq^2}^2 \cdots Y_{i,aq^{2s-2}}^2 Y_{i,aq^{2s}} \prod_{j:a_{ij}=-1} Y_{j,aq^{2s+1}} \cdots Y_{j,aq^{2k-1}} \\ &+ \cdots \end{aligned}$$

The procedure to define $L(P)$ from $M(P)$ is recursive. So we do not need to consider the latter part. \square

Remark 4.2. During the proof, we encountered a simplification of the characterization (2.1), i.e., simply taking constant terms. This is exactly the phenomena appearing when we study the decomposition theorem for a *semi-small* morphism $\pi: X \rightarrow Y$. (See [14, §7], [15, 9.3] for similar arguments.)

It seems natural to expect that $M(P)$ is *semi-small* in the sense of [15, 10.1], i.e., $c_{QR}(t) \in \mathbb{Z}[t^{-1}]$ for all $Q, R \leq P$, although we have checked the assertion for only the first few terms.

Proof of the irreducibility of $W_{k+1,a}^{(i)} \otimes W_{k-1,aq^2}^{(i)}$. We consider

$$t^{-\varepsilon(P_{k+1,a}^{(i)}, P_{k-1,aq^2}^{(i)})} \chi_{q,t}(W_{k+1,a}^{(i)} \otimes W_{k-1,aq^2}^{(i)}).$$

Our first task is to classify all l -dominant monomials appearing in it. Take monomials m and m' from $\chi_{q,t}(W_{k+1,a}^{(i)})$ and $\chi_{q,t}(W_{k-1,aq^2}^{(i)})$ respectively such that mm' is not right negative. If m is right negative, then m' is not right negative, hence we have $m' = Y_{i,aq^2} \cdots Y_{i,aq^{2k-2}}$ by Theorem 3.2(1). But then mm' must be right negative as we can see from Theorem 3.2(2), applied to m . Therefore, m is not right negative and we have $m = Y_{i,a} \cdots Y_{i,aq^{2k}}$ by Theorem 3.2(1). Since mm' is not right negative, the factor $Y_{j,aq^{r(m')}}$ must be one of $Y_{i,a}^{-1}, \dots, Y_{i,aq^{2k}}^{-1}$. (In fact, $Y_{i,a}^{-1}$ is impossible.) By Theorem 3.2(2) we have

$$\begin{aligned} mm' &= Y_{i,a} \cdots Y_{i,aq^{2k}} \cdot Y_{i,aq^2} \cdots Y_{i,aq^{2s-2}} Y_{i,aq^{2s+2}}^{-1} \\ &\quad \cdots Y_{i,aq^{2k}}^{-1} \prod_{j:a_{ij}=-1} Y_{j,aq^{2s+1}} \cdots Y_{j,aq^{2k-1}} \\ &= Y_{i,a} Y_{i,aq^2}^2 \cdots Y_{i,aq^{2s-2}}^2 Y_{i,aq^{2s}} \prod_{j:a_{ij}=-1} Y_{j,aq^{2s+1}} \cdots Y_{j,aq^{2k-1}} \end{aligned}$$

for some $s = 1, 2, \dots, k - 1$. This is the classification. Again by Theorem 3.2(2), the coefficient of the above m' in $\chi_{q,t}(W_{k-1,aq^2}^{(i)})$ is 1. A direct computation shows

$$d(m, m_{P_{k+1,a}^{(i)}}; m', m_{P_{k-1,aq^2}^{(i)}}) = d(m', m_{P_{k-1,aq^2}^{(i)}}; m, m_{P_{k+1,a}^{(i)}}) = 0.$$

Therefore, the coefficient of mm' in $t^{-\varepsilon(P_{k+1,a}^{(i)}, P_{k-1,aq^2}^{(i)})} \chi_{q,t}(W_{k+1,a}^{(i)} \otimes W_{k-1,aq^2}^{(i)})$ is also 1.

Let us define $a_Q(t)$ by

$$(4.3) \quad \chi_{q,t}(W_{k+1,a}^{(i)} \otimes W_{k-1,aq^2}^{(i)}) = \sum_Q a_Q(t) \chi_{q,t}(L(Q)),$$

where Q runs all Drinfeld polynomials. We have $a_P(t) = 1$. Let us consider $Q \neq P$. If m_Q is not one of the above classified monomials, then the coefficient in the left-hand side is 0. Since $L_{QQ}(t) = 1$, the positivity of coefficients implies $a_Q(t) = 0$ for such Q . Next, we consider m_Q as above. We have

$$1 = \sum_{Q'} a_{Q'}(t) L_{Q'Q}(t).$$

But Lemma 4.1 means $L_{PQ}(t) = 1$. Therefore, the positivity of $L_{Q'Q}(t)$ implies that $a_{Q'} L_{Q'Q}(t) = 0$ for $Q' \neq P$. Setting $Q' = Q$, we get $a_Q(t) = 0$ since $L_{QQ}(t) = 1$. This means that the right-hand side of (4.3) is a single term $L(P)$; that is, $W_{k+1,a}^{(i)} \otimes W_{k-1,aq^2}^{(i)}$ is irreducible. \square

Remark 4.4. Chari gave a sufficient condition for the irreducibility of the tensor product of Kirillov-Reshetikhin modules [1]. But the above case is not covered by her result.

Completion of the proof of Theorem 1.1. First of all, it is easy to show the irreducibility of the tensor product $\bigotimes_{j:a_{ij}=-1} W_{k,aq}^{(j)}$: Using Theorem 3.2(1),(2), one can prove that its $\chi_{q,t}$ has no l -dominant monomials other than the l -highest weight monomial. This case can be also deduced from Chari’s sufficient condition [1].

The rest of the proof goes along a similar line as the above proof of the irreducibility of $W_{k+1,a}^{(i)} \otimes W_{k-1,aq^2}^{(i)}$. Let us consider $t^{-\varepsilon(P_{k,a}^{(i)}, P_{k,aq^2}^{(i)})} \chi_{q,t}(W_{k,a}^{(i)} \otimes W_{k,aq^2}^{(i)})$. Take monomials m and m' from $\chi_{q,t}(W_{k,a}^{(i)})$ and $\chi_{q,t}(W_{k,aq^2}^{(i)})$ respectively such that mm' is not right negative. By an argument as above, we have

$$mm' = Y_{i,a} Y_{i,aq^2}^2 \cdots Y_{i,aq^{2s-2}}^2 Y_{i,aq^{2s}} \prod_{j:a_{ij}=-1} Y_{j,aq^{2s+1}} \cdots Y_{j,aq^{2k-1}}$$

for some $s = 0, \dots, k - 1$. (We have $m' = Y_{i,aq^2} \cdots Y_{i,aq^{2k}}$.) Note that we have one extra monomial when $s = 0$, i.e.,

$$\prod_{j:a_{ij}=-1} Y_{j,aq} \cdots Y_{j,aq^{2k-1}}.$$

We have

$$\begin{aligned} & d(m, m_{P_{k,a}^{(i)}}; m', m_{P_{k,aq^2}^{(i)}}) - d(m', m_{P_{k,aq^2}^{(i)}}; m, m_{P_{k,a}^{(i)}}) \\ &= v_{i,aq^{2k+1}}(m, m_{P_{k,a}^{(i)}}) - v_{i,aq}(m, m_{P_{k,a}^{(i)}}) = \begin{cases} -1 & \text{if } s = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, the coefficient of mm' in $t^{-\varepsilon(P_{k,a}^{(i)}, P_{k,aq^2}^{(i)})} \chi_{q,t}(W_{k,a}^{(i)} \otimes W_{k,aq^2}^{(i)})$ is t^{-1} ($s = 0$) or 1 ($s \neq 0$). Considering the decomposition of $\chi_{q,t}(W_{k,a}^{(i)} \otimes W_{k,aq^2}^{(i)})$ as above, we get

$$\begin{aligned} & t^{-\varepsilon(P_{k,a}^{(i)}, P_{k,aq^2}^{(i)})} \chi_{q,t}(W_{k,a}^{(i)} \otimes W_{k,aq^2}^{(i)}) \\ &= t^{-\varepsilon(P_{k+1,a}^{(i)}, P_{k-1,aq^2}^{(i)})} \chi_{q,t}(W_{k+1,a}^{(i)} \otimes W_{k-1,aq^2}^{(i)}) + t^{-1-N} \chi_{q,t}\left(\bigotimes_{j:a_{ij}=-1} W_{k,aq}^{(j)}\right), \end{aligned}$$

where $N = \sum_{a < b} \varepsilon(P_{k,aq}^{(j_a)}, P_{k,aq}^{(j_b)})$. Here j_1, j_2, \dots is the ordering used for the definition of the tensor product. If we set $t = 1$, this is simply the T -system.

By [1], $W_{k,a}^{(i)} \otimes W_{k,aq^2}^{(i)}$ is cyclic: it is generated by the tensor product of l -highest weight vectors. Its simple quotient must be isomorphic to $W_{k+1,a}^{(i)} \otimes W_{k-1,aq^2}^{(i)}$ since they have the same Drinfeld polynomials. This shows the existence of the exact sequence. □

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DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

E-mail address: nakajima@kum.kyoto-u.ac.jp

URL: <http://www.kum.kyoto-u.ac.jp/~nakajima>